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Note

# Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, II

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**Abstract**

In [S.G. Samko, B.G. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, J. Math. Anal. Appl. 310 (2005) 229–246], Sobolev-type $p(\cdot) \to q(\cdot)$-theorems were proved for the Riesz potential operator $I^\alpha$ in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with the variable exponent $p(x)$ and a two-parameter power weight fixed to an arbitrary finite point $x_0$ and to infinity, under an additional condition relating the weight exponents at $x_0$ and at infinity. We show in this note that those theorems are valid without this additional condition. Similar theorems for a spherical analogue of the Riesz potential operator in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ are also improved in the same way.


*Keywords:* Weighted Lebesgue spaces; Variable exponent; Riesz potentials; Spherical potentials; Stereographical projection

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## 1. Introduction

We consider the Riesz potential operator

$$I^\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n, \tag{1.1}$$

in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n,\rho)$ with a variable exponent $p(x)$ defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)} = \inf\left\{\lambda>0\colon \int\limits_{\mathbb{R}^n} \rho(x)\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}, \tag{1.2}$$

where

$$\rho(x) = \rho_{\gamma_0,\gamma_\infty}(x) = |x|^{\gamma_0}\big(1+|x|\big)^{\gamma_\infty-\gamma_0}. \tag{1.3}$$

We refer to [3–6] for the basics of the spaces $L^{p(\cdot)}$ with variable exponent.

We assume that the exponent $p(x)$ satisfies the standard conditions

$$1<p_-\leqslant p(x)\leqslant p_+<\infty, \quad x\in\mathbb{R}^n, \tag{1.4}$$

$$\big|p(x)-p(y)\big| \leqslant \frac{A}{\ln\frac{1}{|x-y|}}, \quad |x-y|\leqslant\frac{1}{2},\ x,y\in\mathbb{R}^n, \tag{1.5}$$

and also the following condition at infinity

$$\big|p_*(x)-p_*(y)\big| \leqslant \frac{A_\infty}{\ln\frac{1}{|x-y|}}, \quad |x-y|\leqslant\frac{1}{2},\ x,y\in\mathbb{R}^n, \tag{1.6}$$

where $p_*(x) = p(\frac{x}{|x|^2})$. Conditions (1.5) and (1.6) taken together are equivalent to the following global condition:

$$\big|p(x)-p(y)\big| \leqslant \frac{C}{\ln\big(\frac{2\sqrt{1+|x|^2}\sqrt{1+|y|^2}}{|x-y|}\big)}, \quad x,y\in\mathbb{R}^n. \tag{1.7}$$

Let

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

The following statement was proved in [8].

**Theorem 1.1.** *Under assumptions* (1.4)–(1.6) *and the condition*

$$p_+ < \frac{n}{\alpha} \tag{1.8}$$

*the operator $I^\alpha$ is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n,\rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n,\rho_{\mu_0,\mu_\infty})$, where*

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \text{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty, \tag{1.9}$$

*if*

$$\alpha p(0)-n<\gamma_0<n\big[p(0)-1\big], \qquad \alpha p(\infty)-n<\gamma_\infty<n\big[p(\infty)-1\big], \tag{1.10}$$

*and the exponents $\gamma_0$ and $\gamma_\infty$ are related to each other by the equality*

$$\frac{q(0)}{p(0)}\gamma_0 + \frac{q(\infty)}{p(\infty)}\gamma_\infty = \frac{q(\infty)}{p(\infty)}\big[(n+\alpha)p(\infty) - 2n\big]. \tag{1.11}$$

The goal of this note is to prove that Theorem 1.1 is valid without the additional condition (1.11). We consider also a similar statement for the spherical potential operators

$$\big(K^\alpha f\big)(x) = \int\limits_{\mathbb{S}_n} \frac{f(\sigma)}{|x-\sigma|^{n-\alpha}}\,d\sigma, \quad x \in \mathbb{S}_n,\ 0<\alpha<n, \tag{1.12}$$

in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

## 2. Preliminaries

We need the following theorem for bounded domains proved in [7].

**Theorem 2.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$ and let $p(x)$ satisfy conditions* (1.4)*,* (1.5) *and* (1.8) *in $\Omega$. Then the following estimate*

$$\big\|I^\alpha f\big\|_{L^{q(\cdot)}(\Omega,|x-x_0|^\mu)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega,|x-x_0|^\gamma)} \tag{2.1}$$

*is valid, if*

$$\alpha p(x_0) - n < \gamma < n\big[p(x_0)-1\big] \tag{2.2}$$

*and*

$$\mu \geqslant \frac{q(x_0)}{p(x_0)}\gamma. \tag{2.3}$$

## 3. The case of the spatial potential operator

We prove the following theorem.

**Theorem A.** *Under assumptions* (1.4)–(1.6) *and* (1.8)*, the operator $I^\alpha$ is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \rho_{\mu_0,\mu_\infty})$, where*

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \textit{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty, \tag{3.1}$$

*if*

$$\alpha p(0) - n < \gamma_0 < n\big[p(0)-1\big], \qquad \alpha p(\infty) - n < \gamma_\infty < n\big[p(\infty)-1\big]. \tag{3.2}$$

**Proof.** Let $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)} \leqslant 1$. To estimate the integral $\int_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x)|I^\alpha f(x)|^{q(x)}\,dx$, we split it, as in [8], in the following way:

$$\int\limits_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x)\big|I^\alpha f(x)\big|^{q(x)}\,dx \leqslant c(A_{++} + A_{+-} + A_{-+} + A_{--}),$$

where

$$A_{++} = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_{+-} = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

and

$$A_{-+} = \int\limits_{|x|>1} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_{--} = \int\limits_{|x|>1} |x|^{\mu_\infty} \left| \int\limits_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The boundedness of the terms $A_{++}$ and $A_{--}$ was shown in [8] without condition (1.11). So we only have to treat the terms $A_{+-}$ and $A_{-+}$.

**$1^0$.** *The term $A_{-+}$.* We split $A_{-+}$ as

$$A_{-+} = A_1 + A_2,$$

where

$$A_1 = \int\limits_{1<|x|<2} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

and

$$A_2 = \int\limits_{|x|>2} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term

$$A_1 \leqslant C \int\limits_{1<|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx \leqslant C \int\limits_{|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

is covered by Theorem 2.1. For the term $A_2$ we have

$$|x-y| \geqslant |x| - |y| \geqslant \frac{|x|}{2}.$$

Therefore,

$$A_2 \leqslant C \int\limits_{|x|>2} |x|^{\mu_\infty + (\alpha-n)q(x)} \left( \int\limits_{|y|<1} \left|f(y)\right| dy \right)^{q(x)} dx.$$

It follows from condition (1.6) (see also (1.7)) that

$$\left|p(x) - p(\infty)\right| \leqslant \frac{C}{\ln|x|}, \quad |x| \geqslant 2,$$

and then the same is valid for $q(x)$, so that

$$A_2 \leqslant C \int_{|x|>2} |x|^{\mu_\infty+(\alpha-n)q(\infty)} \Bigg( \int_{|y|<1} \big|f(y)\big|\,dy \Bigg)^{q(x)} dx.$$

Observe that

$$\int_{|y|<1} \big|f(y)\big|\,dy \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)}. \tag{3.3}$$

Indeed, denote $g(y) = [\rho(y)]^{-\frac{1}{p(y)}}$; by the Hölder inequality for variable $L^{p(\cdot)}$-spaces we get

$$\begin{aligned}\int_{|y|<1} \big|f(y)\big|\,dy &= \int_{|y|<1} g(y)\big[\rho(y)\big]^{\frac{1}{p(y)}}\big|f(y)\big|\,dy \\ &\leqslant k\|g\|_{L^{p'(\cdot)}}\big\|\rho^{\frac{1}{p}} f\big\|_{L^{p(\cdot)}} = k\|g\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)}.\end{aligned} \tag{3.4}$$

To arrive at (3.3), we have to show that $\|g\|_{L^{p'(\cdot)}} < \infty$. Under condition (1.4) one has

$$\|g\|_{L^{p'(\cdot)}} < \infty \quad \Longleftrightarrow \quad \int_{|y|<1} \big|g(y)\big|^{p'(y)}\,dy < \infty. \tag{3.5}$$

As is easily seen, the last integral is finite since $\gamma_0 < n[p(0) - 1]$. Therefore, from (3.4) there follows (3.3).

Then $A_2 \leqslant C < \infty$ if we take into account that $\mu_\infty + (\alpha - n)q(\infty) < -n$ under the condition $\gamma_\infty < n[p(\infty) - 1]$.

**$2^0$.** *The term* $A_{+-}$ is estimated similarly to $A_{-+}$: we split $A_{+-}$ as

$$A_{+-} = A_3 + A_4,$$

where

$$A_3 = \int_{|x|<1} |x|^{\mu_0} \Bigg| \int_{1<|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \Bigg|^{q(x)} dx$$

and

$$A_4 = \int_{|x|<1} |x|^{\mu_0} \Bigg| \int_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \Bigg|^{q(x)} dx.$$

The term $A_3$ is covered by Theorem 2.1 similarly to the term $A_1$ in $\mathbf{1^0}$. For the term $A_4$, we have $|x - y| \geqslant |y| - |x| \geqslant \frac{|y|}{2}$. Then

$$\Bigg| \int_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \Bigg| \leqslant C \int_{|y|>2} \frac{|f(y)|\,dy}{|y|^{n-\alpha}} = C \int_{|y|>2} \frac{|f_0(y)|\,dy}{|y|^{n-\alpha+\frac{\gamma_\infty}{p(\infty)}}},$$

where $f_0(y) = |y|^{\frac{\gamma_\infty}{p(\infty)}} f(y)$. It is easily seen that $f_0(y) \in L^{p(\cdot)}(\mathbb{R}^n \backslash B(0,2))$, since $[\rho(y)]^{\frac{1}{p(y)}} \times f(y) \in L^{p(\cdot)}(\mathbb{R}^n)$ and $[\rho(y)]^{\frac{1}{p(y)}} \sim |y|^{\frac{\gamma_\infty}{p(\infty)}}$ for $|y| \geqslant 2$ under the log-condition at infinity. Hence by the Hölder inequality and the same log-condition at infinity,

$$\left|\int\limits_{|y|>2}\frac{f(y)\,dy}{|x-y|^{n-\alpha}}\right|\leqslant C_1\|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus B(0,2))}\left\||y|^{\alpha-n-\frac{\gamma_\infty}{p(\infty)}}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n\setminus B(0,2))}$$
$$\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho_{\gamma_0,\gamma_\infty})}\left\||y|^{\alpha-n-\frac{\gamma_\infty}{p(\infty)}}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n\setminus B(0,2))}$$
$$\leqslant C\left\||y|^{\alpha-n-\frac{\gamma_\infty}{p(\infty)}}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n\setminus B(0,2))},\tag{3.6}$$

where the last norm is finite under the condition $\alpha p(\infty)-n<\gamma_\infty$ (use the argument given in (3.5)). □

**Corollary 3.1.** *Let* $0<\alpha<n$, $p(x)$ *satisfy conditions* (1.4)–(1.6) *and* (1.8). *Then the operator* $I^\alpha$ *is bounded from the space* $L^{p(\cdot)}(\mathbb{R}^n)$ *into the space* $L^{q(\cdot)}(\mathbb{R}^n)$, $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}$.

The statement of the corollary was proved in [1,2] under a weaker than (1.6) version of the log-condition at infinity.

## 4. The case of the spherical potential operator

### *4.1. The space* $L^{p(\cdot)}(\mathbb{S}^n,\rho)$

We consider the weighted space $L^{p(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})$ with a variable exponent on the unit sphere $\mathbb{S}^n=\{\sigma\in\mathbb{R}^{n+1}\colon |\sigma|=1\}$, defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})}=\left\{\lambda>0\colon \int\limits_{\mathbb{S}^n}|\sigma-a|^{\beta_a}\cdot|\sigma-b|^{\beta_b}\left|\frac{f(\sigma)}{\lambda}\right|^{p(\sigma)}d\sigma\leqslant 1\right\},$$

where $\rho_{\beta_a,\beta_b}(\sigma)=|\sigma-a|^{\beta_a}\cdot|\sigma-b|^{\beta_b}$ and $a\in\mathbb{S}^n$ and $b\in\mathbb{S}^n$ are arbitrary points on $\mathbb{S}^n$, $a\neq b$.

We assume that $0<\alpha<n$ and

$$1<p_-\leqslant p(\sigma)\leqslant p_+<\frac{n}{\alpha},\quad \sigma\in\mathbb{S}^n,\tag{4.1}$$

$$\left|p(\sigma_1)-p(\sigma_2)\right|\leqslant\frac{A}{\ln\frac{3}{|\sigma_1-\sigma_2|}},\quad \sigma_1\in\mathbb{S}^n,\ \sigma_2\in\mathbb{S}^n.\tag{4.2}$$

The following theorem is valid.

**Theorem B.** *Let the function* $p:\mathbb{S}^n\to[1,\infty)$ *satisfy conditions* (4.1) *and* (4.2). *The spherical potential operator* $K^\alpha$ *is bounded from the space* $L^{p(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})$ *with* $\rho_{\beta_a,\beta_b}(\sigma)=|\sigma-a|^{\beta_a}\cdot|\sigma-b|^{\beta_b}$, *where* $a\in\mathbb{S}^n$ *and* $b\in\mathbb{S}^n$ *are arbitrary points on the unit sphere* $\mathbb{S}^n$, $a\neq b$, *into the space* $L^{q(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})$ *with* $\rho_{\nu_a,\nu_b}(\sigma)=|\sigma-a|^{\nu_a}\cdot|\sigma-b|^{\nu_b}$, *where* $\frac{1}{q(\sigma)}=\frac{1}{p(\sigma)}-\frac{\alpha}{n}$, *and*

$$\alpha p(a)-n<\beta_a<np(a)-n,\qquad \alpha p(b)-n<\beta_b<np(b)-n,\tag{4.3}$$

$$\nu_a=\frac{q(a)}{p(a)}\beta_a,\qquad \nu_b=\frac{q(b)}{p(b)}\beta_b.\tag{4.4}$$

This theorem was proved in [8] under the additional assumption that the weight exponents $\beta_a$ and $\beta_b$ are related to each other by the connection

$$\frac{q(a)}{p(a)}\beta_a=\frac{q(b)}{p(b)}\beta_b.\tag{4.5}$$

Now Theorem B without this condition follows from Theorem A by means of the stereographic projection exactly in the same way as in [8, Section 5].

**Corollary 4.1.** *Under assumptions* (4.1) *and* (4.2)*, the spherical potential operator* $K^\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{S}^n)$ *into* $L^{q(\cdot)}(\mathbb{S}^n)$, $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$.

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