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RESEARCH PAPER

MIXED NORM SPACES OF ANALYTIC FUNCTIONS AS SPACES OF GENERALIZED FRACTIONAL DERIVATIVES OF FUNCTIONS IN HARDY TYPE SPACES

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Dedicated to Professor Virginia Kiryakova on the occasion of her 65th birthday and the 20th anniversary of FCAA

Abstract

The aim of the paper is twofold. First, we present a new general approach to the definition of a class of mixed norm spaces of analytic functions $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$ on the unit disc $\mathbb{D}$. We study a problem of boundedness of Bergman projection in this general setting. Second, we apply this general approach for the new concrete cases when X is either Orlicz space or generalized Morrey space, or generalized complementary Morrey space. In general, such introduced spaces are the spaces of functions which are in a sense the generalized Hadamard type derivatives of analytic functions having l^q summable Taylor coefficients.

MSC 2010: Primary 30H20; Secondary 46E30, 46E15

Key Words and Phrases: mixed norm, fractional derivatives, Hadamard type integro-differentiation, Bergman type space, Bergman projection, Orlicz space, Morrey space

1. Introduction

Let $X(I) \subseteq L^1(I)$, $I = (0,1)$, denote any Banach space of functions f on interval I containing step functions, and let $\|\cdot\|_{X(I)}$ stand for the norm. Given a function $f(z) = f(r, e^{i\alpha})$ on $\mathbb{D}$ or in general a distribution

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on $\mathbb{D}$ we denote by f_n its distributional Fourier coefficients (see Section 7 for definition). We introduce the mixed norm space $\mathcal{L}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, as the space of distributions on $\mathbb{D}$ such that the (distributional) Fourier coefficients f_n are regular functions $f_n(r) \in X(I)$ and $\sum_{n\in\mathbb{Z}} \|f_n\|^q_{X(I)}$ is finite. The $q-th$ root from this sum gives the norm in $\mathcal{L}^{q;X}(\mathbb{D})$. The mixed norm Bergman space $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, is defined as the subspace in $\mathcal{L}^{q;X}(\mathbb{D})$ of functions analytic in $\mathbb{D}$.

In [20, 21] we studied special cases for the space $X(I)$. The variable exponent Lebesgue space $X(I) = L^{p(\cdot)}(I)$, $1 \leqslant p(r) \leqslant \infty$, was treated in [20], and the cases of classical Morrey space $X(I) = L^{p,\lambda}(I)$, $0 \leqslant \lambda < 1$, $1 \leqslant p < \infty$, and complementary Morrey space $X(I) = {}^{\complement}L^{p,\lambda}(I)$, $0 \leqslant \lambda < p-1$, $1 < p < \infty$, were studied in [21]. This research was inspired by the evident fact that introduction of the mixed norm in the unit disc allows to distinguish between radial and angular behavior of functions, and, hence, to specify the boundary behaviour with more accuracy. In such a way one can reveal the behaviour of a function using variety of norms, including norms of the so called spaces of functions of non standard growth.

A special motivation to introduce new spaces in this paper is in the fact that, for instance, in the case $q = 2$ these Bergman type spaces may be precisely characterize as the range of certain generalized fractional differentiation operator over the Hardy space $H^2(\mathbb{D})$ (see Theorem 4.1). Moreover, the introduced spaces, in general, are the spaces of functions which are the generalized fractional derivatives of analytic functions with l^q summable Taylor coefficients. This important fact sheds a light on the nature of the introduction of the spaces via conditions on Fourier coefficients, which is different from the usual mixed norm space setting. The notion of generalized derivatives of Hadamard type is a wide generalization of the used in the theory of analytic functions so called radial derivatives, Flett's derivatives, etc. We discuss this notion in Section 4. In the particular cases of Orlicz and generalized Morrey type spaces the corresponding form of Hadamard derivative can be explicitly seen from the characterization of functions in such spaces, see Theorems 5.2, 6.2. For a general theory of fractional derivatives and integrals we refer to [23], [36].

In the last two decades the theory of new spaces arising in harmonic analysis of functions with non standard growth and the theory of operators of harmonic analysis in various general spaces with non-standard growth have been intensively developed. These spaces include in particular variable exponent Lebesgue, Hölder, Sobolev spaces, Lorentz spaces, and Orlicz, Morrey-Campanato type, Herz spaces, and others. Within these spaces there were widely considered singular integrals, Riesz and Bessel potentials, maximal and fractional operators, some other classical operators of

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harmonic analysis. As a matter of fact, major attention was paid to real variable settings. We refer to the books [7, 8, 25, 26] (see also review paper [34]).

Therefore it seems natural and fruitful to make use of widely developed methods in the area mentioned above in the case of new spaces of analytic functions. Indeed, the investigation in such a direction already presents very new effects and interesting results. For instance, such a variety of spaces include classical Hardy space and Bergman type space, these very different spaces of functions even within one scale of spaces, i.e. when $X(I) = L^{p(\cdot)}(I)$. Depending on the growth of $p(r)$ as $r \to 1$ we may obtain both Hardy and Bergman type spaces as particular cases of $\mathcal{A}^{q;X}(\mathbb{D})$ with $X(I) = L^{p(\cdot)}(I)$.

Starting with the papers [4], [17], [18], the Bergman spaces, called sometimes the Bergman-Jerbashian spaces, and other spaces of analytic functions attracted attention of many researchers, see the books [5, 10, 16, 39, 40] and references therein. In particular, an important role is played by the boundedness of the Bergman projection. Besides the above cited books we also refer to [1, 3, 9] with respect to such boundedness and related questions. There are known result on the mixed norm Bergman spaces with integral mixed norm with integration in angular and radial variables (boundedness of the Bergman projection, and some functional space properties of the mixed norm Bergman spaces, such as duality, interpolation etc.). We refer to the papers [12, 15, 19, 29, 30] (see also references therein).

The spaces that we study are different from such mixed integration norm spaces. In some cases embedding of our spaces into such spaces may be traced under some concrete choice of the space $X(I)$ (see Theorem 4.2).

Introducing the spaces $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, we in fact suggest a new general approach to the definition of a class of mixed norm Bergman spaces on the unit disc $\mathbb{D}$. We provide some general assumption on the space $X(I)$ under which the Bergman projection is bounded from $\mathcal{L}^{q;X}(\mathbb{D})$ onto $\mathcal{A}^{q;X}(\mathbb{D})$ and reveal the importance of the asymptotical behaviour of the norms $\|r^n\|_{X(I)}$ as $n \to \infty$ for the characterization of the spaces $\mathcal{A}^{q;X}(\mathbb{D})$. This general scheme may be considered as a useful guide. However, the main difficulties start when we treat concrete cases: we have to overcome various problems to verify the above mentioned assumption on the space $X(I)$ and the asymptotic of $\|r^n\|_{X(I)}$ as $n \to \infty$ in the case of this or other concrete case of $X(I)$. Exactly at this step we often have to use very specific properties of the space $X(I)$ or even to obtain new ones, as for instance in [20]. We also present a short review of results previously obtained in [20, 21], i.e. we summarize these results under a general point of view. We apply this general approach for the new concrete cases where $X(I)$ is

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either the Orlicz space $L^{\Phi}(I)$ or the generalized Morrey space $L^{p,\varphi}(I)$, or the generalized complementary Morrey space ${}^{\complement}L^{p,\varphi}(I)$. The fundamentals of Young functions and Orlicz spaces appeared to be very useful in this study. This is a promising area of research, and we plan to explore new cases, for instance, via introducing Besov type norms (see [22]), as well as we keep in mind to study Toeplitz type operators on such new spaces.

The paper is organized as follows. Section 2 contains necessary preliminaries on classical function spaces: Bergman, Morrey and Orlicz. In Section 3 we develop our general approach. In Section 3.1 we give our basic definition and prove the completeness of $\mathcal{L}^{q;X}(\mathbb{D})$, and in Section 3.2 we give general condition on the boundedness of Bergman projection from $\mathcal{L}^{q;X}(\mathbb{D})$ onto $\mathcal{A}^{q;X}(\mathbb{D})$. In Section 3.3 we briefly discuss Toeplitz operators on our new spaces. In Section 4 we discuss the characterization of our spaces in terms of generalized fractional differentiation. A special attention is paid to the space $\mathcal{A}^{2;X}(\mathbb{D})$ which coincides with the range of certain differential operator over the classical Hardy space $H^2(\mathbb{D})$. We also outline certain results previously obtained in [20, 21]. In Sections 5, 6 we realize our approach for $X(I)$ being Orlicz space or generalized Morrey and generalized complementary Morrey space where we manage to find the asymptotic for the norm $\|r^n\|_{X(I)}$ which also leads to the characterization of the spaces under consideration and to the corresponding results on boundeddness of the Bergman projection. In Section 7 we pay a special attention how we interpret distributional Fourier coefficients of traces of functions.

2. Preliminaries

2.1. On the Bergman $\mathcal{A}^p(\mathbb{D})$ space, Hardy $H^p(\mathbb{D})$ space and Bergman projection $B_{\mathbb{D}}$. For the references, see [10, 16, 39, 40]. Let $dA(z)$ stand for the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. As usual $\mathcal{A}^p(\mathbb{D})$ stands for the Bergman space of analytic in $\mathbb{D}$ functions f that belong to $L^p(\mathbb{D}) = L^p(\mathbb{D}; dA(z))$. The corresponding Bergman projection $B_{\mathbb{D}}$ which is defined on $f \in L^1(\mathbb{D})$ as

$$B_{\mathbb{D}}f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} dA(w), \ z \in \mathbb{D}, \tag{2.1}$$

is bounded from $L^p(\mathbb{D})$ onto $\mathcal{A}^p(\mathbb{D})$ for $1 < p < \infty$. For a function f on the unit disc $\mathbb{D}$, and for $0 \leqslant r < 1$, we write $\mathcal{M}_p(f;r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(r, e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}$, for $0 < p < \infty$, and $\mathcal{M}_p(f;r) = \text{ess-sup}_{\,\theta \in [0,2\pi)} |f(r, e^{i\theta})|$, for $p = \infty$. The class of analytic in $\mathbb{D}$ functions f for which $\|f\|_{H^p(\mathbb{D})} \equiv \lim_{r \to 1} \mathcal{M}_p(f;r) < \infty$, $0 < p \leqslant \infty$, is the Hardy class $H^p(\mathbb{D})$.

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2.2. **Generalized Morrey and complementary Morrey spaces $L^{p,\varphi}(I)$, $^{\complement}L^{p,\varphi}(I)$.** For more details on the Morrey type spaces, we refer to [13, 28, 32, 33, 37]. In the definition below we naturally assume that the function $\varphi : I \to \mathbb{R}_+$ is increasing on I and $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$. We treat the Lebesgue space $L^p(I)$, $1 \le p < \infty$, as equipped with the measure $2rdr$.

Let $1 \leqslant p < \infty$. The generalized Morrey space $L^{p,\varphi}(I)$ over the interval I is defined as the set of functions f measurable on I such that

$$\sup_{r,r\pm h\in I,h>0}\frac{1}{\varphi(h)}\int_{r-h}^{r+h}|f(t)|^p\,2tdt < \infty.$$

Let $1 < p < \infty$. The generalized complementary Morrey space $^{\complement}L^{p,\varphi}(I)$ over the interval I is defined as the set of functions f measurable on I such that

$$\sup_{h\in I}\varphi(h)\int_0^{1-h}|f(t)|^p\,2tdt < \infty.$$

The p-th root from the expressions above provides the corresponding norm in $L^{p,\varphi}(I)$ and in $^{\complement}L^{p,\varphi}(I)$. The space $L^{p,\varphi}(I)$ is trivial if $\lim_{t\to 0}\frac{\varphi(t)}{t} = 0$, and $L^{p,\varphi}(I)\Big|_{\varphi(t)=t} = L^\infty(I)$. So we suppose that $\varphi(t) \geqslant Ct$, when $t \to 0$. The spaces $L^{p,\varphi}(I)$, $^{\complement}L^{p,\varphi}(I)$ are non separable. The embedding $L^{p,\varphi}(I) \hookrightarrow L^p(I)$, $1 \leqslant p < \infty$ is obvious. For the embedding $^{\complement}L^{p,\varphi}(I) \hookrightarrow L^1(I)$, $1 < p < \infty$, we need to assume additional condition on φ provided by the following result.

LEMMA 2.1. *Let $1 < p < \infty$ and*

$$\int_I \frac{dt}{(t\varphi(t))^{\frac{1}{p}}} < \infty. \tag{2.2}$$

Then $^{\complement}L^{p,\varphi}(I) \hookrightarrow L^1(I)$.

P r o o f. The proof is straightforward: use dyadic decomposition, which is the standard tool when working with Morrey type norm (see, for instance, [6, 31]) for the interval I and apply Hölder inequality. We have ($\mathbb{Z}_+ \equiv \mathbb{N}\cup\{0\}$):

$$\int_I |f(t)|2tdt \leqslant 2^{\frac{1}{p'}}\sum_{k\in\mathbb{Z}_+}\left(2^{-k-1}\right)^{\frac{1}{p'}}\left(\int_{1-2^{-k}}^{1-2^{-k-1}}|f(t)|^p 2tdt\right)^{\frac{1}{p}}$$

$$\leqslant 2^{\frac{1}{p'}}\sum_{k\in\mathbb{Z}_+}\left(2^{-k-1}\right)^{\frac{1}{p'}}\varphi^{-\frac{1}{p}}(2^{-k-1})\left(\varphi(2^{-k-1})\int_0^{1-2^{-k-1}}|f(t)|^p 2tdt\right)^{\frac{1}{p}}$$

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$$\leqslant 2^{1+\frac{2}{p'}}\|f\|_{{}^{\complement}L^{p,\varphi}(I)} \sum_{k\in\mathbb{Z}_+} \int_{2^{-k-2}}^{2^{-k-1}} \frac{t^{\frac{1}{p'}}}{\varphi^{\frac{1}{p}}(t)} \frac{dt}{t} = 2^{1+\frac{2}{p'}}\|f\|_{{}^{\complement}L^{p,\varphi}(I)} \int_0^{\frac{1}{2}} \frac{t^{\frac{1}{p'}}}{\varphi^{\frac{1}{p}}(t)} \frac{dt}{t}.$$

We refer, for instance, to Lemma 3.2 from [6] for the estimation of the infinite sum via integral. □

Note that in the classical case of complementary Morrey space when $\varphi(t) = t^\lambda$ the condition (2.2) is $\lambda < p-1$. If there exists $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ decreases on I, then the condition (2.2) is satisfied for such φ.

Remark 2.1. In what follows when considering the Morrey space $L^{p,\varphi}(I)$ and complementary Morrey space ${}^{\complement}L^{p,\varphi}(I)$, we always assume the mentioned above natural assumptions on φ : the function $\varphi : I \to \mathbb{R}_+$ is increasing on I and $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$. Additionally, for the Morrey space we suppose that $\varphi(t) \geqslant Ct$, when $t \to 0$, and for the complementary Morrey space ${}^{\complement}L^{p,\varphi}(I)$ we assume validity of (2.2).

Remark 2.2. In the definition of the Morrey space one may want to use a more classical way, writing $\varphi(4rh)$ instead of $\varphi(h)$, since the measure of the interval $(r-h, r+h)$ with respect to the $2rdr$ equals to $4rh$. Such introduced mixed norm space, let us denote it as $\mathcal{L}_*^{q;\varphi}(\mathbb{D})$, will be different from $\mathcal{L}^{q;\varphi}(\mathbb{D})$. However, the corresponding subspaces of analytic functions coincide up to norm equivalence. For instance, the proof of the Theorem 6.3 explicitly shows that we will have similar results in that another setting. We leave the details for the reader. Also note that the usual definition of the Morrey space deals with the supremum over all $r, h \in I$ and intervals $I \cap (r-h, r+h)$. For our goals, for the definition of $L^{p,\varphi}(I)$ we admit only intervals $(r-h, r+h) \subset I$.

2.3. **Young functions and Orlicz space $L^\Phi(I)$.** We refer to [27, 28, 32, 35]. Let $\Phi : [0,\infty] \to [0,\infty]$ be a convex function, $\Phi(0) = 0, \lim_{x\to\infty} \Phi(x) = \Phi(\infty) = \infty$. From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. To each Young function Φ one identifies the complementary function Ψ, which possesses the same properties, by the rule $\Psi(y) = \sup_{x\geqslant 0}\{xy - \Phi(x)\}$. Note that

$$t \leqslant \Phi^{-1}(t)\Psi^{-1}(t) \leqslant 2t, \ t \geqslant 0. \tag{2.3}$$

We say that $\Phi \in \Delta_2$ if there exists $C_{(2)} > 0$ such that $\Phi(2t) \leqslant C_{(2)}\Phi(t)$, $t > 0$. This Δ_2 condition is usually referred as doubling condition. A Young function Φ is said to satisfy the ∇_2 condition, denoted also by $\Phi \in \nabla_2$, if for some $k > 1$ one has $\Phi(kt) \geqslant 2k\Phi(t)$, $t \geqslant 0$. Let, as usual, $L^\Phi(I)$ be the Orlicz space of functions f measurable on I such that $\int_I \Phi(k|f(r)|)2rdr < \infty$ for

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some $k > 0$. The functional $N_\Phi(f) = \inf\left\{\lambda > 0 :\ \int_I \Phi\left(\frac{|f(r)|}{\lambda}\right) 2rdr \leqslant 1\right\}$ defines norm in $L^\Phi(I)$. The following analog of Hölder inequality is valid:

$$\int_I |f(t)g(t)|2tdt \leqslant 2\|f\|_{L^\Phi(I)}\|g\|_{L^\Psi(I)}. \tag{2.4}$$

Let $\chi_{r,h}$ be characteristic function of the interval $(r-h, r+h)$. It is known that

$$\|\chi_{r,h}\|_{L^\Phi(I)} = \frac{1}{\Phi^{-1}\left(\frac{1}{4rh}\right)}. \tag{2.5}$$

The following result may be found in [24], however we present it here, with slight modification, for the sake of completeness. A function φ is said to be almost decreasing for $t > 0$ if there exists $C > 0$ such that $\varphi(t_2) \leqslant C\varphi(t_1)$, $t_2 > t_1 > 0$.

LEMMA 2.2. *Let Φ satisfy the doubling Δ_2 condition for $t > 0$ with the constant $C_{(2)}$. Then given any $\beta \geqslant \log_2 C_{(2)}$ we have*

(1) *$t^{-\beta}\Phi(t)$ is almost decreasing for $t > 0$;*
(2) *$\Phi(At) \leqslant A^\beta C_{(2)}\Phi(t)$, for any constant $A \geqslant 1$, for $t > 0$.*

P r o o f. The statements (1) and (2) are equivalent up to the constants in the inequalities. Let us prove the second one. Naturally we assume that $C_{(2)} > 1$. Fix $N = [\log_2 A]$ so that $2^N \leqslant A < 2^{N+1}$. Then $\Phi(At) \leqslant \Phi(2^{N+1}t) \leqslant C_{(2)}^{N+1}\Phi(t) = C_{(2)}^{[\log_2 A]+1}\Phi(t) \leqslant C_{(2)}^{\log_2 A+1}\Phi(t) = C_{(2)}^{\log_2 A}C_{(2)}\Phi(t) = A^{\log_2 C_{(2)}}C_{(2)}\Phi(t)$. Hence it follows that given any $\beta \geqslant \log_2 C_{(2)}$ we have $\Phi(At) \leqslant A^\beta C_{(2)}\Phi(t)$, $t > 0$. □

3. Mixed norm Bergman type space and boundedness of Bergman projection: general scheme

3.1. Mixed norm space $\mathcal{L}^{q;X}(\mathbb{D})$ and mixed norm Bergman type space $\mathcal{A}^{q;X}(\mathbb{D})$. Let $X(I) \subseteq L^1(I)$, $I = (0,1)$, denote a Banach space of functions f on interval I containing step functions, and let $\|\cdot\|_{X(I)}$ stand for the norm. Given a function $f(z) = f(r, e^{i\alpha})$ on $\mathbb{D}$ or in general a distribution on $\mathbb{D}$ we denote by f_n its distributional Fourier coefficients. The notion of distributional Fourier coefficients needs a certain precise definition. In order not interrupt the presentation of the main results we refer the reader to Section 7 where all the necessary definitions are given. Introduce the mixed norm space $\mathcal{L}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, as the space of distributions f on $\mathbb{D}$ such that the (distributional) Fourier coefficients f_n are regular functions $f_n \in X(I)$, and the following norm is finite:

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$$\|f\|_{\mathcal{L}^{q;X}(\mathbb{D})} = \left(\sum_{n\in\mathbb{Z}} \|f_n\|^q_{X(I)}\right)^{\frac{1}{q}}. \tag{3.1}$$

THEOREM 3.1. *The space $\mathcal{L}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, is complete.*

P r o o f. Let $f^k \in \mathcal{L}^{q;X}(\mathbb{D})$, $k = 0, 1, \dots$, be a Cauchy sequence. Then for each $n \in \mathbb{Z}$ the sequence of Fourier coefficients f^k_n, $k = 0, 1, \dots$, is Cauchy sequence in $X(I)$ and converges in $X(I)$ to some element $f_n \in X(I)$. From the inequality

$$\sum_{n\in\mathbb{Z}} \left| \|f^k_n\|_{X(I)} - \|f^m_n\|_{X(I)} \right|^q \leqslant \sum_{n\in\mathbb{Z}} \|f^k_n - f^m_n\|^q_{X(I)}$$

it follows that the numerical sequence of elements $\{\|f^k_n\|_{X(I)}\}_{n\in\mathbb{Z}}$, $k = 0, 1, \dots$, is Cauchy sequence in l^q, $1 \leqslant q < \infty$, and, hence, converges to some numerical sequence $\{a_n\}_{n\in\mathbb{Z}} \in l^q$. Obviously, $a_n = \|f_n\|_{X(I)}$, and therefore the distribution $f = \sum_{n\in\mathbb{Z}} f_n(r)e^{in\alpha}$ is the limit of the sequence $f^k \in \mathcal{L}^{q;X}(\mathbb{D})$, $k = 0, 1, \dots$, and its distributional Fourier coefficients are nothing but f_n, by Lemma 7.1. Hence, $f \in \mathcal{L}^{q;X}(\mathbb{D})$ and the space $\mathcal{L}^{q;X}(\mathbb{D})$ is complete. □

We introduce the mixed norm Bergman space $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, as the space of functions from $\mathcal{L}^{q;X}(\mathbb{D})$ which are analytic in $\mathbb{D}$. Hence, the norm of a function $f \in \mathcal{A}^{q;X}(\mathbb{D})$ is given by $\|f\|_{\mathcal{A}^{q;X}(\mathbb{D})} = \left(\sum_{n\in\mathbb{Z}_+} \|f_n\|^q_{X(I)}\right)^{\frac{1}{q}}$.

REMARK 3.1. From the definition of the space $\mathcal{A}^{q;X}(\mathbb{D})$, it can be derived that the Fourier coefficients $f_n = f_n(r)$, $n \in \mathbb{Z}$, of a function f in Bergman space $\mathcal{A}^{q;X}(\mathbb{D})$ may be represented as

$$f_n(r) = \begin{cases} a_n \|r^n\|^{-1}_{X(I)} r^n, & n \in \mathbb{Z}_+, \\ 0, & \text{otherwice}, \end{cases} \tag{3.2}$$

where $\{a_n\}_{n\in\mathbb{Z}_+} \in l^q_+$, $|a_n| = \|f_n\|_{X(I)}$, $n \in \mathbb{Z}_+$, moreover, $\|f\|_{\mathcal{A}^{q;X}(\mathbb{D})} = \|\{a_n\}_{n\in\mathbb{Z}_+}\|_{l^q_+}$.

It is evident that the multipliers $\|r^n\|^{-1}_{X(I)}$, $n \in \mathbb{Z}_+$, in (3.2) characterize the functions in $\mathcal{A}^{q;X}(\mathbb{D})$. Their behaviour when $n \to \infty$ is a crucial point in the whole study. This behavior depends only on the choice of the space $X(I)$. So to characterize the introduced space in each particular case of $X(I)$ we should examine the asymptotic behavior of the numbers $\|r^n\|^{-1}_{X(I)}$.

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This is a quite difficult issue for spaces of functions with special norm, such as, for instance, variable exponent Lebesgue norm, Morrey space norm, etc.

Remark 3.2. Our main goal is the study of the Bergman space of analytic functions $\mathcal{A}^{q;X}(\mathbb{D})$. The space $\mathcal{L}^{q;X}(\mathbb{D})$ plays only a background role: the space $\mathcal{A}^{q;X}(\mathbb{D})$ will be obtained from $\mathcal{L}^{q;X}(\mathbb{D})$ via the Bergman projection. In the definition of the space $\mathcal{L}^{q;X}(\mathbb{D})$ in view of the property of the Bergman projection (see Lemma 3.1) negative entries may be replaced for instance by $\|f_n\|_{Y(I)}$ with an arbitrary Banach space $Y(I)$, or even more generally $\|\{f_n\}_{n\in\mathbb{Z}\setminus\mathbb{Z}_+}\|_{Z(I)}$, where $Z(I)$ is an arbitrary Banach space of sequences of functions. The resulting subspace of analytic functions $\mathcal{A}^{q;X}(\mathbb{D})$ will be the same independently what kind of norm is used for negative entries. Theorem 3.1 remains also true under such changes if the space $Y(I)$ or $Z(I)$ is complete. For simplicity of presentation we keep the definition of the space $\mathcal{L}^{q;X}(\mathbb{D})$ as given in (3.1).

3.2. **Boundedness of Bergman projection from $\mathcal{L}^{q;X}(\mathbb{D})$ onto $\mathcal{A}^{q;X}(\mathbb{D})$.** The proof of the following result is straightforward (see for instance [20]).

Lemma 3.1. *Given a function f in $L^1(\mathbb{D})$ let $f_n = f_n(r)$, $n \in \mathbb{Z}$, denote the Fourier coefficients of the function f. Then the Fourier coefficients of the function $B_{\mathbb{D}}f$ are*

$$\begin{aligned} (B_{\mathbb{D}}f)_n(r) &= \vartheta_n(f)\ r^n, \quad n \in \mathbb{Z}_+, \\ (B_{\mathbb{D}}f)_n(r) &= 0, \quad n \in \mathbb{Z} \setminus \mathbb{Z}_+, \end{aligned} \tag{3.3}$$

where $\vartheta_n(f) = (n+1)\int_I \tau^n f_n(\tau)\ 2\tau\ d\tau$, $n \in \mathbb{Z}_+$.

Let $\mathcal{S}_0^X(\mathbb{D})$ denote the set of functions $f(z) = f(r, e^{i\alpha}) = \sum\limits_{n=-N}^{N} f_n(r)e^{in\alpha}$, $f_n \in X(I)$, where $N \in \mathbb{Z}_+$ is arbitrary. Since $X(I) \subset L^1(I)$, then $\mathcal{S}_0^X(\mathbb{D}) \subset L^1(\mathbb{D})$ and therefore the Bergman projection $B_{\mathbb{D}}$ is well defined on functions of such type as integral operator (2.1). It is evident that $\mathcal{S}_0^X(\mathbb{D})$ is a dense subset in $\mathcal{L}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$. The Bergman projection $B_{\mathbb{D}}$ on $\mathcal{L}^{q;X}(\mathbb{D})$ is understood as a continuous extension from this dense subset (see the proof of Theorem 3.2).

Using (3.3) we get the following expression for the $X(I)$ - norm for $(B_{\mathbb{D}}f)_n$, $n \in \mathbb{Z}_+$ when $f \in L^1(\mathbb{D})$: $\|(B_{\mathbb{D}}f)_n\|_{X(I)} = |\vartheta_n(f)|\,\|r^n\|_{X(I)}$, $n \in \mathbb{Z}_+$. This allows us to formulate the following condition on the space $X(I)$ that ensures boundedness of the corresponding Bergman projection as a projection from $\mathcal{L}^{q;X}(\mathbb{D})$ onto $\mathcal{A}^{q;X}(\mathbb{D})$: there exists $C_0 > 0$ such that

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$$n\left|\int_I \tau^n g(\tau)2\tau d\tau\right| \|r^n\|_{X(I)} \leqslant C_0\|g\|_{X(I)},\ n\to\infty,\ g\in X(I). \tag{3.4}$$

THEOREM 3.2. *Let $1\leqslant q<\infty$, and let condition (3.4) be satisfied. The operator $B_{\mathbb{D}}$ is bounded as a projection from $\mathcal{L}^{q;X}(\mathbb{D})$ onto $\mathcal{A}^{q;X}(\mathbb{D})$.*

P r o o f. For $f\in\mathcal{S}_0^X(\mathbb{D})$ we obtain

$$\|B_{\mathbb{D}}f\|^q_{\mathcal{L}^{q;X}(\mathbb{D})} = \sum_{-N}^{N}\|\,(B_{\mathbb{D}}f)_n\,\|^q_{X(I)} \leqslant C_0^q\sum_{-N}^{N}\|f_n\|^q_{X(I)} = C_0^q\|f\|^q_{\mathcal{L}^{q;X}(\mathbb{D})},$$

where the constant C_0 comes from condition (3.4) and does not depend on f. Making use of the Banach-Steinghaus theorem we finish the proof. □

COROLLARY 3.1. *Let $1\leqslant q<\infty$, and let the condition (3.4) be satisfied. The space $\mathcal{A}^{q;X}(\mathbb{D})$ is a closed subspace of $\mathcal{L}^{q;X}(\mathbb{D})$.*

3.3. **On Toeplitz operators with radial symbols on $\mathcal{A}^{q;X}(\mathbb{D})$.** Given a function $a=a(|z|)\in L^1(\mathbb{D})$ consider the Toeplitz operator T_a on $\mathcal{A}^{q;X}(\mathbb{D})$ which acts on polynomials $f\in\mathcal{A}^{q;X}(\mathbb{D})$ as follows: $T_af(z)=(B_{\mathbb{D}}af)(z)$. For an analytic function $f(z)=\sum_{n\in\mathbb{Z}_+}c_nz^n$ in $\mathcal{A}^{q;X}(\mathbb{D})$ the following formula is true:

$$(T_af)_n(r)=\gamma_a(n)c_nr^n,\quad n\in\mathbb{Z}_+,\quad \text{and}\quad (T_af)_n(r)=0,\quad n\in\mathbb{Z}\setminus\mathbb{Z}_+,$$

where $\{c_n\|r^n\|_{X(I)}\}_{n\in\mathbb{Z}_+}\in l^q_+$, and $\gamma_a(n)=(n+1)\int_I\tau^{2n}a(\tau)\,2\tau\,d\tau$, $n\in\mathbb{Z}_+$. From Remark 3.1 and the definition of the norm in $\mathcal{A}^{q;X}(\mathbb{D})$ it is follows that the operator T_a is bounded on $\mathcal{A}^{q;X}(\mathbb{D})$ if and only if the sequence $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is bounded. The operator T_a is compact on $\mathcal{A}^{q;X}(\mathbb{D})$ if and only if $\gamma_a(n)\to 0$, $n\to\infty$.

There are known many sufficient and, in some cases, necessary conditions for boundedness and vanishing of the sequence $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$. These conditions were obtained (see, for instance, [14]) in terms of behavior of some means (averages) of the symbol a when $r\to 1$. There also many examples of badly behaved oscillating and unbounded symbols that generate even compact operators. We refer to the book [38], and also references therein, for recent development of the theory of Toeplitz operators with special non standard symbols on classical weighted Bergman spaces over unit disc and half plain. We call attention to the fact that in the setting of our spaces $\mathcal{A}^{q;X}(\mathbb{D})$ the boundedness and compactness conditions do not depend on the choice of the space $X(I)$. We suppose to give more consideration to the study of Toeplitz operators in another paper.

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4. The spaces $\mathcal{A}^{q;X}(\mathbb{D})$ and the range of generalized fractional differentiation operator over Hardy type space

We will use the notion of the Hadamard product composition. Let the functions $b(z) = \sum_{k\in\mathbb{Z}_+} b_k z^k$, $g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n$ be analytic in $\mathbb{D}$. Consider the expression $\mathcal{D}(b,g)(z) = b \circ g(z) = \sum_{n\in\mathbb{Z}_+} b_n g_n z^n$, known as Hadamard product composition of functions b and g. This general notion includes in particular operation of fractional integro-differentiation. It generalizes fractional differentiation of analytic function g if $b_n \to \infty$, when $n \to \infty$. For g analytic in $\mathbb{D}$,

$$\mathcal{D}(b,g)(z) = \frac{1}{2\pi i}\int_{|u|=r} b\left(\frac{z}{u}\right) g(u)\frac{du}{u}, \quad |z| < r < 1.$$

We refer to [23], [36] for instance. We will use the following operator

$$\mathcal{D}_X g(z) = \sum_{n\in\mathbb{Z}_+} \|r^n\|^{-1}_{X(I)} g_n z^n, \quad g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n,$$

defined in terms of Hadamard product composition.

The case $q = 2$ is of a special interest. Basing on Remark 3.1 for the case $q = 2$ we arrive at the following theorem.

Theorem 4.1. *The space $\mathcal{A}^{2;X}(\mathbb{D})$ coincides with the range of the operator $\mathcal{D}_X$ over the Hardy space $H^2(\mathbb{D}) : \mathcal{A}^{2;X}(\mathbb{D}) = \mathcal{D}_X(H^2(\mathbb{D}))$.*

In view of Theorem 4.1, the distinction between $\mathcal{A}^{2;X}(\mathbb{D})$ and $H^2(\mathbb{D})$ is determined by the behavior of $\|r^n\|^{-1}_{X(I)}$ when $n \to \infty$. In our general case these asymptotics may be quite different varying, for instance, from very slow logarithmical to very high exponential type. As we showed in [20], we may even meet the non trivial situation when $\|r^n\|_{X(I)} = 1$. In that case $\mathcal{A}^{2;X}(\mathbb{D}) = H^2(\mathbb{D})$. This situation is realized for the case of variable exponent Lebesgue space $X(I) = L^{p(\cdot)}(I)$ when $p = p(r)$ grows to infinity fast enough, so that

$$\lim_{r\to 1} \frac{p(r)}{\ln \frac{A}{(1-r)\ln\frac{1}{1-r}}} = \infty. \tag{4.1}$$

If again $X(I) = L^{p(\cdot)}(I)$ and $p(r)$ is still growing but slower, then in (4.1), like $p(r) \leqslant C_1 \ln^\alpha \frac{1}{1-r}$, in a neighborhood of the point $r = 1$, for some $0 < \alpha < 1$, $C_1 > 0$, then $\|r^n\|_{X(I)} \leqslant C_2\, e^{-C_3(\ln n)^{1-\alpha}}$, $n \to \infty$, with some positive C_2, C_3.

At least for $1 \leqslant q \leqslant 2$ we can provide the information of embedding of $\mathcal{A}^{q;X}(\mathbb{D})$ into mixed integral norm space. Following [11] we introduce the mixed norm space $H(s,t,\gamma)$, $s > 0, t > 0, \gamma > 0$ of measurable on $\mathbb{D}$ functions with the norm:

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$$\|f\|_{H(s,t,\gamma)} = \left\{ \int_I (1-r)^{t\gamma-1}\mathcal{M}_s^t(f;r)\,dr \right\}^{\frac{1}{t}}, \quad 0<t<\infty,$$
$$\|f\|_{H(s,\infty,\gamma)} = \text{ess-sup}_{\,r\in I}\left\{(1-r)^{\gamma}\mathcal{M}_s(f;r)\right\}, \quad t=\infty.$$

Most resent information about such spaces, including embedding theorems, may be found in [2]. By D^α, $\alpha > 0$, we denote the Flett's fractional derivative whose action on analytic function is defined by the multiplication by $(n+1)^\alpha$ of its nth Taylor coefficient.

THEOREM 4.2. *Let $1 \leqslant q \leqslant 2$ and there exist $C > 0$ and $\alpha > 0$ such that $\|r^n\|^{-1}_{X(I)} \leqslant Cn^\alpha$, $n \to \infty$. Then the continuous embedding $\mathcal{A}^{q,X}(\mathbb{D}) \hookrightarrow H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha) \hookrightarrow L^1(\mathbb{D})$ holds, $\alpha < \frac{1}{2}+\frac{1}{s}$, $2 < s \leqslant \infty$, $2 \leqslant t \leqslant \infty$.*

P r o o f. We provide a sketch of proof. Due to $\mathcal{A}^{q;X}(\mathbb{D}) \hookrightarrow \mathcal{A}^{2;X}(\mathbb{D})$, $1 \leqslant q \leqslant 2$ it suffices to prove the theorem for $q = 2$. According to Flett's result (see [11], Theorem B and Theorem 6), we have the estimate $\|D^\alpha f\|_{H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha)} \leqslant C\|f\|_{H^2(\mathbb{D})}$, which shows that the fractional derivative $g = D^\alpha f$ of a function $f \in H^2(\mathbb{D})$ belongs to the weighted mixed norm space $H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha)$ and proves the embedding $\mathcal{A}^{2;X}(\mathbb{D}) \hookrightarrow H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha)$. To prove $H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha) \hookrightarrow L^1(\mathbb{D})$ one has to repeatedly use the Hölder inequality under the condition on parameter $s : \alpha < \frac{1}{2}+\frac{1}{s}$, $2 < s \leqslant \infty$. □

Note that in Theorem 4.2 there is no restriction on t except $2 \leqslant t \leqslant \infty$. Though it is useful to note that the minimal space in the scale $H(s,t,\frac{1}{2}-\frac{1}{s}+\alpha)$ with respect to the parameter t, $2 \leqslant t \leqslant \infty$, is achieved when $t = 2$ (see [2]).

In the following theorem we use the notation of the following classical fractional derivative of analytic functions:

$$\mathcal{D}^\alpha g(z) = \frac{\Gamma(1+\alpha)}{2\pi i}\int_{|u|=r} \frac{g(u)}{\left(1-\frac{z}{u}\right)^{1+\alpha}}\frac{du}{u}, \quad |z|<r<1, \quad \alpha>0. \tag{4.2}$$

THEOREM 4.3. *Let $1 \leqslant q < \infty$ and there exist $C > 0$ and $\alpha > 0$ such that $\|r^n\|^{-1}_{X(I)} \sim Cn^\alpha$, $n \to \infty$. Each function $f \in \mathcal{A}^{q;X}(\mathbb{D})$ is represented as $f = \mathcal{D}^\alpha g$ with some analytic function $g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n$, such that $\{g_n\}_{n\in\mathbb{Z}_+} \in l^q_+$.*

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P r o o f. We have $n^\alpha \sim \Gamma(1+\alpha)(-1)^n \begin{pmatrix} -1-\alpha \\ n \end{pmatrix}$, $n \to \infty$. Replacing $\|r^n\|^{-1}_{X(I)}$ by equivalent expression via binomial coefficients we arrive at the kernel $b(z) = \Gamma(1+\alpha)(1-z)^{-1-\alpha}$, and, consequently at the operator (4.2). □

The condition $\|r^n\|^{-1}_{X(I)} \sim Cn^\alpha$, $n \to \infty$, is satisfied with some $\alpha > 0$, for instance, for the following cases of the space $X(I)$ considered in [20], [21]:

(1) $\alpha = \frac{1}{p(1)}$ for the case of the variable exponent Lebesgue space $X(I) = L^{p(\cdot)}(I)$, $p(1) = \lim_{r\to 1-} p(r) < \infty$, p satisfies logarithmic decay condition in a neighborhood of $r = 1$, i.e. there exist $\delta > 0$, $K > 0$ such that $|p(r) - p(1)| \ln \frac{e}{1-r} \leqslant K, \quad r \in (1-\delta, 1)$;

(2) $\alpha = \frac{1-\lambda}{p}$ for the case of the classical Morrey space $X(I) = L^{p,\lambda}(I)$, $0 \leqslant \lambda < 1$, $1 \leqslant p < \infty$;

(3) $\alpha = \frac{1+\lambda}{p}$ for the case of the classical complementary Morrey space $X(I) = {}^{\complement}L^{p,\lambda}(I)$, $0 \leqslant \lambda < p-1$, $1 < p < \infty$.

5. Mixed norm Bergman - Orlicz space $\mathcal{A}^{q;\Phi}(\mathbb{D})$

Let Φ be a Young function and $X(I) = L^\Phi(I)$. Here instead of writing $\mathcal{L}^{q;L^\Phi(I))}(\mathbb{D})$, as we did previously for an abstract space $X(I)$, we use $\mathcal{L}^{q;\Phi}(\mathbb{D})$ for simplicity. The same applies to $\mathcal{A}^{q;\Phi}(\mathbb{D})$.

5.1. **Characterization of functions in $\mathcal{A}^{q;\Phi}(\mathbb{D})$.** We first provide the description of functions in $\mathcal{A}^{q;\Phi}(\mathbb{D})$ in terms of Taylor coefficients and in terms of fractional derivatives.

THEOREM 5.1. *Let Φ be a Young function, $\Phi \in \Delta_2 \cap \nabla_2$. Then for an analytic in $\mathbb{D}$ function $f(z) = \sum_{n\in\mathbb{Z}_+} c_n z^n \in \mathcal{A}^{q;\Phi}(\mathbb{D})$, $z \in \mathbb{D}$ the norm $\|f\|_{\mathcal{A}^{q;\Phi}(\mathbb{D})}$ is equivalent to $\left(\sum_{n\in\mathbb{Z}_+} \left(\frac{|c_n|}{\Phi^{-1}(n)}\right)^q\right)^{\frac{1}{q}}$.*

THEOREM 5.2. *Under the conditions of Theorem 5.1, each function $f \in \mathcal{A}^{q;\Phi}(\mathbb{D})$ has the form $f = \mathcal{D}_X g$ with some analytic function $g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n$, such that $\{g_n\}_{n\in\mathbb{Z}_+} \in l^q_+$, where $\mathcal{D}_X g(z) = \sum_{n\in\mathbb{Z}_+} \Phi^{-1}(n) g_n z^n$, $z \in \mathbb{D}$.*

Theorem 5.2 follows form Theorem 5.1. The later, in view of Remark 3.1, is the corollary of the following result which will be also crucial for the proof of the boundedness of the Bergman projection in the next section.

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THEOREM 5.3. *Let* Φ *be a Young function. Then*

$$\|r^s\|_{L^\Phi(I)} \leqslant \frac{2}{\Phi^{-1}(s)}, \quad s>0. \tag{5.1}$$

If, in addition, Φ *satisfies the* Δ_2 *doubling condition with the constant* $C_{(2)}$*, then*

$$\frac{1}{\Phi^{-1}(\gamma s)} \leqslant \|r^s\|_{L^\Phi(I)}, \quad s>0, \tag{5.2}$$

where $\gamma = C_{(2)}\left(\frac{1}{2}\log_2 C_{(2)} + \frac{1}{s}\right)$. *If moreover* $\Phi \in \nabla_2$*, then*

$$\frac{\frac{2}{\gamma}}{\Phi^{-1}(s)} \leqslant \frac{1}{\Phi^{-1}(\gamma s)} \leqslant \|r^s\|_{L^\Phi(I)}, \quad s>0. \tag{5.3}$$

P r o o f. Due to the convexity of Φ :

$$\int_I \Phi\left(r^s \frac{1}{2}\Phi^{-1}(s)\right) 2rdr \leqslant \int_I r^{s+1}\Phi(\Phi^{-1}(s))dr = \frac{s}{s+2} < 1,$$

so the estimate (5.1) follows by the definition of the norm in $L^\Phi(I)$.

Put $A = r^{-s}$, $\beta = \log_2 C_{(2)}$, $\gamma = C_{(2)}\left(\frac{1}{2}\log_2 C_{(2)} + \frac{1}{s}\right)$. We assume $C_{(2)} > 1$. Use the second statement of Lemma 2.2 ($s>0$):

$$\int_I \Phi\left(r^s\Phi^{-1}(\gamma s)\right) 2rdr \geqslant \frac{2}{C_{(2)}}\int_I r^{s\beta+1}\Phi\left(\Phi^{-1}(\gamma s)\right)dr = \frac{2s\gamma}{C_{(2)}(s\beta+2)} = 1.$$

Again by the definition of the norm in $L^\Phi(I)$ we obtain the estimate (5.2). To prove (5.3) we have to show that $\gamma s \leqslant \Phi\left(\frac{\gamma}{2}\Phi^{-1}(s)\right)$. Setting $\Phi^{-1}(s) = t$, we get $\gamma\Phi(t) \leqslant \Phi\left(\frac{\gamma}{2}t\right)$, which is exactly the ∇_2 condition for Φ. □

We find it convenient for future references to outline the following bilateral estimate which is proved in Theorems 5.3: if a Young function $\Phi \in \Delta_2 \cap \nabla_2$, then

$$\frac{2}{C_{(2)}\left(\frac{1}{2}\log_2 C_{(2)} + 1\right)}\frac{1}{\Phi^{-1}(n)} \leqslant \|r^n\|_{L^\Phi(I)} \leqslant 2\frac{1}{\Phi^{-1}(n)}, \quad n \in \mathbb{N}.$$

REMARK 5.1. Note that Theorems 5.1, 5.2 are obtained under the condition: $\Phi \in \Delta_2 \cap \nabla_2$, which eliminates such Young functions which have exponential type growth. This case undoubtedly presents a great interest and we formulate as an open problem: to prove similar results for the mixed norm Bergman-Orlicz type spaces $\mathcal{A}^{q;\Phi}(\mathbb{D})$ when the condition $\Phi \in \Delta_2 \cap \nabla_2$ is either omitted or at least weakened.

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5.2. **Boundedness of the Bergman projection.**

Theorem 5.4. *Let Φ be a Young function. Then the condition (3.4) with $X(I) = L^{\Phi}(I)$ is satisfied.*

P r o o f. Due to the Hölder inequality,

$$\left|\int_I t^n g(t) 2t dt\right| \leqslant 2\|g\|_{L^{\Phi}(I)}\|t^n\|_{L^{\Psi}(I)}.$$

Therefore,

$$n\left|\int_I t^n g(t) 2t dt\right| \ \|r^n\|_{L^{\Phi}(I)} \leqslant 2n\frac{4\|g\|_{L^{\Phi}(I)}}{\Phi^{-1}(n)\Psi^{-1}(n)} \leqslant 8\|g\|_{L^{\Phi}(I)}, \quad n \in \mathbb{N}.$$

The ultimate inequality follows by (2.3). □

As an immediate consequences of Theorem 3.2 we obtain the following results.

Theorem 5.5. *Let Φ be a Young function. The operator $B_{\mathbb{D}}$ is bounded as a projection from $\mathcal{L}^{q;\Phi}(\mathbb{D})$ onto $\mathcal{A}^{q;\Phi}(\mathbb{D})$, $1 \leqslant q < \infty$.*

Corollary 5.1. *Let Φ be a Young function. The space $\mathcal{A}^{q;\Phi}(\mathbb{D})$ is the closed subspace of $\mathcal{L}^{q;\Phi}(\mathbb{D})$, $1 \leqslant q < \infty$.*

Remark 5.2. Note that Theorem 5.5 holds for an arbitrary Young function, i.e. it admits, in particular, exponential growth of Φ, like $\Phi(t) = e^t - 1$. In this case the norms $\|r^n\|_{L^{\Phi}(I)}$ will be bounded by log-type estimate: $\frac{2}{\ln(n+2)}$.

6. Mixed norm Bergman-Morrey type spaces

Here and below the space $X(I)$ is either the generalized Morrey space $L^{p,\varphi}(I)$, or the generalized complementary Morrey space ${}^{\complement}L^{p,\varphi}(I)$. Similarly to the previous section, if $X(I) = L^{p,\varphi}(I)$, instead of writing $\mathcal{L}^{q;L^{p,\varphi}}(\mathbb{D})$, $\mathcal{A}^{q;L^{p,\varphi}}(\mathbb{D})$ we use the notations $\mathcal{L}^{q;p,\varphi}(\mathbb{D})$, $\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ for simplicity. Also we will denote by ${}^{\complement}\mathcal{L}^{q;p,\varphi}(\mathbb{D})$, ${}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ the spaces that correspond to $X(I) = {}^{\complement}L^{p,\varphi}(I)$.

6.1. **Characterization of functions in $\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ and ${}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$.** We first provide the description of functions in $\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ and ${}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ in terms of Taylor coefficients and in terms of fractional derivatives.

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THEOREM 6.1. *For an analytic in $\mathbb{D}$ function $f(z) = \sum_{n\in\mathbb{Z}_+} c_n z^n$, $z \in \mathbb{D}$ the following statements hold*:

(1) *Let $1 \leqslant p < \infty$. Let the function φ be concave on I, the function $\frac{\varphi(t)}{t}$ be decreasing on I, $\lim_{t\to 0}\frac{\varphi(t)}{t} = \infty$, and the function $t\varphi(\frac{1}{t})$ be concave for $t > 1$. Then the norm $\|f\|_{\mathcal{A}^{q;p,\varphi}(\mathbb{D})}$ of $f \in \mathcal{A}^{q;p,\varphi}(\mathbb{D})$ is equivalent to $\left(\sum_{n\in\mathbb{Z}_+} |c_n|^q \left(np\,\varphi\left(\frac{1}{np}\right)\right)^{-\frac{q}{p}}\right)^{\frac{1}{q}}$.*

(2) *Let $1 < p < \infty$. Let there exists $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ is decreasing on I. Then the norm $\|f\|_{{}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})}$ of $f \in {}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ is equivalent to $\left(\sum_{n\in\mathbb{Z}_+} |c_n|^q \left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$.*

THEOREM 6.2. *Under the conditions of Theorem 6.1, each function $f \in \mathcal{A}^{q;p,\varphi}(\mathbb{D})$ is represented as $f = \mathcal{D}_X g$ with some analytic function $g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n$, such that $\{g_n\}_{n\in\mathbb{Z}_+} \in l^q_+$, where $\mathcal{D}_X g(z) = \sum_{n\in\mathbb{Z}_+}\left(np\,\varphi\left(\frac{1}{np}\right)\right)^{\frac{1}{p}} g_n z^n$, $z \in \mathbb{D}$. Analogously, each function $f \in {}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ is represented as $f = \mathcal{D}_X g$ with some analytic function $g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n$, such that $\{g_n\}_{n\in\mathbb{Z}_+} \in l^q_+$, where $\mathcal{D}_X g(z) = \sum_{n\in\mathbb{Z}_+}\left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{-\frac{1}{p}} g_n z^n$, $z \in \mathbb{D}$.*

Theorem 6.2 follows form Theorem 6.1. The latter, in view of Remark 3.1, is the corollary of the estimates obtained below for the norms $\|r^n\|_{L^{p,\varphi}(I)}$ and $\|r^n\|_{{}^{\complement}L^{p,\varphi}(I)}$, which will be also crucial for the proof of the boundedness of the Bergman projection in the next section.

THEOREM 6.3. *Let $1 \leqslant p < \infty$. If the function $\frac{\varphi(t)}{t}$ is decreasing on I, $\lim_{t\to 0}\frac{\varphi(t)}{t} = \infty$, and the function $t\varphi(\frac{1}{t})$ is concave for $t > 1$, then $\|r^n\|_{L^{p,\varphi}(I)} \leqslant C_1\left(np\,\varphi\left(\frac{1}{np}\right)\right)^{-\frac{1}{p}}$, $n \to \infty$. If the function φ is concave on I, then $C_2\left(np\,\varphi\left(\frac{1}{np}\right)\right)^{-\frac{1}{p}} \leqslant \|r^n\|_{L^{p,\varphi}(I)}$, $n \to \infty$. Here C_1, C_2 are some positive constants which do not depend on n.*

P r o o f. Apply the Hölder inequality (2.4) with Young function Φ which will be determined below ($h > 0, r \pm h \in I$): $\frac{1}{\varphi(h)}\int_{r-h}^{r+h} t^{np} 2t\,dt \leqslant \frac{2}{\varphi(h)}\|t^{np}\|_{L^\Phi(I)}\|\chi_{r,h}\|_{L^\Psi(I)}$, where $\chi_{r,h}$ is the characteristic function of $(r-h, r+h)$ and Ψ is the conjugate of Φ. Using (2.5), (2.3) and (5.1) we obtain

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$$\frac{1}{\varphi(h)}\int_{r-h}^{r+h} t^{np}2tdt \leqslant \frac{4}{\varphi(h)\Phi^{-1}(np)\Psi^{-1}\left(\frac{1}{4rh}\right)} \leqslant 4\frac{4rh}{\varphi(h)}\frac{\Phi^{-1}\left(\frac{1}{4rh}\right)}{\Phi^{-1}(np)}.$$

By assumption, the function $t\varphi\left(\frac{1}{t}\right)$ is concave and increasing for $t>1$, so for $t>1$ we set now $\Phi^{-1}(t)=t\varphi\left(\frac{1}{t}\right)$, $t>1$. We have to take care about a possibility of concave continuation of Φ^{-1} to the interval $I=(0,1)$ with the condition $\Phi^{-1}(0)=0$. Note that the behavior of φ when t approaches $t=1$ is not of importance for the Morrey space up to the equivalence of norms. Therefore, if needed, we may modify the function φ in a left-sided neighbourhood of $t=1$ so that the function Φ^{-1} will always have a concave continuation for $t\leqslant 1$. To this end, it suffices to change the function φ so that $\varphi\in C^1(1-\delta,1]$, and $\varphi'(1)>0$.

We claim that $4rh\Phi^{-1}\left(\frac{1}{4rh}\right)\leqslant 4\ h\Phi^{-1}\left(\frac{1}{h}\right)$. Indeed, for $4r<1$ we use the fact that φ increases on I, and for $4r>1$ we apply the fact that $\frac{\varphi(t)}{t}$ decreases on I.

Taking into account the made change of the function φ, up to equivalence of norms we have $\|r^n\|^p_{L^{p,\varphi}(I)}\leqslant C\left(np\varphi\left(\frac{1}{np}\right)\right)^{-1}$, $n\in\mathbb{N}$.

Now we prove the estimate from below. Again, let Φ be a Young function and Ψ – its conjugate. Denote

$$E_n=\left\{r,h\in I:\ r+h=1,\ \frac{1}{4}<(1-2h)^{np}<\frac{1}{2}\right\}.$$

Applying (2.3), we have

$$\begin{aligned}
&\|r^n\|^p_{L^{p,\varphi}(I)}=\sup\nolimits_{r,r\pm h\in I,h>0}\frac{1}{\varphi(h)}\int_{r-h}^{r+h}t^{np}2tdt\\
&\geqslant \sup\nolimits_{r,r\pm h\in I,h>0}\frac{\frac{1}{8}}{\varphi(h)}\Phi^{-1}\left(4\int_{r-h}^{r+h}t^{np}2tdt\right)\Psi^{-1}\left(4\int_{r-h}^{r+h}t^{np}2tdt\right)\\
&\geqslant \sup\nolimits_{r,h\in E_n}\frac{\frac{1}{8}}{\varphi(h)}\Phi^{-1}\left(4(1-2h)^{np}\int_{1-2h}^{1}2tdt\right)\Psi^{-1}\left(8\frac{1-(1-2h)^{np+2}}{np+2}\right)\\
&\geqslant \frac{1}{8}\sup\nolimits_{r,h\in E_n}\frac{1}{\varphi(h)}\Phi^{-1}(4rh)\,\Psi^{-1}\left(\frac{4}{np+2}\right).
\end{aligned}$$

The function φ is concave and increasing on I, so we set now $\Phi^{-1}(t)=\varphi(t)$, $0<t<1$. Again, we need to guarantee the existence of a concave continuation of Φ^{-1} for $t\geqslant 1$ with the condition $\Phi^{-1}(\infty)=\infty$. As in the previous case we free to modify φ in a left sided neighbourhood of $t=1$ so that $\varphi\in C^1(1-\delta,1]$, and $\varphi'(1)>0$.

Since $4rh>h$ on E_n, and $\frac{4}{np+2}>\frac{1}{np}$, $n\neq 0$, then

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$$\|r^n\|^p_{L^{p,\varphi}(I)} \geqslant C\sup_{r,h\in E_n} \frac{1}{\varphi(h)} \Phi^{-1}(h)\, \Psi^{-1}\left(\frac{1}{np}\right) \geqslant C\frac{1}{np\ \Phi^{-1}(\frac{1}{np})},\ n \in \mathbb{N}.$$

This finishes the proof. □

THEOREM 6.4. *Let $1 < p < \infty$. Then $C_1 \left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{\frac{1}{p}} \leqslant \|r^n\|_{\mathtt{C}L^{p,\varphi}(I)}$, $n \to \infty$. If there exists $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ is decreasing on I, then $\|r^n\|_{\mathtt{C}L^{p,\varphi}(I)} \leqslant C_2 \left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{\frac{1}{p}}$, $n \to \infty$. Here C_1, C_2 are some positive constants which do not depend on n.*

P r o o f. In is obvious that

$$\|r^n\|^p_{\mathtt{C}L^{p,\varphi}(I)} = \sup_{h\in I} \frac{\varphi(h)\,(1-h)^{np+2}}{np+2} \geqslant \frac{\varphi(\frac{1}{n})(1-\frac{1}{n})^{np+2}}{np+2} \geqslant C\frac{1}{n}\varphi\left(\frac{1}{n}\right).$$

Since φ is increasing on I, then $\sup_{0<h<\frac{1}{n}} \frac{\varphi(h)(1-h)^{np+2}}{np+2} \leqslant C_1 \frac{1}{n}\varphi\left(\frac{1}{n}\right)$, and since there exists $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ is decreasing on I, then

$$\begin{aligned} &\sup_{\frac{1}{n}<h<1} \frac{\varphi(h)}{h^\beta} \frac{h^\beta\,(1-h)^{np+2}}{np+2} \\ &\leqslant C_2 \frac{1}{n} \frac{\varphi(\frac{1}{n})}{\left(\frac{1}{n}\right)^\beta} \sup_{\frac{1}{n}<h<1} h^\beta\,(1-h)^{np+2} \leqslant C_3 \frac{1}{n}\varphi\left(\frac{1}{n}\right). \end{aligned}$$

□

We find it convenient for further references to outline the following bilateral estimates which are proved in Theorems 6.3, 6.4:

(1) Let $1 \leqslant p < \infty$. Let the function φ be concave on I, the function $\frac{\varphi(t)}{t}$ be decreasing on I, $\lim_{t\to 0} \frac{\varphi(t)}{t} = \infty$, and the function $t\varphi(\frac{1}{t})$ be concave for $t > 1$. Then $C_1 \left(np\ \varphi\left(\frac{1}{np}\right)\right)^{-\frac{1}{p}} \leqslant \|r^n\|_{L^{p,\varphi}(I)} \leqslant C_2 \left(np\ \varphi\left(\frac{1}{np}\right)\right)^{-\frac{1}{p}}$, $n \to \infty$, where C_1, C_2 are some positive constants which do not depend on n.

(2) Let $1 < p < \infty$. Let there exists $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ is decreasing on I. Then $C_1 \left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{\frac{1}{p}} \leqslant \|r^n\|_{\mathtt{C}L^{p,\varphi}(I)} \leqslant C_2 \left(\frac{1}{n}\varphi\left(\frac{1}{n}\right)\right)^{\frac{1}{p}}$, $n \to \infty$, where C_1, C_2 are some positive constants which do not depend on n.

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6.2. **Boundedness of the Bergman projection.**

THEOREM 6.5. *Let $1 \leqslant p < \infty$. Let the function $\frac{\varphi(t)}{t}$ be decreasing on I, $\lim_{t\to 0} \frac{\varphi(t)}{t} = \infty$, and the function $t\varphi(\frac{1}{t})$ be concave for $t > 1$. Then the condition (3.4) is valid with $X(I) = L^{p,\varphi}(I)$.*

P r o o f. Passing to the dyadic decomposition over the intervals $I_k = (1-2^{-k}, 1-2^{-k-1})$, $k \in \mathbb{Z}_+$, we have: $\int_I \tau^n g(\tau) 2\tau d\tau = \sum_{k\in\mathbb{Z}_+} \int_{I_k} \tau^n g(\tau) 2\tau d\tau$. Let as usual $\frac{1}{p} + \frac{1}{p'} = 1$. Below we will proceed with the case $p > 1$. Using the Hölder inequality we obtain

$$\left|\int_{I_k} \tau^n g(\tau) 2\tau d\tau\right| \leqslant \left(\int_{I_k} \tau^{np'} \varphi^{\frac{p'}{p}}(1-\tau) 2\tau d\tau\right)^{\frac{1}{p'}} \left(\int_{I_k} \frac{1}{\varphi(1-\tau)} |g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}}.$$

For each $k \in \mathbb{Z}_+$ we have $I_k = (1-2^{-k}, 1-2^{-k-1}) = (r_k - h_k, r_k + h_k)$, where $r_k = 1 - 2^{-k-1} - 2^{-k-2}$, $h_k = 2^{-k-2}$. Since φ increases on I, we get

$$\left(\int_{I_k} \frac{1}{\varphi(1-\tau)} |g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}} \leqslant \left(\frac{1}{\varphi(\frac{1}{2^{k+1}})} \int_{I_k} |g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}}$$
$$\leqslant \left(\frac{1}{\varphi(\frac{1}{2^{k+2}})} \int_{I_k} |g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}} \leqslant \|g\|_{L^{p,\varphi}(I)}.$$

It is convenient to introduce the notation $\varphi_*(t) = \dfrac{\varphi(t)}{t}$, $t \in I$. Direct estimation gives

$$\left(\int_{I_k} \tau^{np'} \varphi^{\frac{p'}{p}}(1-\tau) 2\tau d\tau\right)^{\frac{1}{p'}} = \left(\int_{I_k} \tau^{np'} \varphi_*^{\frac{p'}{p}}(1-\tau)(1-\tau)^{\frac{p'}{p}} 2\tau d\tau\right)^{\frac{1}{p'}}$$
$$\leqslant (1-2^{-k-1})^n \varphi^{\frac{1}{p}}\left(\frac{1}{2^{k+1}}\right) 2^{-k} \leqslant 4 \int_{I_{k+1}} \tau^n \varphi_*^{\frac{1}{p}}(1-\tau) d\tau. \tag{6.1}$$

Hence, $\left|\int_I \tau^n g(\tau) 2\tau d\tau\right| \leqslant 4\|g\|_{L^{p,\varphi}(I)} \int_I (1-t)^n \varphi^{\frac{1}{p}}(t) t^{-\frac{1}{p}} dt$. Further,

$$\int_I (1-t)^n \varphi^{\frac{1}{p}}(t) t^{-\frac{1}{p}} dt = \int_0^{np} \left(1 - \frac{t}{np}\right)^n \left(\frac{\varphi\left(\frac{t}{np}\right)}{\frac{t}{np}}\right)^{\frac{1}{p}} \frac{dt}{np}$$
$$\leqslant (np)^{-\frac{1}{p'}} \varphi^{\frac{1}{p}}\left(\frac{1}{np}\right) \left(\int_0^1 \frac{dt}{t^{\frac{1}{p}}} + \int_1^{np} e^{-\frac{t}{p}} dt\right).$$

Here for the estimate over the interval $I = (0,1)$ we use that φ is increasing on I, and for the estimate over $(1, np)$ we use that $\frac{\varphi(t)}{t}$ is decreasing on I.

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Let now $p = 1$. In that case in (6.1) we use the $L^\infty(I_k)$ norm instead of the integral p'-norm. The analogue to (6.1) is as follows $\|\tau^n\varphi(1-\tau)\|_{L^\infty(I_k)} = \|\tau^n\varphi_*(1-\tau)(1-\tau)\|_{L^\infty(I_k)} \leqslant (1-2^{-k-1})^n\varphi\left(\frac{1}{2^{k+1}}\right)2^{-k} \leqslant 4\int_{I_{k+1}}\tau^n\varphi_*(1-\tau)d\tau$, and the rest of the proof is similar to the case $p > 1$.

Collecting all the estimates we finally obtain:

$$n\left|\int_I \tau^n g(\tau)2\tau d\tau\right| \|r^n\|_{L^{p,\varphi}(I)} \leqslant C_0\|g\|_{L^{p,\varphi}(I)},\ n\in\mathbb{N}, g\in L^{p,\varphi}(I), 1\leqslant p<\infty,$$

where the constant C_0 does not depend on either g or n. □

THEOREM 6.6. *Let $1 < p < \infty$. Let there exist $\beta \in (0, p-1)$ such that $\frac{\varphi(t)}{t^\beta}$ decreases on I. Then the condition (3.4) is valid with $X(I) = {}^{\complement}L^{p,\varphi}(I)$.*

P r o o f. We provide a sketch of the proof since it follows the steps of the proof of Theorem 6.5. Use the dyadic decomposition and estimate each integral over I_k:

$$\left|\int_{I_k}\tau^n g(\tau)2\tau d\tau\right| \leqslant \left(\int_{I_k}\tau^{np'}\varphi^{-\frac{p'}{p}}(1-\tau)2\tau d\tau\right)^{\frac{1}{p'}}\left(\int_{I_k}\varphi(1-\tau)|g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}}.$$

For each $k \in \mathbb{Z}_+$ we have $I_k = (1-2^{-k}, 1-2^{-k-1}) \subset \widetilde{I_k} = (0, 1-2^{-k-1}) = (0, 1-h_k)$, where $h_k = 2^{-k-1}$. Since φ increases on I we get

$$\left(\int_{I_k}\varphi(1-\tau)|g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}} \leqslant \left(\varphi\left(\frac{1}{2^{k+1}}\right)\int_{\widetilde{I_k}}|g(\tau)|^p 2\tau d\tau\right)^{\frac{1}{p}} \leqslant \|g\|_{{}^{\complement}L^{p,\varphi}(I)}.$$

Further,

$$\begin{aligned}\left(\int_{I_k}\tau^{np'}\varphi^{\frac{p'}{p}}(1-\tau)2\tau d\tau\right)^{\frac{1}{p'}} &\leqslant (1-2^{-k-1})^n\varphi^{-\frac{1}{p}}\left(\frac{1}{2^{k+1}}\right)2^{-\frac{k}{p'}}\\ \leqslant\ 2^{1+\frac{1}{p'}}&\int_{I_{k+1}}\frac{\tau^n}{\varphi^{\frac{1}{p}}(1-\tau)(1-\tau)^{\frac{1}{p}}}d\tau,\end{aligned}$$

and therefore, $\left|\int_I\tau^n g(\tau)2\tau d\tau\right| \leqslant 2^{1+\frac{1}{p'}}\|g\|_{{}^{\complement}L^{p,\varphi}(I)}\int_I(1-t)^n\varphi^{-\frac{1}{p}}(t)t^{-\frac{1}{p}}dt$.
Finally,

$$\begin{aligned}&\int_I(1-t)^n(t\varphi(t))^{-\frac{1}{p}}dt \leqslant \frac{1}{n}\int_0^1\left(\left(\frac{t}{n}\right)^{-\beta}\varphi\left(\frac{t}{n}\right)\right)^{-\frac{1}{p}}\left(\frac{t}{n}\right)^{-\frac{\beta+1}{p}}dt\\ &+\frac{1}{n}\int_1^n e^{-t}\left(\frac{t}{n}\varphi\left(\frac{t}{n}\right)\right)^{-\frac{1}{p}}dt \leqslant Cn^{-\frac{1}{p'}}\varphi^{-\frac{1}{p}}\left(\frac{1}{n}\right)\left(\int_0^1\frac{dt}{t^{\frac{\beta+1}{p}}}+\int_1^n t^{-\frac{1}{p}}e^{-t}dt\right).\end{aligned}$$

Here for the estimate over the interval $I = (0,1)$ we use that $t^{-\beta}\varphi(t)$ is decreasing on I, and for the estimate over $(1,n)$ we use that $\varphi(t)$ is increasing on I. Collecting all the estimates will finish the proof. □

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Now in view of Theorem 3.2 we have the following result.

THEOREM 6.7. *The following statements are true*:

(1) *Let the function* $\frac{\varphi(t)}{t}$ *be decreasing on* I, $\lim_{t\to 0}\frac{\varphi(t)}{t}=\infty$, *and the function* $t\varphi(\frac{1}{t})$ *be concave for* $t>1$. *The operator* $B_{\mathbb{D}}$ *is bounded as a projection from* $\mathcal{L}^{q;p,\varphi}(\mathbb{D})$ *onto* $\mathcal{A}^{q;p,\varphi}(\mathbb{D})$, $1\leqslant p<\infty$, $1\leqslant q<\infty$;

(2) *Let there exists* $\beta\in(0,p-1)$ *such that* $\frac{\varphi(t)}{t^\beta}$ *decreases on* I. *The operator* $B_{\mathbb{D}}$ *is bounded as a projection from* ${}^{\complement}\mathcal{L}^{q;p,\varphi}(\mathbb{D})$ *onto* ${}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$, $1<p<\infty$, $1\leqslant q<\infty$.

COROLLARY 6.1. *Under the conditions of Theorem 6.7 the spaces* $\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ *and* ${}^{\complement}\mathcal{A}^{q;p,\varphi}(\mathbb{D})$ *are correspondingly the closed subspaces of* $\mathcal{L}^{q;p,\varphi}(\mathbb{D})$ *and* ${}^{\complement}\mathcal{L}^{q;p,\varphi}(\mathbb{D})$.

7. Appendix: On distributional Fourier coefficients

Given a function $f(z)=f(r,e^{i\alpha})\in L^1(\mathbb{D})$ its Fourier coefficients $f_n(r) =\frac{1}{2\pi}\int_0^{2\pi}f(r,e^{i\alpha})e^{-in\alpha}d\alpha$, $n\in\mathbb{Z}$, exist for almost all $r\in I$. The distribution of Fourier series of 2π periodic functions, i.e., Fourier analysis on the unit circle, is well known. For our goals we need the notion of distributional Fourier coefficient of a distribution f on $\mathbb{D}$. This coefficient are treated as a distribution on the interval I. We do not touch the study of the distributional Fourier transform in full extend and rather restrict ourselves to necessary facts for our needs.

We define the test function space $\mathfrak{S}=\mathfrak{S}(\mathbb{D})$ as the set of functions $\omega=\omega(r,e^{i\alpha})\in C^\infty(\mathbb{D})$ such that $\Lambda_{m_1,m_2}(\omega)=\sup_{r,\alpha}\left|\partial_r^{m_1}\partial_\alpha^{m_2}\omega(r,e^{i\alpha})\right|<\infty$, $m_1,m_2\in\mathbb{Z}_+$. The set $\mathfrak{S}$ is a linear topological space with the topology defined by the countable set of seminorms $\Lambda_{m_1,m_2}(\cdot)$. For any $\gamma>0$ Fourier coefficients of $\omega\in\mathfrak{S}$ satisfy the estimates

$$\sup_{r\in I}|\omega_n(r)|\leqslant C|n|^{-\gamma} \tag{7.1}$$

with the constant $C>0$ depending only on ω and γ. By $\mathfrak{S}'=\mathfrak{S}'(\mathbb{D})$ we denote the set of all linear continuous functionals (distributions) on $\mathfrak{S}$. By (f,ω), $f\in\mathfrak{S}'$, $\omega\in\mathfrak{S}$, we denote the value of functional f on test function ω choosing such a bilinear form for that which coincides for $f\in L^1(\mathbb{D})$ with

$$(f,\omega)=\int_{\mathbb{D}}f(z)\,\overline{\omega(z)}\,dA(z)=\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}f(r,e^{i\alpha})\,\overline{\omega(r,e^{i\alpha})}\,d\alpha\right)2rdr.$$

Let now $\sigma=\sigma(I)$ be the set of test functions $v\in C^\infty(I)$ such that $\lambda_m(v)=\sup_{r\in I}\left|v^{(m)}(r)\right|<\infty$, $m\in\mathbb{Z}_+$. Thus the space of test functions σ is a linear topological space with the topology defined by the countable set of seminorms $\lambda_m(\cdot)$. By $\sigma'=\sigma'(I)$ we denote the space

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of linear continuous functionals (distributions) on σ. Similarly, $\langle g, v\rangle$ will represent the corresponding bilinear form in the case of "nice" functionals $g : \langle g, v\rangle = \int_0^1 g(r)\ \overline{v(r)}\ 2rdr$. Given a distribution $f \in \mathfrak{S}'$ we define its distributional Fourier coefficient $f_n \in \sigma'$ by the rule

$$\langle f_n, v\rangle = (f, ve^{in\alpha}), \quad v \in \sigma,\ n \in \mathbb{Z}. \tag{7.2}$$

The function $ve^{in\alpha}$ belongs to $\mathfrak{S}$ so the right side of this equality is well defined for any $f \in \mathfrak{S}'$. If $f \in L^1(\mathbb{D})$ then the equality (7.2) is valid in the regular sense when both sides replaced with corresponding integrals. This fact justifies our definition (7.2).

LEMMA 7.1. *Let $f_n \in L^1(I)$, $n \in \mathbb{Z}$, and suppose that $\|f_n\|_{L^1(I)} \leqslant C|n|^\gamma$ for some $\gamma \geqslant 0$ and absolute constant $C > 0$. Then the series $\sum_{n\in\mathbb{Z}} f_n(r)e^{in\alpha}$ converges to a distribution f in $\mathfrak{S}'$. The distributional Fourier coefficients of f are nothing but f_n, i.e. the distributional expansion of f into Fourier series is unique.*

P r o o f. In view of (7.1) we have (use $\gamma + 2$ instead of γ there):

$$\begin{aligned} \left|(f_n(r)e^{in\alpha}, \omega)\right| &= \left|\int_I f_n(r) 2rdr \int_0^{2\pi} e^{in\alpha}\ \overline{\omega(r, e^{i\alpha})}\ \frac{d\alpha}{2\pi}\right| \\ &= \left|\int_I f_n(r)\ \overline{\omega_n(r)}\ 2rdr\right| \leqslant C|n|^\gamma |n|^{-\gamma-2} = C|n|^{-2},\ n \neq 0. \end{aligned}$$

Hence, the limit

$$\lim_{N\to\infty}\left(\sum_{n=-N}^{N} f_n(r)e^{in\alpha}, \omega\right) = \lim_{N\to\infty}\sum_{n=-N}^{N}\left(f_n(r)e^{in\alpha}, \omega\right) \tag{7.3}$$

exists for any $\omega \in \mathfrak{S}$, and so it defines a distribution f in $\mathfrak{S}'$. Let us show that f_n is the distributional Fourier coefficient of f. Indeed, let $\omega(r, e^{i\alpha}) = v(r)e^{il\alpha} \in \mathfrak{S}$ for some fixed $l \in \mathbb{Z}$. Due to the orthogonality and (7.3) we obtain:

$$(f, \omega) = \left(\lim_{N\to\infty}\sum_{n=-N}^{N} f_n(r)e^{in\alpha}, \omega\right) = \int_0^1 f_l(r)\ \overline{v(r)}\ 2rdr = \langle f_l, v\rangle.$$

The distributional coincidence of nth Fourier coefficients of f with f_n is then obvious. □

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