Computing Domains of Attraction for Planar Dynamics

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Abstract. In this note we investigate the problem of computing the domain of attraction of a flow on $\mathbb{R}^2$ for a given attractor. We consider an operator that takes two inputs, the description of the flow and a cover of the attractors, and outputs the domain of attraction for the given attractor. We show that: (i) if we consider only (structurally) stable systems, the operator is (strictly semi-)computable; (ii) if we allow all systems defined by C^1-functions, the operator is not (semi-)computable. We also address the problem of computing limit cycles on these systems.

Many problems about dynamical systems (DSs) are concerned with their long term behavior. For example, given some trajectory, where will it end up? Which are the invariant sets of a DS? Which are its attractors?

Recently, with the advent of increasingly powerful digital computers, numerous new ideas and concepts related to these question have appeared (e.g. sensitive dependence on initial conditions, chaos, strange attractors, Mandelbrot set). However it is interesting to note that the computer, while being an invaluable tool to get some intuition about a DS, is rarely used to prove results. Usually the formal analysis of DSs is done analytically (but often relies on information provided by numerical simulations), using heavy mathematics with little reliance on the computer. A notable exception is the proof that the Lorenz strange attractor exists and is robust under small perturbations [1], [2].

One of the reasons for this phenomenon is that computers introduce truncation errors which, in conjunction with other properties such as sensitive dependency on initial conditions, is likely to produce simulated trajectories that cannot be proved accurate. Of course, there are many results exhibiting that these simulations are valuable; the foremost of such results is perhaps the Shadowing Lemma [3], [4]. However the accuracy of a particular simulation, especially when we are interested in global properties, can usually be put into question.

In this paper we deal with a particular type of the problems mentioned above: is it possible to conceive a computer program that, given an input that describes a dynamical system (DS) as well as an attractor of this DS, computes (rigourously) the basin of attraction of the given attractor?

Since there are many open questions about general classes of dynamical system, we restrict ourselves to a well-studied case, the continuous-time DS defined on the plane $\mathbb{R}^2$ by

$$x' = f(x), \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and t is the independent variable.

Some techniques introduced on this paper are based on [5]. They are essentially adaption and enhancements, that allow to correct some results of [5].

1 Differential equations

Here we give a summary of results concerning the ODE (1). For more details, the reader is referred to [6], [7], [8].

Definition 1. *Let $\phi(t, x_0)$ denote the solution of (1) corresponding to the initial condition $x(0) = x_0$. The function $\phi(\cdot, x_0) : \mathbb{R}^2 \to \mathbb{R}^2$ is called a* solution curve*,* trajectory*, or* orbit *of (1) through the point x_0.*

1. *A point y is called an* equilibrium point *of (1) if $f(y) = 0$. An equilibrium point y_0 is called* hyperbolic *if none of the eigenvalues of the gradient matrix $Df(y_0)$ of f at y_0 has zero real part.*
2. *An equilibrium point x_0 of (1) is called* stable *if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\phi(t, \tilde{x}) - x_0| < \varepsilon$ for all $t \geq 0$, provided $|\tilde{x} - x_0| < \delta$. Furthermore, x_0 is called* asymptotically stable *if it is stable and there exists a (fixed) $\delta_0 > 0$ such that $\lim_{t\to\infty} \phi(t, \tilde{x}) = x_0$ for all $\tilde{x}$ satisfying $|\tilde{x} - x_0| < \delta_0$.*

Given a trajectory $\Gamma_{x_0} = \phi(\cdot, x_0)$, we define the positive half-trajectory as $\Gamma^+_{x_0} = \{\phi(t, x_0)|t \geq 0\}$. When the context is clear, we often drop the subscript x_0 and write simply Γ and Γ^+.

It is not difficult to see that if x_0 is an equilibrium point of (1), then $\phi(t, x_0) = x_0$ for all $t \geq 0$. It is also known that if all eigenvalues of $Df(x_0)$ of an hyperbolic equilibrium point x_0 are negative, then x_0 is asymptotically stable; in this case x_0 is called a *sink*.

While many results about the long term dynamics of (1) focus on fixed points, especially hyperbolic ones, since this is the easiest case to tackle, fixed points are not the sole objects to which trajectories converge as we now will see.

Definition 2. 1. *A point $p \in \mathbb{R}^n$ is an ω-*limit point *of the trajectory $\phi(\cdot, x)$ of the system (1) if there is a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \phi(t_n, x) = p$.*

Definition 3. *The set of all ω-limit points of the trajectory Γ is called the ω-*limit set *of Γ; written as $\omega(\Gamma)$ or $\omega(x)$ if $\Gamma = \phi(\cdot, x)$.*

Definition 4. *A* cycle *or* periodic orbit *of (1) is any closed solution curve of (1) which is not an equilibrium point. A cycle Υ is stable if for each $\varepsilon > 0$ there is a neighborhood U of Υ such that for all $x \in U$, $d(\Gamma^+_x, \Upsilon) < \varepsilon$. A cycle Υ is asymptotically stable (we also say that Υ is a* limit cycle*) if for all points x_0 in some neighborhood U of Υ one has $\lim_{t\to\infty} d(\phi(t, x_0), \Upsilon) = 0$.*

We can define hyperbolic limit cycles in terms of characteristics exponents (see [8, Section 3.5]), in a way similar to Definition 1. However, to avoid technical issues that are not relevant to our discussion, we simply state one of their properties (see [8, Section 3.5]): in a suitable neighborhood, the convergence rate to the hyperbolic limit cycle is exponential ($\sim e^{-\lambda t}$, for $\lambda > 0$).

Definition 5. *A set $A \subseteq \mathbb{R}^2$ is* invariant *if $\phi(t,x) \in A$ for all $t \in [0,+\infty)$ and $x \in A$. If A is a closed invariant set, its* domain of attraction *(or basin of attraction) is the set*

$$\{x \in \mathbb{R}^2 : \omega(x) \subseteq A\}.$$

Domains of attraction are separated by curves in $\mathbb{R}^2$ (stable manifolds - see [7, p. 34]) and therefore are open sets.

The following result can be found in e.g. [8].

Proposition 1 (Poincaré-Bendixson). *Suppose that $f \in C^1(E)$, where $E \subseteq \mathbb{R}^n$ is open and (1) has a trajectory Γ such that Γ^+ is contained in a compact $F \subseteq E$. Then if $\omega(\Gamma)$ contains no singularity of (1), $\omega(\Gamma)$ is a periodic orbit of (1).*

For this paper it will be of special relevance to consider structurally stable DSs. A dynamical system defined by (1) is structurally stable if for any vector field g close to f, the vector fields f and g are topologically equivalent (see [6], [8] for further details). In practice this means that if we perturb the system (1) by a small amount, the resulting DS will still be close to the one defined by (1). According to Peixoto's theorem (see e.g. [8]) if $M \subseteq \mathbb{R}^2$ is a compact, then the set of structurally stable vector fields in $C^1(M)$ is an open, dense subset of $C^1(M)$. It can be proved (see e.g. [8]) that the only limit sets for a structurally stable DS defined on a compact $M \subseteq \mathbb{R}^2$ are hyperbolic equilibrium points and hyperbolic limit cycles, and that these appear in a finite number.

We now need estimates on the error committed on the computation of a trajectory when the system is perturbed. Let x be a solution of the equation (1). Let y be a solution of the ODE

$$y' = g(y).$$

The following result is classical and can be found in e.g. [6]

Lemma 1. *In the above conditions, and supposing that x and y are defined on a region $\mathcal{D}$ where f and g satisfy a Lipschitz condition, with Lipschitz constant L, then on $\mathcal{D}$*

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| \, e^{Lt} + \frac{\varepsilon}{L}(e^{Lt} - 1),$$

provided $\|f - g\| \leq \varepsilon$ there.

We also set up the following notation. A trajectory $\phi : \mathbb{R}_0^+ \to \mathbb{R}^2$ is a solution of the differential inclusion

$$x' \in f_\varepsilon(x),$$

where f_ε is a set valued function defined by $f_\varepsilon(x) = B(f(x), \varepsilon)$ (the ball of center $f(x)$ and radius ε) if $\phi'(t) \in f_\varepsilon(\phi(t))$ almost everywhere. See [9], [10] for further details on differential inclusions.

2 Computable analysis

This section introduces concepts and results from computable analysis. Although computable analysis can be adapted to other topological spaces, here we restrict it to $\mathbb{R}^n$, which is our case of interest. For more details the reader is referred to [11], [12], [13]. The idea underlying computable analysis to compute over a set A is to encode each element a of A into a countable sequence of symbols, called ρ-name. Each sequence must encode at most an element of A. From this point of view, we can forget the set A, and work only over sequences of symbols. Usually each sequence should converge to an element a: the more elements we have from a sequence encoding a, the more precisely we can pinpoint a. To compute with the sequences, we use Type-2 machines, which are similar to Turing machines, but (i) have a read-only tape, where the input (i.e. the sequence encoding it) is written; (ii) have a write-only output tape, where the head cannot move back and the sequence encoding the output is written.

At any finite amount of time we can halt the computation, and we will have a partial result on the output tape. The more time we wait, the more accurate this result will be. We now introduce notions of computability over $\mathbb{R}^n$.

Definition 6. *1. A sequence $\{r_n\}$ of rational numbers is called a ρ-name of a real number x if there are three functions a, b and c from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $r_n = (-1)^{a(n)} \frac{b(n)}{c(n)+1}$ and*

$$|r_n - x| \leq \frac{1}{2^n}. \tag{2}$$

2. A double sequence $\{r_{n,k}\}_{n,k\in\mathbb{N}}$ of rational numbers is called a ρ-name for a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers if there are three computable functions a, b, c from $\mathbb{N}^2$ to $\mathbb{N}$ such that, for all $k, n \in \mathbb{N}$, $r_{n,k} = (-1)^{a(k,n)} \frac{b(k,n)}{c(k,n)+1}$ and

$$|r_{n,k} - x_n| \leq \frac{1}{2^k}.$$

3. A real number x (a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers) is called computable if it has a computable ρ-name, i.e. there is a Type-2 machine that computes the ρ-name without any input.

The notion of the ρ-name can be extended to points in $\mathbb{R}^l$ as follows: a sequence $\{(r_{1n}, r_{2n}, \ldots, r_{ln})\}_{n\in\mathbb{N}}$ of rational vectors is called a ρ-name of $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$ if $\{r_{jn}\}_{n\in\mathbb{N}}$ is a ρ-name of x_j, $1 \leq j \leq l$. Having ρ-names, we can define computable functions.

Definition 7. *Let A, B be sets, where ρ-names can be defined for elements of A and B. A function $f : A \to B$ is computable if there is a Type-2 machine such that on any ρ-name of $x \in A$, the machine computes as output a ρ-name of $f(x) \in B$.*

Next we present a notion of computability for open and closed subsets of $\mathbb{R}^l$ (cf. [13], Definition 5.1.15). We implicitly use ρ-names. For instance, to obtain names of open subsets of $\mathbb{R}^n$, we note that the set of rational balls $B(a,r) = \{x \in \mathbb{R}^l : |x-a| < r\}$ where $a, r \in \mathbb{Q}$ is a basis for the standard topology over $\mathbb{R}^n$. Depending on the ρ-names used, we obtain different notions of computability. We omit further details for reasons of space.

Definition 8. *1. An open set $E \subseteq \mathbb{R}^l$ is called recursively enumerable (r.e. for short) open if there are computable sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in E$ and $r_n \in \mathbb{Q}$ such that*

$$E = \cup_{n=0}^{\infty} B(a_n, r_n).$$

Without loss of generality one can also assume that for any $n \in \mathbb{N}$, the closure of $B(a_n, r_n)$, denoted as $\overline{B(a_n, r_n)}$, is contained in E, where $B(a_n, r_n) = \{x \in \mathbb{R}^l : |x - a_n| < r_n\}$.

2. *A closed subset $K \subseteq \mathbb{R}^l$ is called r.e. closed if there exist computable sequences $\{b_n\}$ and $\{s_n\}$, $b_n \in \mathbb{Q}^l$ and $s_n \in \mathbb{Q}$, such that $\{B(b_n, s_n)\}_{n \in \mathbb{N}}$ lists all rational open balls intersecting K.*
3. *An open set $E \subseteq \mathbb{R}^l$ is called computable (or recursive) if E is r.e. open and its complement E^c is r.e. closed. Similarly, a closed set $K \subseteq \mathbb{R}^l$ is called computable (or recursive) if K is r.e. closed and its complement K^c is r.e. open.*

When dealing with open sets in $\mathbb{R}^n$, we identify a special case of computability, that we call semi-computability. Let $\mathcal{O}(\mathbb{R}^n) = \{O | O \subseteq \mathbb{R}^n$ is open in the standard topology$\}$.

Definition 9. *A function $f : A \to \mathcal{O}(\mathbb{R}^n)$ is called semi-computable if there is a Type-2 machine such that on any ρ-name of $x \in A$, the machine computes as output two sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in \mathbb{R}^n$ and $r_n \in \mathbb{Q}$ such that*

$$f(x) = \cup_{n=0}^{\infty} B(a_n, r_n).$$

Without loss of generality one can also assume that for any $n \in \mathbb{N}$, the closure of $B(a_n, r_n)$ is contained in $f(x)$.

We call this function semi-computable because we can tell in a finite amount of time if a point belongs to $f(x)$, but we have to wait an infinite amount of time to know that it does not belong to $f(x)$.

Before closing this section, we present some useful results from [14]. Recall that a function $f : E \to \mathbb{R}^m$, $E \subseteq \mathbb{R}^l$, is said to be locally Lipschitz on E if it satisfies a Lipschitz condition on every compact set $V \subset E$. The following definition gives a computable analysis analog of this condition.

Definition 10. *Let* $E = \cup_{n=0}^{\infty} B(a_n, r_n) \subseteq \mathbb{R}^l$*, where* $\overline{B(a_n, r_n)} \subseteq E$*, be a r.e. open set. A function* $f : E \to \mathbb{R}^m$ *is called effectively locally Lipschitz on* E *if there exists a computable sequence* $\{K_n\}$ *of positive integers such that*

$$|f(x) - f(y)| \leq K_n |x - y| \text{ whenever } x, y \in \overline{B(a_n, r_n)}.$$

The following result is from [14].

Theorem 1. *Assume that* $f : E \to \mathbb{R}^m$ *is a computable function in* $C^1(E)$ *(meaning that both* f *and its derivative* f' *are computable). Then* f *is effectively locally Lipschitz on* E*.*

3 Results

In this section we show that domains of attraction of (1) for hyperbolic attractors can be computed from the former for C^1-computable functions f (meaning that both f and Df are computable). We also extend a result of Zhong [15] for the case of $\mathbb{R}^2$, showing that not only hyperbolic sinks have r.e. domains of attraction, but also hyperbolic limit cycles. Our results rely on the procedure introduced below.

3.1 Main construction

The idea underlying the main construction is as follows. We pick an $n \in \mathbb{N}$, construct a $n \times n$ square, and divide it into n^4 smaller squares of size $\frac{1}{n} \times \frac{1}{n}$. Next we replicate the dynamics of (1) over these small squares, obtaining a finite automata on which we can decide things in finite time. Then we increment n and the accuracy of the simulation, repeating this procedure indefinitely. In the limit we get the exact dynamics of (1), but in between we obtain correct partial results that allow us to compute domains of attraction for (1).

The procedure is an adaptation (we restrict ourselves to differential equations and not inclusions) and extension (this construction works for unbounded domains) of a technique introduced in [5].

The construction. We consider only structurally stable systems. Therefore limit sets are limit cycles or equilibrium points. We will see later why this construction cannot be carried over to unstable systems.

Suppose that our attractor is a computable fixed point x_0. We want to compute its domain of attraction. First pick an $n = n_0 \in \mathbb{N}$ such that $x_0 \in S_n = (-n, n)^2$. In general, we ought to iterate the algorithm by incrementing n, thus capturing more and more of the dynamics of (1) in each step. But for simplicity, we fix some n for now. Suppose, without loss of generality, that the distance of x_0 to $\overline{S_n}$ (= complement of S_n) is bigger than $\frac{1}{n}$. Let $L_n \in \mathbb{N}$ be a Lipschitz constant valid over S_n (L_n can be computed by Theorem 1). Now divide the big square S_n into smaller squares of size $\frac{1}{2L_n n} \times \frac{1}{2L_n n}$ (later we will see why we use the $2L_n$ factor). For simplicity, we call these squares $\frac{1}{n} \times \frac{1}{n}$ squares. Let

s be a $\frac{1}{n} \times \frac{1}{n}$ square. Since f is computable, one can compute in finite time a rational polytope $P_{s,n}$ (the polytope of directions) such that $f(s) \subset P_{s,n}$ ($P_{s,n}$ is an over-approximation) and $\operatorname{dist}(f(s), P_{s,n}) < \frac{1}{n}$, where $\operatorname{dist}(f(s), P_{s,n})$ is the Hausdorff distance between $f(s)$ and $P_{s,n}$. Now we define a function Θ by the formula that, for each $\frac{1}{n} \times \frac{1}{n}$ square s, $\Theta(s) = \bigcup A_s$, where A_s is the collection of all $\frac{1}{n} \times \frac{1}{n}$ squares which are adjacent to one the faces of s (including the square s) and intersect with the following rational polytope

$$R_{s,n} = \{\alpha + tP_{s,n} | t \in \mathbb{R}_0^+,\ \alpha \text{ is a face of } s\}. \tag{3}$$

(cf. Fig. 1)

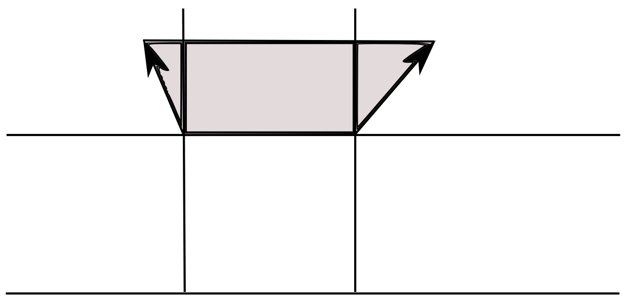

Fig. 1. Flow from a $\frac{1}{n} \times \frac{1}{n}$-square

There is a special case to be dealt with (this differs from the paper [5], where it doesn't deal with the case when the flow leaves the square S_n): when α is part of the boundary of S_n (i.e. s is not completely surrounded by $\frac{1}{n} \times \frac{1}{n}$ squares). In this case, we check if the set (3) leaves the face α to go directly to the set $\overline{S_n}$. If this is the case, proceed as before (don't add $\frac{1}{n} \times \frac{1}{n}$ squares outside of S_n to the definition of $\Theta(s)$) but mark s with some symbol, say ♣ (formally this equals to define a function over $\frac{1}{n} \times \frac{1}{n}$ squares). This ♣ indicates that the trajectory is incomplete from this point, since it leaves S_n.

Given a $\frac{1}{n} \times \frac{1}{n}$ square $s \subseteq S_n$, its n-polygonal trajectory is $Traj_n(s) = \cup_{i=0}^{\infty} \Theta^i(s)$ which can be computed in finite time by the following sketch of algorithm (remark that the number of $\frac{1}{n} \times \frac{1}{n}$ squares is finite):

$R_0 = s$
$R_{i+1} = \Theta(R_i)$

We will have $Traj_n(s) = R_j$ for some $j \in \mathbb{N}$. In any step, if $\Theta(R_i)$ is marked with ♣, then mark s also with ♣ to indicate that $Traj_n(s)$ is incomplete (it goes out of the $n \times n$ square S_n). The following lemma follows immediately from the previous construction.

Lemma 2. *Assume the conditions as above, let $y \in s \subseteq S_n$. If $\phi(t, y) \in S_n$ for all $t \in [0, T)$, where $0 \leq T \leq +\infty$, then*

1. $\{\phi(t,y)|t \in [0,T)\} \subseteq Traj_n(s)$.
2. For each $\varepsilon > 0$ there exists $n_0 \geq n$ such that $Traj_{n_0}(s) \subseteq \{\phi(t,s)|\phi'(t) \in f_\varepsilon(\phi(t))\}$.

Proof. The claim 1 is clear from the previous construction.

For the claim 2, consider a $\frac{1}{n} \times \frac{1}{n}$ square s. Since s is a square of size $\frac{1}{2L_n n} \times \frac{1}{2L_n n}$, two points $A, B \in s$ are at most within distance $\frac{\sqrt{2}}{2L_n n} \leq \frac{1}{L_n n}$. Since L_n is a Lipschitz constant, $|f(A) - f(B)| \leq \frac{1}{n}$. This implies that the over-approximation rational polytope $P_{s,n}$ of $f(s)$ has diameter $\leq \frac{2}{n}$. Therefore it suffices to use an n_0 such that $\frac{2}{n_0} \leq \varepsilon$ to prove the second claim.

Some simple facts can be obtained from this construction:

Lemma 3. *Assume the conditions as above:*

1. Let $\gamma \subseteq S_n$ be a limit cycle. If s intersects γ and $\gamma \nsubseteq s$ then there is some $i > 0$ such that $s \subseteq \Theta(R_i - s)$.
2. If $Traj(s)$ is not marked with ♣, then:
   (a) If $Traj(s)$ doesn't include a square containing the sink p, then s is not contained in the basin of attraction of p.
   (b) If $Traj(s)$ doesn't include a square intersecting a limit cycle γ, then s is not contained in the basin of attraction of γ.
   (c) If $Traj(s)$ include just *one* square with an equilibrium point p, which is also a sink, and does not include any limit cycle, then s is in the basin of attraction of p.
   (d) If $Traj(s)$ doesn't include any square containing an equilibrium point p, and includes squares that intersect only one limit cycle γ, then s is in the basin of attraction of γ.
3. If $E \subseteq S_n$ is a union of $\frac{1}{n} \times \frac{1}{n}$ squares such that $Traj(E)$ is not marked with ♣ and $Traj(E) = E$ then E contains an invariant set.

Proof. For the claim 1, remark that the flow will leave s, but it will reenter s again. For the claim 2, it suffices to notice that (i) $Traj(s)$ is an overapproximation of the real flow starting on s (this fact can also be used to prove the claim 3); (ii) since the flow doesn't go to infinity (it is the case because $Traj(s)$ is not marked with ♣), it must converge to a sink (s has non-empty interior, and therefore cannot be contained on the stable manifold of a saddle point) or to a limit cycle.

3.2 Results

This section introduces the main results. In general, it is not yet known whether the number of attractors in a given compact set can be decided effectively; this is due to the possibility of having unknown number of nested limit cycles with unknown sizes (an interesting problem would be to prove or disprove if this problem is undecidable). However, in practical applications, it is usually more important to know that a certain compact C indeed contains one or more attractors, and to determine the union of all domains of attraction for attractors in C. This is of interest, for example, to control theory. For instance, imagine that we are dealing with a nuclear reactor and want to decide which are the initial conditions that lead to a critical domain of operation (i.e. these are points to be avoided). We pick a compact set that covers all these critical situations and then determine the domain of attraction as described above within this compact set.

Usually it is assumed that C is invariant, but this assumption is not necessary. The set C may have some parts to which the flow circulates before reaching attractors outside of C. This does not pose a problem and henceforth we do not assume that C is invariant. Nevertheless, the procedure of the previous subsection could be used to semi-decide if a set is invariant, as long as the flow enters the invariant set in a direction that is not tangent to the frontier of the set. We do not delve further in this subject since it is secondary to our objectives.

Formally, let $\mathcal{S}(\mathbb{R}^2)$ be the set of C^1 structurally stable vector fields on $\mathbb{R}^2$, $f \in \mathcal{S}(\mathbb{R}^2)$, and A_f the class of compact sets $A \subseteq \mathbb{R}^2$ containing at least one hyperbolic attractor (in this case, hyperbolic sink or limit cycle) of (1) such that no limit set of (1) intersects simultaneously A and its complement. Note that $\mathcal{S}(\mathbb{R}^2) \subseteq C^1(\mathbb{R}^2)$ and that functions in $C^1(\mathbb{R}^2)$ admit names, e.g. by means of rational polygonal functions over increasing rational cubes over $\mathbb{R}^2$ — see [13] for details.

The following result shows that the operator that gives the basin of attraction for attractors in A, given dynamics of (1), is semi-computable.

Theorem 2. *Assume the conditions as above. The function* $F : \{(A, f)|f \in \mathcal{S}(\mathbb{R}^2), A \in A_f\} \to \mathcal{O}(\mathbb{R}^2)$ *defined by* $F(A, f)$ = *union of the domains of attraction for attractors in* A *for (1) is semi-computable, but not computable.*

Proof. (Sketch) The fact that F is not computable follows from a result of Zhong [15], that shows that there is a C^1 function f such that (1) has a sink at the origin, which domain of attraction is not computable (if F is computable then this domain of attraction would be computable).

For the semi-computability part, let x be an interior point of the domain of attraction of A. Suppose that the trajectory does not converge to the boundary of the domain of attraction (i.e. part of the boundary is not itself an attractor — we will see that this condition is of uttermost importance for the theorem to hold). Then the n-polygonal trajectory of a square s containing x will converge to a sink or limit cycle contained in the interior of A for n large enough (use Lemma 1, the fact that attractors inside A are stable, and the hypothesis that Γ_x^+ does not converge to the boundary of the domain of attraction). As long as this does not happen (the flow converges to an equilibrium point outside A, flows outside the current box S_n, or converges towards a limit cycle — see below a sketch of how to obtain equilibrium points and limit cycles), do not list this square as being inside the basin of attraction of A. When it happens, list s.

The previous result no longer is true if we allow all C^1 functions f, instead of only of those who give structurally stable systems.

Theorem 3. *Assume the conditions as above, the function* $F : \{(A_f, f) | f \in C^1(\mathbb{R}^2)\} \to \mathcal{O}(\mathbb{R}^2)$ *defined by* $F(A, f) =$ *domain of attraction of* A *for (1) is not semi-computable.*

Proof. Consider the following system, defined by polar coordinates

$$\begin{cases} r' = -r\left[(r^2-1)^2 - \mu\right] \\ \theta' = 1 \end{cases}$$

adapted from [8, p. 319, Example 2], which can easily be written in (x, y)-coordinates. When $\mu < 0$, the system has just one attractor, the fixed point 0 (see Fig. 2). When $\mu = 0$, the point 0 still is a sink, but a cycle, with center

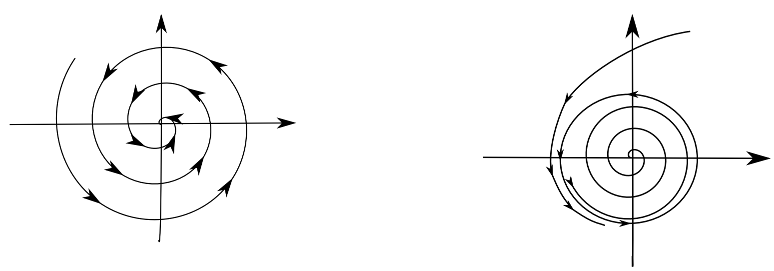

Fig. 2. Phase portrait for $\mu < 0$ (left) and $\mu = 0$ (right)

at 0 and radius 1 appears (see Fig. 2). Therefore the domain of attraction of the sink at 0 is $\mathbb{R}^2$ for $\mu < 0$, and $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1 - \sqrt{\mu}\}$ for $\mu \geq 0$. Therefore the map F is not continuous over its domain, and since computable functions are continuous [13], F cannot be computable (semi-computability is in essence computability with respect to ρ-names for open sets formed by a subbasis of the topology — so semi-computable functions are still continuous).

It is not hard to see that our previous construction does not work when $\mu = 0$ in the example of Fig. 2. Each time a square outside the cycle approaches the latter, it will eventually overlap the cycle and our square will overlap the domain of attraction of 0 and will falsely be drawn to this equilibrium. Therefore we cannot compute the domain of attraction of the cycle.

As a corollary of Theorem 2, we obtain a result already stated in [15], but with a different proof and for structurally stable systems.

Corollary 1. *Assume the system (1) is structurally stable and that* x_0 *is an hyperbolic sink. If* f *is computable, then the domain of attraction of* x_0 *is an r.e. open set of* $\mathbb{R}^2$.

We remark that the equilibrium points of (1) can be computed by computing the zeros of f (this is computable — see [13]). Those zeros whose eigenvalues of Df have all negative real part are the hyperbolic sinks. Moreover, given a compact set $M \subseteq \mathbb{R}^2$ not containing equilibrium points, we can decide whether or not it contains at least one limit cycle γ. To see this it suffices to apply the claim 1 of Lemma 3, for all squares $s \subseteq M$. At the beginning we may obtain the existence of possible cycles — these candidates are cycles of adjacent $\frac{1}{n} \times \frac{1}{n}$ squares (it might be the case that the cycle has just one $\frac{1}{n} \times \frac{1}{n}$ square — the case where the cycle is inside this square). If no limit cycle exists in M, after some time, with increasing n, we will be able to rule out the behavior described by the claim 1 of Lemma 3, and thus conclude that no cycle exists. If we see a pseudo-cycle inside a $\frac{1}{n} \times \frac{1}{n}$ squares, for n big enough the flow will enter the square in a narrow direction and will not reenter the square, the same happening with the polygonal flow, and we will then conclude at a certain point that no "small" cycle exists if none is present on this $\frac{1}{n} \times \frac{1}{n}$ square. On the other hand, if a cycle exists, we keep having cycles of $\frac{1}{n} \times \frac{1}{n}$ squares. But since the flow converges exponentially to the limit cycle, when the perturbation error on the flow is made small enough, the flow from s will not grow, and will hit s again in a "thinning cycle". The argument is very sketchy; indeed, due to the page limit the details have to be omitted.

The point is that we can compute hyperbolic limit cycles, the catch being that if the resolution is not enough at a given point, our approximation can consist in reality of 2 or more cycles very near to each other and that we where unable to distinguish. But if we use enough resolution, we will be able to separate limit cycles.

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