

Spherical 2-Categories and 4-Manifold Invariants

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INTRODUCTION

In this paper I give a formal construction of a 4-manifold invariant out of what I call a *non-degenerate finitely semi-simple semi-strict spherical 2-category of non-zero dimension*.

For some time now people have had the feeling that 4-manifold invariants and certain kinds of monoidal 2-categories have a relation with each other similar to that of 3-manifold invariants and certain kinds of monoidal categories.

The first evidence for this feeling can be found in the work of Crane and collaborators. In [16] Crane and Frenkel give a formal construction of 4-manifold invariants out of Hopf categories and indicate where one should look for such algebraic objects, namely in the work on crystal bases by Saito and the work on canonical bases and perverse sheaves by Lusztig. One could argue that it should be possible to use the 2-category of representations of a Hopf category instead of the Hopf category itself; the reason for this thought being that for the construction of 3-manifolds one can use Hopf algebras, as Kuperberg [29] and Chung, Fukuma, and Shapere did [15], or the category of representations of Hopf algebras as people like Turaev and Viro [37], Yetter [40], and Barrett and Westbury [9] did. Recently Neuchl [31] showed that the representations of a Hopf category do form a monoidal 2-category indeed.

In [18] Crane and Yetter proposed a construction of 4-manifold invariants out of the semi-simple sub-quotient of the category of finite dimensional representations of the quantum group $U_q(sl(2))$ for q a principal $4r$ -th root of unity. In [17] Crane, Kauffman, and Yetter generalized this construction for any finitely semi-simple tortile category and gave detailed proofs. These Crane–Yetter invariants can be seen as a special case in which the authors use a 2-category of a certain type with only one object.

Another piece of evidence for the aforementioned “feeling” is the work of Baez and collaborators. In a series of papers [1–4] they have tried to convince people of the importance of n -categories. Among other important applications n -categories seem to be the right context in which to study Topological Quantum Field Theories, which in this context can be defined as n -functors from the n -category of n -cobordisms to the n -category of n -vector spaces. If we take a 3-category with one object we get a monoidal 2-category, which is roughly speaking the case we are concerned with in this paper.

Based on the work of Carter, Rieger, and Saito [13, 14], several people [5, 6, 22, 28, 30] have shown that braided 2-categories with duals form the right algebraic context in which to study invariants of 2-tangles. This should be closely related to the construction of 4-manifold invariants, at least formally.

So altogether one could say that there is more than enough reason to believe that it is possible to construct 4-manifold invariants out of certain kinds of 2-categories. But nowhere in the literature can one find a paper with an explicit construction. This paper is meant to fill this gap.

In this paper I use triangulations of 4-manifolds for the construction of a state sum. I will show that this state sum is independent of the chosen triangulation by using Pachner’s theorem [32], that relates triangulations of piece-wise linear homeomorphic manifolds. The whole construction should be considered as a lift to the fourth dimension of the construction by Barrett and Westbury [9] of 3-manifold invariants out of non-degenerate finitely semi-simple spherical categories. For that reason I have given the kind of 2-categories that I use the name above.

The simplest example of a 2-category as used in this paper is the semi-strict version of 2Hilb , the 2-category of 2-Hilbert spaces. In [3] Baez defined the weak version of 2Hilb . In this paper I define the completely coordinatized version, 2Hilb_{cc} , which is semi-strict.

In Section 6, I show how any finite group gives rise to a 2-category of the right kind and I give an explicit formula for the invariant. This invariant looks like a four dimensional version of the Dijkgraaf–Witten invariant [21].

It is likely that the 2-category of representations of the right kind of Hopf category will be such a 2-category too. But we have only one Hopf category that has been worked out in detail, namely the categorification $C(D(G))$ of the quantum double of a finite group [19]. Probably $C(D(G))$ -mod, the 2-category of finite dimensional representations of $C(D(G))$, is an example of a non-degenerate finitely semi-simple semi-strict spherical 2-category. But I have not worked out the details yet. Carter, Kauffman, and Saito are working out the Crane–Frenkel invariant for this particular example [12]. In [16] Crane and Frenkel indicate that it is possible to construct more examples of Hopf categories $C(U_0(g))$ using the crystal

bases of quantum groups at $q=0$. However, they do not define the categorification of the antipode for a general Hopf category. In the aforementioned case of $C(D(G))$ this antipode looks to be straight forward. In the case of $C(U_0(g))$ it is not known how to define the right categorification of the antipode.

Also Carter, Kauffman, and Saito found that it is necessary to impose extra conditions on the cocycles defining the structure isomorphisms of the Hopf category $C(D(G))$ in order to obtain invariance of the state sum under permutation of the vertices of a chosen triangulation of the 4-manifold. This was not foreseen in [16] and, as far as I know, it is not known how to obtain invariance under permutation of vertices for an arbitrary Hopf category.

In a future paper Crane and Yetter show how to build a monoidal 2-category out of the modules of a quantum group at $q=0$ using their crystal bases. Here they avoid the Hopf categories and build the 2-categories directly, which makes a construction of 4-manifold invariants out of a certain kind of 2-categories, as presented in this paper, even more desirable. It is definitely a good place to look for interesting examples of monoidal 2-categories, but it is not likely that these 2-categories are already the ones we are looking for. It is like having the non-finitely semi-simple “trivially” spherical category $U(g)\text{-mod}$, where g is a finite dimensional semi-simple Lie algebra, and having an abstract Turaev–Viro-like construction of 3-manifold invariants which requires finitely semi-simple spherical categories. In that case the missing link comes from the deformation theory of $U(g)$ which shows that there are deformations $U_q(g)$, where q is a certain root of unity, such that there is a certain non-degenerate quotient of the category of *tilting modules* of $U_q(g)$ that is a finitely semi-simple spherical category. For details about this see [9] and some references therein. Likewise, in the case of 2-categories, one should first define the deformation theory of monoidal 2-categories analogously to what Crane and Yetter have done for monoidal categories [20]. Then one has to find actual deformations of the aforementioned 2-categories and finally one has to “melt” them, i.e., get back to generic q . These last two problems are of a very deep nature and certainly far beyond the scope of this paper. In the meanwhile it is worthwhile, I think, to study the kind of 2-category we are looking for from an abstract point of view.

1. THE BASIC IDEA

Throughout this paper a manifold means an oriented piece-wise linear compact 4-manifold without boundary. A triangulated manifold (M, T) is a manifold M together with a given simplicial 4-complex T , the triangulation,

such that its underlying PL-manifold is PL-homeomorphic to M . Throughout this paper we will always assume that there is a total ordering on the vertices of the triangulation of a manifold. A combinatorial isomorphism between two triangulated manifolds (M, T) and (M', T') will always mean an isomorphism between the simplicial complexes T and T' . A simplicial isomorphism between two triangulated manifolds (M, T) and (M', T') will always mean a combinatorial isomorphism which preserves the ordering on the vertices. I also want to fix some notation. The letter \mathbf{F} will always denote a fixed field of characteristic 0 and any vector space in this paper will be a finite dimensional vector space over \mathbf{F} . My notation for the simplices follows Barrett and Westbury's convention in [9]. The standard n -simplex $(012\dots n)$ with vertices $\{0, 1, 2, \dots, n\}$ has the standard orientation $(+)$. The opposite orientation is denoted by $(-)$. The standard 4-simplex $+(01234)$ has boundary

$$(1234) - (0234) + (0134) - (0124) + (0123).$$

The 4-simplex $-(01234)$ has the same boundary but with the opposite signs. The sign with which a tetrahedron appears in the boundary of a 4-simplex I will call the induced orientation of the tetrahedron. The total ordering on the vertices of the simplicial complex defining the triangulation of a manifold induces an ordering on all 4-simplices and the ordering on the vertices of a 4-simplex induces an orientation on its underlying polytope. If the orientation of the underlying polytope of a 4-simplex induced by the orientation of the manifold is equal to the orientation of this polytope induced by the total ordering on the vertices of the 4-simplex, then our convention will be that the 4-simplex has the positive orientation, as a simplex, and if the two induced orientations of the polytope are opposite, we take the 4-simplex to be negatively oriented, as a simplex.

Remark 1.1. Notice that in a triangulated manifold each tetrahedron lies in the boundary of exactly two 4-simplices, occurring once with induced orientation $(+)$ and once with $(-)$.

Let (M, T) be a triangulated manifold. For the definition of my state sum I need two sets of labels, \mathcal{E} and \mathcal{F} , respectively. The edge (ij) with vertices i, j is labelled with $e_{ij} \in \mathcal{E}$; the face (ijk) with vertices i, j, k is labelled with $f_{ijk} \in \mathcal{F}$. Let $T((0123), e, f)$ be the labelled standard oriented tetrahedron $+(0123)$. We do not take the different orientations of the edges and the faces into account for the labelling yet.

Notation 1.2. The state space of this labelled tetrahedron is a certain vector space

$$2H((0123), e, f).$$

The state space of $-(0123)$ is defined to be the dual vector space

$$2H((0123), e, f)^*.$$

Notation 1.3. Let $+(01234)$ be the standard oriented 4-simplex with the triangle (ijk) labelled by $f_{ijk} \in \mathcal{F}$ and the edge (ij) by $e_{ij} \in \mathcal{E}$. Denote the state space of the labelled tetrahedron $(ijkl)$ by $2H(ijkl)$ just as a shorthand. The partition function of this labelled 4-simplex is a certain linear map

$$Z(+01234): 2H(0234) \otimes 2H(0124) \rightarrow 2H(1234) \otimes 2H(0134) \otimes 2H(0123).$$

The partition function of $-(01234)$ is also a certain linear map

$$Z(-01234): 2H(1234) \otimes 2H(0134) \otimes 2H(0123) \rightarrow 2H(0234) \otimes 2H(0124).$$

Notice that $Z(\pm(01234))$ is defined for a fixed labelling of T , although this dependence does not show up in the notation. This is a deliberate choice, or a deliberate flaw in the notation, that I want to allow myself in order to write down formulas that are not too polluted by a high number of sub- and superscripts. I think that the context will leave no doubt of what depends on what in my formulas.

Assume that $M = (M, T)$ is labelled with a fixed labelling ℓ . Notice that the ordering on the vertices of T induces a natural total ordering on the tetrahedra, by means of the boundary operator, within each 4-simplex. For example, in the ordered 4-simplex $(abcde)$ the ordering is given by 1. $(bcde)$, 2. $(acde)$, 3. $(abde)$, 4. $(abce)$, 5. $(abcd)$, which is independent of the orientation. In the same way we get a fixed ordering on the triangles within each tetrahedron, etc. Fix also a total ordering on the 4-simplices, for example, the one induced by the total ordering on the vertices of the whole triangulation. Take out of each 4-simplex the tetrahedra that appear with a negative sign in its boundary in the induced order described above. Together with the chosen ordering on the 4-simplices this fixes an ordering of all the tetrahedra of M . Notice that each tetrahedron appears exactly once in this way by Remark 1.1.

DEFINITION 1.4. Let $V(M, T, \ell)$ be the tensor product of the state spaces of all the tetrahedra of T in which the ordering of the factors is as described above. The tensor product of the respective partition functions applied to $V(M, T, \ell)$ has its image in a permuted tensor product $V(M, T, \ell)'$. Again Remark 1.1 is vital. Now compose this linear map with the linear map $P(M, T, \ell): V(M, T, \ell)' \rightarrow V(M, T, \ell)$ induced by the standard transposition $P: x \otimes y \rightarrow y \otimes x$. The result is a linear map $L(M, T, \ell): V(M, T, \ell) \rightarrow V(M, T, \ell)$. The element $Z(M, T, \ell) \in \mathbf{F}$ is defined to be the trace of $L(M, T, \ell)$.

Notice that $Z(M, T, \ell)$ does not depend on the ordering on the 4-simplices, because it is defined by a conjugation invariant trace. In the next section we will prove that $Z(M, T, \ell)$ is a combinatorial invariant of M .

The state sum $I(M, T)$ is a certain weighted sum over all labellings of the numbers $Z(M, T, \ell)$.

So far we have only sketched the basic idea of our approach without telling anyone where to get these state spaces and these partition functions. In Section 3 we will show how they appear naturally out of a certain kind of 2-categories. Therefore we have to study this kind of 2-categories in the next section first.

2. SPHERICAL 2-CATEGORIES

In this Section I define what I call a spherical 2-category. The underlying 2-category will always be assumed to be strict. This means that the composition is strictly associative and the composite of a 1-morphism with an identity 1-morphism is equal to the 1-morphism itself. I will denote the composite of two 1-morphisms f, g by fg , the vertical composite of two 2-morphisms α, β by $\alpha \cdot \beta$, and their horizontal composite by $\alpha \circ \beta$. If $f, f': A \rightarrow B$ and $g, g': B \rightarrow C$ are 1-morphisms and $\alpha: f \rightarrow f'$ and $\beta: g \rightarrow g'$ are 2-morphisms, then $f \circ \beta: fg \rightarrow fg'$ denotes $1_f \circ \beta$ and $\alpha \circ g: fg \rightarrow f'g$ denotes $\alpha \circ 1_g$. In this paper $\text{Hom}(A, B)$ will always denote the category whose objects are all 1-morphisms with source A and target B and whose morphisms are all 2-morphisms between such 1-morphisms, where the composition of the morphisms is defined by the vertical composition of the 2-morphisms. The notation $2\text{Hom}(f, g)$ will always denote the set of 2-morphisms with source f and target g . When the source and target are equal we will also use the notations $\text{End}(A) = \text{Hom}(A, A)$ and $2\text{End}(f) = 2\text{Hom}(f, f)$.

So let C be a strict 2-category. This is not too restrictive an assumption because a weak 2-category can always be strictified, see [27]. We also assume that C has a semi-strict monoidal structure. Loosely speaking this means that for every pair of objects A, B in C there is a unique object $A \otimes B$. For every object A and every 1-morphism $f: X \rightarrow Y$ there is a unique 1-morphism $A \otimes f: A \otimes X \rightarrow A \otimes Y$ and a unique 1-morphism $f \otimes A: X \otimes A \rightarrow Y \otimes A$. For every object A and every 2-morphism $\alpha: f \Rightarrow g$ there is a unique 2-morphism $A \otimes \alpha: A \otimes f \Rightarrow A \otimes g$ and a unique 2-morphism $\alpha \otimes A: f \otimes A \Rightarrow g \otimes A$. Also there is an identity object I such that $I \otimes X = X \otimes I = X$ for all objects, 1- and 2-morphisms X . All the usual structural 2-isomorphisms are identities except one: given a pair of

1-morphisms $f: A \rightarrow C$, $g: B \rightarrow D$ in C there is a 2-isomorphism called the *tensorator*

$$\bigotimes_{f,g}: (f \otimes B)(C \otimes g) \Rightarrow (A \otimes g)(f \otimes D).$$

This 2-isomorphism is required to satisfy some conditions. These conditions guarantee that $\bigotimes_{f,g}$ behaves well under the tensor product and composition. The obvious condition which tells us how to obtain $\bigotimes_{fg,h}$ by pasting $\bigotimes_{f,h}$ and $\bigotimes_{g,h}$, and analogously how to obtain $\bigotimes_{f,gh}$ by pasting $\bigotimes_{f,g}$ and $\bigotimes_{f,h}$, resembles the condition defining a braiding in a monoidal category. For the exact definition of a semi-strict monoidal structure see [27].

DEFINITION 2.1. A semi-strict monoidal 2-category is a strict 2-category with a semi-strict monoidal structure.

Let C always be a semi-strict monoidal 2-category. Again this is a legitimate assumption, since every weak monoidal 2-category is equivalent in a well defined sense to a semi-strict one [26]. The following lemma is well known. For a proof see [27].

LEMMA 2.2. *Let I be the identity object in C . Then the category $\text{End}(I)$ is a braided monoidal category with the tensor product being defined by the horizontal composition of 1- and 2-morphisms and the braiding being defined by the tensorator $\bigotimes_{\bullet, \bullet}$.*

In order to get to the spherical condition I first have to define duality in C . For this I will copy the definition given by Baez and Langford in [5, 6, 30].

DEFINITION 2.3. C is called a *semi-strict monoidal 2-category with duals* if it is equipped with the following structures:

1. For every 2-morphism $\alpha: f \Rightarrow g$ there is a 2-morphism $\alpha^*: g \Rightarrow f$ called the *dual* of α .
2. For every 1-morphism $f: A \rightarrow B$ there is a 1-morphism $f^*: B \rightarrow A$ called the *dual* of f , and 2-morphisms $i_f: 1_A \Rightarrow ff^*$ and $e_f: f^*f \Rightarrow 1_B$, called the *unit* and *counit* of f , respectively.
3. For any object A , there is an object A^* called the *dual* of A , 1-morphisms $i_A: I \rightarrow A \otimes A^*$ and $e_A: A^* \otimes A \rightarrow I$ called the *unit* and *counit* of A , respectively, and a 2-morphism $T_A: (i_A \otimes A)(A \otimes e_A) \Rightarrow 1_A$ called the *triangulator* of A .

We say that a 2-morphism α is *unitary* if it is invertible and $\alpha^{-1} = \alpha^*$. Given a 2-morphism $\alpha: f \Rightarrow g$, we define the *adjoint* $\alpha^\dagger: g^* \Rightarrow f^*$ by

$$\alpha^\dagger = (g^* \circ i_f) \cdot (g^* \circ \alpha \circ f^*) \cdot (e_g \circ f^*).$$

In addition, the structures above are also required to satisfy the following conditions:

- (1) $X^{**} = X$ for any object, 1-morphism or 2-morphism.
- (2) $1_X^* = 1_X$ for any object or 1-morphism X
- (3) For all objects A, B , 1-morphisms f, g , and 2-morphisms α, β for which both sides of the following equations are well-defined, we have

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*,$$

$$(\alpha \circ \beta)^* = \alpha^* \circ \beta^*,$$

$$(fg)^* = g^* f^*,$$

$$(A \otimes \alpha)^* = A \otimes \alpha^*,$$

$$(A \otimes f)^* = A \otimes f^*,$$

$$(\alpha \otimes A)^* = \alpha^* \otimes A$$

$$(f \otimes A)^* = f^* \otimes A,$$

$$(A \otimes B)^* = B^* \otimes A^*.$$

- (4) For all 1-morphisms f, g the 2-isomorphism $\bigotimes_{f, g}$ is unitary.
- (5) For any object or 1-morphism X we have $i_{X^*} = e_X^*$ and $e_{X^*} = i_X^*$.
- (6) For any object A , the 2-morphism T_A is unitary.
- (7) For any objects A and B we have

$$i_{A \otimes B} = i_A \cdot (A \otimes i_B \otimes A^*),$$

$$e_{A \otimes B} = (B^* \otimes e_A \otimes B) \cdot e_B,$$

$$T_{A \otimes B} = \left[(i_A \otimes A \otimes B) \left(A \otimes \bigotimes_{i_B, e_A}^{-1} \otimes B \right) (A \otimes B \otimes e_B) \right] \\ \cdot [(T_A \otimes B) \circ (A \otimes T_B)].$$

- (8) $T_I = 1_{1_I}$.

(9) For any object A and 1-morphism f , we have

$$i_{A \otimes f} = A \otimes i_f,$$

$$i_{f \otimes A} = i_f \otimes A,$$

$$e_{A \otimes f} = A \otimes e_f,$$

$$e_{f \otimes A} = e_f \otimes A.$$

(10) For any 1-morphisms f, g , $i_{fg} = i_f \cdot (f \circ i_g \circ f^*)$ and $e_{fg} = (g^* \circ e_f \circ g) \cdot e_g$.

(11) For any 1-morphism f , $(i_f \circ f) \cdot (f \circ e_f) = 1_f$ and $(f^* \circ i_f) \cdot (e_f \circ f^*) = 1_{f^*}$.

(12) For any 2-morphism α , $\alpha^{\dagger*} = \alpha^{*\dagger}$.

(13) For any object A we have

$$[i_A \circ (A \otimes T_A^\dagger)] \cdot \left[\bigotimes_{i_A, i_A}^{-1} (A \otimes e_A \otimes A^*) \right] \cdot [i_A \circ (T_A \otimes A^*)] = 1_{i_A}.$$

Note that the first two identities in condition (7) imply $i_I = e_I = 1_I$ so that condition (8) makes sense; the source of T_I is equal to 1_I . It would be worthwhile to study weaker notions of duality and prove a coherence theorem that allows one to strictify 2-categories with such duality up to a semi-strict monoidal 2-category with duals as defined above. In [7] Barrett and Westbury prove such a coherence theorem for monoidal categories with duals. In this paper we always work with semi-strict monoidal 2-categories with duals as defined above.

For our purpose we need a little bit of extra structure. Let $f: A \rightarrow B$ be a 1-morphism in C . We define the 1-morphism ${}^\#f: B^* \rightarrow A^*$ by

$$B^* \xrightarrow{i_{A^*} \otimes B^*} A^* \otimes A \otimes B^* \xrightarrow{A^* \otimes f \otimes B^*} A^* \otimes B \otimes B^* \xrightarrow{A^* \otimes e_{B^*}} A^*.$$

Analogously we define $f^\#: B^* \rightarrow A^*$ by

$$B^* \xrightarrow{B^* \otimes i_A} B^* \otimes A \otimes A^* \xrightarrow{B^* \otimes f \otimes A^*} B^* \otimes B \otimes A^* \xrightarrow{e_B \otimes A^*} A^*.$$

Notice that given a 2-morphism $\alpha: f \Rightarrow g$ in C we also have the 2-morphism $\alpha^\#: f^\# \Rightarrow g^\#$ defined by

$${}^\#\alpha = 1_{(i_{A^*} \otimes B^*)} \circ (A^* \otimes \alpha \otimes B^*) \circ 1_{(A^* \otimes e_{B^*})},$$

and the 2-morphism ${}^\#\alpha: {}^\#f \Rightarrow {}^\#g$ defined by

$$\alpha^\# = 1_{(B^* \otimes i_A)} \circ (B^* \otimes \alpha \otimes A^*) \circ 1_{(e_B \otimes A^*)}.$$

It is not difficult to show that for any 1-morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ there exist unitary 2-morphisms

$$\#(fg) \cong \#g \#f$$

and

$$(fg)^\# \cong g^\# f^\#.$$

The first 2-isomorphism is given by

$$\begin{aligned} & (1_{(i_{A^*} \otimes C^*)(A^* \otimes f \otimes C^*)} \circ (A^* \otimes T_B^\dagger \otimes C^*) \circ 1_{(A^* \otimes g \otimes C^*)(A^* \otimes e_{C^*})}) \\ & \cdot \left(1_{(i_{A^*} \otimes C^*)(A^* \otimes f \otimes C^*)(A^* \otimes B \otimes i_{B^*} \otimes C^*)} \circ \left(A^* \otimes \bigotimes_{e_{B^*}, g} \otimes C^* \right) \circ 1_{(A^* \otimes e_{C^*})} \right) \\ & \cdot \left(\bigotimes_{(i_{A^*} \otimes C^*)(A^* \otimes f \otimes C^*), (i_{B^*} \otimes C^*)(B^* \otimes g \otimes C^*)} \circ 1_{(A^* \otimes e_{B^*} \otimes C \otimes C^*)(A^* \otimes e_{C^*})} \right) \\ & \cdot \left(1_{(i_{B^*} \otimes C^*)(B^* \otimes g \otimes C^*)} \circ \bigotimes_{(i_{A^*} \otimes B^*)(A^* \otimes f \otimes B^*)(A^* \otimes e_{B^*}), e_{C^*}} \right). \end{aligned}$$

The second 2-isomorphism is given by

$$\begin{aligned} & (1_{(C^* \otimes i_A)(C^* \otimes f \otimes A^*)} \circ (C^* \otimes T_B^{-1} \otimes A^*) \circ 1_{(C^* \otimes g \otimes A^*)(e_C \otimes A^*)}) \\ & \cdot \left(1_{(C^* \otimes i_A)(C^* \otimes f \otimes A^*)(C^* \otimes i_B \otimes B \otimes A^*)} \circ \left(C^* \otimes \bigotimes_{g, e_B}^{-1} \otimes A^* \right) \circ 1_{(e_C \otimes A^*)} \right) \\ & \cdot \left(\bigotimes_{(C^* \otimes i_B)(C^* \otimes g \otimes B^*), (C^* \otimes i_A)(C^* \otimes f \otimes A^*)}^{-1} \circ 1_{(C^* \otimes C \otimes e_B \otimes A^*)(e_C \otimes A^*)} \right) \\ & \cdot \left(1_{(C^* \otimes i_B)(C^* \otimes g \otimes B^*)} \circ \bigotimes_{e_C, (B^* \otimes i_A)(B^* \otimes f \otimes A^*)(e_B \otimes A^*)}^{-1} \right). \end{aligned}$$

If $A = B = C = I$ we have $\#f = f^\# = f$ and $\#g = g^\# = g$ and the 2-isomorphisms above become simply equal to $\bigotimes_{f, g}$ and $\bigotimes_{g, f}^{-1}$, respectively, since $i_I = e_I = 1_I$, $T_I = 1_{1_I}$ and $\bigotimes_{X, 1_I} = \bigotimes_{1_I, X} = 1_X$ for any 1-morphism X .

The following definition reminds us of the definition of a pivotal category (for the definition of a pivotal category see [24]).

DEFINITION 2.4. A semi-strict pivotal 2-category is a semi-strict monoidal 2-category with duals C such that for any 1-morphism $f: A \rightarrow B$ there exists a unitary 2-morphism

$$\phi_f: {}^\# f \Rightarrow f^\#$$

satisfying the following conditions:

- (1) For any 2-morphism $\alpha: f \Rightarrow g$, we have

$${}^\# \alpha \cdot \phi_g = \phi_f \cdot \alpha^\#.$$

- (2) $\phi_f^\dagger = \phi_{f^*}$.

- (3) The following diagram commutes

$$\begin{array}{ccc} {}^\#(fg) & \xrightarrow{\phi_{fg}} & (fg)^\# \\ \downarrow & & \downarrow \\ {}^\#g^\#f & \xrightarrow{\phi_g \circ \phi_f} & g^\#f^\#. \end{array}$$

- (4) For any 2-morphism α , we have

$${}^\#(\alpha^\dagger) = {}^\dagger(\alpha^\#) \quad \text{and} \quad ({}^\# \alpha)^\dagger = ({}^\dagger \alpha)^\#.$$

Note that, by definition, we have

$${}^\#(f^*) = (f^\#)^* \quad \text{and} \quad ({}^\# f)^* = (f^*)^\#$$

for any 1-morphism f , so these conditions make sense. Note also that for $A = B = C = I$ the third condition becomes

$$\phi_{fg} = \bigotimes_{f, g} (\phi_g \circ \phi_f) \otimes \bigotimes_{g, f}.$$

Together with the second condition this reminds us of the conditions defining a *twist* in a ribbon category (see [35]). As a matter of fact Lemma 2.14 shows that this is exactly what we should have in mind. Note finally that, by definition, any 2-morphism α satisfies

$$(\alpha^\dagger)^* = {}^\dagger(\alpha^*) \quad \text{and} \quad ({}^\dagger \alpha)^* = (\alpha^*)^\dagger,$$

where

$${}^\dagger \alpha = (i_{f^*} \circ g^*) \cdot (f^* \circ \alpha \circ g^*) \cdot (f^* \circ e_{g^*}).$$

This, together with conditions (1) and (12) in the definition of duality (Definition 2.3), implies the equalities

$$\dagger\alpha = \dagger(\alpha^{**}) = ((\alpha^*)^\dagger)^* = (\alpha^\dagger)^{**} = \alpha^\dagger.$$

Condition (11) in the same definition implies

$$(\dagger\alpha)^\dagger = \dagger(\alpha^\dagger) = \alpha,$$

so we also get

$$\dagger\dagger\alpha = \alpha^{\dagger\dagger} = (\dagger\alpha)^\dagger = \alpha.$$

The next thing to define is the notion of a *trace-functor* in a semi-strict pivotal 2-category. Let C be such a 2-category for the rest of this section.

DEFINITION 2.5. For any object A in C there is a left trace functor $\text{Tr}_L: \text{End}(A) \rightarrow \text{End}(I)$. For any object $f \in \text{End}(A)$ define $\text{Tr}_L(f)$ by

$$I \xrightarrow{i_{A^*}} A^* \otimes A \xrightarrow{A \otimes f} A^* \otimes A \xrightarrow{e_A} I.$$

For a morphism $\alpha: f \Rightarrow g$ in $\text{End}(A)$ define $\text{Tr}_L(\alpha)$ by

$$1_{i_{A^*}} \circ (A^* \otimes \alpha) \circ 1_{e_A}.$$

Analogously there is a right trace functor Tr_R defined by

$$I \xrightarrow{i_A} A \otimes A^* \xrightarrow{f \otimes A} A \otimes A^* \xrightarrow{e_{A^*}} I,$$

and

$$1_{i_A} \circ (\alpha \otimes A^*) \circ 1_{e_{A^*}}.$$

The following lemma is the analogue of the well known lemma that says that traces are conjugation invariant in pivotal categories.

LEMMA 2.6. For any two objects A, B in C and any two 1-morphisms $f: A \rightarrow B$ and $g: B \rightarrow A$ we have a unitary 2-morphism $\theta_{f,g}^R: \text{Tr}_R(fg) \Rightarrow \text{Tr}_R(gf)$. For any 1-morphisms $f': A \rightarrow B$ and $g': B \rightarrow A$ and any 2-morphisms $\alpha: f \Rightarrow f'$ and $\beta: g \Rightarrow g'$ we have

$$\theta_{f,g}^R \cdot \text{Tr}_R(\beta \circ \alpha) = \text{Tr}_R(\alpha \circ \beta) \cdot \theta_{f',g'}^R.$$

This last property of θ_R can be called its naturality property.

Proof. The proof is identical to the proof in the case of pivotal categories except that the essential identities are now 2-isomorphisms. Explicitly we find

$$\begin{aligned}
\theta_{f,g}^R &= (1_{i_A(f \otimes A^*)} \circ (T_B^\dagger \otimes A^*) \circ 1_{(g \otimes A^*) e_{A^*}}) \\
&\quad \cdot \left(1_{i_A(f \otimes A^*)(B \otimes i_{B^*} \otimes A^*)} \circ \left(\bigotimes_{e_{B^*}, (g \otimes A^*) e_{A^*}} \right) \right) \\
&\quad \cdot (1_{i_A(f \otimes A^*)} \circ \phi_g \circ 1_{e_{B^*}}) \cdot \left(\bigotimes_{i_A(f \otimes A^*), i_B(g \otimes B^*)} \circ 1_{(B \otimes e_A \otimes B^*) e_{B^*}} \right) \\
&\quad \cdot \left(1_{i_B(g \otimes B^*)(i_A \otimes A \otimes B^*)} \circ \left(\bigotimes_{f, e_A} \otimes B^* \right) \circ 1_{e_{B^*}} \right) \\
&\quad \cdot (1_{i_B(g \otimes B^*)} \circ (T_A \otimes B^*) \circ 1_{(f \otimes B^*) e_{B^*}}).
\end{aligned}$$

Since all the 2-morphisms in this formula are unitary, it follows that $\theta_{f,g}^R$ is unitary also. The “naturality” of θ follows immediately from the formula above and the “naturality” of the triangulator and the tensorator. ■

Of course there is also a family of natural unitary 2-morphism θ^L for the left trace functor. Now if we take $g = f^*$ then we can write

$$\mathrm{Tr}_R(ff^*) = i_A(f \otimes A^*)(f^* \otimes A^*) i_A^* = i_A(f \otimes A^*)(i_A(f \otimes A^*))^*,$$

so we have the 2-morphism

$$\bigcap_f^R = i_{i_A(f \otimes A^*)} : 1_I \Rightarrow \mathrm{Tr}_R(ff^*).$$

Analogously we have the 2-morphisms

$$\bigcup_f^R = i_{i_A^*(f \otimes A^*)} : \mathrm{Tr}_R(ff^*) \Rightarrow 1_I.$$

We will call these 2-morphisms *cap* and *cup* respectively. Analogously there exist caps and cups with respect to the left trace functor,

$$\bigcap_f^L = i_{i_{A^*}(A^* \otimes f)} : 1_I \Rightarrow \mathrm{Tr}_L(ff^*)$$

and

$$\bigcup_f^L = i_{i_{A^*}^*(A^* \otimes f)} : \mathrm{Tr}_L(f^*f) \Rightarrow 1_I.$$

These names are of course inspired by what these 2-morphisms stand for in the 2-category of 2-tangles (see [5, 6, 30]). For our purposes we want these cups and caps to be compatible with the pivotal condition.

DEFINITION 2.7. A semi-strict pivotal 2-category is called *consistent* if the following conditions are satisfied

- (1) $\cap_f^R \cdot \theta_{f, f^*}^R = \cap_{f^*}^R$
- (2) $\theta_{f, f^*}^R \cdot \cup_{f^*}^R = \cup_f^R$
- (3) $\cap_f^L \cdot \theta_{f, f^*}^L = \cap_{f^*}^L$
- (4) $\theta_{f, f^*}^L \cdot \cup_{f^*}^L = \cup_f^L$.

In the sequel pivotal 2-categories will always be assumed to be consistent.

DEFINITION 2.8. A *semi-strict spherical 2-category* is a semi-strict consistent pivotal 2-category C such that for any object A in C and any 1-morphism $f \in \text{End}(A)$ we have a unitary 2-morphism $\sigma_f: \text{Tr}_L(f) \Rightarrow \text{Tr}_R(f)$. For any 1-morphism $g \in \text{End}(A)$ and any 2-morphism $\alpha: f \Rightarrow g$ these 2-isomorphisms are required to satisfy

$$\sigma_f \cdot \text{Tr}_R(\alpha) = \text{Tr}_L(\alpha) \cdot \sigma_g.$$

Furthermore we require the following identities to be satisfied:

$$\cap_{1_A}^L \cdot \sigma_{1_A} = \cap_{1_A}^R \quad \text{and} \quad \sigma_{1_A} \cdot \cup_{1_A}^R = \cup_{1_A}^L.$$

The last two conditions just mean that “cupping” and “capping” are compatible with the spherical condition.

In a semi-strict spherical 2-category C we can define a symmetric pairing

$$\langle \cdot, \cdot \rangle: 2\text{Hom}(f, g) \otimes 2\text{Hom}(g, f) \rightarrow 2\text{End}(1_I).$$

DEFINITION 2.9. Let A, B be any objects in C and $f, g: A \rightarrow B$ be any 1-morphisms. For any $\alpha \in 2\text{Hom}(f, g)$ and any $\beta \in 2\text{Hom}(g, f)$ we define a *pairing* by

$$\langle \alpha, \beta \rangle = \cap_f^R \cdot \text{Tr}_R((\alpha \cdot \beta) \circ 1_{f^*}) \cdot \cup_f^R.$$

Notice that the analogous definition of a pairing, using the left trace functor, gives exactly the same pairing because C is assumed to be spherical. Another way to understand this pairing is by noting that for any 2-morphism $\alpha: f \Rightarrow f$, with $f: A \rightarrow B$ an arbitrary 1-morphism, there is “a kind of trace” defined by

$$1_A \xRightarrow{i_f} ff^* \xRightarrow{\alpha \circ f^*} ff^* \xRightarrow{i_f^*} 1_A.$$

In order to get a real trace we have to map this to $2\text{End}(1_I)$. The right way to do this is by first applying the trace-functor to the “traced” 2-morphism above and then close this up by adding cups and caps. This leads precisely

to our definition of the pairing. The following lemmas show that the pivotal condition implies that one gets the same trace if one uses

$$1_B \xRightarrow{i_f^*} f^*f \xRightarrow{f^* \circ \alpha} f^*f \xRightarrow{i_f^*} 1_B.$$

LEMMA 2.10. $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ for any $\alpha \in 2\text{Hom}(f, g)$ and any $\beta \in 2\text{Hom}(g, f)$.

Proof. The proof is identical to the proof in the case of pivotal categories. Just as in that case it is easy to show that

$$\begin{aligned} \langle \alpha, \beta \rangle &= \bigcap_f^R \cdot \text{Tr}_R(\alpha \circ {}^\dagger \beta) \cdot \bigcup_f^R \\ &= \bigcap_f^R \cdot \text{Tr}_R(\alpha \circ \beta^\dagger) \cdot \bigcup_f^R = \langle \beta, \alpha \rangle. \quad \blacksquare \end{aligned}$$

LEMMA 2.11. $\langle \alpha, \beta \rangle^* = \langle \alpha^*, \beta^* \rangle$ for any $\alpha \in 2\text{Hom}(f, g)$ and any $\beta \in 2\text{Hom}(g, f)$.

Proof. From the definition of the pairing we see

$$\langle \alpha, \beta \rangle^* = \langle \beta^*, \alpha^* \rangle.$$

Now use the previous lemma. \blacksquare

LEMMA 2.12. $\langle \alpha, \beta \rangle = \langle \alpha^\dagger, \beta^\dagger \rangle$ for any $\alpha \in 2\text{Hom}(f, g)$ and any $\beta \in 2\text{Hom}(g, f)$.

Proof. From the pivotal condition we get

$$\langle \alpha, \beta \rangle = \bigcap_f^R \cdot \text{Tr}_R(1_f \circ (\beta^\dagger \cdot \alpha^\dagger)) \cdot \bigcup_f^R.$$

Now use $\text{Tr}_R(ff^*) \cong \text{Tr}_R(f^*f)$ and compatibility with cupping and capping. We get

$$\bigcap_{f^*}^R \cdot \text{Tr}_R((\beta^\dagger \cdot \alpha^\dagger) \circ 1_f) \cdot \bigcup_{f^*}^R = \langle \beta^\dagger, \alpha^\dagger \rangle.$$

Finally use Lemma 2.10 again. \blacksquare

LEMMA 2.13. $\langle \alpha^\#, \beta^\# \rangle = \langle {}^\# \alpha, {}^\# \beta \rangle = \langle \alpha, \beta \rangle$ for any $\alpha \in 2\text{Hom}(f, g)$ and any $\beta \in 2\text{Hom}(g, f)$.

Proof. If $\alpha = \beta = 1_A$, then this follows immediately from condition (13) in Definition 2.3 and the spherical condition. The general case now follows from this particular case by using the pivotal condition. \blacksquare

The following lemma shows that our setup is really a generalization of the setup of Crane, Kauffman, and Yetter in [17] (see [35] for the definition of a ribbon category).

LEMMA 2.14. *Let C be a semi-strict consistent pivotal 2-category. Then $\text{End}(I)$ is a ribbon category.*

Proof. We already know that $\text{End}(I)$ is a braided monoidal category (see Lemma 2.2). Duality of course follows from the duality on C . The dual object of $f \in \text{End}(I)$ is $f^* \in \text{End}(I)$. The evaluation on f is defined by i_f and the coevaluation by e_f and the identities they should satisfy are exactly those of condition (11) in Definition 2.3. The dual morphism of $\alpha \in 2\text{Hom}(f, g)$ is $\alpha^\dagger \in 2\text{Hom}(g^*, f^*)$.

$\text{End}(I)$ is pivotal since we have imposed the condition $\alpha^\dagger = {}^\dagger\alpha$, which is equivalent to condition (12) in Definition 2.3, as we explained.

The ribbon structure in $\text{End}(I)$ is defined by the family of unitary 2-morphisms

$$\phi_f: f = {}^\# f \Rightarrow f^\# = f$$

indexed by all $f \in \text{End}(I)$, which define the pivotal structure of C (see Definition 2.4). As we already mentioned below Definition 2.4, conditions (2) and (3) in that definition are exactly the conditions that make ϕ_f into a twist. Note that the consistency of the pivotal structure as defined in Definition 2.7 restricted to the case $f \in \text{End}(I)$ translates into the condition that i_{f^*} and e_{f^*} , which define the structure of the right dual f^* , be compatible with the structure of the left dual f^* , defined by i_f and e_f , and the twist ϕ_f . In order to see this one should realise that in this case $\theta_{f^*}^R$ is equal to $(f \otimes \phi_{f^*}) \cdot \bigotimes_{f, f^*}$ (see the formula in the proof of Lemma 2.6). ■

Before we go on let us have a look at an example of a spherical 2-category. In [3] Baez defines the 2-category of 2-Hilbert spaces, 2Hilb , and equips it with a tensor product and duals. Unfortunately 2Hilb is not a semi-strict monoidal 2-category with duals: the composition of the 1-morphisms is not strict, the monoidal structure is not semi-strict and the duals do not satisfy all the required identities up to the nose either. Let us recall the definitions from [3] and some of Baez's results and then indicate how to obtain a semi-strict version of 2Hilb , the 2-category of completely coordinatized 2-Hilbert spaces, 2Hilb_{cc} . We also define duals in 2Hilb_{cc} and show that the pivotal and the spherical conditions are satisfied. It is not difficult to show that 2Hilb_{cc} is equivalent to 2Hilb , but we omit the proof here.

EXAMPLE 2.15. The objects in 2Hilb , which are called 2-Hilbert spaces, are all finite dimensional abelian H^* -categories. An abelian H^* -category H is an abelian category such that the Hom-spaces are Hilbert spaces and composition is bilinear, and additionally H is equipped with antilinear maps $*$: $\text{hom}(x, y) \rightarrow \text{hom}(y, x)$ for all objects x, y in H such that

- (1) $f^{**} = f$,
- (2) $(fg)^* = g^*f^*$,
- (3) $\langle fg, h \rangle = \langle g, f^*h \rangle$,
- (4) $\langle fg, h \rangle = \langle f, hg^* \rangle$

for all $f: x \rightarrow y$, $g: y \rightarrow z$, and $h: x \rightarrow z$. It is shown in [3] that any abelian H^* -category H is semi-simple as an abelian category and if it is finitely semi-simple then its dimension is defined as the number of objects in a basis of H . It is also shown that any basis of a 2-Hilbert space has the same number of objects, so its dimension is well defined. Furthermore Baez shows that two 2-Hilbert spaces are equivalent if and only if they have the same dimension.

A 1-morphism $F: H \rightarrow H'$ in 2Hilb is an exact functor such that $F: \text{hom}(x, y) \rightarrow \text{hom}(F(x), F(y))$ is linear and $F(f^*) = F(f)^*$ for all $f \in \text{hom}(x, y)$.

A 2-morphism $\alpha: F \Rightarrow F'$ in 2Hilb is a natural transformation.

This all looks a little abstract, but, since we can always choose bases in 2-Hilbert spaces, 1-morphisms correspond to matrices with integer coefficients. This correspondence is reliable because any 2-Hilbert space is unitarily equivalent to a skeletal 2-Hilbert space, which is one where isomorphic objects are equal. So given a basis $\{a_i\}$ in a 2-Hilbert space A and a basis $\{b_j\}$ in a 2-Hilbert space B any 1-morphism $F: A \rightarrow B$ can be presented by the matrix (F_i^j) with coefficients $F_i^j \in \mathbb{N}$, where

$$F(a_i) = \bigoplus_j F_i^j b_j.$$

The 2-morphisms now correspond to matrices of complex matrices. If (F_i^j) presents the 1-morphism $F: A \rightarrow B$ and (G_i^j) the 1-morphism $G: A \rightarrow B$, then we can write a 2-morphism $\alpha: F \Rightarrow G$ as the matrix (α_i^j) , where α_i^j is a $G_i^j \times F_i^j$ matrix with complex coefficients. Note that in our notation (X_i^j) denotes the matrix with coefficients X_i^j .

The tensor product is a little bit complicated in 2Hilb and we will not define it here in a basis invariant way. Roughly speaking the tensor product of two 2-Hilbert spaces A and B can be obtained in the obvious way: choose a basis $\{a_i\}$ in A and a basis $\{b_j\}$ in B and “define” $A \otimes B$ as the 2-Hilbert space with basis $\{a_i \otimes b_j\}$ and define $\text{hom}(a_i \otimes b_j, a_i \otimes b_j) = \text{hom}(a_i, a_i) \otimes \text{hom}(b_j, b_j)$. It is obvious that Hilb , the category of finite dimensional Hilbert spaces, is the identity object. The braiding comes from the ordinary transposition of factors in the tensor product and we will not say more about it because it is not important for our purpose.

Let us now have a look at the duality in 2Hilb . The dual of a 2-Hilbert space H is the 2-Hilbert space $H^* = \text{Hom}(H, \text{Hilb})$. There is always a

unique dual basis in H^* for each basis in H , up to isomorphism of course. The coevaluation i_H and evaluation e_H are now defined as usual. Let us assume that H is skeletal. Given a basis $\{h_i\}$ in H and its dual basis $\{h^i\}$ in H^* we define $i_H: \text{Hilb} \rightarrow H \otimes H^*$ by

$$\mathbf{C} \mapsto \bigoplus_i h_i \otimes h^i$$

and $e_H: H^* \otimes H \rightarrow \text{Hilb}$ by

$$h^i \otimes h_j \mapsto \delta_j^i \mathbf{C},$$

where δ_j^i is the Kronecker-delta. If H is skeletal, then T_H , the triangulator, is trivial. Since, as already mentioned, any 2-Hilbert space is unitarily equivalent to a skeletal one, this defines the triangulator in general.

The dual of a 1-morphism $F: A \rightarrow B$ presented by the matrix (F_i^j) is defined by the transpose of this matrix, i.e.,

$$F^*(b_j) = \bigoplus_i F_i^j a_i.$$

Baez shows that this really defines a left and right adjoint to F , so the 2-coevaluation i_F and the 2-evaluation e_F are easy to define. The Hilbert space $\text{hom}(F(a_i), F(a_i))$ is isomorphic to $\text{hom}(a_i, FF^*(a_i))$ for every i , so $i_F: 1_A \Rightarrow FF^*$ is simply defined as the natural transformation corresponding to the identity on $F(a_i)$ for each basis element a_i under this isomorphism. In the same way we obtain the 2-evaluation $e_F: F^*F \Rightarrow 1_B$ by the isomorphism $\text{hom}(F^*(b_i), F^*(b_i)) \cong \text{hom}(F^*F(b_i), b_i)$. Note that here we continue to use the convention under which FF^* means first F and then F^* and not the other way around as is the more usual convention for functors and natural transformations, though not for 2-categories. In order to understand what i_H is in more concrete terms one should note that the matrix corresponding to FF^* has diagonal coefficients that are sums of squares of coefficients of F and that 1_A is just a diagonal matrix with all diagonal coefficients equal to 1. So i_H is defined by the sums of the coevaluation maps on the terms in the diagonal coefficients and by the zero map for all non-diagonal coefficients of FF^* . In the same way we see that e_H is defined by the “ordinary” evaluation maps on the terms of the diagonal coefficients of F^*F , which are squares also of course, and by the zero map on all the non-diagonal coefficients.

The dual of a 2-morphism $\alpha: F \Rightarrow G$ presented by the matrix (α_i^j) is the 2-morphism $\alpha^*: G \Rightarrow F$ presented by the matrix $((\alpha^*)_i^j)$, where $(\alpha^*)_i^j$ is the adjoint of α_i^j , obtained by taking the transpose and then the complex conjugate of each coefficient. A little thinking shows that α^\dagger is presented by

the adjoint of the whole matrix (α_i^j) . It is now obvious that $\alpha^{\dagger*} = \alpha^{* \dagger}$ corresponds to the matrix $((\alpha^{\dagger*})_i^j)$ where $(\alpha^{\dagger*})_i^j = \alpha_j^i$.

We now define 2Hilb_{cc} . The objects are the non-negative integers. A 1-morphism between n and m is an $m \times n$ matrix with non-negative integer coefficients. A 2-morphism between two 1-morphisms $(F_i^j), (G_i^j): n \rightarrow m$ is an $m \times n$ matrix (α_i^j) , where α_i^j is a $G_i^j \times F_i^j$ matrix with complex coefficients. The various compositions are defined as for the matrices representing 1- and 2-morphisms in 2Hilb .

The tensor product of two objects n and m is defined by $n \otimes m = nm$, the identity object being 1. The tensor product of a 1- or 2-morphism (X_i^j) and an object n is defined by the matrix $((X \otimes n)_{ik}^{jl})$ with coefficients $(X \otimes n)_{ik}^{jl} = X_i^j \delta_k^l$. Likewise we define $(n \otimes X)_{ik}^{jl} = \delta_i^j X_k^l$. Note that for any 1-morphisms $(F_i^j): n \rightarrow m$ and $(G_k^l): r \rightarrow s$ we have

$$[(F \otimes r)(m \otimes G)]_{ik}^{jl} = F_i^j G_k^l$$

and

$$[(n \otimes G)(F \otimes s)]_{ik}^{jl} = G_k^l F_i^j.$$

The tensorator is non-trivial,

$$\left(\bigotimes_{F, G} \right)_{ik}^{jl} = P_{F_i^j G_k^l},$$

where $P_{F_i^j G_k^l}$ is the $F_i^j G_k^l \times F_i^j G_k^l$ permutation matrix with coefficients $(P_{F_i^j G_k^l})_{xy}^{wz} = \delta_x^z \delta_y^w$. This matrix clearly corresponds to the permutation of the factors in the tensor product $F_i^j \otimes G_k^l = F_i^j G_k^l$. It is clear that 2Hilb_{cc} is a semi-strict monoidal 2-category. Note that so far the definition of 2Hilb_{cc} coincides with the definition of 2Vect_{cc} , the totally coordinatized version of 2Vect defined by Kapranov and Voevodsky (see Section 5.21 in [27]). Additionally we are going to define duals on 2Hilb_{cc} , which is the extra structure that distinguishes 2Hilb_{cc} from 2Vect_{cc} .

The dual of the object n is n itself. The coevaluation $i_n: 1 \rightarrow n^2$ is defined by the $n^2 \times 1$ matrix with coefficients $(i_n)_i^1 = 1$ if $i = (k-1)n + k$ for all $1 \leq k \leq n$ and $(i_n)_i^1 = 0$ otherwise. In order to find this matrix we used the equivalence $\text{Hom}(1, n \otimes n) \simeq \text{Hom}(n, n)$ and the fact that i_n corresponds to the $n \times n$ identity matrix under this equivalence. Loosely speaking one just renumbers the coefficients of this identity matrix according to the following principle: the coefficient on the k th row and the l th column of the identity matrix becomes the coefficient on the $((k-1)n + l)$ th row of i_n . In an analogous way we see that the evaluation e_n is given by the $1 \times n^2$ matrix that is the transpose of i_n . The rest of the duality structure on 2Hilb_{cc} can be defined as for the matrices representing 1- and 2-morphisms in 2Hilb .

With these definitions 2Hilb_{cc} becomes a semi-strict monoidal 2-category with duals as defined in Definition 2.3. It is easy to check that 2Hilb_{cc} is a semi-strict spherical 2-category. As a matter of fact it turns out that for any 1-morphism $F: n \rightarrow m$ the 1-morphism $F^\#$ also corresponds to the transpose of (F_i^j) . The same is true for ${}^\#F$, so we see that $F^\# = {}^\#F$, which suffices to conclude that 2Hilb is pivotal. For any 2-morphism $\alpha: F \Rightarrow G$ we find that $\alpha^\#$ corresponds to the matrix $((\alpha^\#)_i^j)$ where $(\alpha^\#)_i^j = \alpha_j^i$, as for $\alpha^{\dagger*}$. Note that this does not mean that $\alpha^\#$ is equal to $\alpha^{\dagger*}$, because $\alpha^\#$ is a 2-morphism from $F^\#$ to $G^\#$ and $\alpha^{\dagger*}$ is a 2-morphism from F^* to G^* .

The left trace-functor applied to FF^* gives the same result as when it is applied to F^*F , since this is the ordinary trace of the matrices corresponding to FF^* and F^*F , respectively. For the same reason compatibility with cupping and capping is guaranteed. As a matter of fact it is easy to see directly what the cup and cap are for $\text{Tr}_L(FF^*) = \text{Tr}_L(F^*F)$. The 2-morphism 1_I is just equal to 1, corresponding to the identity on \mathbb{C} , and $\text{Tr}_L(FF^*) = \text{Tr}_L(F^*F)$ is just the sum of the squares of the coefficients of F , so the cup $1_I \Rightarrow \text{Tr}_L(FF^*)$ is nothing but the sum of the respective coevaluations and the cap $\text{Tr}_L(FF^*) \Rightarrow 1_I$ is nothing but the sum of the respective evaluations.

The left and the right trace functor are equal when applied to 1-morphisms in 2Hilb_{cc} , again because these are just ordinary matrix traces, so the spherical condition is also satisfied.

Having defined a semi-strict spherical 2-category we now have to go a little further and add some linear structure. In order to define this linear structure we can work with a semi-strict 2-category C first.

DEFINITION 2.16. C is called *Vect-linear* if $\text{Hom}(A, B)$ for any two objects A, B in C is a 2-vector space of finite rank and the composition and the tensor product are Vect-bilinear, with Vect the ring category of finite dimensional vector spaces over the fixed field \mathbb{F} .

For the definitions of a 2-vector space and its rank and the definition of a ring category see [27]. The following lemma is straightforward.

LEMMA 2.17. *The category $\text{End}(I)$ is a braided monoidal ring category and $2\text{End}(1_I)$ is a commutative ring.*

Each category $\text{Hom}(A, B)$ becomes an $\text{End}(I)$ -module category with the action

$$\text{End}(I) \times \text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(A, B)$$

on objects defined by

$$f \times g \mapsto f \triangleright g = (f \otimes A)(I \otimes g).$$

The isomorphism $(f_1 f_2) \triangleright g \cong f_1 \triangleright (f_2 \triangleright g)$ is defined by $\bigotimes_{f_1 f_2, g}$ in the definition of a semi-strict monoidal 2-category, see condition $(\rightarrow \rightarrow \otimes \cdot)$ in [27]. Notice that

$$f \times g \mapsto (I \otimes g)(f \otimes B)$$

defines another action. This action is isomorphic to the one we have chosen by means of the isomorphisms $\bigotimes_{f, g}$. The action on morphisms is defined in an obvious way now. For the rest of this paper I will assume that $\text{End}(I)$ is equivalent to Vect and that $2\text{End}(1_I)$ is a field isomorphic to \mathbf{F} . It is now easy to prove that the composition and the tensor product in C are $\text{End}(I)$ -bilinear. It is obvious that the action of $\mathbf{F} = 2\text{End}(1_I)$ on $2\text{Hom}(f, g)$ for any $f, g: A \rightarrow B$ and for any A and B is the one induced by the action above. Notice that it makes sense to write $\text{Hom}(A, B) \oplus \text{Hom}(C, D)$ as the direct sum of two $\text{End}(I)$ -module categories and $\text{Hom}(A, B) \otimes \text{Hom}(C, D)$ as the tensor product of two $\text{End}(I)$ -module categories, see [27].

The following condition that we should impose on our 2-categories concerns the non-degeneracy of the pairing in Definition 2.9. Assume that C is a Vect -linear semi-strict spherical 2-category. Notice that the pairing is bi-linear.

DEFINITION 2.18. We call C *non-degenerate* if the pairing in Definition 2.9 is non-degenerate.

As in the case of \mathbf{F} -linear spherical categories [7, 9] one can always take a non-degenerate quotient of an additive semi-strict spherical 2-category.

LEMMA 2.19. *Let C be as above. Let J be the Vect -linear subcategory with the same objects and 1-morphisms, but with $2\text{Hom}_J(f, g)$ being the sub-vector space of $2\text{Hom}_C(f, g)$ defined by*

$$2\text{Hom}_J(f, g) = \{\alpha \in 2\text{Hom}_C(f, g) \mid \langle \alpha, \beta \rangle = 0 \ \forall \beta \in 2\text{Hom}_C(g, f)\}.$$

Then C/J is a Vect -linear semi-strict spherical 2-category.

Proof. It is clear that the vertical composition of 2-morphisms is well defined in C/J .

Let us now prove that the horizontal composition is well defined also. Let $f, g: X \rightarrow Y$ and $f', g': Y \rightarrow Z$ be 1-morphisms in C . For any

$\alpha \in 2\text{Hom}_J(f, g)$, any $\beta \in 2\text{Hom}_C(f', g')$ and any $\gamma \in 2\text{Hom}_C(gg', ff')$ we have

$$\langle \alpha \circ \beta, \gamma \rangle = \langle \alpha, (g \circ i_{f'}) \cdot (g \circ \beta \circ (f')^*) \cdot (\gamma \circ (f')^*) \cdot (f \circ i_{f'}^*) \rangle = 0.$$

Next we show that the tensor product is well defined in C/J . If $\alpha \in 2\text{Hom}_J(f, g)$, then $\alpha \otimes W \in 2\text{Hom}_J(f \otimes W, g \otimes W)$ and $W \otimes \alpha \in 2\text{Hom}_J(W \otimes f, W \otimes g)$ for any object W . The proof of these facts is not difficult, but is a bit cumbersome because we have to keep track of both the vertical and the horizontal composition. Writing out everything carefully gives

$$\begin{aligned} \langle \alpha \otimes W, \beta \rangle &= \left\langle \alpha, (g \circ i_{(B \otimes i_W)}) \cdot \left(\bigotimes_{g, i_W} (B \otimes 1_{i_W}^*) \right) \right. \\ &\quad \cdot ((A \otimes 1_{i_W}) \circ (\beta \otimes W^*) \circ (B \otimes 1_{i_W}^*)) \\ &\quad \left. \cdot \left(\bigotimes_{f, i_W}^{-1} (B \otimes 1_{i_W}^*) \right) \cdot (f \circ i_{(B \otimes i_W)}^*) \right\rangle = 0. \end{aligned}$$

Notice that the long formula defining the second 2-morphism in the second pairing is really a 2-morphism from g to f , so that its pairing with α makes sense. This shows our first assertion. The proof of the second is analogous and we leave the details to the reader.

Lemma 2.11 shows that $\alpha^* \in 2\text{Hom}_J(g, f)$ if and only if $\alpha \in 2\text{Hom}_J(f, g)$, Lemma 2.12 shows that $\alpha^\dagger \in 2\text{Hom}_J(g^*, f^*)$ if and only if $\alpha \in 2\text{Hom}_J(f, g)$, and Lemma 2.13 shows that $\alpha^\# \in 2\text{Hom}_J(f^\#, g^\#)$ if and only if $\alpha \in 2\text{Hom}_J(f, g)$.

Now take the quotients in C/J of all the structural 2-morphisms involved in the definition of the tensor product, the duality, and the pivotal and the spherical conditions. Then C/J becomes a Vect-linear non-degenerate semi-strict spherical 2-category. ■

Now let us define semi-simplicity for Vect-linear 2-categories. A non-zero object A in C is an object for which we have $\text{End}(A) \neq 0$.

DEFINITION 2.20. Let C be a Vect-linear semi-strict monoidal 2-category. We say that C is finitely semi-simple if the following condition is satisfied: There is a finite set of non-equivalent non-zero objects \mathcal{E} such that for any pair of objects A, B in C we have

$$\bigoplus_{X \in \mathcal{E}} \text{Hom}(A, X) \otimes \text{Hom}(X, B) \simeq \text{Hom}(A, B).$$

The equivalence is given by the obvious composition of 1- and 2-morphisms.

Note that each Hom-space is finitely semi-simple as an \mathbf{F} -linear category because it is a 2-vector space of finite rank. Let us fix a basis $\mathcal{F}_{A, B} \in \text{Hom}(A, B)$ for any objects A and B . We will usually refer to this semi-simplicity of the Hom-spaces as “vertical semi-simplicity.” The semi-simplicity

in the definition above we will always refer to as “horizontal semi-simplicity.” Note that $2\text{Hilb}_{\mathbb{C}}$ is finitely semi-simple with the unique simple non-zero object 1 . It is also non-degenerate, because the pairing of a 2-morphism α with its dual is equal to the sum of the squares of the absolute values of the complex coefficients of the matrix representing α , which of course is non-zero if α is non-zero.

DEFINITION 2.21. Let C be a Vect-linear semi-strict monoidal 2-category. An object A in C is called simple if $\text{rk}(\text{End}(A)) = 1$.

Here $\text{rk}(\text{End}(A))$ is the rank of $\text{End}(A)$ as a 2-vector space, see [27].

LEMMA 2.22. Assume that C is a Vect-linear semi-strict monoidal finitely semi-simple 2-category. An object A in C is simple if and only if A is equivalent to one in \mathcal{E} .

Proof. The proof is identical to the one that proves the analogous statement about finitely semi-simple categories. It follows from the facts

$$\text{rk}(X \oplus Y) = \text{rk}(X) + \text{rk}(Y), \quad \text{rk}(X \otimes Y) = \text{rk}(X) \text{rk}(Y),$$

for any 2-vector spaces X and Y . These identities can be found in [27]. ■

Let us define the quantum dimension of objects and 1-morphisms in a finitely semi-simple non-degenerate semi-strict spherical 2-category.

DEFINITION 2.23. Let A be an object in C . Then we define its quantum dimension to be

$$\dim_q(A) = \langle 1_{1_A}, 1_{1_A} \rangle.$$

DEFINITION 2.24. Let f be a 1-morphism in C . Then we define its quantum dimension to be

$$\dim_q(f) = \langle 1_f, 1_f \rangle.$$

LEMMA 2.25. For any simple objects A, B, C, D, E and any 1-morphisms $f: A \rightarrow B \otimes C$, $g: B \rightarrow D \otimes E$ and $h: C \rightarrow D \otimes E$ we have

$$\dim_q(f(g \otimes C)) = \dim_q(f) \dim_q(B)^{-1} \dim_q(g),$$

and

$$\dim_q(f(B \otimes h)) = \dim_q(f) \dim_q(C)^{-1} \dim_q(h).$$

Proof. Let us first look closely at $\dim_q(f)$. The 1-morphism $ff^*: A \rightarrow A$ is just a “multiple” of the identity, because A is simple. “Multiple” here means that there is a vector space $V(f)$ such that $ff^* = V(f)1_A$. So we get

$$\dim_q(f) = \dim(V(f)) \dim_q(A).$$

Of course the same holds for g and h .

Using $f(g \otimes C)(g^* \otimes C)f^* = V(g)ff^*$, we get

$$\dim_q(f(g \otimes C)) = \dim_q(f) \dim(V(g)) = \dim_q(f) \dim_q(B)^{-1} \dim_q(g).$$

In the same way we get

$$\dim_q(f(B \otimes h)) = \dim_q(f) \dim(V(h)) = \dim_q(f) \dim_q(C)^{-1} \dim_q(h). \quad \blacksquare$$

Note also that $\dim_q(A^*) = \dim_q(A)$ for any object A by the spherical condition, that $\dim_q(A \otimes B) = \dim_q(A) \otimes \dim_q(B)$ for any objects A and B by the pivotal and the spherical conditions, that $\dim_q(f^*) = \dim_q(f)$ for any 1-morphism f by Lemma 2.12, and the fact that $1_f^\dagger = 1_{f^*}$, and that $\dim_q(f^\#) = \dim_q(\#f) = \dim_q(f)$ for any 1-morphism f by Lemma 2.13.

Finally we have to define the dimension of C .

DEFINITION 2.26. Let C be a finitely semi-simple non-degenerate semi-strict spherical 2-category. Its dimension K is defined as the number of equivalence classes of non-zero simple objects in C .

This definition may seem rather surprising at first. In the proof of invariance under the $1 \rightleftharpoons 5$ Pachner move in Lemma 5.4 and the proof of the auxiliary Lemma 5.6 we show that this defines the right weight for the vertices of the triangulation of a manifold. Probably it has to do with the fact that we define a simple object to be an object A such that $\text{End}(A) \simeq \text{Vect}$, which has dimension 1 as a category, and not just any finitely semi-simple non-degenerate spherical category. In our concluding remarks we will say a little more about this.

3. THE DEFINITIONS OF $Z_C(\pm(IJKLM))$ AND $I_C(M, T)$

In this section $M = (M, T)$ still denotes a triangulated manifold. We also assume that there is a total ordering on the vertices of the simplicial complex defining the triangulation T and a total ordering on the 4-simplices of the same complex. Let C be a finitely semi-simple semi-strict non-degenerate spherical 2-category with a fixed finite basis of simple objects \mathcal{E} and for any objects A and B a fixed finite basis of simple 1-morphisms $\mathcal{F}_{A,B} \in \text{Hom}(A, B)$. The linear maps $Z_C(\pm(ijklm))$, the number $Z_C(M, T, \ell)$,

and the state sum $I_C(M, T)$ obviously depend on the given 2-category C , but we will suppress the subscript C at all places where this does not lead to any confusion.

We now label the edges (ij) of the triangulation with simple objects e_{ij} in \mathcal{C} and the triangles (ijk) with simple 1-morphisms $f_{ijk} \in \mathcal{F}_{ijk} = \mathcal{F}_{e_{ik}, e_{jk} \otimes e_{ij}}$, the chosen basis in $\text{Hom}(e_{ik}, e_{jk} \otimes e_{ij}) = H(ijk)$. We will use the following notation for the different compositions of the 1-morphisms:

$$f_{(ijk)l} = f_{ikl}(e_{kl} \otimes f_{ijk}),$$

$$f_{i(jkl)} = f_{ijl}(f_{jkl} \otimes e_{ij}),$$

$$f_{((ijk)l)m} = f_{(ikl)m}(e_{lm} \otimes e_{kl} \otimes f_{ijk}) = f_{ilm}(e_{lm} \otimes f_{ikl})(e_{lm} \otimes e_{kl} \otimes f_{ijk}),$$

$$f_{i(j(klm))} = f_{i(jkm)}(f_{klm} \otimes e_{jk} \otimes e_{ij}) = f_{ijm}(f_{jkm} \otimes e_{ij})(f_{klm} \otimes e_{jk} \otimes e_{ij}).$$

Let $+(ijklm)$ be the positively oriented standard 4-simplex labelled as described above. We are going to define the linear map

$$\begin{aligned} Z(+ (ijklm)): 2\text{Hom}(f_{(ikl)m}, f_{i(klm)}) \otimes 2\text{Hom}(f_{(ijk)m}, f_{i(jkm)}) \\ \rightarrow 2\text{Hom}(f_{(jkl)m}, f_{j(klm)}) \otimes 2\text{Hom}(f_{(ijl)m}, f_{i(jlm)}) \otimes 2\text{Hom}(f_{(ijk)l}, f_{i(jkl)}) \end{aligned}$$

using the pairing $\langle \cdot, \cdot \rangle$ described in the previous section. Consider the linear map

$$2\text{Hom}(f_{(ikl)m}, f_{i(klm)}) \otimes 2\text{Hom}(f_{(ijk)m}, f_{i(jkm)}) \rightarrow 2\text{Hom}(f_{((ijk)l)m}, f_{i(j(klm))})$$

defined by

$$\beta \otimes \delta \mapsto (\beta \circ (e_{lm} \otimes e_{kl} \otimes f_{ijk})) \cdot \left(f_{ikm} \circ \bigotimes_{f_{klm}, f_{ijk}} \right) \cdot (\delta \circ (f_{klm} \otimes e_{jk} \otimes e_{ij})).$$

Let us write the right-hand side as $\beta\delta$ as a shorthand. Consider also the linear map

$$\begin{aligned} 2\text{Hom}(f_{j(klm)}, f_{(jkl)m}) \otimes 2\text{Hom}(f_{i(jlm)}, f_{(ijl)m}) \otimes 2\text{Hom}(f_{i(jkl)}, f_{(ijk)l}) \\ \rightarrow 2\text{Hom}(f_{i(j(klm))}, f_{((ijk)l)m}) \end{aligned}$$

defined by

$$\alpha \otimes \gamma \otimes \varepsilon \mapsto (f_{ijm} \circ (\alpha \otimes e_{ij})) \cdot (\gamma \circ (e_{lm} \otimes f_{ijk} \otimes e_{ij})) \cdot (f_{ilm} \circ (e_{lm} \otimes \varepsilon)).$$

Let us write the right-hand side as $\alpha\gamma\varepsilon$ as a shorthand. Now define the linear map

$$\begin{aligned} 2\text{Hom}(f_{j(klm)}, f_{(jkl)m}) \otimes 2\text{Hom}(f_{(ikl)m}, f_{i(klm)}) \otimes 2\text{Hom}(f_{i(jlm)}, f_{(ijl)m}) \\ \otimes 2\text{Hom}(f_{(ijk)m}, f_{i(jkm)}) \otimes 2\text{Hom}(f_{i(jkl)}, f_{(ijk)l}) \rightarrow \mathbf{F} \end{aligned}$$

by

$$\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \mapsto \langle \beta\delta, \alpha\gamma\varepsilon \rangle.$$

Using $2\text{Hom}(g, f) = 2\text{Hom}(f, g)^*$, following from the non-degeneracy of $\langle \cdot, \cdot \rangle$, this gives us $Z(+ (ijklm))$.

Before we go on, let us have a look at a diagrammatic picture of $Z(+ (01234))$. In Fig. 1 we have depicted a diagram that resembles a 15j symbol as defined in [17]. Here we have taken the dual 1-skeleton of the boundary of a 4-simplex, a graph with 5 4-valent vertices and 10 edges, and “split” each 4-valent vertex into 2 trivalent vertices connected by an edge which we call a “dumbbell.” Each dumbbell in the resulting graph corresponds to a tetrahedral face of the 4-simplex; the remaining 10 edges in this graph correspond to the triangular faces of the 4-simplex. Here we have labelled the dumbbells by the 2-morphisms $\alpha, \beta, \gamma, \delta, \varepsilon$, and labelled the remaining edges by triples (ijk) corresponding to the 1-morphisms f_{ijk} . Finally, the crossing is just $\bigotimes_{f_{234}, f_{012}}$ (the tensorator that appears in the definition of $\beta\delta$). In this diagram we do not see the objects involved and so we do not see the effect of the trace functor and the final caps and cups either. But we do see the “kind of trace” mentioned after the definition of the pairing. Remember, if $f, g: A \rightarrow B$ are 1-morphisms and $\zeta: f \Rightarrow g$ and $\xi: g \Rightarrow f$ are 2-morphisms, then one obtains $\langle \zeta, \xi \rangle$ by first taking a “kind of trace,” namely

$$1_A \xrightarrow{i_f} ff^* \xrightarrow{\zeta \circ f^*} gf^* \xrightarrow{\xi \circ f^*} ff^* \xrightarrow{i_f^*} 1_A.$$

This is exactly the diagram we see when we take $\zeta = \beta\delta$ and $\xi = \alpha\gamma\varepsilon$, the respective 2-morphisms involved in the definition of $Z(+ (01234))$. The properties satisfied by the duality of the 1-morphisms, condition (11) in Definition 2.3, and the properties of the composition of \bigotimes, \dots , analogous to the properties of the braiding in a braided category, allow us to apply isotopies to this diagram, corresponding to Reidemeister moves 0, 2, and 3, without changing the linear map $Z(+ (01234))$ defined by the diagram. Since our 2-category C is also assumed to be spherical, we can also exchange closing strands on the left-hand side for closing strands to the right-hand side of the diagram. The pivotal condition alone justifies these manipulations of the diagrams, but the spherical condition is needed for the different ways of nesting the cups and the caps that cannot be seen in the diagrams. These are the ones we put on our pairing after applying the trace functor. This nice behaviour under different ways of nesting turns out to be necessary for our purposes, as can be seen in the proof of Lemma 4.4. There this kind of diagram allows us to prove invariance of our invariant under any permutation of the vertices of the chosen triangulation. The formulas, which I had written out completely by hand at first, are far too

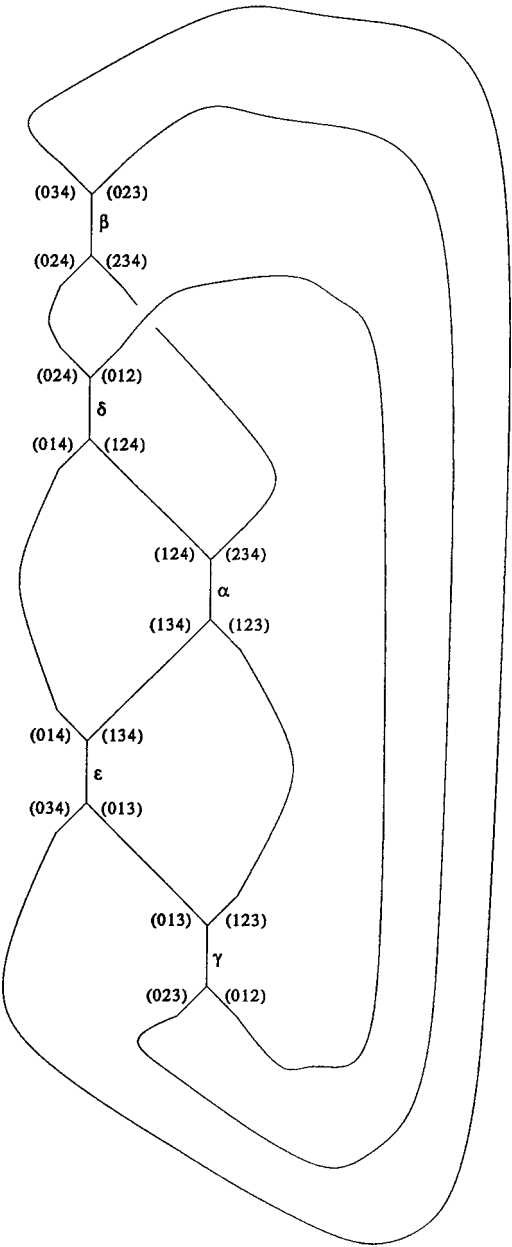


FIGURE 1

large to fit on ordinary sheets of paper. So, although these diagrams do not define a complete diagrammatic calculus for my invariants, since they do not show the objects, they are certainly very helpful to see what is going on. Better diagrams would be very welcome, but so far I have not been able to invent them.

Let us now define

$$\begin{aligned} Z(-(ijklm)): 2\text{Hom}(f_{(jkl)m}, f_{j(klm)}) \otimes 2\text{Hom}(f_{(ijl)m}, f_{i(jlm)}) \\ \otimes 2\text{Hom}(f_{(ijk)l}, f_{i(jkl)}) \\ \rightarrow 2\text{Hom}(f_{(ikl)m}, f_{i(klm)}) \otimes 2\text{Hom}(f_{(ijk)m}, f_{i(jkm)}). \end{aligned}$$

Consider the linear map

$$\begin{aligned} 2\text{Hom}(f_{(jkl)m}, f_{j(klm)}) \otimes 2\text{Hom}(f_{(ijl)m}, f_{i(jlm)}) \otimes 2\text{Hom}(f_{(ijk)l}, f_{i(jkl)}) \\ \rightarrow 2\text{Hom}(f_{((ijk)l)m}, f_{i(j(klm))}) \end{aligned}$$

defined by

$$\alpha \otimes \gamma \otimes \varepsilon \mapsto (f_{ilm} \circ (e_{lm} \otimes \varepsilon)) \cdot (\gamma \circ (e_{lm} \otimes f_{jkl} \otimes e_{ij})) \cdot (f_{ijm} \circ (\alpha \otimes e_{ij})).$$

Call this $\varepsilon\gamma\alpha$. Consider also the linear map

$$2\text{Hom}(f_{i(klm)}, f_{(ikl)m}) \otimes 2\text{Hom}(f_{i(jkm)}, f_{(ijk)m}) \rightarrow 2\text{Hom}(f_{i(j(klm))}, f_{((ijk)l)m})$$

by

$$\begin{aligned} \beta \otimes \delta \mapsto (\delta \circ (f_{klm} \otimes e_{jk} \otimes e_{ij})) \cdot \left(f_{024} \circ \bigotimes_{f_{klm}, f_{ijk}}^{-1} \right) \\ \cdot (\beta \circ (e_{lm} \otimes e_{kl} \otimes f_{ijk})). \end{aligned}$$

Call this $\delta\beta$. Now define the linear map

$$\begin{aligned} 2\text{Hom}(f_{(jkl)m}, f_{j(klm)}) \otimes 2\text{Hom}(f_{i(klm)}, f_{(ikl)m}) \otimes 2\text{Hom}(f_{(ijl)m}, f_{i(jlm)}) \\ \otimes 2\text{Hom}(f_{i(jkm)}, f_{(ijk)m}) \otimes 2\text{Hom}(f_{(ijk)l}, f_{i(jkl)}) \rightarrow \mathbf{F} \end{aligned}$$

defined by

$$\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \mapsto \langle \delta\beta, \varepsilon\gamma\alpha \rangle.$$

Using $2\text{Hom}(g, f) = 2\text{Hom}(f, g)^*$ again this gives us $Z(-(ijklm))$.

In the sequel let us write $2H(ijkl)$ for $2\text{Hom}(f_{(ijk)l}, f_{i(jkl)})$. Likewise let us write $2H(ijklm)$ for $2\text{Hom}(f_{((ijk)l)m}, f_{i(j(klm))})$. The next lemma is the analogue of the Crossing Lemma 5.4 in [9].

LEMMA 3.1. (crossing). *The following diagram is commutative:*

$$\begin{array}{ccc}
 \bigoplus_{f_{024}} 2H(0234) \otimes 2H(0124) & \xrightarrow{\Phi_{01234}} & \bigoplus_{\substack{e_{13}, f_{013}, \\ f_{123}, f_{134}}} 2H(1234) \otimes 2H(0134) \otimes 2H(0123) \\
 \downarrow & & \downarrow \\
 2H(01234) & \xlongequal{\hspace{2cm}} & 2H(01234)
 \end{array}$$

Here the vertical linear maps are the isomorphisms defined by the composition of the respective 2-morphisms and Φ_{01234} is defined by

$$\begin{aligned}
 \Phi_{01234} = & \bigoplus_{\substack{e_{13}, f_{024}, f_{013}, \\ f_{123}, f_{134}}} Z(+ (01234)) \dim_q(e_{13})^{-1} \dim_q(f_{013}) \\
 & \times \dim_q(f_{123}) \dim_q(f_{134}).
 \end{aligned}$$

Furthermore the inverse of Φ_{01234} is given by

$$\Phi_{01234}^{-1} = \bigoplus_{\substack{e_{13}, f_{024}, f_{013}, \\ f_{123}, f_{134}}} Z(- (01234)) \dim_q(f_{024}).$$

Proof. First of all let us explain why the vertical maps are isomorphisms. Using the same notation as above we must show that for any 2-morphism $\alpha \in 2\text{Hom}(f_{((012)3)4}, f_{0(1(234))})$ there exist 2-morphisms $\beta_{024} \in 2\text{Hom}(f_{(023)4}, f_{0(234)})$ and $\gamma_{024} \in 2\text{Hom}(f_{(012)4}, f_{0(124)})$ such that

$$\sum_{f_{024}} (\beta_{024} \circ (e_{34} \otimes e_{23} \otimes f_{012})) \cdot \left(f_{024} \circ \bigotimes_{f_{234}, f_{012}} \right) \cdot (\gamma_{024} \circ (f_{234} \otimes e_{12} \otimes e_{01})) = \alpha.$$

By (vertical) semi-simplicity we get the following (vertical) decomposition of α :

$$\begin{array}{ccc}
 f_{034}(e_{34} \otimes f_{023})(e_{34} \otimes e_{23} \otimes f_{012}) & & f_{034}(e_{34} \otimes f_{023})(e_{34} \otimes e_{23} \otimes f_{012}) \\
 \Downarrow & & \Downarrow \\
 & & \sum_{f_{024}} f_{024}(f_{234} \otimes e_{02})(e_{34} \otimes e_{23} \otimes f_{012}) \\
 = & & \Downarrow \otimes_{f_{234}, f_{012}}^{-1} \\
 & & \sum_{f_{024}} f_{024}(e_{24} \otimes f_{012})(f_{234} \otimes e_{12} \otimes e_{01}) \\
 & & \Downarrow \\
 f_{014}(f_{124} \otimes e_{01})(f_{234} \otimes e_{12} \otimes e_{01}) & & f_{014}(f_{124} \otimes e_{01})(f_{234} \otimes e_{12} \otimes e_{01})
 \end{array}$$

Notice that the 1-morphisms $f_{024}(f_{234} \otimes e_{02})(e_{34} \otimes e_{23} \otimes f_{012})$ do not form a basis of simple 1-morphisms of $2\text{Hom}(e_{04}, e_{34} \otimes e_{23} \otimes e_{12} \otimes e_{01})$ in

general, but they are generators by (horizontal) semi-simplicity and the fact that the f_{024} , the f_{234} , and the f_{012} form bases of their respective Hom-spaces. So by summing the projections on the simple components of each $f_{024}(f_{234} \otimes e_{02})(e_{34} \otimes e_{23} \otimes f_{012})$ we get the decomposition in the diagram. By (horizontal) semi-simplicity we can now decompose the 2-morphisms on the right side in the diagram above and we get the β_{024} and γ_{024} we were looking for. This proves that the left vertical map in the lemma is an isomorphism. In an analogous way one proves that the other vertical map in the lemma is an isomorphism.

The commutativity of the diagram in the lemma now follows by taking arbitrary elements in $H(0234)$ and in $H(0124)$, respectively, and pairing it with an arbitrary element in $H(01234)^*$. In order to get the multiplicative factors in the definition of Φ we have to use the identity

$$\dim_q(f_{ikl}(e_{kl} \otimes f_{ijk})) = \dim_q(f_{ikl}) \dim_q(e_{ik})^{-1} \dim_q(f_{ikl}).$$

This identity follows from Lemma 2.25.

The inverse of Φ can be computed by reading the diagram the other way around and using similar arguments. ■

We now define our state sum. Let C be a finitely semi-simple non-degenerate semi-strict spherical 2-category with non-zero dimension.

DEFINITION 3.2. With the data and notations as above we define for every closed compact piece-wise linear 4-manifold M with triangulation T the state sum

$$I_C(M, T) = K^{-v} \sum_{\ell} Z_C(M, T, \ell) \prod_e \dim_q(\ell(e))^{-1} \prod_f \dim_q(\ell(f)).$$

Here v is the number of vertices in T . We sum over all the labellings ℓ and take the products over all the edges e and all the faces f in the triangulation.

4. $Z_C(M, T, \ell)$ IS A COMBINATORIAL INVARIANT

In this section we show that $Z(M, T, \ell) = Z_C(M, T, \ell)$ is equal for all “isomorphic” labellings and $Z(M, T, \ell) = Z(M', T', \ell')$ for any pair of triangulated manifolds M and M' with triangulations T and T' that are isomorphic under a combinatorial isomorphism and any pair of “compatible” labellings ℓ and ℓ' .

First of all we have to show that $Z(M, T, \ell)$ does not depend on the choice of representatives $f_{ijk} \in \mathcal{F}_{ijk}$ nor on the choice of representatives $e_{ij} \in \mathcal{E}_{ij}$.

LEMMA 4.1. *Let ℓ and ℓ' be labellings with the same basis of objects \mathcal{E}_{ij} but different bases of 1-morphisms in \mathcal{F}_{ijk} . Then we have $Z(M, T, \ell) = Z(M, T, \ell')$.*

Proof. Let us denote the labels in ℓ by f_{ijk} and the labels in ℓ' by f'_{ijk} . They are representatives of the same isomorphism classes so for any triple ijk there is a 2-isomorphism $\phi_{ijk} : f_{ijk} \Rightarrow f'_{ijk}$. These 2-isomorphisms induce an isomorphism of vector spaces

$$2\mathrm{Hom}(f_{(ijk)l}, f_{i(jkl)}) \cong 2\mathrm{Hom}(f'_{(ijk)l}, f'_{i(jkl)})$$

given by

$$\alpha \mapsto \phi_{(ijk)l}^{-1} \alpha \phi_{i(jkl)}.$$

Here $\phi_{(ijk)l}$ stands for $\phi_{ikl}(e_{kl} \otimes f_{ijk})$ and $\phi_{i(jkl)}$ for $\phi_{ijl}(\phi_{jkl} \otimes e_{ij})$. Likewise there is an isomorphism between $2\mathrm{Hom}(f_{((ijk)l)m}, f_{i(j(klm))})$ and $2\mathrm{Hom}(f'_{((ijk)l)m}, f'_{i(j(klm))})$.

Now without writing out the explicit formulas, which is not very difficult but extremely tedious, one can see immediately the result of the lemma. The crossing lemma implies that the following diagram is commutative.

$$\begin{array}{ccc} \bigoplus_{f_{ikm}} 2H(iklm) \otimes 2H(ijkm) & \xrightarrow{Z(ijklm)} & \bigoplus_{\substack{e_{jl}, f_{ijl}, \\ f_{jkl}, f_{jlm}}} 2H(jklm) \otimes 2H(ijlm) \otimes 2H(ijkl) \\ \downarrow & & \downarrow \\ \bigoplus_{f'_{ikm}} 2H'(iklm) \otimes 2H'(ijkm) & \xrightarrow{Z'(ijklm)} & \bigoplus_{\substack{e'_{jl}, f'_{ijl}, \\ f'_{jkl}, f'_{jlm}}} 2H'(jklm) \otimes 2H'(ijlm) \otimes 2H'(ijkl) \end{array}$$

So $Z'(ijklm)$ is a conjugate of $Z(ijklm)$. Taking the respective traces over the tensor product of all the partition functions now shows that $Z(M, T, \ell) = Z(M, T, \ell')$. ■

Now suppose we take a different basis of simple objects in C . In other words, suppose we have two labellings ℓ and ℓ' such that $\ell((ij)) = e_{ij}$ is equivalent to $\ell'((ij)) = e'_{ij}$ for every edge. Let us assume that there is an isomorphism $\phi_{ij} : e_{ij} \rightarrow e'_{ij}$ for every i, j actually. In general ϕ_{ij} fails to be an isomorphism just by an invertible scalar, whose existence is not important for the following arguments. These isomorphisms induce a linear isomorphism

$$\phi_{ijk} : \mathrm{Hom}(e_{ik}, e_{jk} \otimes e_{ij}) \rightarrow \mathrm{Hom}(e'_{ik}, e'_{jk} \otimes e'_{ij})$$

defined by

$$f_{ijk} \mapsto \phi_{ik}^{-1} f_{ijk}(\phi_{jk} \otimes \phi_{ij})$$

for any ijk . Denote these 1-morphisms by f'_{ijk} . It is clear that the simplicity of f_{ijk} implies the simplicity of f'_{ijk} for any ijk . Since we have proved that $Z(M, T, \ell)$ is independent of the choice of basis of simple 1-morphisms in Lemma 4.1 we may assume that $\ell'((ijk)) = f'_{ijk}$.

LEMMA 4.2. *Let ℓ and ℓ' be two labellings as described above. Then $Z(M, T, \ell) = Z(M, T, \ell')$.*

Proof. The identities

$$\begin{aligned} f'_{ikl}(e'_{kl} \otimes f'_{ijk}) &= \phi_{il}^{-1} f_{ikl}(\phi_{kl} \otimes \phi_{ik})(\phi_{kl}^{-1} \otimes \phi_{ik}^{-1}(e_{kl} \otimes f_{ijk}))(\phi_{kl} \otimes \phi_{jk} \otimes \phi_{ij}) \\ &= \phi_{il}^{-1} f_{ikl}(e_{kl} \otimes f_{ijk})(\phi_{kl} \otimes \phi_{jk} \otimes \phi_{ij}) \end{aligned}$$

and

$$\begin{aligned} f'_{ijl}(f'_{jkl} \otimes e'_{ij}) &= \phi_{jl}^{-1} f_{ijl}(\phi_{jl} \otimes \phi_{ij})(\phi_{jl}^{-1} \otimes \phi_{ij}^{-1}(f_{jkl} \otimes e_{ij}))(\phi_{kl} \otimes \phi_{jk} \otimes \phi_{ij}) \\ &= \phi_{ij}^{-1} f_{ijl}(f_{jkl} \otimes e_{ij})(\phi_{kl} \otimes \phi_{jk} \otimes \phi_{ij}) \end{aligned}$$

show the existence of a linear isomorphism

$$2H(ijkl) = 2\text{Hom}(f_{(ijk)l}, f_{i(jkl)}) \cong 2\text{Hom}(f'_{(ijk)l}, f'_{i(jkl)}) = 2H'(ijkl).$$

Again by applying the crossing lemma we get the commutative diagram of the previous lemma, although the vertical arrows now represent different linear maps. So again $Z(ijklm)$ and $Z'(ijklm)$ are conjugates, which implies that $Z(M, T, \ell)$ and $Z(M, T, \ell')$ are equal. ■

Finally let us prove that $Z(M, T, \ell)$ does not depend on the ordering of the vertices in the triangulation T of M .

DEFINITION 4.3. Let $\phi: T \rightarrow T'$ be a combinatorial isomorphism of two triangulations of M . Let the edge (ij) of T be labelled with a simple object e_{ij} for all pairs of different vertices i and j of T and the edge (ij) of T' with a simple object e'_{ij} for all pairs of different vertices i and j of T' . Let a triangle (ijk) of T be labelled with a simple 1-morphism f_{ijk} for all triples of different vertices i, j, k of T and a triangle (ijk) of T' with a simple 1-morphism f'_{ijk} for all triples of different vertices i, j, k of T' . We say that ϕ is compatible with the labellings if the following conditions are satisfied:

(1) $e'_{\phi(i)\phi(j)} = e_{ij}$ if ϕ preserves the orientation of the edge and $e'_{\phi(i)\phi(j)} = e_{ij}^*$ if it reverses the orientation.

(2) If ϕ decomposes into the transposition $(ijk) \mapsto (ikj)$ and a simplicial isomorphism, then $f'_{\phi(i)\phi(j)\phi(k)}$ is unitarily isomorphic to $(e_{kj} \otimes e_{ij})(e_{kj} \otimes f_{ijk}^*)$. If the transposition in the decomposition of ϕ is $(ijk) \mapsto (kji)$, then $f'_{\phi(i)\phi(j)\phi(k)}$ is unitarily isomorphic to $(f_{ijk}^\#)^* = \#(f_{ijk}^*)$.

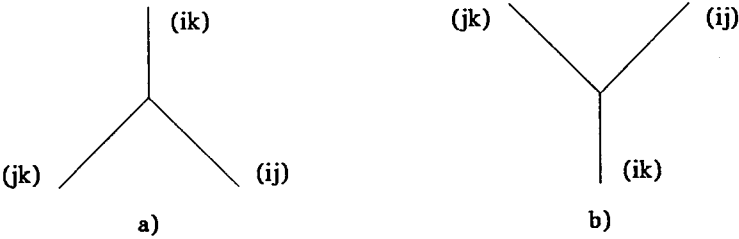


FIG. 2. (a) f_{ijk} . (b) f_{ijk}^* .

If we represent f_{ijk} and f_{ijk}^* by the diagrams in Fig. 2, then we can represent $(i_{e_{kj}} \otimes e_{ij})(e_{kj} \otimes f_{ijk}^*)$ as in Fig. 3 and $(f_{ijk}^\#)^*$ as in Fig. 4. Note that any combinatorial isomorphism ϕ , when restricted to a triangle (ijk) in T , always decomposes into a permutation of (ijk) and a simplicial isomorphism. Since S_3 is generated by the two transpositions in condition (2), we have really defined compatibility for any combinatorial isomorphism. Note also that in condition (2) we have only required the isomorphy of the corresponding 1-morphisms. The reason for this is that we want the composition of a combinatorial isomorphism compatible with the labellings ϕ with its inverse ϕ^{-1} to be compatible with the labellings. This would not be the case if we required the corresponding 1-morphisms to be equal because $^\#(f_{ijk}^\#)$ is only isomorphic to f_{ijk} . For our purpose these isomorphisms do not matter, because we have already shown that $Z(M, T, \ell)$ is independent of the choice of representative simple 1-morphisms in each isomorphism class. So, given a combinatorial isomorphism ϕ and a labelling of T , there is a unique compatible labelling only up to isomorphism.

Since $Z(M, T, \ell)$ does not depend on the chosen ordering on the 4-simplices themselves, as we have explained already at the end of Section 1, we can restrict our attention to what happens when we permute the vertices of a 4-simplex. The symmetric group on 5 elements, S_5 , is generated by the transpositions interchanging i and 5 for $1 \leq i \leq 4$, so we will restrict ourselves to showing that $Z(M, T, \ell)$ is invariant under the

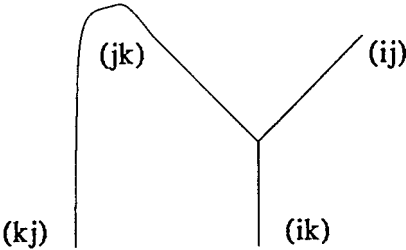
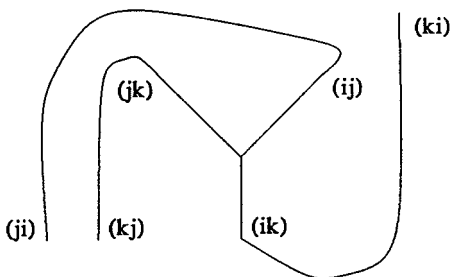


FIG. 3. $(i_{e_{kj}} \otimes e_{ij})(e_{kj} \otimes f_{ijk}^*)$.

FIG. 4. $(f_{ijk}^\#)^*$.

transposition $\sigma: (01234) \mapsto (01432)$. The cases of the other transpositions are similar. Remember the definition of the boundary of $-(01432)$

$$-\partial(01432) = -(1432) + (0432) - (0132) + (0142) - (0143).$$

We have to show that σ induces linear isomorphisms $\sigma_{0432}: 2H(0234) \rightarrow 2H(0432)^*$ and $\sigma_{0142}: 2H(0124) \rightarrow 2H(0142)$ and linear isomorphisms $\sigma'_{1432}: 2H(1234)^* \rightarrow 2H(1432)$, $\sigma'_{0143}: 2H(0134)^* \rightarrow 2H(0143)$ and $\sigma'_{0132}: 2H(0123)^* \rightarrow 2H(0132)$ such that the following diagram

$$\begin{array}{ccc} 2H(+ (01234)) & \xrightarrow{\sigma_{01432}} & 2H(- (01432)) \\ \downarrow & & \downarrow \\ \mathbf{F} & \xlongequal{\quad\quad\quad} & \mathbf{F} \end{array} \quad (4.4)$$

commutes. In this diagram $2H(+ (01234))$ stands for

$$2H(1234)^* \otimes 2H(0234) \otimes 2H(0134)^* \otimes 2H(0124) \otimes 2H(0123)^*$$

and $2H(- (01234))$ for

$$2H(1432) \otimes 2H(0432)^* \otimes 2H(0132) \otimes 2H(0142)^* \otimes 2H(0143).$$

The linear map σ_{01432} is the tensor product of the $\sigma_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}$ and the $\sigma'_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}$ composed with some transpositions $P: x \otimes y \mapsto y \otimes x$ so that the respective factors in the tensor product appear in the right order. The vertical linear maps are the ones defining $Z(+ (01234))$ and $Z(- (01432))$ respectively by means of the pairing (see Section 3).

First of all we will show what the $\sigma_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}$ and the $\sigma'_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}$ are. Using the diagrammatic conventions as established by the diagrams in

the Figs. 2, 3, and 4, we are now going to give the definition of σ'_{1432} . Remember that $2H(1234)^*$ is the vector space of 2-morphisms

$$\alpha: f_{124}(f_{234} \otimes e_{12}) \mapsto f_{134}(e_{34} \otimes f_{123}).$$

Diagrammatically we denote such a 2-morphism by the diagram in Fig. 5. The vector space $2H(1432)$ was defined by

$$2\text{Hom}(f_{132}(e_{32} \otimes f_{143}), f_{142}(f_{432} \otimes e_{14})).$$

In Fig. 6 we show how to get a 1-morphism

$$e_{12} \mapsto e_{42} \otimes e_{14} \mapsto e_{32} \otimes e_{43} \otimes e_{14}$$

from $f_{124}(f_{234} \otimes e_{12})$.

Of course the cups and caps represent coevaluations i_* and evaluations e_* , respectively, defined by the duality on the objects in C . The way we have drawn our diagrams already shows that we assume that the 1-morphism we obtain in this way is equal to $f_{142}(f_{432} \otimes e_{14})$. This is justified by two arguments. In the first place we can decompose the 1-morphism into simple 1-morphisms in $\text{Hom}(e_{12}, e_{42} \otimes e_{14})$ and $\text{Hom}(e_{42}, e_{32} \otimes e_{43})$ by semi-simplicity. Since we have shown in Lemma 4.1 that $Z(M, T, \ell)$ does not depend on the particular choice of simple 1-morphisms in the various isomorphism classes, we may assume that the decomposition gives f_{142} and f_{432} actually.

In the same way we obtain a 1-morphism

$$e_{12} \mapsto e_{32} \otimes e_{13} \mapsto e_{32} \otimes e_{43} \otimes e_{14}$$

from $f_{134}(e_{34} \otimes f_{123})$, which we take to be equal to $f_{132}(e_{32} \otimes f_{143})$ (see Fig. 7).

Now by horizontal composition of the identity 2-morphisms on the respective coevaluations and evaluations and α^\dagger we get the image of α under σ'_{1432} . Explicitly we get the 2-morphism

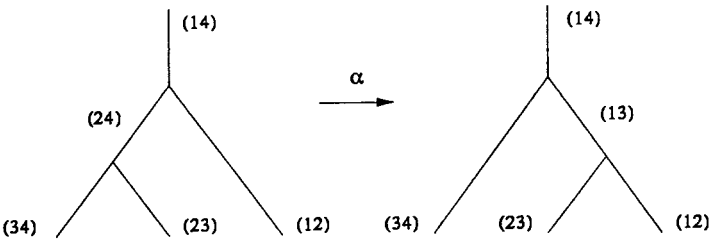
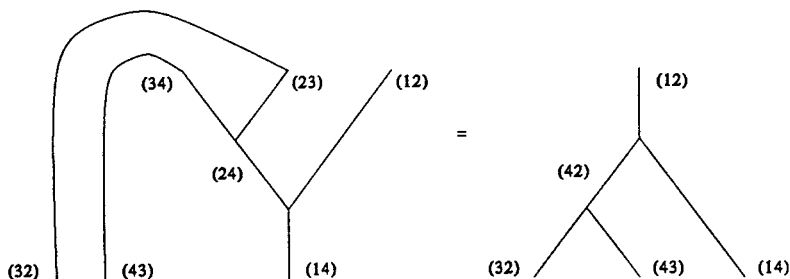
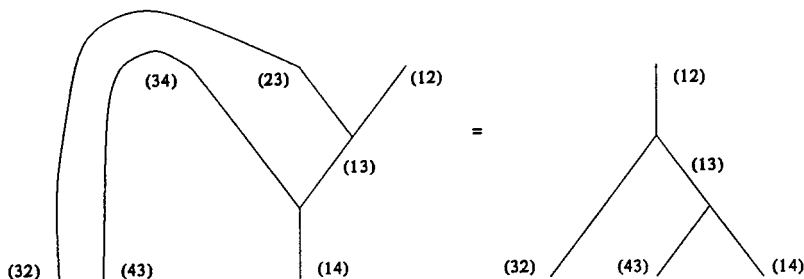


FIGURE 5

FIG. 6. $f_{142}(f_{234} \otimes e_{14})$.

$$\begin{aligned}
 & (i_{e_{32}} \otimes e_{12})(e_{32} \otimes i_{e_{43}} \otimes e_{23} \otimes e_{12})(e_{32} \otimes e_{43} \otimes f_{234}^* \otimes e_{12}) \\
 & \quad \times (e_{32} \otimes e_{43} \otimes e_{24} \otimes i_{e_{42}} \otimes e_{12})(e_{32} \otimes e_{43} \otimes i_{e_{24}}^* \otimes e_{24} \otimes e_{12}) \\
 & \quad \times (e_{32} \otimes e_{43} \otimes f_{124}^*) \\
 & \xRightarrow{T_{e_{42}}} (i_{e_{32}} \otimes e_{12})(e_{32} \otimes i_{e_{43}} \otimes e_{23} \otimes e_{12})(e_{32} \otimes e_{43} \otimes f_{234}^* \otimes e_{12}) \\
 & \quad \times (e_{32} \otimes e_{43} \otimes f_{124}^*) \\
 & \xRightarrow{\alpha^\dagger} (i_{e_{32}} \otimes e_{12})(e_{32} \otimes i_{e_{43}} \otimes e_{23} \otimes e_{12})(e_{32} \otimes e_{43} \otimes e_{34} \otimes f_{123}^*) \\
 & \quad \times (e_{32} \otimes e_{43} \otimes f_{134}^*).
 \end{aligned}$$

Note that by taking the dual of the 1-morphisms in the diagrams we have to use α^\dagger , which corresponds to the fact that σ is an odd permutation. It is clear that σ'_{1432} is a linear isomorphism, although it is only an involution up to a 2-isomorphism. But, in the end when we take the pairing that defines our partition function $Z(ijklm)$, these isomorphisms do not harm us because the pairing is defined by means of a trace that is invariant under conjugation, so we always get the same result anyhow. The linear isomorphism σ_{0432} is defined in an analogous way. As a matter of fact one only has to reverse the morphisms in the definition of σ'_{1432} and substitute

FIG. 7. $f_{132}(e_{32} \otimes f_{143})$.

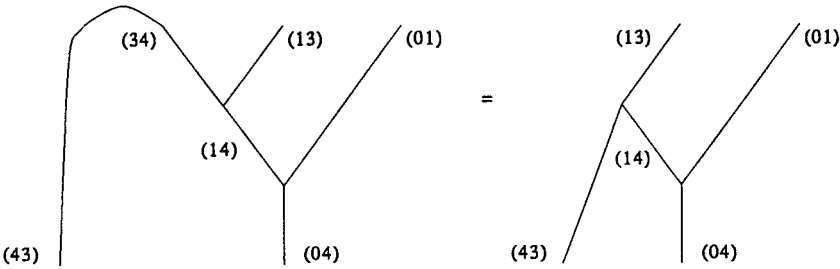


FIG. 8. $(f_{143} \otimes e_{01})(e_{43} \otimes f_{014}^*)$

1 by 0. Let us now define the linear isomorphism $\sigma'_{0143}: 2H(0134)^* \rightarrow 2H(0143)$. Remember that the vector space $2H(0134)^*$ was defined by

$$2\text{Hom}(f_{014}(f_{134} \otimes e_{01}), f_{034}(e_{34} \otimes f_{013}))$$

and $2H(0143)$ by

$$2\text{Hom}(f_{043}(e_{43} \otimes f_{014}), f_{013}(f_{143} \otimes e_{01})).$$

Figure 8 shows how we map $f_{014}(f_{134} \otimes e_{01})$ to $(f_{143} \otimes e_{01})(e_{43} \otimes f_{014}^*)$. Figure 9 shows how we map $f_{034}(e_{34} \otimes f_{013})$ to $f_{013}^* f_{043}$.

So given $\varepsilon \in 2H(0134)^*$ we get a 2-morphism

$$f_{013}^* f_{043} \Rightarrow (f_{143} \otimes e_{01})(e_{43} \otimes f_{014}^*).$$

Again we have composed the identities on the coevaluations and evaluations with ε^\dagger . Now the image of ε under σ'_{0143} is defined by the obvious composition

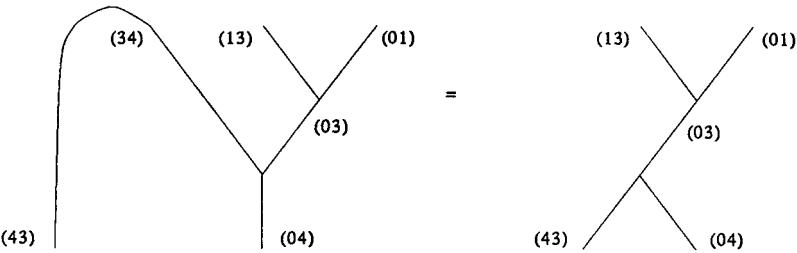


FIG. 9. $f_{013}^* f_{043}$

$$\begin{aligned}
f_{043}(e_{43} \otimes f_{014}) &\Rightarrow f_{013} f_{013}^* f_{043}(e_{43} \otimes f_{014}) \\
&\Rightarrow f_{013}(f_{143} \otimes e_{01})(e_{43} \otimes f_{014}^*)(e_{43} \otimes f_{014}) \\
&\Rightarrow f_{013}(f_{143} \otimes e_{01}).
\end{aligned}$$

Explicitly we get

$$\begin{aligned}
f_{043}(e_{43} \otimes f_{014}) &= (i_{e_{43}} \otimes e_{03})(e_{43} \otimes f_{034}^*)(e_{43} \otimes f_{014}) \\
&\xrightarrow{i_{f_{013}}} f_{013} f_{013}^*(i_{e_{43}} \otimes e_{03})(e_{43} \otimes f_{034}^*)(e_{43} \otimes f_{014}) \\
&\xrightarrow{\otimes i_{43}^{-1}, f_{013}} f_{013}(i_{43} \otimes e_{13} \otimes e_{01})(e_{43} \otimes e_{34} \otimes f_{013}^*) \\
&\quad \times (e_{43} \otimes f_{034}^*)(e_{43} \otimes f_{014}) \\
&\xrightarrow{\varepsilon^*} f_{013}(i_{43} \otimes e_{13} \otimes e_{01})(e_{43} \otimes f_{134}^* \otimes e_{01}) \\
&\quad \times (e_{43} \otimes f_{014}^*)(e_{43} \otimes f_{014}) \\
&\xrightarrow{e_{f_{014}}} f_{013}(i_{43} \otimes e_{13} \otimes e_{01})(e_{43} \otimes f_{134}^* \otimes e_{01}) \\
&= f_{013}(f_{143} \otimes e_{01}).
\end{aligned}$$

Again this is an isomorphism, because C is pivotal. Again σ'_{0143} is only involutive up to a 2-isomorphism. The other linear isomorphisms σ_{0142} and σ'_{0132} are defined analogously. So now all linear maps in diagram (4.4) are defined. Let us prove the commutativity of the diagram.

LEMMA 4.4. *With the definitions as above diagram (4.4) is commutative.*

Proof. As I already announced, the proof is essentially diagrammatic. I actually worked out all the formulas by hand, but these are far too large to fit on ordinary paper. Since I have given all the explicit isomorphisms and identifications used in the horizontal linear isomorphism of our diagram, the reader can work out the explicit formulas from them and check that my diagrammatics are correct in that way.

When one works out the image of an element $\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \in 2H(01234)$ under σ_{01432} , as described above, and the right vertical linear map in our diagram explicitly, one can read off the diagram in Fig. 10.

We have explained the diagrammatics after our definition of $Z(+ (ijklm))$ in Section 3 and we have explained what kind of moves we can apply to them. It is now immediately clear that the diagram in Fig. 10 can be transformed into the diagram in Fig. 11. We have already explained how these transformations work, but let us explain it again in this particular case. For the transformation of the diagram in Fig. 10 to the diagram in Fig. 11 one only has to use the rules for duality on 1-morphisms (condition (11) in Definition 2.3), which resemble the same properties for duality on objects in a monoidal category, and the evident properties of

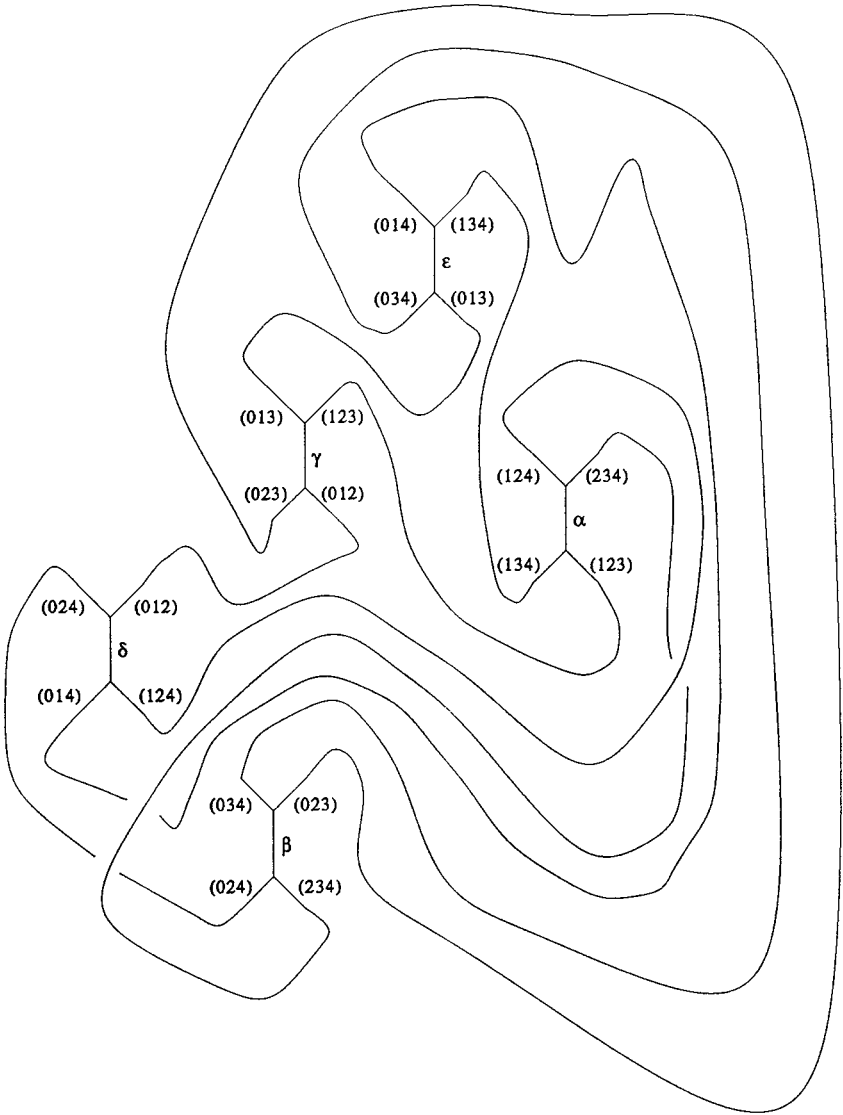


FIGURE 10

\otimes , which precisely resemble the properties of a braiding in a monoidal category. So far we do not need the condition that C is spherical. But in order to transform the diagram in Fig. 11 into the one in Fig. 1 one definitely needs it. “Swinging” around the closing strands of our diagram requires both the pivotal condition and the spherical condition. In order to

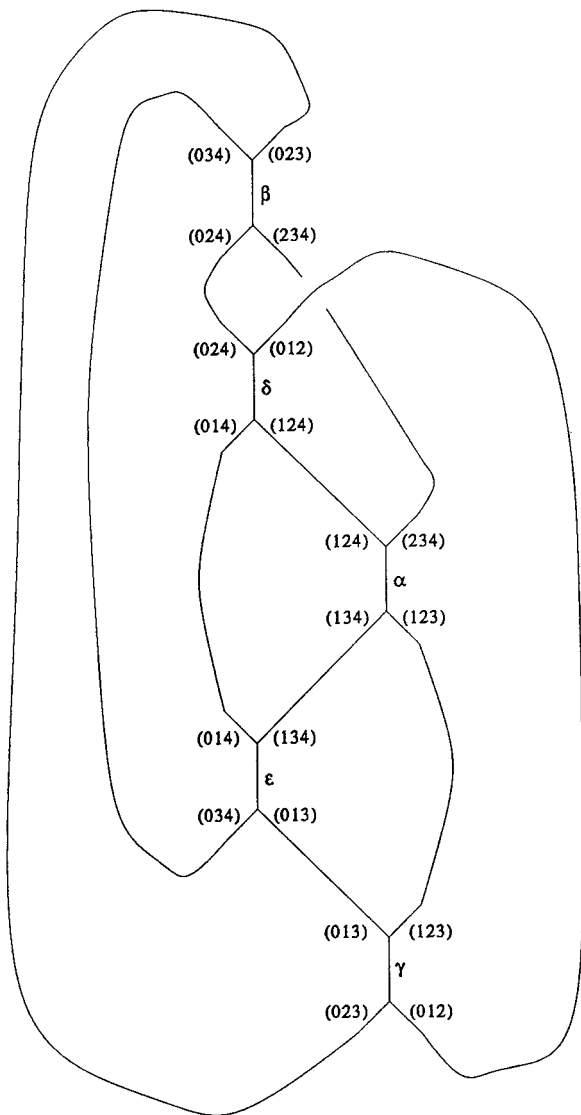


FIGURE 11

get the strands labelled by f_{023}^* and f_{034}^* to the other side of the diagram one needs the pivotal condition. The spherical condition is needed because the way in which the caps and cups are nested differs in the last two diagrams. This change of nesting can only be obtained by changing the left trace functor into the right trace functor on some levels of the diagram. All

these operations of course need to be compatible with the cupping and capping, but we have included these compatibilities in the axioms of the pivotal and the spherical conditions. The proof now finishes with the observation that the last diagram is exactly the one describing $Z(01234)$, i.e., the left arrow in the diagram of this lemma. ■

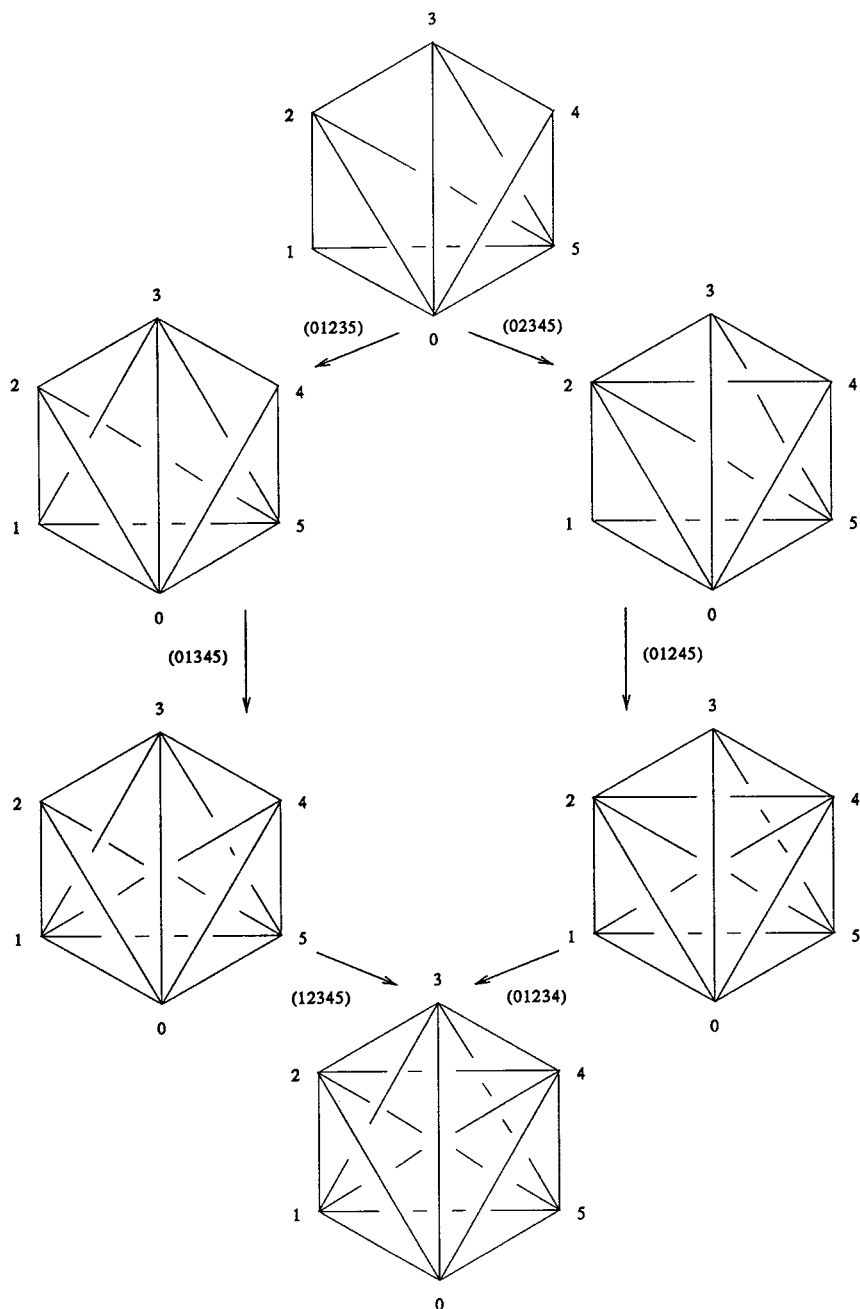
The following theorem is now an immediate result of the previous lemmas in this section.

THEOREM 4.5. *Let $\Phi: (M, T, \ell) \rightarrow (M', T', \ell')$ be a combinatorial isomorphism of labelled triangulated manifolds that is compatible with the labellings. Then $Z(M, T, \ell) = Z(M', T', \ell')$.*

5. INVARIANCE UNDER THE PACHNER MOVES

For the proof of invariance under the Pachner moves we should really look at the 4D Pachner moves as equalities between series of 3D Pachner moves. This idea has been worked out in [11] and can be seen in the Figs. 12, 13, and 14. In these figures an arrow should be interpreted as the boundary of the 4-simplex representing a 3D Pachner move. The source diagram contains the simplicial 3-complex defining one side of the Pachner move and the target diagram contains the simplicial 3-complex defining the other side. For example, the arrow labelled by (01235) in the first Figure represents the $2 \rightleftharpoons 3$ 3D Pachner move that inserts the edge (13) in the 3-complex given by the two tetrahedra (0125) and (0235) glued over the triangle (025).

These observations about the Pachner moves show that the algebraic categorification going from a certain kind of category up to a certain kind of 2-category, as first predicted and sketched in [16], goes hand in hand with a geometrical kind of categorification. From a very abstract point of view we have substituted the identities in the categories which are equivalent to the 3D Pachner moves by isomorphisms in the 2-categories which we will prove to satisfy identities equivalent to the 4D Pachner moves. The Φ in the crossing Lemma 3.1 is the isomorphism that substitutes the identity which is equivalent to the $2 \rightleftharpoons 3$ move of the 3D Pachner moves. Its inverse is of course the substitute of the inverse move. The isomorphism substituting the $1 \rightleftharpoons 4$ move is not so easy to describe but will come out of our calculations below. However vague these remarks may seem, they describe the deeper reason of why everything works as nicely as it does. The notion of categorification is really central in this whole setup and causes the proofs of invariance under the 4D Pachner moves to become almost tautological.

FIG. 12. Pachner move $3 \rightleftharpoons 3$.

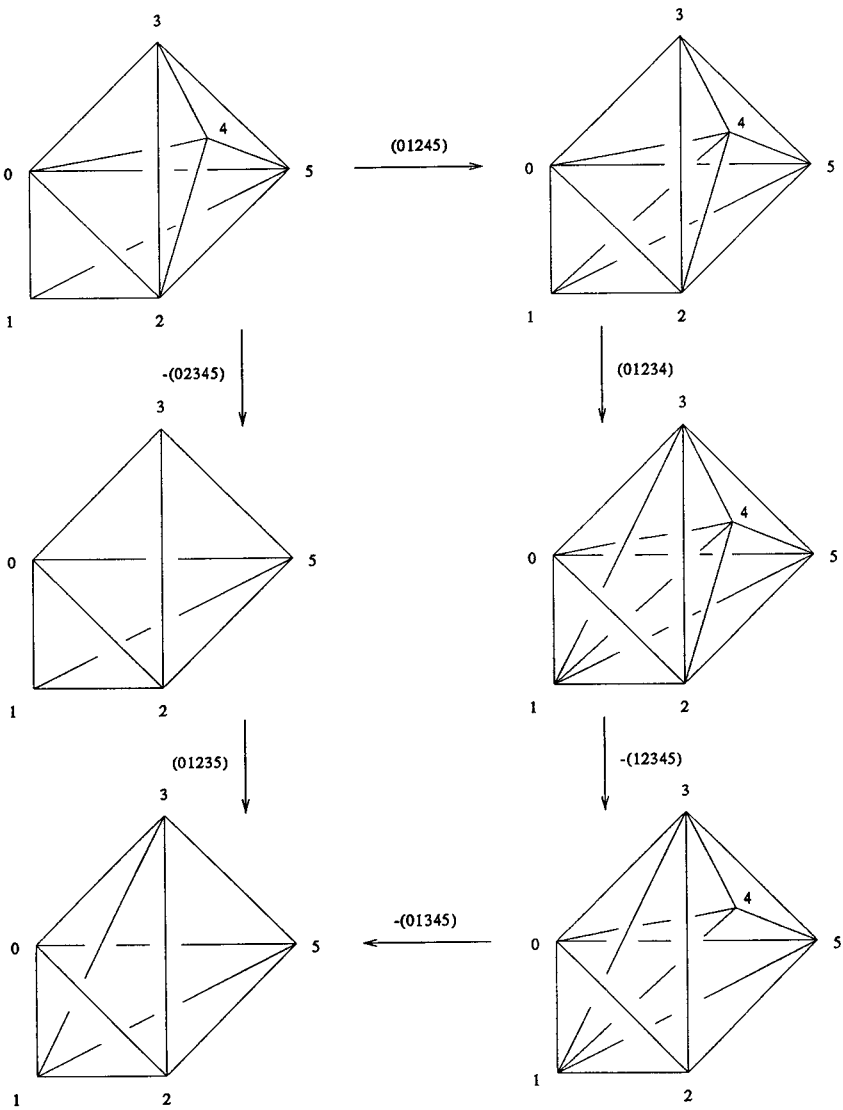
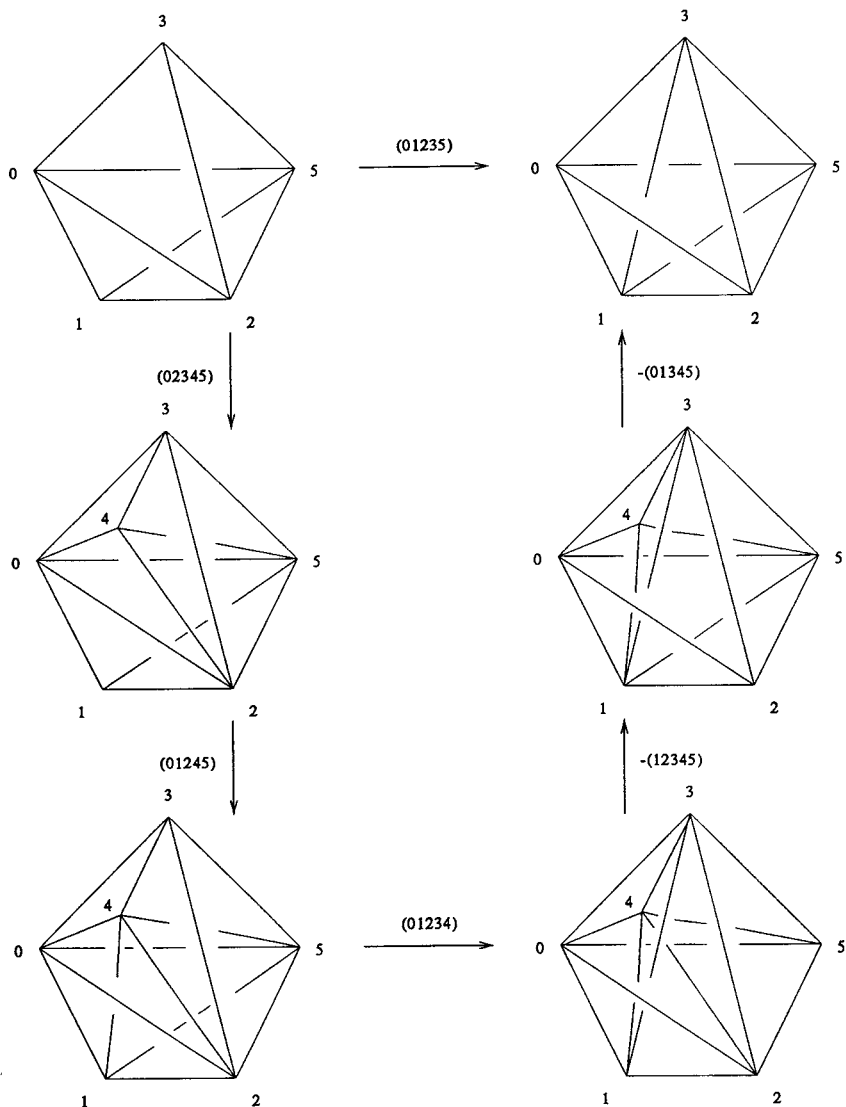


FIG. 13. Pachner move $2 \rightleftharpoons 4$.

Let T_1 and T_2 be two triangulations of M that can be obtained from one another by one 4D Pachner move. Let $D_1 \subseteq T_1$ be the simplicial 4-complex on one side of the Pachner move and $D_2 \subseteq T_2$ the simplicial 4-complex on the other. We denote the complement of the interior of D_1 in T_1 by X , which by definition is equal to the complement of the interior of D_2 in T_2 .

FIG. 14. Pachner move $1 \rightleftharpoons 5$.

Notice that $\partial X = \partial D_1 = \partial D_2$. Also $D_1 \cup D_2$ is the boundary of a 5-simplex (012345) . Now any labelling of $X \cup \partial(012345)$ defines a labelling ℓ_X on X , a labelling ℓ_1 on D_1 and a labelling ℓ_2 on D_2 which are equal on intersections. We define $Z(X)$ as the linear map obtained by taking the partial trace over all the state spaces of all the tetrahedra in the interior of X of

the tensor product of the partition functions for each labelled 4-simplex in X . Define $Z(D_1)$ and $Z(D_2)$ analogously. Then we can decompose our state sum $I(M, T_i) = I_C(M, T_i)$ for $i = 1, 2$ in the following way:

$$\begin{aligned} I(M, T_i) &= K^{-v_X} \operatorname{tr} \left(\sum_{l_X} Z(X) \otimes \left(K^{-v'_i} \sum_{\ell'_i} Z(D'_i) \right. \right. \\ &\quad \times \prod_{e_{D'_i}} \dim_q(\ell(e_{D'_i}))^{-1} \prod_{f_{D'_i}} \dim_q(\ell(f_{D'_i})) \left. \left. \right) \right) \\ &\quad \times \prod_{e_X} \dim_q(\ell(e_X))^{-1} \prod_{f_X} \dim_q(\ell(f_X)). \end{aligned}$$

The number v_X is the number of vertices of X , v'_i is the number of vertices internal to D_i for $i = 1, 2$. The first summation is over all labellings on X ; the second is over all labellings on D_i fixed on ∂D_i but ranging over all the simple objects in \mathcal{E} and all the simple 1-morphisms in all the \mathcal{F}_{ijk} for all the edges and faces internal to D_i . The trace is of course the trace over all the state spaces of all the tetrahedra in ∂X . Therefore proving invariance of our state sum under a 4D Pachner move means showing that the following identity holds

$$\begin{aligned} &K^{-v'_1} \sum_{\ell'_1} Z(D'_1) \prod_{e_{D'_1}} \dim_q(\ell(e_{D'_1}))^{-1} \prod_{f_{D'_1}} \dim_q(\ell(f_{D'_1})) \\ &= K^{-v'_2} \sum_{\ell'_2} Z(D'_2) \prod_{e_{D'_2}} \dim_q(\ell(e_{D'_2}))^{-1} \prod_{f_{D'_2}} \dim_q(\ell(f_{D'_2})). \end{aligned}$$

The lemmas in the rest of this section prove this identity for all the 4D Pachner moves.

The equation proving invariance under the $3 \rightleftharpoons 3$ move is the analogue of the Biedernharn–Elliot equation. In the following lemma P is the transposition $P: x \otimes y \mapsto y \otimes x$.

LEMMA 5.1. ($3 \rightleftharpoons 3$).

$$\begin{aligned} &\sum_{f_{135}} \dim_q(f_{135})(1 \otimes Z(+ (01235)))(P \otimes 1 \otimes 1)(1 \otimes Z(+ (01345)) \otimes 1) \\ &\quad \times (P \otimes 1 \otimes 1 \otimes 1)(Z(+ (12345)) \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes P \otimes 1 \otimes 1) \\ &= \sum_{f_{024}} \dim_q(f_{024})(Z(+ (02345))(1 \otimes 1 \otimes P)(1 \otimes Z(+ (01245)) \otimes 1) \\ &\quad \times (1 \otimes 1 \otimes 1 \otimes P)(1 \otimes 1 \otimes 1 \otimes Z(+ (01234)))). \end{aligned}$$

Proof. Just write down the hexagon of which the left-hand side is

$$\begin{array}{c}
 \oplus_{f_{035}, f_{025}} 2H(0345) \otimes 2H(0235) \otimes 2H(0125) \\
 \downarrow 1 \otimes \Phi_{01235} \\
 \oplus_{e_{13}, f_{035}, f_{135}, f_{123}, f_{013}} 2H(0345) \otimes 2H(1235) \otimes 2H(0135) \otimes 2H(0123) \\
 \downarrow (P \otimes 1 \otimes 1)(1 \otimes \Phi_{01345} \otimes 1) \\
 \oplus_{e_{13}, e_{14}, f_{135}, f_{123}, f_{013}, f_{145}, f_{134}, f_{014}} 2H(1235) \otimes 2H(1345) \otimes 2H(0145) \otimes 2H(0134) \otimes 2H(0123) \\
 \downarrow (P \otimes 1 \otimes 1 \otimes 1)(\Phi_{12345} \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes P \otimes 1 \otimes 1) \\
 \oplus_{e_{13}, e_{14}, e_{24}, f_{245}, f_{234}, f_{145}, f_{124}, f_{014}, f_{134}, f_{123}, f_{013}} 2H(2345) \otimes 2H(1245) \otimes 2H(0145) \otimes 2H(1234) \otimes 2H(0134) \otimes 2H(0123),
 \end{array}$$

and the right-hand side

$$\begin{array}{c}
 \oplus_{f_{035}, f_{025}} 2H(0345) \otimes 2H(0235) \otimes 2H(0125) \\
 \downarrow \Phi_{02345} \otimes 1 \\
 \oplus_{e_{24}, f_{025}, f_{245}, f_{024}, f_{234}} 2H(2345) \otimes 2H(0245) \otimes 2H(0234) \otimes 2H(0125) \\
 \downarrow (1 \otimes 1 \otimes P)(1 \otimes \Phi_{01245} \otimes 1) \\
 \oplus_{e_{24}, e_{14}, f_{245}, f_{024}, f_{234}, f_{145}, f_{124}, f_{014}} 2H(2345) \otimes 2H(1245) \otimes 2H(0145) \otimes 2H(0124) \otimes 2H(0234) \\
 \downarrow (1 \otimes 1 \otimes 1 \otimes P)(1 \otimes 1 \otimes 1 \otimes \Phi_{01234}) \\
 \oplus_{e_{13}, e_{14}, e_{24}, f_{245}, f_{234}, f_{145}, f_{124}, f_{014}, f_{014}, f_{134}, f_{123}, f_{013}} 2H(2345) \otimes 2H(1245) \otimes 2H(0145) \otimes 2H(1234) \otimes 2H(0134) \otimes 2H(0123).
 \end{array}$$

Applying the crossing lemma (Lemma 3.1) six times, i.e., for each Φ_{ijklm} separately, shows that the hexagon is commutative. The result now follows from restriction to the summands which appear in the lemma. ■

In the next lemma we prove the analogue of the orthogonality equation.

LEMMA 5.2. (Orthogonality).

$$\begin{aligned}
& \dim_q(f_{024}) \sum_{\substack{e_{13}, f_{013}, \\ f_{123}, f_{134}}} Z(+ (01234)) Z(- (01234)) \dim_q(e_{13})^{-1} \\
& \quad \times \dim_q(f_{013}) \dim_q(f_{123}) \dim_q(f_{134}) \\
& = \text{id}_{2H(0234) \otimes 2H(0124)}, \\
& \dim_q(e_{13})^{-1} \dim_q(f_{013}) \dim_q(f_{123}) \dim_q(f_{134}) \sum_{f_{024}} Z(- (01234)) \\
& \quad \times Z(+ (01234)) \dim_q(f_{024}) \\
& = \text{id}_{2H(1234) \otimes 2H(0134) \otimes 2H(0123)}.
\end{aligned}$$

Proof. This follows from the formulas in the crossing lemma (Lemma 3.1) for Φ and Φ^{-1} . ■

Now the other two Pachner moves follow from the $3 \rightleftharpoons 3$ lemma and the orthogonality lemma.

LEMMA 5.3. ($2 \rightleftharpoons 4$).

$$\begin{aligned}
& (Z(- (02345)) \otimes 1)(1 \otimes Z(+ (01235))) \\
& = \sum_{e_{14}, f_{014}, f_{124}, f_{145}, f_{134}} \dim_q(e_{14})^{-1} \dim_q(f_{014}) \dim_q(f_{124}) \\
& \quad \times \dim_q(f_{145}) \dim_q(f_{134}) \\
& \quad \times (1 \otimes 1 \otimes P)(1 \otimes Z(+ (01245)) \otimes 1)(1 \otimes 1 \otimes 1 \otimes P) \\
& \quad \times (1 \otimes 1 \otimes 1 \otimes Z(+ (01234)))(1 \otimes 1 \otimes P \otimes 1 \otimes 1) \\
& \quad \times (Z(- (12345)) \otimes 1 \otimes 1 \otimes 1) \\
& \quad \times (P \otimes 1 \otimes 1 \otimes 1)(1 \otimes Z(- (01345)) \otimes 1)(P \otimes 1 \otimes 1).
\end{aligned}$$

Proof. Note that the expressions in this lemma do not relate to Fig. 13 just as directly as the expressions in the lemmas about the invariance under the other two 4D Pachner moves do to their respective figures. What I mean is that $2H(0345)$ is in the targets of $Z(- (02345))$ and $Z(- (01345))$ rather than in their sources. Since the partial trace over the state spaces in ∂X in the decomposition of $I(M, T_i)$ is the composition of the coevaluation on these state spaces with

$$\left(\sum_{I_X} Z(X) \otimes \left(K^{-v'_i} \sum_{\ell'_i} Z(D'_i) \prod_{e_{D'_i}} \dim_q(\ell(e_{D'_i}))^{-1} \prod_{f_{D'_i}} \dim_q(\ell(f_{D'_i})) \right) \right)$$

and with the evaluation on $2H(0345)$ we can just as well multiply both sides of the equation in this lemma by

$$(\text{ev}_{2H(0345)} \otimes 1 \otimes 1 \otimes 1).$$

That is what I had actually done in an earlier version of this paper, because then the expressions are exactly the ones one can read off from the diagram. However, in that case the expressions do not exactly correspond to the decomposition of $I(M, T_i)$ into partial traces and so I have decided to change it in this version to avoid confusion. The essence of the lemma remains the same.

Let us prove the lemma now. Multiply each side of the $3 \rightleftharpoons 3$ equation by

$$\begin{aligned} & \dim_q(e_{24})^{-1} \dim_q(e_{14})^{-1} \dim_q(f_{234}) \dim_q(f_{245}) \dim_q(f_{014}) \dim_q(f_{124}) \\ & \times \dim_q(f_{145}) \dim_q(f_{134}) \dim_q(f_{035})(Z(-(02345)) \otimes 1) \end{aligned}$$

on the left and multiply by

$$\begin{aligned} & (1 \otimes 1 \otimes P \otimes 1 \otimes 1)(Z(-(12345)) \otimes 1 \otimes 1 \otimes 1)(P \otimes 1 \otimes 1 \otimes 1) \\ & (1 \otimes Z(-(01345)) \otimes 1)(P \otimes 1 \otimes 1) \end{aligned}$$

on the right and sum over all the edges, i.e., simple objects, and all the faces, i.e., simple 1-morphisms, involved. Using the orthogonality lemma once on the left-hand side and twice on the right-hand side we get the $2 \rightleftharpoons 4$ equation. ■

LEMMA 5.4. ($1 \rightleftharpoons 5$).

$$\begin{aligned} Z(+ (01235)) &= K^{-1} \sum_{\substack{e_{04}, e_{14}, e_{24}, e_{34}, e_{45}, \\ f_{014}, f_{024}, f_{034}, f_{045}, f_{124}, \\ f_{134}, f_{145}, f_{234}, f_{245}, f_{345}}} \text{tr}_1[(Z(+ (02345)) \otimes 1)(1 \otimes 1 \otimes P) \\ & \times (1 \otimes Z(+ (01245)) \otimes 1)(1 \otimes 1 \otimes 1 \otimes P) \\ & \times (1 \otimes 1 \otimes 1 \otimes Z(+ (01234)))(1 \otimes 1 \otimes P \otimes 1 \otimes 1) \\ & \times (Z(-(12345)) \otimes 1 \otimes 1 \otimes 1)(P \otimes 1 \otimes 1 \otimes 1) \\ & \times (1 \otimes Z(-(01345)) \otimes 1)(P \otimes 1 \otimes 1)] \\ & \times \prod_{\substack{i < j \\ i=4 \vee j=4}} \dim_q(e_{ij})^{-1} \prod_{\substack{i < j < k \\ j=4 \vee k=4}} \dim_q(f_{ijk}). \end{aligned}$$

Proof. Multiply each side of the the $3 \rightleftharpoons 3$ equation on the right by

$$\begin{aligned} & K^{-1} \prod_{\substack{i < j \\ i=4 \vee j=4}} \dim_q(e_{ij})^{-1} \prod_{\substack{i < j < k \\ j=4 \vee k=4 \\ (ijk) \neq (024)}} \dim_q(f_{ijk})(1 \otimes 1 \otimes P \otimes 1 \otimes 1) \\ & \times (Z(-(12345)) \otimes 1 \otimes 1 \otimes 1)(P \otimes 1 \otimes 1 \otimes 1) \\ & \times (1 \otimes Z(-(01345)) \otimes 1)(P \otimes 1 \otimes 1), \end{aligned}$$

take the trace on the first factor, and sum over all the edges and faces involved.

The right-hand side is now equal to the right-hand side of the $1 \rightleftharpoons 5$ equation.

Using the orthogonality lemma the left-hand side becomes

$$\begin{aligned} & \frac{K^{-1}}{\dim_q(f_{035})} \sum_{\substack{e_{04}, e_{34}, e_{45}, \\ f_{034}, f_{045}, f_{345}}} \dim_q(e_{04})^{-1} \dim_q(e_{34})^{-1} \dim_q(e_{45})^{-1} \dim_q(f_{034}) \\ & \times \dim_q(f_{045}) \dim_q(f_{345}) \operatorname{tr}_1(1 \otimes Z(+ (01235))). \end{aligned}$$

Now use the identity

$$\begin{aligned} & \operatorname{tr}_1(1 \otimes Z(+ (01235))) \\ & = Z(+ (01235)) \dim(2\operatorname{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03}))) \end{aligned}$$

and the identity

$$\begin{aligned} & \sum_{e_{04}, f_{034}, f_{045}} \dim_q(e_{04})^{-1} \dim_q(f_{034}) \dim_q(f_{045}) \\ & \times \dim(2\operatorname{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03}))) \\ & = \dim_q(e_{35})^{-1} \dim_q(f_{035}) \dim_q(f_{345}), \end{aligned}$$

which will follow from Lemma 5.5. Finally we get

$$\begin{aligned} & \frac{K^{-1}}{\dim_q(f_{035})} \sum_{\substack{e_{34}, e_{45}, \\ f_{345}}} \dim_q(e_{34})^{-1} \dim_q(e_{35})^{-1} \dim_q(e_{45})^{-1} \dim_q^2(f_{345}) \\ & \times \dim_q(f_{035}) Z(+ (01235)) \\ & = Z(+ (01235)). \end{aligned}$$

This equality will follow from Lemma 5.6. ■

LEMMA 5.5. *With the same notation as everywhere in this section we have*

$$\begin{aligned} & \sum_{e_{04}, f_{034}, f_{045}} \dim_q(e_{04})^{-1} \dim_q(f_{034}) \dim_q(f_{045}) \\ & \quad \times \dim(2\text{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03}))) \\ & = \dim_q(e_{35})^{-1} \dim_q(f_{035}) \dim_q(f_{345}). \end{aligned}$$

Proof. Consider $f_{035}(f_{345} \otimes e_{03})$. Its quantum dimension is equal to

$$\dim_q(f_{035}) \dim_q(e_{35})^{-1} \dim_q(f_{345})$$

by Lemma 2.25. Now use semi-simplicity to write

$$1_{f_{035}(f_{345} \otimes e_{03})} = \sum_{\substack{e_{04} \\ f_{034}, f_{045}}} \alpha_{0345} \cdot \bar{\alpha}_{0345},$$

where

$$\alpha_{0345} \in 2\text{Hom}(f_{035}(f_{345} \otimes e_{03}), f_{045}(e_{45} \otimes f_{034}))$$

and

$$\bar{\alpha}_{0345} \in 2\text{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03})).$$

We can always take the α 's to be the projections and $\bar{\alpha}$ the inclusions, so

$$\bar{\alpha}_{0345} \cdot \alpha_{0345} = 1_{f_{045}(e_{45} \otimes f_{034})}.$$

Therefore we get

$$\begin{aligned} \dim_q(f_{035}(f_{345} \otimes e_{03})) &= \sum_{\substack{e_{04}, \\ f_{034}, f_{045}}} \langle \alpha_{0345}, \bar{\alpha}_{0345} \rangle \\ &= \sum_{\substack{e_{04}, \\ f_{034}, f_{045}}} \langle \bar{\alpha}_{0345}, \alpha_{0345} \rangle \\ &= \sum_{\substack{e_{04}, \\ f_{034}, f_{045}}} \dim_q(f_{045}(e_{45} \otimes f_{034})) \\ & \quad \times \dim(2\text{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03}))) \\ &= \sum_{\substack{e_{04}, \\ f_{034}, f_{045}}} \dim_q(e_{04})^{-1} \dim_q(f_{034}) \dim_q(f_{045}) \\ & \quad \times \dim(2\text{Hom}(f_{045}(e_{45} \otimes f_{034}), f_{035}(f_{345} \otimes e_{03}))). \quad \blacksquare \end{aligned}$$

LEMMA 5.6. *Let A be any simple object in \mathcal{E} (see Definition 2.20). Then we obtain the following expression for K ,*

$$K = \sum_{\substack{B, C, \\ f_{A, B \otimes C}}} \dim_q(A)^{-1} \dim_q(B)^{-1} \dim_q(C)^{-1} \dim_q^2(f_{A, B \otimes C}),$$

where we sum over all the objects $B, C \in \mathcal{E}$ and all 1-morphisms $f_{A, B \otimes C} \in \mathcal{F}_{A, B \otimes C}$, the finite basis of non-isomorphic 1-morphisms in $\text{Hom}(A, B \otimes C)$. Note that this implies that this expression is independent of the choice of $A \in \mathcal{E}$.

Proof. First let us rewrite the expression in this lemma.

$$\begin{aligned} & \sum_{\substack{B, C, \\ f_{A, B \otimes C}}} \dim_q(A)^{-1} \dim_q(B)^{-1} \dim_q(C)^{-1} \dim_q^2(f_{A, B \otimes C}) \\ &= \sum_B \dim_q(A)^{-1} \dim_q(B)^{-1} \sum_{C, f_{A, B \otimes C}} \dim_q(C)^{-1} \dim_q(f_{A, B \otimes C})^2 \\ &= \sum_B \dim_q(A^*)^{-1} \dim_q(B)^{-1} \sum_{C, f_{A, B \otimes C}} \dim_q(C^*)^{-1} \dim_q(f_{C^*, A^* \otimes B})^2. \end{aligned}$$

The third equality is justified by the results in Section 4 where we show that there is a bijection between the isomorphism classes of simple 1-morphisms $f_{A, B \otimes C}: A \rightarrow B \otimes C$ and the isomorphism classes of simple 1-morphisms $f_{C^*, A^* \otimes B}: C^* \rightarrow A^* \otimes B$. The definition of the bijection is such that the quantum dimensions of corresponding 1-morphisms are equal.

The lemma now follows if we can show the identity

$$\sum_{X, f_{X, Y \otimes Z}} \dim_q(X)^{-1} \dim_q(f_{X, Y \otimes Z})^2 = \dim_q(Y) \dim_q(Z).$$

Note first of all that $\dim_q(Y) \dim_q(Z)$ is the quantum dimension of $Y \otimes Z$ by the pivotal and spherical conditions, which is by definition equal to $\langle 1_{Y \otimes Z}, 1_{Y \otimes Z} \rangle$. By horizontal semi-simplicity we can decompose $1_{Y \otimes Z}$ by

$$1_{Y \otimes Z} = \sum_{\substack{i, \\ f_X, \bar{f}_X}} f_X \bar{f}_X,$$

where $f_X: Y \otimes Z \rightarrow X$ and $\bar{f}_X: X \rightarrow Y \otimes Z$ are the projections onto X and the inclusions of X into $Y \otimes Z$, respectively. We have to sum over all f_X and \bar{f}_X because the multiplicity of X in $Y \otimes Z$ may be greater than 1 in general. By vertical semi-simplicity we can decompose 1_{f_X} and $1_{\bar{f}_X}$ for each f_X and each \bar{f}_X , respectively. This can be written as

$$1_{f_X} = \sum_{\substack{i, \\ \alpha_X^i, \bar{\alpha}_X^i}} \alpha_X^i \cdot \bar{\alpha}_X^i,$$

and

$$1_{\bar{f}_X} = \sum_{\substack{j, \\ \beta_X^j, \bar{\beta}_X^j}} \beta_X^j \cdot \bar{\beta}_X^j.$$

In the first decomposition each $\alpha_X^i: f_X \rightarrow f_X^i$ is a projection on a simple 1-morphism f_X^i and each $\bar{\alpha}_X^i: f_X^i \rightarrow f_X$ is an inclusion. The same holds for the second decomposition where each $\beta_X^j: \bar{f}_X \rightarrow \bar{f}_X^j$ is a projection and each $\bar{\beta}_X^j: \bar{f}_X^j \rightarrow \bar{f}_X$ is an inclusion. We can assume that the $\alpha_X^i \cdot \bar{\alpha}_X^i$ and the $\beta_X^j \cdot \bar{\beta}_X^j$ form “dual” bases, in the sense that

$$(\beta_X^j \circ \alpha_X^i) \cdot (\bar{\beta}_X^j \circ \bar{\alpha}_X^i) = (\beta_X^j \cdot \bar{\beta}_X^j) \circ (\alpha_X^i \cdot \bar{\alpha}_X^i) = \delta_i^j 1_{f_X},$$

where δ_i^j is the Kronecker delta. We can make this assumption because there is the non-degenerate pairing

$$2\mathrm{Hom}(f_X, f_X) \otimes 2\mathrm{Hom}(\bar{f}_X, \bar{f}_X) \rightarrow \mathbf{F}$$

defined by

$$\alpha \otimes \beta \mapsto \langle \alpha \circ \beta, 1_{\bar{f}_X f_X} \rangle.$$

This last expression is just “the trace” of $\alpha \circ \beta$, where “trace” means the vertical trace-like map followed by the trace functor and cupping and capping as everywhere in this paper.

Now we can write

$$\begin{aligned} 1_{1_{Y \otimes Z}} &= \sum (\alpha_X^i \cdot \bar{\alpha}_X^i) \circ (\beta_X^j \cdot \bar{\beta}_X^j) \\ &= \sum (\alpha_X^i \circ \beta_X^j) \cdot (\bar{\alpha}_X^i \circ \bar{\beta}_X^j). \end{aligned}$$

As a result we obtain the equalities

$$\begin{aligned} \langle 1_{1_{Y \otimes Z}}, 1_{1_{Y \otimes Z}} \rangle &= \sum \langle \alpha^i_X \circ \beta^j_X, \bar{\alpha}^i_X \circ \bar{\beta}^j_X \rangle \\ &= \sum \langle \beta^j_X \circ \alpha^i_X, \bar{\beta}^j_X \circ \bar{\alpha}^i_X \rangle \\ &= \sum \langle \beta^i_X \circ \alpha^i_X, \bar{\beta}^i_X \circ \bar{\alpha}^i_X \rangle \\ &= \sum \langle \bar{\beta}^i_X \circ \bar{\alpha}^i_X, \beta^i_X \circ \alpha^i_X \rangle. \end{aligned}$$

These equalities all follow from the pivotal and spherical conditions and the fact that the 2-morphisms involved are only inclusions and projections. Now it is not hard to see that the last expression is equal to

$$\begin{aligned} \sum \dim(V((f^i_X)^*))^2 \langle 1_X, 1_X \rangle &= \sum \dim_q(X) \dim(V((f^i_X)^*))^2 \\ &= \sum \dim_q(X)^{-1} \dim_q((f^i_X)^*)^2. \end{aligned}$$

Here $V((f^i_X)^*)$ is the vector space that appears in the equality

$$(f^i_X)^* f^i_X = V((f^i_X)^*) 1_X.$$

It is easy to see that the dimension of this vector space is equal to the dimension of $V(\bar{f}^i_X)$, with

$$\bar{f}^i_X(\bar{f}^i_X)^* = V(\bar{f}^i_X) 1_X,$$

because f^i_X and \bar{f}^i_X are “dual” to each other.

The result now follows from the observation that the $(f^i_X)^*$ form a basis of $\text{Hom}(X, Y \otimes Z)$. ■

Thus we can conclude this section by the following theorem, summarizing all the results obtained in this paper.

THEOREM 5.7. *Let C be a non-degenerate finitely semi-simple semi-strict spherical 2-category of non-zero dimension. For any compact closed piece-wise linear oriented 4-manifold M there exists a state sum $I_C(M)$. For any two such manifolds M and N that are PL-homeomorphic we have $I_C(M) = I_C(N)$.*

6. EXAMPLE

Let G be a finite group. In this section we explain how to obtain a finitely semi-simple non-degenerate semi-strict spherical 2-category

$2\text{Hilb}[G]$ of non-zero dimension. The reason for using this notation is that if $\#G = 1$ then this 2-category will be just 2Hilb_{cc} (see Example 2.15). The invariant $I_{2\text{Hilb}[G]}(M)$ looks very much like a 4-dimensional version of the Dijkgraaf–Witten invariant [21].

The objects of $2\text{Hilb}[G]$ are the elements of the group rig $\mathbf{N}[G]$, which are just formal finite linear combinations of the elements of G with non-negative integer coefficients. The elements of G are taken to be the simple objects in $2\text{Hilb}[G]$. So $\text{Hom}(g, h) = \mathbf{N}$ if $g = h$ and $\{0\}$ otherwise. Here we identify a non-negative integer n with the 1×1 matrix with coefficient n . This means that a 1-morphism F between $x = n_{i_1}g_{i_1} + \cdots + n_{i_k}g_{i_k}$ and $y = m_{j_1}g_{j_1} + \cdots + m_{j_l}g_{j_l}$ is a $(m_{j_1} + \cdots + m_{j_l}) \times (n_{i_1} + \cdots + n_{i_k})$ matrix (F_i^j) where the coefficient F_i^j is just a non-negative integer. The composition of two 1-morphisms F and G is defined by the matrix product of F and G . Here we use the product and the sum of the non-negative integers as the operations on the coefficients of the matrices. A 2-morphism α between $F: x \rightarrow y$ and $G: x \rightarrow y$ is a $(m_{j_1} + \cdots + m_{j_l}) \times (n_{i_1} + \cdots + n_{i_k})$ matrix (α_i^j) where the coefficient α_i^j is a matrix with complex coefficients representing a linear map between F_i^j and G_i^j . The vertical composition of two 2-morphisms α and β is defined by the matrix $(\alpha \cdot \beta)$ where the coefficient $\alpha \cdot \beta_i^j$ is the matrix product of α_i^j and β_i^j . The horizontal composition of α and β is defined by the matrix product of α and β , where the matrix product and the matrix sum are the operations on the coefficients.

The tensor product of two objects x and y is just their product in $\mathbf{N}[G]$. The tensor products on the 1-morphisms and the 2-morphisms are defined by the tensor products of the respective matrices, just as in 2Hilb_{cc} . The tensor product is not semi-strict in general. Between $g(hk)$ and $(gh)k$ we define the *associator* to be the 1-isomorphism represented by the 1-dimensional matrix with coefficient 1. This means that we consider $g(hk)$ and $(gh)k$ only to be isomorphic and not identical! The pentagon in the following diagram is not commutative, but holds only up to a 2-isomorphism $\pi(g, h, k, l) \in \mathbf{C}$, which is called the *pentagonator*.

$$\begin{array}{ccccc}
 g(h(kl)) & \longrightarrow & (gh)(kl) & \longrightarrow & ((gh)k)l \\
 \downarrow & & & & \uparrow \\
 g((hk)l) & \longrightarrow & & \longrightarrow & (g(hk))l
 \end{array}$$

When ranging over all quadruples of elements in G this is equivalent to a \mathbf{N} -linear function $\pi: G \times G \times G \times G \rightarrow \mathbf{C}$. The next order coherence relation that this pentagonator has to satisfy, which can be found in [27], for example, is exactly the condition that π be a 4-cocycle on the group. We do not write down explicitly this coherence condition for a general monoidal 2-category because below we give a direct definition of the state-sum

$I_{2\text{Hilb}[G]}(M)$ and one can verify by hand that invariance under the $3 \rightrightarrows 3$ Pachner move is equivalent to the cocycle condition. In order to apply our construction in this particular example one has to strictify this tensor product in order to obtain a semi-strict monoidal 2-category. But as we have mentioned before this can always be done, see [26], and we will not work this out in detail here. Note that the strictification does not eliminate the pentagonator but “hides” it somehow inside the monoidal structure of the 2-functor which defines the equivalence between the weak monoidal 2-category and the strictified monoidal 2-category. The coherence condition satisfied by the pentagonator, which in this particular example is the cocycle condition, allows us to “forget” the pentagonator while we think abstractly about the construction of the state-sum. But once one writes down an explicit example one has to unpack the abstract definition of the state-sum and calculate where the pentagonator shows up. Of course there are different ways of unpacking, according to the different ways of reparenthesizing the tensor product, but the coherence condition guarantees that they all give the same answer in the end. Below we show how this works out in this example. Of course the same happens when one considers the Dijkgraaf–Witten invariant as an example of the general construction by Barrett and Westbury in [9].

The dual of an object $x = n_{i_1}g_{i_1} + \cdots + n_{i_k}g_{i_k}$ is defined by $x^* = n_{i_1}g_{i_1}^{-1} + \cdots + n_{i_k}g_{i_k}^{-1}$. The coevaluation $i_g : 1 \rightarrow gg^{-1}$ on a simple object g is just the 1-dimensional matrix with coefficient 1. The evaluation $e_g : g^{-1}g \rightarrow 1$ is also the 1-dimensional matrix with coefficient 1. For arbitrary objects we obtain the coevaluation and evaluation by extending these definitions \mathbf{N} -linearly. The duality on 1-morphisms and 2-morphisms is defined as in 2Hilb_∞ (see Example 2.15).

Just as in 2Hilb_∞ it is easy to show that this defines the right kind of duality and that $2\text{Hilb}[G]$ is non-degenerate and spherical. $2\text{Hilb}[G]$ is finitely semi-simple by construction and its dimension is equal to $|G|$.

So we know that our construction gives an invariant $I_{2\text{Hilb}[G]}(M)$, but of course it is easy to write down a more explicit formula for it in this case. Let $M = (M, T)$ be a triangulated manifold. Label the edges with elements of G in such a way that $\ell(ij)\ell(jk) = \ell(ik)$ for any triangle (ijk) . If this rule is not satisfied for a certain labelling, then the Hom-category associated to this triangle is zero, so these labellings do not contribute anything to the state-sum. The 4-simplex $(ijklm)$ gets the weight $\pi(\ell(ij), \ell(jk), \ell(kl), \ell(lm))$, where π is the 4-cocycle defining the pentagonator in $2\text{Hilb}[G]$. Our state-sum is now equal to

$$I_{2\text{Hilb}[G]}(M) = |G|^{-v} \sum_{\ell} \prod_i \pi(S_i)^{e_i},$$

where v is the number of vertices in T , the sum is taken over all labellings, in each term of the state-sum the product is taken over all 4-simplices S_i in T with a fixed labelling, and ε_i is $+1$ if the orientation on S_i induced by the ordering on its vertices and the one induced by the orientation of M coincide, and -1 otherwise. Its easy to show that invariance under the $3 \Rightarrow 3$ move is equivalent to the cocycle condition and that invariance under the $2 \Rightarrow 4$ move also follows directly from that. Invariance under the $1 \Rightarrow 5$ move follows from the identity

$$\begin{aligned} \pi(g, h, k, l) = & |G|^{-1} \sum_{m \in G} \pi(h, k, l, m)^{-1} \pi(gh, k, l, m) \pi(g, hk, l, m)^{-1} \\ & \times \pi(g, h, kl, m) \pi(g, h, k, lm)^{-1}. \end{aligned}$$

For this calculation one should reorder the vertices so that the “new” vertex in the $1 \Rightarrow 5$ move becomes the last one in the 5-simplex. This is because in this example we have chosen a slightly different convention for the indices of our partition function for convenience. We know that the state-sum is independent of the ordering of the vertices so we are allowed to do this.

If M is connected and one takes the trivial 4-cocycle, i.e., $\pi(g, h, k, l) = 1$ for all $g, h, k, l \in G$, then this invariant just counts the number of group homomorphisms of $\pi_1(M)$ into G , as in the three dimensional case (see [21, 33, 34, 38, 39]). For a non-trivial cocycle it might become more interesting, although for $G = \mathbf{Z}/p\mathbf{Z}$ the results so far have been disappointing. In [10] Birmingham and Rakowski study state-sums that appear as models for *lattice gauge theories with finite gauge group $\mathbf{Z}/p\mathbf{Z}$* . In dimension 4 they study the state-sum that is similar to $I_{2\text{Hilb}[\mathbf{Z}/p\mathbf{Z}]}(M)$ except that the 4-cocycle takes values in $U(1)$ and $\mathbf{Z}/p\mathbf{Z}$, respectively. Unfortunately there are no non-trivial 4-cocycles of $\mathbf{Z}/p\mathbf{Z}$ with values in $U(1)$ and for any 4-cocycle π with values in $\mathbf{Z}/p\mathbf{Z}$ we get $\prod_i \pi(S_i)^{\varepsilon_i} = 1$ for a fixed labelling, so the invariant is not very interesting either (see Section 5 in [10]). For other groups I do not know any results about $I_{2\text{Hilb}[G]}(M)$. In [23] Freed and Quinn provided a detailed analysis of the topological quantum field theories (TQFTs) suggested by Dijkgraaf and Witten in [21], of which the restriction to closed manifolds is the Dijkgraaf–Witten invariant. Along the same line one could try to analyse $I_{2\text{Hilb}[G]}(M)$. I must thank John Baez for pointing this out to me.

Although in some cases the invariant defined by $I_{2\text{Hilb}[G]}(M)$ might not be very sophisticated, it defines a non-trivial invariant in general and one could interpret $2\text{Hilb}[G]$ with a non-trivial pentagonator as a finite deformation of $2\text{Hilb}[G]$ with the trivial pentagonator. This is remarkable because we know that all the interesting 3-manifold state-sum invariants,

including the so called *quantum invariants*, come from deformations of monoidal categories. Maybe other interesting (quantum) 4-manifold state-sum invariants will come from deformations of monoidal 2-categories. Does categorification strike again?

As one can notice from the remarks above the example $I_{2\text{Hilb}[G]}(M)$ is not new (see [10, 33, 34, 38, 39]). However, it is interesting to see that this invariant is a nice example of the general construction presented in this paper, just as the Dijkgraaf–Witten invariant is a nice example of the general construction presented in [9].

7. CONCLUDING REMARKS

In this last section I want to sketch my plans for further research based on some remarks concerning the results in this paper.

First of all let me address the question of examples. Although I have given one example of a 2-category that satisfies my conditions (Section 6) and suggested that there is probably another one related to the categorification of the quantum double of a finite group, we need to find more interesting, and more complicated, examples. It is extremely hard to find interesting examples of Hopf categories or 2-categories that are likely to give a new kind of invariant. Any “simple” attempt seems to be doomed to lead to the Euler characteristic or, if one is a bit luckier, the signature of the 4-manifolds (see [17, 34]), or some other homotopy invariant. Crane, Yetter, and I are working on the definition of monoidal 2-categories built from representations of quantum groups at $q=0$ using their crystal bases and their deformation theory. These results will be published in a separate paper. It might well be that, in order to get the desired examples of 2-categories, we have to find actual deformations of these 2-categories and then find our way back to a generic q . Although this is the hard way of trying to find examples it certainly is worth a try.

Next there is the question of the relation of the construction presented in this paper and the ones in [18, 16]. In order to get back the Crane–Yetter invariant out of a construction like the one in this paper we would have to relax the definition of a simple object in a 2-category somehow. The way we define things in this paper implies that a Vect-linear semi-simple monoidal 2-category with one simple object is just Vect. The surprising definition of the dimension of a non-degenerate finitely semi-simple semi-strict 2-category is probably a consequence of this restriction, although I do not know how exactly. One could try to take $\text{End}(I)$ to be the free Vect-module category generated by an arbitrary finitely semi-simple monoidal category R , assume that each Hom-category is a finite dimensional R -module category equivalent to R^n for some non-negative integer n , and

define a simple object to be an object A such that $\text{End}(A) \simeq R$ (here we denote the free Vect-module category generated by R also by R). With such a relaxation we would get the Crane–Yetter invariant via our construction for a non-degenerate finitely semi-simple semi-strict spherical 2-category with one object (see lemma 2.14). However not all the definitions in this paper would still be adequate; for example, one would have to take the tensor product over R rather than Vect in Definition 2.20. Furthermore I do not know if all the proofs in this paper would still hold after such a relaxation. At this point it is not even clear to me that such a relaxation is going to be relevant for our search for quantum 4-manifold invariants. In the three dimensional case one gets all the known quantum invariants out of constructions involving monoidal categories where $\text{End}(I)$ is a field and not just a commutative ring. A field gives only the trivial invariant of 2-dimensional manifolds when one uses the construction of a state-sum out of a finite dimensional semi-simple algebra as defined in [25]. Their state-sum is equal to 1 for every manifold if the semi-simple algebra is a field. In a way the Barrett–Westbury invariant [9] is a lift to the third dimension of the invariant defined in [25]. So it is not clear yet whether, in the process of categorification, one needs the most general structure for $\text{End}(I)$ on the next level or one can assume that it is the simplest example possible, a one dimensional n -Vector space on level n , in order to find “new” $n + 2$ -manifold invariants.

Now let me say something about the possible relation between the state-sum invariant in this paper and the *Tornado Formula* in [16]. First of all let me recall that in the construction of 4-manifold invariants out of Hopf categories there is this difficult question of the right categorified notion of an antipode. Crane and Frenkel do not address this question in [16]. Neither do they prove invariance of their state-sum under the permutation of vertices. In [31] Neuchl gives a definition of an antipode in a Hopf category. It is not clear though, at least to me, that the *Tornado Formula* is invariant under the permutation of vertices, even if one uses Neuchl’s definition. As I already mentioned in my introduction, the only example of a Hopf category of which we are sure that it gives an invariant is the categorification of the quantum double of a finite group [11, 19]. In that case one has to impose some cyclical conditions on the cocycles that define the structure isomorphisms. In order to resolve the problem of invariance under the permutation of vertices in general and to establish a concrete relation between the construction in this paper and the one in [16], I think one should first try to reconstruct the Hopf category out of a spherical 2-category. By Neuchl’s result [31] we know that it is going to be a bitensor category. The duality used in the construction in this paper, which is the right one for the purpose of 4-manifold invariants, will lead to the reconstruction of the right notion of the antipode, hopefully the same as

Neuchl's, and the proof of invariance of the Crane–Frenkel state-sum. When we really understand the Crane–Frenkel state-sum, we can try to see if the invariant coming from the Hopf category is the same as the one coming from the 2-category of its representations. For involutory Hopf algebras and their category of representations Barrett and Westbury show in [8] that the respective 3-manifold invariants are the same. This result was my original motivation for the “categorification” of Barrett and Westbury construction of 3-manifold invariants.

Finally I want to say something about diagrams. If we ever want to compute a real quantum invariant of 4-manifolds we had probably better start looking for some diagrammatic way of doing this. Every reader will remember the major advantage that the “skein approach” brought to the computation of 3-manifold invariants. It even enabled Roberts to show that the Crane–Yetter 4-manifold invariant for $U_q(sl(2))$ was “just” the signature of the 4-manifolds [36]. In this paper I have used some diagrams, but, as I already admitted, they are rather poor diagrams and certainly not good enough for computational aims. For both the construction and the computation of 3-manifold invariants it is very convenient to have the correspondence between pivotal categories and labelled oriented planar graphs as shown in [24]. Let us formulate this correspondence precisely. A C -labelled graph is an oriented planar graph with its edges labelled by objects of C and its vertices labelled by morphisms of C , such that the source of such a morphism is the tensor product of the labels of the ingoing edges and the target the tensor product of the labels of the outgoing edges. If a graph contains two edges with the same vertices labelled by the objects X and Y respectively, then we can substitute them by one single edge labelled by $X \otimes Y$, and vice versa. If a graph contains an edge labelled by X , then we can substitute this edge by the edge with opposite orientation labelled by X^* . We call two C -labelled graphs *essentially equivalent* if one can be obtained from the other by a finite number of such substitutions. We now define two C -labelled graphs G_1 and G_2 to be equivalent if and only if there are two C -labelled graphs G'_1 and G'_2 such that G'_i is isotopic to G_i for $i = 1, 2$ and G'_1 is essentially equivalent to G'_2 . The main theorem of Freyd and Yetter in [24] can now be formulated as follows: the equivalence classes of C -labelled graphs form a pivotal category equivalent to C . This result enables one to translate algebraic manipulations with morphisms in C into diagrammatic moves. As a matter of fact Barrett and Westbury [9] do assume, without using it explicitly, a similar result for spherical categories and labelled oriented graphs in S^2 . The question arises what kind of graphs do correspond to spherical 2-categories. If one dualizes a tetrahedron and looks at the 2-skeleton of the dual complex, one gets a four-valent vertex of a graph with 2-cells. So maybe the answer to my question is “labelled oriented graphs with 2-cells

in S^3 ." Here I am deliberately being sloppy in my definition of the sort of graph I am speaking about. Probably one has to define it by something like a set of elements, the vertices, and a family of two-element subsets, the edges, and a family of three- or more-element subsets, the 2-cells. But I am not trying to make a precise conjecture here. I just want to point out a possible topic for further research that can lead to a better insight in the relation between 4-dimensional topology and combinatorics and algebra. In order to get such a result we could try to study this kind of graph, or hyper-graph, in a way similar to that in which Carter, Rieger, and Saito [13,14] have studied 2-tangles, and see if they provide the diagrammatical tools for the computation of 4-manifold invariants.

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