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Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators

S. Samko [a,*], B. Vakulov [b]

[a] *University of Algarve, Portugal*
[b] *Rostov State University, Russia*

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Abstract

We prove Sobolev-type $p(\cdot) \to q(\cdot)$-theorems for the Riesz potential operator I^{α} in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with the variable exponent $p(x)$ and a two-parametrical power weight fixed to an arbitrary finite point and to infinity, as well as similar theorems for a spherical analogue of the Riesz potential operator in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

Keywords: Weighted Lebesgue spaces; Variable exponent; Riesz potentials; Spherical potentials; Stereographical projection

1. Introduction

Recently, an obvious interest to the operator theory in the generalized Lebesgue spaces with variable exponent $p(x)$ could be observed in a variety of papers, the main objects being the maximal operator, Hardy operators, singular operators and potential type operators, we refer, in particular to surveys [13,24].

* Corresponding author.
E-mail addresses: ssamko@ualg.pt (S. Samko), vakulov@math.rsu.ru (B. Vakulov).

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In the case of maximal operators, we refer to L. Diening [5] for bounded domains in $\mathbb{R}^n$ and to D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer [3] and A. Nekvinda [22,23] for unbounded domains, and to V. Kokilashvili and S. Samko [17] for weighted boundedness on bounded domains.

Hardy operators, including the weighted case were treated in [17].

Singular operators were studied by L. Diening and M. Ružička [8–10] in the non-weighted case and by V. Kokilashvili and S. Samko [15,16] in the weighted case.

Sobolev $p(\cdot) \to q(\cdot)$-theorem for potential operators on bounded domains was considered in S.G. Samko [25] and L. Diening [6], in [6] there being also treated the case of unbounded domains under the assumption that the maximal operator is bounded. Some version of the Sobolev-type theorem for unbounded domain was given in V. Kokilashvili and S. Samko [14]. The Sobolev theorem for unbounded domains in its natural form was proved by C. Capone, D. Cruz-Uribe and A. Fiorenza [1]. Another proof may be found in D. Cruz-Uribe, A. Fiorenza, J.M. Martell, and C. Perez [2] where there are also given new insights into the problems of boundedness of singular and maximal operators in variable exponent spaces.

A weighted statement on $p(\cdot) \to p(\cdot)$-boundedness for the Riesz potential operators on bounded domains was obtained in V. Kokilashvili and S.G. Samko [17], limiting inequalities for bounded domains having been recently proved in S. Samko [27] (Hardy type inequality, $p(\cdot) \to p(\cdot)$-setting) and [28] (Stein–Weiss type inequality, $p(\cdot) \to q(\cdot)$-setting).

In this paper we prove a weighted Sobolev-type theorem for the Riesz potential operator

$$I^\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n, \tag{1.1}$$

over the whole space $\mathbb{R}^n$, in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with the variable exponent $p(x)$ and power weight fixed to the origin and infinity.

We prove also a similar theorem for the spherical analogue

$$(K^\alpha f)(x) = \int\limits_{\mathbb{S}_n} \frac{f(\sigma)}{|x-\sigma|^{n-\alpha}}\,d\sigma, \quad x \in \mathbb{S}_n,\ 0<\alpha<n, \tag{1.2}$$

of the Riesz potential in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

The main results are formulated in Theorems 3.1 and 3.5. Theorem 3.5 for the spherical potential operators is derived from Theorem 3.1 for spatial potentials, while the proof of Theorem 3.1 is based on usage of the estimates obtained in [28].

2. Preliminaries

2.1. *The space* $L^{p(\cdot)}(\mathbb{R}^n, \rho)$

By $L^{p(\cdot)}(\Omega, \rho)$ we denote the weighted space of functions $f(x)$ on Ω such that

$$\int\limits_\Omega \rho(x)\big|f(x)\big|^{p(x)}\,dx < \infty,$$

where $p(x)$ is a measurable function on Ω with values in $[1,\infty)$ and $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, $x \in \Omega$ and ρ is the weight function. This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\rho)} = \inf\left\{\lambda > 0\colon \int\limits_{\Omega} \rho(x)\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\} \tag{2.1}$$

(see, e.g., [18]). We refer to [11,12,18,25] for basics of the spaces $L^{p(\cdot)}$ with variable exponent.

We deal with $\Omega = \mathbb{R}^n$ and consider the weight fixed to the origin and infinity:

$$\rho(x) = \rho_{\gamma_0,\gamma_\infty}(x) = |x|^{\gamma_0}\big(1+|x|\big)^{\gamma_\infty-\gamma_0}. \tag{2.2}$$

We assume that the exponent $p(x)$ satisfies the conditions

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \quad x \in \mathbb{R}^n, \tag{2.3}$$

$$\big|p(x)-p(y)\big| \leqslant \frac{A}{\ln\frac{1}{|x-y|}}, \quad |x-y| \leqslant \frac{1}{2},\ x, y \in \mathbb{R}^n; \tag{2.4}$$

observe that from (2.4) there follows that

$$\big|p(x)-p(y)\big| \leqslant \frac{NA}{\ln\frac{N}{|x-y|}} \tag{2.5}$$

for $x, y \in \bar{\Omega}$, where Ω is any bounded domain in $\mathbb{R}^n$ and $N = 2\operatorname{diam}\Omega$.

We treat $p(x)$ as a function on $\dot{\mathbb{R}}^n$ where $\dot{\mathbb{R}}^n$ is the compactification of $\mathbb{R}^n$ by the unique infinite point. To manage with the weighted case under the consideration, we introduce an assumption on $p(x)$ at infinity stronger than the usually considered assumption

$$\big|p(x)-p(\infty)\big| \leqslant \frac{A_\infty}{\ln(e+|x|)}, \quad x \in \mathbb{R}^n \tag{2.6}$$

(see, for instance, [3,26]); namely, we suppose that

$$\big|p_*(x)-p_*(y)\big| \leqslant \frac{A_\infty}{\ln\frac{1}{|x-y|}}, \quad |x-y| \leqslant \frac{1}{2},\ x, y \in \mathbb{R}^n, \tag{2.7}$$

where $p_*(x) = p\big(\frac{x}{|x|^2}\big)$. Condition (2.7) will be essentially used in the proof of Theorem 3.1, see the part "*The term A_{--}*" in Section 4. Namely, to be able to apply Theorem 2.3 given below, we will need the fact that after the inversion change of variables $x \to x^* = \frac{x}{|x|^2}$, the new exponent $p_*(x) = p(x_*)$ satisfies the local log-condition.

Conditions (2.4) and (2.7) taken together are equivalent to the unique global condition

$$\big|p(x)-p(y)\big| \leqslant \frac{C}{\ln\big(\frac{2\sqrt{1+|x|^2}\sqrt{1+|y|^2}}{|x-y|}\big)}, \quad x, y \in \mathbb{R}^n \tag{2.8}$$

(observe that $\inf_{x,y\in\mathbb{R}^n} \frac{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}{|x-y|} = 1$, see (2.25)).

From (2.7) it follows that there exists the limit $p(\infty) := \lim_{x\to\infty} p(x)$ and (2.6) holds.

Remark 2.1. Condition (2.7) is indeed stronger than condition (2.6), that is, there exist functions $p(x)$ (and even radial ones) such that both the local log-condition and condition (2.6) are satisfied, but condition (2.7) does not hold. This is proved in Appendix A.

The Riesz–Thorin interpolation theorem is valid for the spaces $L^{p(\cdot)}$, as observed by L. Diening [4, p. 20] (see also [7, p. 5]) and proved in a more general setting for Musielak–Orlicz spaces in [21, Theorem 14.16]. Namely, the following statement holds.

Theorem 2.2. *Let $p_j : \Omega \to [1,\infty)$ be bounded measurable functions, $j = 1,2$, and A a linear operator defined on $L^{p_1(\cdot)}(\Omega) \cup L^{p_2(\cdot)}(\Omega)$ and $\|Af\|_{L^{p_j(\cdot)}} \leqslant C_j\|f\|_{L^{p_j(\cdot)}}$, $j = 1,2$. Then A is also bounded in $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} = \frac{1-\theta}{p_1(\cdot)} + \frac{\theta}{p_2(\cdot)}$ and $\|A\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \leqslant C_1^{1-\theta} C_2^{\theta}$.*

Let $q(x)$ be the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}; \tag{2.9}$$

we assume that

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha} \tag{2.10}$$

so that $q(x)$ also satisfies conditions (2.3), (2.4), (2.6).

Weighted $p(\cdot) \to q(\cdot)$-estimates for the operator I^α in the case of bounded domains were proved in [28]. Namely, the following statement holds (in [28] it was proved in the case when the order $\alpha = \alpha(x)$ is variable as well).

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \bar{\Omega}$ and let $p(x)$ satisfy conditions* (2.3) *and* (2.4) *in Ω and* $\operatorname{ess\,sup}_{x\in\Omega} p(x) < \frac{n}{\alpha}$. *Then the following estimate*

$$\|I^\alpha f\|_{L^{q(\cdot)}(\Omega,|x-x_0|^\mu)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega,|x-x_0|^\gamma)} \tag{2.11}$$

is valid, if

$$\alpha p(x_0) - n < \gamma < n\big[p(x_0) - 1\big] \tag{2.12}$$

and

$$\mu = \frac{q(x_0)}{p(x_0)}\gamma. \tag{2.13}$$

2.2. On the inversion $x_* = \frac{x}{|x|^2}$

Lemma 2.4. *Let $x, y \in \mathbb{R}^n$. The following properties hold*:

$$|x_* - y_*| = \frac{|x-y|}{|x|\cdot|y|}, \qquad |x_* - x| = \frac{|1-|x|^2|}{|x|}, \tag{2.14}$$

$$|x_* - y|^2 = \frac{|x-y|^2 + (1-|x|^2)(1-|y|^2)}{|x|^2}, \tag{2.15}$$

and

$$|x_* - y| \geqslant \frac{|x-y|}{|x|} \quad \textit{for } |x| \leqslant 1,\ |y| \leqslant 1. \tag{2.16}$$

Proof. Both the relations in (2.14) and (2.15) are verified directly:

$$|x_* - y_*|^2 = \frac{1}{|x|^2} - 2\frac{x \cdot y}{|x|^2|y|^2} + \frac{1}{|y|^2} = \frac{|x-y|^2}{|x|^2|y|^2},$$

and similarly for the second relation in (2.14) and formula (2.15). The inequality in (2.16) is a consequence of (2.15). □

Lemma 2.5. *Let p satisfy condition* (2.4). *Then in the spherical layer $\frac{1}{2} \leqslant |x| \leqslant 2$ the inequality*

$$\left|p(x_*) - p(x)\right| \leqslant \frac{C}{\ln \frac{2}{|1-|x|^2|}} \tag{2.17}$$

is valid, where $C > 0$ does not depend on x.

Proof. By (2.5), we have

$$\left|p(x_*) - p(x)\right| \leqslant \frac{4A}{\ln \frac{4}{|x-x_*|}} = \frac{4A}{\ln \frac{4|x|}{|1-|x|^2|}},$$

where we have used the second of the relations in (2.14). Hence (2.17) easily follows since $\frac{1}{2} \leqslant |x| \leqslant 2$. □

2.3. The space $L^{p(\cdot)}(\mathbb{S}^n, \rho)$

We consider a similar weighted space with variable exponent on the unit sphere $\mathbb{S}^n = \{\sigma \in \mathbb{R}^{n+1}\colon |\sigma| = 1\}$:

$$L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b}) = \left\{ f\colon \int_{\mathbb{S}^n} \rho_{\beta_a,\beta_b}(\sigma)\left|f(\sigma)\right|^{p(\sigma)}\,d\sigma < \infty \right\},$$

where $\rho_{\beta_a,\beta_b}(\sigma) = |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b}$ and $a \in \mathbb{S}^n$ and $b \in \mathbb{S}^n$ are arbitrary points on the unit sphere $\mathbb{S}^n$.

For the variable exponent $p(\sigma)$ defined on $\mathbb{S}^n$ we assume that

$$1 < p_- \leqslant p(\sigma) \leqslant p_+ < \infty, \quad \sigma \in \mathbb{S}^n, \tag{2.18}$$

$$\left|p(\sigma_1) - p(\sigma_2)\right| \leqslant \frac{A}{\ln \frac{3}{|\sigma_1 - \sigma_2|}}, \quad \sigma_1 \in \mathbb{S}^n,\ \sigma_2 \in \mathbb{S}^n, \tag{2.19}$$

$$\operatorname{ess\,sup}_{\sigma \in \mathbb{S}^n} p(\sigma) < \frac{n}{\alpha}. \tag{2.20}$$

Under assumption (2.18), this is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b})} = \left\{ \lambda > 0\colon \int_{\mathbb{S}^n} |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b} \left|\frac{f(\sigma)}{\lambda}\right|^{p(\sigma)} d\sigma \leqslant 1 \right\}.$$

2.4. Stereographic projection

We use the stereographic projection (see, for instance, [19, p. 36]) of the sphere $\mathbb{S}^n$ onto the space $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1}\colon\ x_{n+1} = 0\}$ generated by the following change of variables in $\mathbb{R}^{n+1}$:

$$\xi = s(x) = \big\{s_1(x), s_2(x), \dots, s_{n+1}(x)\big\}, \tag{2.21}$$

where

$$s_k(x) = \frac{2x_k}{1+|x|^2}, \quad k = 1, 2, \dots, n, \quad \text{and} \quad s_{n+1}(x) = \frac{|x|^2-1}{|x|^2+1},$$

$x \in \mathbb{R}^{n+1}$, $|x| = \sqrt{x_1^2 + \dots + x_{n+1}^2}$.

We remind some useful formulas of passage from $\mathbb{R}^n$ to $\mathbb{S}^n$:

$$|x| = \frac{|\xi + e_{n+1}|}{|\xi - e_{n+1}|}, \qquad \sqrt{1+|x|^2} = \frac{2}{|\xi - e_{n+1}|}, \tag{2.22}$$

$$|x-y| = \frac{2|\sigma - \xi|}{|\sigma - e_{n+1}| \cdot |\xi - e_{n+1}|}, \qquad dy = \frac{2^n\, d\sigma}{|\sigma - e_{n+1}|^{2n}}, \tag{2.23}$$

and inverse formulas of passage from $\mathbb{S}^n$ to $\mathbb{R}^n$:

$$|\xi - e_{n+1}| = \frac{2}{\sqrt{1+|x|^2}}, \qquad |\xi + e_{n+1}| = \frac{2|x|}{\sqrt{1+|x|^2}}, \tag{2.24}$$

$$|\xi - \sigma| = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \qquad d\sigma = \frac{2^n\, dy}{(1+|y|^2)^n}, \tag{2.25}$$

where $\xi = s(x)$, $\sigma = s(y)$, $x, y \in \mathbb{R}^{n+1}$ and $e_{n+1} = (0, 0, \dots, 0, 1)$.

Lemma 2.6. *If the spatial exponent $p(x)$ defined on $\mathbb{R}^n$ satisfies the logarithmic conditions* (2.4) *and* (2.7)*, then the spherical exponent $p[s^{-1}(\sigma)]$ satisfies the logarithmic condition* (2.19) *on $\mathbb{S}^n$. Inversely, if a function $p(\sigma), \sigma \in \mathbb{S}^n$ satisfies condition* (2.19)*, then the function $p[s(x)]$, $x \in \mathbb{R}^n$, satisfies conditions* (2.4) *and* (2.7)*.*

Proof. The proof is direct. □

3. The main statements

Theorem 3.1. *Under assumptions* (2.3), (2.4), (2.6) *and* (2.10) *the spatial potential type operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \rho_{\mu_0,\mu_\infty})$, where*

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \textit{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty, \tag{3.1}$$

if

$$\alpha p(0) - n < \gamma_0 < n\big[p(0) - 1\big], \qquad \alpha p(\infty) - n < \gamma_\infty < n\big[p(\infty) - 1\big], \tag{3.2}$$

and the exponents γ_0 and γ_∞ are related to each other by the equality

$$\frac{q(0)}{p(0)}\gamma_0+\frac{q(\infty)}{p(\infty)}\gamma_\infty=\frac{q(\infty)}{p(\infty)}\big[(n+\alpha)p(\infty)-2n\big]. \tag{3.3}$$

In the case of constant $p(x)\equiv p=\text{const}$, the $(p\to q)$-boundedness of the Riesz potential operator with the power weight $|x|^{\gamma_0}$ is due to E.M. Stein and G. Weiss [29] without the additional condition (3.3). The general weighted case for constant p is due to B. Muckenhoupt and R. Wheeden [20]. The inequalities for the exponents γ_0 and γ_∞ in (3.2), as is well known, are necessary and sufficient for power weight to belong to the Muckenhoupt–Wheeden A_{pq}-class.

Corollary 3.2. *Let* $0<\alpha<n$, $p(x)$ *satisfy conditions* (2.3), (2.4), (2.7) *and* (2.10), *and suppose that*

$$-\frac{1}{2}\left(1-\frac{1}{p_-}\right)<\frac{1}{p(\infty)}-\frac{n+\alpha}{2n}<\frac{1}{2}\left(\frac{1}{p_+}-\frac{\alpha}{n}\right). \tag{3.4}$$

Then the operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$, $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}$.

The statement of the corollary was proved in [1] and [2] without assumption (3.4) and under weaker assumption (2.6) instead of (2.7).

Remark 3.3. In the non-weighted case of Corollary 3.2 there are given bounds for the difference $\frac{1}{p(\infty)}-\frac{n+\alpha}{2n}$, which is more general than just to write the assumption $\frac{1}{p(\infty)}=\frac{n+\alpha}{2n}$ which follows from condition (3.3) of Theorem 3.1. There might be similarly written some inequalities instead of just equality (3.3) in the weighted case in Theorem 3.1 as well, but the bounds of the corresponding intervals are not expressed in "nice" terms.

Remark 3.4. Theorem 3.1 is obviously valid also for the case of the weight $\rho_{\mu_{x_0},\mu_\infty}(x)=|x-x_0|^{\gamma_{x_0}}(1+|x|)^{\gamma_\infty-\gamma_{x_0}}$ fixed to an arbitrary point $x_0\in\mathbb{R}^n$; in conditions (3.2) and (3.3) the values $p(0)$ and $q(0)$ should be replaced in this case by $p(x_0)$ and $q(x_0)$, respectively.

Theorem 3.5. *Let the function $p:\mathbb{S}^n\to[1,\infty)$ satisfy conditions* (2.18)–(2.20). *The spherical potential operator K^α is bounded from the space $L^{p(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})$ with $\rho_{\beta_a,\beta_b}(\sigma)=|\sigma-a|^{\beta_a}\cdot|\sigma-b|^{\beta_b}$, where $a\in\mathbb{S}^n$ and $b\in\mathbb{S}^n$ are arbitrary points on the unit sphere $\mathbb{S}^n$, $a\neq b$, into the space $L^{q(\cdot)}(\mathbb{S}^n,\rho_{\beta_a,\beta_b})$ with $\rho_{\nu_a,\nu_b}(\sigma)=|\sigma-a|^{\nu_a}\cdot|\sigma-b|^{\nu_b}$, where $\frac{1}{q(\sigma)}=\frac{1}{p(\sigma)}-\frac{\alpha}{n}$, and*

$$\alpha p(a)-n<\beta_a<np(a)-n,\qquad \alpha p(b)-n<\beta_b<np(b)-n, \tag{3.5}$$

$$\nu_a=\frac{q(a)}{p(a)}\beta_a,\qquad \nu_b=\frac{q(b)}{p(b)}\beta_b \tag{3.6}$$

and the weight exponents β_a and β_b are related to each other by the connection

$$\frac{q(a)}{p(a)}\beta_a=\frac{q(b)}{p(b)}\beta_b. \tag{3.7}$$

Corollary 3.6. *Under assumptions* (2.18)–(2.20) *the spherical potential operator* K^α *is bounded from* $L^{p(\cdot)}(\mathbb{S}^n)$ *into* $L^{q(\cdot)}(\mathbb{S}^n)$, $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$.

4. Proof of Theorem 3.1

Proof. We denote

$$A^p_{\mu_0,\mu_\infty}(f) = \int\limits_{\mathbb{R}^n} |x|^{\mu_0}\big(1+|x|\big)^{\mu_\infty-\mu_0}\big|f(x)\big|^{p(x)}\,dx.$$

We have to show that $A^q_{\mu_0,\mu_\infty}(I^\alpha\varphi) \leqslant c < \infty$ for all φ with $A^p_{\gamma_0,\gamma_\infty}(\varphi) \leqslant 1$, where $c > 0$ does not depend on φ.

Let

$$B_+ = \big\{x \in \mathbb{R}^n\colon\ |x| < 1\big\} \quad \text{and} \quad B_- = \big\{x \in \mathbb{R}^n\colon\ |x| > 1\big\}.$$

In view of (2.3) it is easily seen that

$$A^q_{\mu_0,\mu_\infty}(I^\alpha\varphi) \leqslant c(A_{++} + A_{+-} + A_{-+} + A_{--}), \tag{4.1}$$

where

$$A_{++} = \int\limits_{B_+} |x|^{\mu_0}\left|\int\limits_{B_+} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx,$$

$$A_{+-} = \int\limits_{B_+} |x|^{\mu_0}\left|\int\limits_{B_-} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx,$$

and

$$A_{-+} = \int\limits_{B_-} |x|^{\mu_\infty}\left|\int\limits_{B_+} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx,$$

$$A_{--} = \int\limits_{B_-} |x|^{\mu_\infty}\left|\int\limits_{B_-} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx$$

so that we may separately estimate these terms. We note that the relation (3.3) will be used only in the estimation of the “mixed” terms A_{+-} and A_{-+}.

The term A_{++}. This term is covered by Theorem 2.3, the condition (2.12) of Theorem 2.3 being fulfilled by the first assumption in (3.2).

The term A_{--}. The estimation of A_{--} is reduced to that of A_{++} by means of the simultaneous change of variables (inversion):

$$x = \frac{u}{|u|^2}, \qquad dx = \frac{du}{|u|^{2n}}, \qquad y = \frac{v}{|v|^2}, \qquad dy = \frac{dv}{|v|^{2n}}. \tag{4.2}$$

As a result, we obtain

$$A_{--} = \int_{B_+} |x|^{-\mu_\infty - 2n} \left| \int_{B_+} \frac{\varphi(y_*)\,dy}{|y|^{2n}|x_* - y_*|^{n-\alpha}} \right|^{q_*(x)} dx,$$

where we denoted

$$q_*(x) = q(x_*) = q\left(\frac{x}{|x|^2}\right).$$

By (2.14), we obtain

$$A_{--} = \int_{B_+} |x|^{-\mu_\infty - 2n} \left| |x|^{(n-\alpha)q_*(x)} \int_{B_+} \frac{|y|^{-n-\alpha}\varphi(y_*)\,dy}{|x-y|^{n-\alpha}} \right|^{q_*(x)} dx.$$

Since $q(x)$ satisfies the logarithmic condition (2.6) at infinity, the function $q_*(x)$ satisfies the local logarithmic condition (2.4) near the origin, so that $|x|^{(n-\alpha)q_*(x)} \leqslant c|x|^{(n-\alpha)q_*(0)} = c|x|^{(n-\alpha)q(\infty)}$ and we get

$$A_{--} = \int_{B_+} |x|^{\mu_1} \left| \int_{B_+} \frac{\psi(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q_*(x)} dx, \tag{4.3}$$

where

$$\mu_1 = (n-\alpha)q(\infty) - 2n - \mu_\infty \quad \text{and} \quad \psi(y) = |y|^{-n-\alpha}\varphi\left(\frac{y}{|y|^2}\right). \tag{4.4}$$

It is easily checked that

$$\int_{B_+} |x|^{\gamma_1} \left|\psi(x)\right|^{p_*(x)} dx = \int_{B_-} |x|^{\gamma_\infty} \left|\varphi(x)\right|^{p(x)} dx < \infty \tag{4.5}$$

under the choice $\gamma_1 = (n+\alpha)p(\infty) - 2n - \gamma_\infty$, and conditions

$$\alpha p_*(0) - n < \gamma_1 < n\big[p_*(0) - 1\big] \quad \text{and} \quad \mu_1 = \frac{q_*(0)}{p_*(0)}\gamma_1$$

hold. By (2.7), the exponent q_* satisfies the local log-condition. Therefore, Theorem 2.3 is applicable in (4.3) and then $A_{--} \leqslant c < \infty$.

Estimation of the terms A_{-+} and A_{+-} is less direct and requires condition (3.3) which was not used when we estimated the terms A_{++} and A_{--}.

The term A_{-+}. By the inversion change $x \to x_*$ of the variable x, we have

$$\begin{aligned} A_{-+} &= \int_{B_+} |x|^{-\mu_\infty - 2n} \left| \int_{B_+} \frac{\varphi(y)\,dy}{|x_* - y|^{n-\alpha}} \right|^{q_*(x)} dx \\ &\leqslant \int_{B_+} |x|^{(n-\alpha)q_*(0) - \mu_\infty - 2n} \left|h(x)\right|^{q_*(x)} dx = \int_{B_+} |x|^{\mu_1} \left|h(x)\right|^{q_*(x)} dx, \end{aligned}$$

where

$$h(x)=\int\limits_{B_+}\frac{\varphi(y)\,dy}{(|x|\cdot|x_*-y|)^{n-\alpha}}.$$

In contrast to the case of the terms A_{++} and A_{--}, now the information about the integrability of $\varphi(x)$ is known in terms of $p(x)$, while $h(x)$ should be integrated to the power $q_*(x)$, not $q(x)$ (in the symmetrical term A_{+-}, on the contrary, we will have to deal with $q(x)$ preserved, but $p(x)$ replaced by $p_*(x)$). Fortunately, we may pass to $q_*(x)$ thanks to the properties of the inversion $x_*=\frac{x}{|x|^2}$ and the logarithmic smoothness of $q(x)$ when x passes through the unit sphere. We proceed as follows. First we observe that

$$|x|\cdot|x_*-y|\geqslant|x-y|\quad\text{and}\quad|x|\cdot|x_*-y|\geqslant 1-|x| \tag{4.6}$$

for $|x|\leqslant 1$ and $|y|\leqslant 1$. The former of the inequalities in (4.6) was given in (2.16), the latter follows from the fact that $|x_*|\geqslant 1$ and $|y|\leqslant 1$ and then $|x_*-y|\geqslant|x_*|-|y|=\frac{1}{|x|}-|y|\geqslant\frac{1}{|x|}-1$.

Let $E_1=\{x\in B_+:\ q_*(x)\leqslant q(x)\}$ and $E_2=\{x\in B_+:\ q_*(x)\geqslant q(x)\}$. We have

$$A_{-+}\leqslant\int\limits_{x\in E_1,\,|h(x)|\leqslant 1}|x|^{\mu_1}\,dx+\int\limits_{x\in E_1,\,|h(x)|\geqslant 1}|x|^{\mu_1}\big|h(x)\big|^{q(x)}\,dx$$
$$+\int\limits_{E_2}|x|^{\mu_1}\big|h(x)\big|^{q_*(x)}\,dx=:A_1+A_2+A_3.$$

Here the term A_1 is finite since $\mu_1>-n$. For the term A_3 we have

$$A_3\leqslant\int\limits_{x\in E_2,\,|x|\leqslant\frac12}|x|^{\mu_1}\big|h(x)\big|^{q_*(x)}\,dx$$
$$+\int\limits_{x\in E_2,\,|x|\geqslant\frac12}|x|^{\mu_1}\big|h(x)\big|^{q(x)}\big|h(x)\big|^{q_*(x)-q(x)}\,dx=:A_{31}+A_{32}.$$

The term A_{31} is finite since $|x|\cdot|x_*-y|\geqslant\frac12$ for $|x|\leqslant\frac12$ by (4.6) and then $|h(x)|\leqslant c\|\varphi\|_{L^1}\leqslant c_1\|\varphi\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho_{\gamma_0,\gamma_\infty})}$. For the term A_{32} we have to show that

$$\sup_{x\in E_2,\,|x|\geqslant\frac12}\big|h(x)\big|^{q_*(x)-q(x)}<\infty.$$

To this end, we make use of the second inequality in (4.6) and obtain

$$\big|h(x)\big|\leqslant\big(1-|x|\big)^{\alpha-n}\int\limits_{B_+}\big|\varphi(t)\big|\,dt=c\big(1-|x|\big)^{\alpha-n}$$

and then

$$\big|h(x)\big|^{q_*(x)-q(x)}\leqslant e^{(\alpha-n)[q_*(x)-q(x)]\ln(1-|x|)},$$

which is bounded for $|x| \geqslant \frac{1}{2}$ by Lemma 2.5.

Gathering the estimates, we obtain

$$A_{-+} \leqslant c + c\int\limits_{B_+} |x|^{\mu_1} \big|h(x)\big|^{q(x)}\,dx$$
$$\leqslant c + c\int\limits_{B_+} |x|^{\mu_1} \left(\int\limits_{B_+} \frac{|\varphi(y)|\,dy}{(|x|\cdot|x_* - y|)^{n-\alpha}}\right)^{q(x)} dx.$$

Hence, by the first inequality in (4.6),

$$A_{-+} \leqslant c + c\int\limits_{B_+} |x|^{\mu_1} \left(\int\limits_{B_+} \frac{|\varphi(y)|\,dy}{|x-y|^{n-\alpha}}\right)^{q(x)} dx$$

and we are able now to apply Theorem 2.3. However, this requires the condition $\mu_1 \geqslant \mu_0 = \frac{q(0)}{p(0)}\gamma_1$, that is,

$$\mu_0 + \mu_\infty \leqslant (n-\alpha)q(\infty) - 2n$$

or equivalently,

$$\frac{q(0)}{p(0)}\gamma_0 + \frac{q(\infty)}{p(\infty)}\gamma_\infty \leqslant \frac{q(\infty)}{p(\infty)}\big[(n+\alpha)p(\infty) - 2n\big]. \tag{4.7}$$

Therefore, by (3.3) we may apply Theorem 2.3 which provides the necessary estimation $A_{-+} \leqslant c < \infty$.

The term A_{+-}. After the inversion change of variables in the inner integral in A_{+-} we have

$$A_{+-} = \int\limits_{B_+} |x|^{\mu_0} \left|\int\limits_{B_+} \frac{|y|^{-2n}\varphi(\frac{y}{|y|})\,dy}{|x - \frac{y}{|y|}|^{n-\alpha}}\right|^{q(x)} dx$$
$$= \int\limits_{B_+} |x|^{\mu_0} \left|\int\limits_{B_+} \frac{\psi(y)\,dy}{(|y|\cdot|x-y_*|)^{n-\alpha}}\right|^{q(x)} dx, \tag{4.8}$$

where

$$\psi(y) = |y|^{-n-\alpha}\varphi(y_*) \in L^{p_*(\cdot)}\big(B_+, |x|^{\gamma_1}\big)$$

is the same function as in (4.4). We distinguish the cases $|y| \leqslant \frac{1}{2}$ and $|y| \geqslant \frac{1}{2}$. In the first case we make use the second of the inequalities in (4.6) in the form $|y|\cdot|x-y_*| \geqslant 1-|y| \geqslant \frac{1}{2}$ and then the estimation becomes trivial. In the case $|y| \geqslant \frac{1}{2}$ we make use of the first inequality in (4.6): $|y|\cdot|x-y_*| \geqslant |x-y|$ which gives a possibility to make use of Theorem 2.3, the passage to the exponent $q_*(x) = \frac{np_*(x)}{n-\alpha p_*(x)}$ in (4.8) is done in the same way as in the estimation of A_{-+} by distinguishing the cases where $q(x) \leqslant q_*(x)$ and $q(x) \geqslant q_*(x)$:

$$A_{+-} \leqslant c + \int\limits_{B_+} |x|^{\mu_0} \left(\int\limits_{B_+} \frac{|\psi(y)|\,dy}{|x-y|^{n-\alpha}}\right)^{q_*(x)} dx;$$

we omit details of that passage to the exponent $q_*(x)$, they are symmetrical to those in the case of A_{-+} when we passed from $q_*(x)$ to $q(x)$. We only mention that when proving the uniform boundedness of

$$\left| \int\limits_{y\in B_+,\, |y|>\frac{1}{2}} \frac{|\psi(y)|\,dy}{(|y|\cdot|x-y_*|)^{n-\alpha}} \right|^{q(x)-q_*(x)}$$

with $q(x) \geqslant q_*(x)$, we may use the obvious inequality $|x-y_*| \geqslant 1-|x|$.

When applying Theorem 2.3 with the exponents $p_*(x)$ and $q_*(x)$, according to condition (2.12) we have to assume that

$$\mu_0 \geqslant \frac{q_*(0)}{p_*(0)}\gamma_1 = \frac{q(\infty)}{p(\infty)}\gamma_1,$$

which gives the condition

$$\frac{q(0)}{p(0)}\gamma_0 + \frac{q(\infty)}{p(\infty)}\gamma_\infty \geqslant \frac{q(\infty)}{p(\infty)}\big[(n+\alpha)p(\infty)-2n\big] \tag{4.9}$$

contrary to (4.7), which holds because of condition (3.3). Therefore the application of Theorem 2.3 ends the proof. □

Proof of Corollary 3.2. The statement of this corollary follows immediately from Theorem 3.1 under condition (3.3) which in the non-weighted case takes the form

$$p(\infty) = \beta, \qquad \beta = \frac{2n}{n+\alpha} > 1.$$

We make use of the Riesz–Thorin interpolation theorem, see Theorem 2.2, to show that the boundedness holds if instead of $p(\infty)=\beta$ we require that the value of $\frac{1}{p(\infty)}$ does not differ much from $\frac{1}{\beta}$, namely, $-\frac{1}{2}(1-\frac{1}{p_-}) < \frac{1}{p(\infty)} - \frac{1}{\beta} < \frac{1}{2}(\frac{1}{p_+} - \frac{\alpha}{n})$ which is condition (3.4).

To avoid the condition $p(\infty)=\beta$, we may interpolate between a constant $p_0 > 1$ and some $r(\cdot)$ for which the condition $r(\infty)=\beta$ holds. That is, we have to find $\theta \in (0,1)$ and $p_0 \in (1, \frac{n}{\alpha})$ such that

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0} + \frac{\theta}{r(x)},$$

where $r(x)$ satisfies the conditions

$$\inf_{x\in\mathbb{R}^n} r(x) > 1, \qquad \sup_{x\in\mathbb{R}^n} r(x) < \frac{n}{\alpha} \quad \text{and} \quad r(\infty)=\beta \tag{4.10}$$

(note that any log-condition for $r(x)$ follows from the same log-condition of $p(x)$). Conditions (4.10) take the form

$$\frac{1-\theta}{p_0} + \theta > \frac{1}{p_-}, \qquad \frac{1-\theta}{p_0} + \frac{\alpha}{n}\theta < \frac{1}{p_+} \quad \text{and} \quad \frac{1-\theta}{p_0} + \frac{\theta}{\beta} = \frac{1}{p(\infty)}, \tag{4.11}$$

respectively. By direct calculations, it can be proved that conditions (4.11) may be satisfied jointly with conditions $p_0 \in (1, \frac{n}{\alpha})$ and $\theta \in (0,1)$ if and only if assumption (3.4) holds. We prove this in Appendix B. □

5. Proof of Theorem 3.5

The statement of Theorem 3.5 is derived from that of Theorem 3.1 by means of the stereographic projection. By Remark 3.4 and an appropriate rotation on the sphere, we reduce the proof to the case where $a = e_{n+1} = (0, 0, \dots, 0, 1)$ and $b = -e_{n+1}$.

Formulas (2.22)–(2.25) give the relations

$$\int\limits_{\mathbb{R}^n} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}} = 2^\alpha \int\limits_{S_n} \frac{\varphi_*(\sigma)\,d\sigma}{|\xi-\sigma|^{n-\alpha}}, \tag{5.1}$$

where $\xi = s(x)$, $\sigma = s(y)$ and

$$\varphi_*(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma - e_{n+1}|^{n+\alpha}}.$$

We have also the modular equivalence

$$\begin{aligned}&\int\limits_{\mathbb{S}^n} |\sigma - e_{n+1}|^{\beta_a} \cdot |\sigma + e_{n+1}|^{\beta_b} \bigl|\varphi_*(\sigma)\bigr|^{p(\sigma)}\,d\sigma \\ &\quad\sim \int\limits_{\mathbb{R}^n} |y|^{\gamma_0} \cdot \bigl(1+|y|\bigr)^{\gamma_\infty - \gamma_0} \bigl|\varphi(y)\bigr|^{\tilde{p}(y)}\,dy,\end{aligned} \tag{5.2}$$

where

$$\tilde{p}(y) = p\bigl[s(y)\bigr], \qquad \beta_a = -\gamma_\infty + (n+\alpha)\tilde{p}(\infty) - 2n \quad \text{and} \quad \beta_b = \gamma_0.$$

The direct verification shows that the corresponding intervals for the spherical weight exponents β_a and β_b coincide with the corresponding intervals for the spatial weight exponents γ_0, γ_∞:

$$\begin{cases} \gamma_0 \in (\alpha\tilde{p}(0) - n, n\tilde{p}(0) - n), \\ \gamma_\infty \in (\alpha\tilde{p}(\infty) - n, n\tilde{p}(\infty) - n), \end{cases}$$

$$\iff \quad \begin{cases} \beta_b \in (\alpha p(-e_{n+1}) - n, np(-e_{n+1}) - n), \\ \beta_a \in (\alpha p(e_{n+1}) - n, np(e_{n+1}) - n). \end{cases}$$

Similarly we have an equivalence between the relation (3.3) for spatial weight exponents γ_0 and γ_∞ and the relation (3.7) for spherical weight exponents, which in our case has the form

$$\frac{q(e_{n+1})}{p(e_{n+1})}\beta_b = \frac{q(-e_{n+1})}{p(-e_{n+1})}\beta_a,$$

where $q(\sigma) = \frac{np(\sigma)}{n-\alpha p(\sigma)}$ is the Sobolev limiting exponent on the sphere.

In view of the relation (5.1) and equivalence (5.2) of norms, we then easily derive Theorem 3.5 from Theorem 3.1 after obvious recalculations.

Appendix A

To prove what was stated in Remark 2.1, it suffices to consider the case $n = 1$. The question we have to treat, is whether from the conditions

$$\left|p(x) - p(y)\right| \leqslant \frac{A}{\ln \frac{1}{|x-y|}} \quad \text{for } |x-y| \leqslant \frac{1}{2} \quad \text{and} \quad \left|p(x) - p(\infty)\right| \leqslant \frac{B}{\ln(1+|x|)}$$

there follows that

$$\left|p\left(\frac{1}{x}\right) - p\left(\frac{1}{y}\right)\right| \leqslant \frac{C}{\ln \frac{1}{|x-y|}},$$

where $x, y \in \mathbb{R}^1_+$. This is equivalent to the following question. Let a continuous on $[0, \frac{1}{2}]$ function $f(x)$ $(= p(\frac{1}{x}))$, satisfy the log-condition everywhere beyond the origin:

$$\left|f(x) - f(y)\right| \leqslant \frac{C_\delta}{\ln \frac{1}{|x-y|}} \quad \text{for all } x, y \in \left[\delta, \frac{1}{2}\right],\ \delta > 0, \tag{A.1}$$

and

$$\left|f(x) - f(0)\right| \leqslant \frac{C_0}{\ln \frac{1}{x}}. \tag{A.2}$$

Do conditions (A.1) and (A.2) guarantee that

$$\left|f(x) - f(y)\right| \leqslant \frac{C}{\ln \frac{1}{|x-y|}} \quad \text{for all } x, y \in \left[0, \frac{1}{2}\right]? \tag{A.3}$$

The answer to this question is negative, because the only condition (A.2) may not prevent from the constant C_δ in (A.1) to be tending to infinity when $\delta \to 0$. The corresponding counterexample is given in the lemma below.

Lemma A.1. *There exists a function $f(x)$ continuous on $[0, \frac{1}{2}]$ such that conditions* (A.1) *and* (A.2) *are satisfied, but* (A.3) *is not valid.*

Proof. Let $\mu(x) \in C^\infty(\mathbb{R}^1)$ be an even smooth “cap” with support in $(-1, 1)$, $0 \leqslant \mu(x) \leqslant 1$, such that $\mu(0) = 1$ and $\mu(\frac{1}{2}) = \frac{1}{2}$.

Let also $\{b_n\}_{n=1}^\infty$ be a monotonically decreasing sequence of points in $[0, \frac{1}{2}]$ tending to 0 as $n \to \infty$. We construct the “narrow” caps

$$\mu_n(x) = \mu\left(\frac{x - a_{n+1}}{\lambda_n}\right), \quad \text{where } a_{n+1} = \frac{b_n + b_{n+1}}{2} \text{ and } \lambda_n = \frac{b_n - b_{n+1}}{2},$$

supported on (b_{n+1}, b_n). By the choice of $\mu(x)$, we have

$$\mu_n(\beta_n) = \frac{1}{2}, \qquad \beta_n = \frac{b_{n+1} + 3b_n}{4} \in (b_{n+1}, b_n). \tag{A.4}$$

We denote $\omega(x) = \frac{1}{\ln \frac{1}{x}}$ for brevity and construct the function $f(x)$ in the form

$$f(x) = \omega(x) G(x), \quad \text{where } G(x) = \sum_{k=1}^\infty A_k \mu_k(x) \omega\big(|x - a_{k+1}|\big) \tag{A.5}$$

and the positive constants A_k will be chosen later. Obviously, for any fixed x, the series in (A.5) contains one term only

$$f(x) = A_n\omega(x)\mu_n(x)\omega\big(|x-a_{n+1}|\big), \quad x \in (b_{n+1}, b_n). \tag{A.6}$$

Under any choice of A_n condition (A.1) is satisfied automatically, because for $x \in [\delta, \frac{1}{2}]$, the series in (A.5) contains a finite number of terms and

$$\big|\omega\big(|x-a_{k+1}|\big) - \omega\big(|y-a_{k+1}|\big)\big| \leqslant \omega\big(|x-y|\big)$$

(where we took into account that the function $\omega(x) = \frac{1}{\ln\frac{1}{x}}$ is the continuity modulus, that is, $\omega(x) - \omega(y) \leqslant \omega(x-y),\ x > y$).

To satisfy condition (A.2) and show that (A.3) does not hold, we have to show that

$$\sup_{x\in[0,\frac{1}{2}]} G(x) < \infty \quad \text{and} \quad \sup_{x,y\in[0,\frac{1}{2}]} \frac{|f(x)-f(y)|}{\omega(|x-y|)} = \infty. \tag{A.7}$$

To this end, we have to properly choose both the coefficients A_n and the points b_n. For the former of conditions in (A.7) we need to show that

$$\sup_n \sup_{x\in[b_{n+1},b_n]} A_n\mu_n(x)\omega\big(|x-a_{n+1}|\big) < \infty,$$

for which it suffices to choose A_n so that

$$\sup_n A_n\omega(b_n - a_{n+1}) < \infty. \tag{A.8}$$

As regards the latter of the conditions in (A.7), we have

$$\begin{aligned}
\sup_{x,y\in[0,\frac{1}{2}]} \frac{|f(x)-f(y)|}{\omega(|x-y|)} &\geqslant \sup_n \sup_{x\in[0,\frac{1}{2}]} \frac{|f(x)-f(a_{n+1})|}{\omega(|x-a_{n+1}|)} \\
&= \sup_n \sup_{x\in[b_{n+1},b_n]} \frac{A_n\omega(x)\mu_n(x)\omega(|x-a_{n+1}|)}{\omega(|x-a_{n+1}|)} \\
&\geqslant \sup_n A_n\omega(\beta_n)\mu_n(\beta_n).
\end{aligned}$$

Then, by (A.4) we obtain

$$\sup_{x,y\in[0,\frac{1}{2}]} \frac{|f(x)-f(y)|}{\omega(|x-y|)} \geqslant \frac{1}{2}\sup_n A_n\omega(\beta_n) = \frac{1}{2}\sup_n \frac{A_n}{\ln\frac{1}{\beta_n}}. \tag{A.9}$$

Now we choose

$$A_n = \ln^2\frac{1}{\beta_n}.$$

Then by (A.9), $\sup_{x,y\in[0,\frac{1}{2}]} \frac{|f(x)-f(y)|}{\omega(|x-y|)} = \infty$ so that condition (A.3) is not satisfied, independently of the choice of the points b_n. It remains to show that there exists a choice of these points such that (A.8) holds. Under our choice of A_n we have

$$A_n\omega(b_n - a_{n+1}) = \frac{\ln^2\frac{1}{\beta_n}}{\ln\frac{1}{b_n-a_{n+1}}} = \frac{\ln^2\frac{4}{b_{n+1}+3b_n}}{\ln\frac{2}{b_n-b_{n+1}}} = \frac{(\ln\frac{1}{b_n} + \ln\frac{4}{3+t_n})^2}{\ln 2 + \ln\frac{1}{b_n} + \ln\frac{1}{1-t_n}},$$

where we have denoted $t_n = \frac{b_{n+1}}{b_n} \leqslant 1$. Hence

$$A_n\omega(b_n - a_{n+1}) \leqslant c \frac{\ln^2 \frac{1}{b_n}}{\ln \frac{1}{b_n} + \ln \frac{1}{1-t_n}} \tag{A.10}$$

with some positive constant c. Now we wish to make a choice of b_n so that $\ln \frac{1}{1-t_n} = \ln^2 \frac{1}{b_n}$, that is, $t_n = 1 - b_n^{\ln \frac{1}{b_n}}$ and we arrive at the recurrent relation for b_n:

$$b_{n+1} = b_n\bigl(1 - b_n^{\ln \frac{1}{b_n}}\bigr)$$

(whence it follows that b_n tends monotonously to zero as $n \to \infty$). Then under this choice of b_n, from (A.10) there follows that

$$A_n\omega(b_n - a_{n+1}) \leqslant c \frac{\ln^2 \frac{1}{b_n}}{\ln \frac{1}{b_n} + \ln^2 \frac{1}{b_n}} \leqslant c$$

with c not depending on n which proves (A.8) and the lemma. □

Appendix B

Lemma B.1. *Let $p_+ \geqslant p_- > 1$, $p(\infty) > 1$ and $\beta = \frac{2n}{n+\alpha}$. The numbers $\theta \in (0,1)$ and $p_0 \in (1, \frac{n}{\alpha})$ satisfying conditions* (4.11) *exist if and only if assumption* (3.4) *holds.*

Proof. Since $\frac{1}{p_0}$ and θ are related by the linear relation, the last one in (4.11), after excluding $\frac{1}{p_0}$, we see that our problem is equivalent to the problem of existence of $\theta \in (0,1)$ such that

$$\frac{1}{p_-} - \frac{1}{p(\infty)} < \theta\left(1 - \frac{1}{\beta}\right), \qquad \frac{1}{p(\infty)} - \frac{\theta}{\beta} < \frac{1}{p_+} - \frac{\alpha\theta}{n} \tag{B.1}$$

and

$$\frac{1}{1-\theta}\left[\frac{1}{p(\infty)} - \frac{\theta}{\beta}\right] \in \left(\frac{\alpha}{n}, 1\right). \tag{B.2}$$

The restrictions on θ in (B.1) together are equivalent to the condition

$$\theta > a := \max\left(\frac{\frac{1}{p_-} - \frac{1}{p(\infty)}}{1 - \frac{1}{\beta}}, \frac{\frac{1}{p(\infty)} - \frac{1}{p_+}}{1 - \frac{1}{\beta}}\right). \tag{B.3}$$

Observe (in the “only if” part) that this lower bound a must be less than 1, which gives the conditions

$$\frac{1}{p_-} - 1 < \frac{1}{p(\infty)} - \frac{1}{\beta} < \frac{1}{p_+} - \frac{\alpha}{n}. \tag{B.4}$$

It remains to take care about condition (B.2). After direct calculations we obtain that (B.2) is equivalent to the following restriction on θ from above:

$$\theta < b := \min\left(\frac{1 - \frac{1}{p(\infty)}}{1 - \frac{1}{\beta}}, \frac{\frac{1}{p(\infty)} - \frac{\alpha}{n}}{1 - \frac{1}{\beta}}\right). \tag{B.5}$$

Finally, according to (B.3) and (B.5) the required θ exists if and only if $a < b$. This gives the condition

$$\frac{1}{p_-} + \frac{\alpha}{n} < \frac{2}{p(\infty)} < 1 + \frac{1}{p_+}$$

which is nothing else but (3.4). $\square$

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