SOLVING SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS BY USING INTERACTIVE SOFTWARE

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ABSTRACT

Differential equations constitute an area of great theoretical research and applications in several branches of science and technology. The scope of this work is to present new software that is able to show all the steps in the process of solving a linear second-order ordinary differential equation with constant coefficients.

Keywords: Linear Ordinary Differential Equations, Symbolic Computation, Wolfram Mathematica, Computable Document Format.

1. INTRODUCTION

In the present work, we present interactive software that can be used to aid in the process of teaching linear second-order ordinary differential equations (ODEs) with constant coefficients. This work can be seen as a natural complement to our previous work (Coelho & Marreiros, 2013). In (Coelho & Marreiros, 2013) we studied the second-order linear differential equations with constant coefficients. In what relates to the implementation of the computational tool, in (Coelho & Marreiros, 2013) we started with the homogeneous case and then extended it to the construction of a particular solution by using the variation of parameters method (the "general method") in the nonhomogeneous case. In the present work, we also consider the undetermined coefficients method to construct a particular solution for the same type of equations. As we see bellow, this method can only be applied to a special form that the right-hand side (rhs) of the equation can assume.

This work is organized as follows. We begin shortly recalling the basic concepts of the theory of the second-order linear ODE; for references we address the reader to our cited work (Coelho & Marreiros, 2013), and, for instance, (Ross, 1984), (Krasnov, Kiselev, & Makarenko, 1981) and (Marreiros, 2012). Then follows the main section where we describe the implementation and the use of the software.

2. BASIC CONCEPTS

2.1 Basic theory

Below we recall the basic concepts of the theory of the second-order linear differential equation.

Definition 1. A second-order linear ordinary differential equation in the dependent variable y and the independent variable x is an equation that can be written in the form

$$y'' + ay' + by = f, \tag{1}$$

where *a, b* and *f* are continuous real functions on a real interval *I*, i.e., $x \in I \subset \mathbb{R}$.

If *f* is not identically zero on *I*, then the equation (1) is said to be nonhomogeneous.

If *f* is identically zero on *I*, then we obtain the so called corresponding homogeneous equation to (1), being its form naturally given by

$$y'' + ay' + by = 0. \tag{2}$$

The following result takes place.

Theorem 1. Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation (2) on an interval *I*. Then the general solution of (2) is given by

$$y = C_1 y_1 + C_2 y_2$$

where $C_1, C_2 \in \mathbb{R}$.

The general solution of the nonhomogeneous equation (1) is given in the next theorem.

Theorem 2. Let y be the general solution of the nonhomogeneous equation (1), y_h be the general solution of the homogeneous equation (2) and y_p a particular solution of the nonhomogeneous equation (1). Then

$$y = y_h + y_p.$$

First, let us consider the homogeneous equation with constant coefficients, i.e.,

$$y'' + ay' + by = 0. \quad (3)$$

where $a, b \in \mathbb{R}$.. Associated to the equation (3) we define the characteristic equation,

$$r^2 + ar + b = 0. \quad (4)$$

We know that the roots of (4) have to fall into one of the following three cases: two real distinct roots; one real root with multiplicity two; or two complex conjugate roots. Let r_1, r_2 be the roots of (4) and $C_1, C_2 \in \mathbb{R}$; according to each case, the general solution of the equation (3) is written in one of the following forms.

Case 1. Distinct real roots, $r_1 \neq r_2$; then

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Case 2. Repeated real roots, $r_1 = r_2 =$; then

$$y(x) = (C_1 + C_2\, x) e^{rx}.$$

Case 3. Complex conjugate roots, $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \alpha, \beta \in \mathbb{R}, \beta \neq 0$; then

$$y(x) = (C_1 \cos(\beta x) + C_2 \sin(\beta x)) e^{\alpha x}.$$

Next, we consider the nonhomogeneous equation with constant coefficients, i.e.,

$$y'' + ay' + by = f, \quad (5)$$

where $f(x) \neq 0$ and $a, b \in \mathbb{R}$. By theorem 2 we know that the general solution of this equation is given by $y = y_h + y_p$, where y_h is the general solution of the associated homogeneous equation and y_p is a particular solution of the given nonhomogeneous equation. The process to obtain y_h was explained for equation (3); therefore, we only need to know how to obtain one particular solution for (5). There are two methods to perform this task: the undetermined coefficients method and the variation of parameters method.

The undetermined coefficients method. This method can only be applied to a special form that the function on the rhs of the equation (5) can assume; the general form of f allowing the use of this method is

$$f(x) = (P_m(x) \cos(\beta x) + Q_n(x) \sin(\beta x)) e^{\alpha x},$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of degree m and n, respectively.

In this case, a particular solution $y_p(x)$ of the equation (5) is sought in the form

$$y_p(x) = x^s (A_k(x) \cos(\beta x) + B_k(x) \sin(\beta x)) e^{\alpha x},$$

where $k = max\{m,n\}$, A_k and B_k are polynomials of degree k of the general form with undetermined coefficients, and s is the multiplicity of the root $r = \alpha \pm i\beta$ of the characteristic equation of the corresponding homogenous differential equation to (5) (if $\alpha \pm i\beta$ is not a root of the characteristic equation, then $s = 0$). Let us briefly explain the use of this method.

Depending on the function f we seek a particular solution $y_p(x)$ of the equation (5) in a similar form with undetermined coefficients. Then we substitute the expression for in the given equation. Equating coefficients of the similar terms of the first and the second member of (5), we obtain a linear system of equations, to find the undetermined coefficients. Solving this system, we get the particular solution sought $y_p(x)$.

The variation of parameters method. For sake of self-contained present work, we briefly recall this "general method" (see our previous work (Coelho & Marreiros, 2013)) that we have to use in the general case. Consider the nonhomogeneous linear differential equation

$$y'' + ay' + by = f,$$

where a,b and f are continuous functions on some interval I. Let y_1 and y_2 be two linearly independent solutions of the corresponding homogeneous equation

$$y'' + ay' + by = 0,$$

on the interval I. Then a particular solution of the nonhomogeneous equation is given by

$$y_p = uy_1 + vy_2,$$

provided that *u* and *v* are functions that satisfy the following conditions

$$\begin{cases} u'y_1 + v'y_2 = 0, \\ u'y_1' + v'y_2' = f. \end{cases}$$

These conditions represent a linear algebraic system of equations for the unknowns *u'* and *v'*. Note that this system has a unique solution because,

$$W = y_1y_2' - y_1'y_2 \neq 0,$$

since y_1 and y_2 are two linearly independent solutions of the corresponding homogeneous differential equation. Therefore, applying Cramer's rule, we can write,

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W} = -\frac{fy_2}{W}, \quad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W} = \frac{fy_1}{W}.$$

Performing the integration, we obtain

$$u = -\int \frac{fy_2}{W}dx, \quad v = \int \frac{fy_1}{W}dx.$$

The superposition principle. It is convenient to use the following result when the rhs *f(x)* of the nonhomogeneous equation is given by a sum of several functions. If $y_{pk}(x)$ is a solution of the equation

$$y'' + ay' + by = f_k(x), \quad k = 1, \ldots, n,$$

then the function

$$y_p(x) = \sum_{k=1}^{n} y_{pk}(x)$$

is a solution of the following equation

$$y'' + ay' + by = \sum_{k=1}^{n} f_k(x).$$

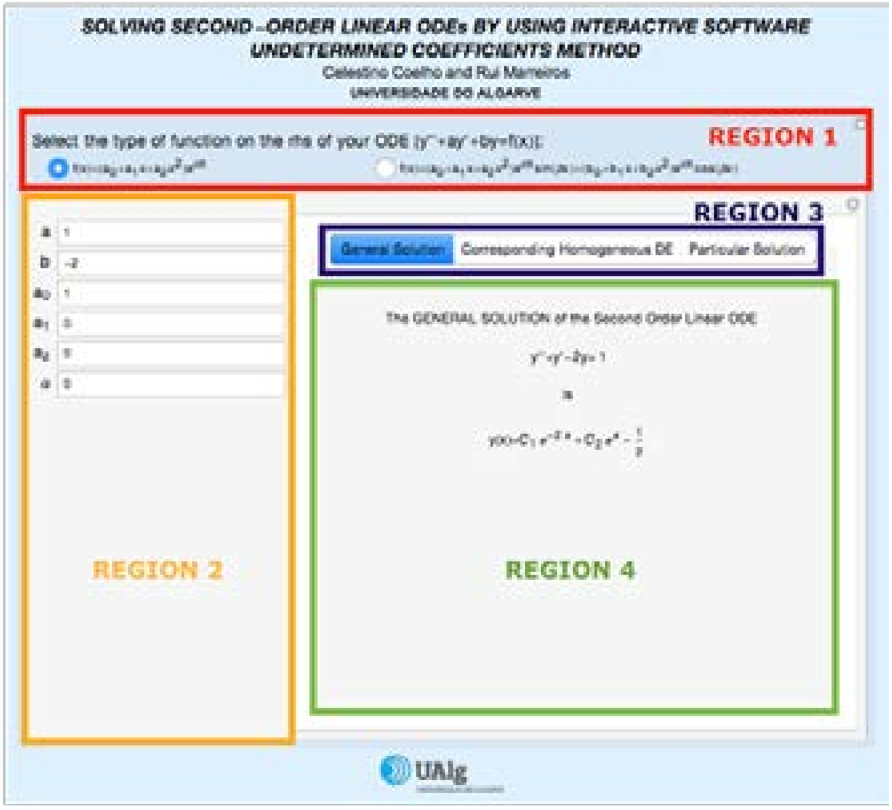
3. SOFTWARE IMPLEMENTATION AND USE

This section is used to show the outlook as well as to present a brief description of the software developed to solve second-order ODEs using the undetermined coefficient method and the variation of parameter method, according to the works (Coelho & Marreiros, 2013) and (Coelho & Marreiros, 2015) respectively. First, we decided to create the one using the variation of parameter, but, due to user input restrictions, we were forced to return to this problem and create one tool that can be used by students when learning the linear second-order ODEs with constant coefficients without having to pay any license. In both situations, it was used the Wolfram's software, more precisely we implement the code to create the Computable Document Format (CDF) file with the program Mathematica® following the references (Torrence & Torrence, 2009), (Ruskeepää, 2009), (Trott, 2006) and (Wolfram, 2003).

The CDF file type is a type of file that, as the name itself indicates, has the ability of doing calculus within the document. It is worth mentioning that this type of file requires a previous installation of the CDF Player program that can be downloaded from Wolfram's webpage[1]. This intrinsic characteristic of the file gives the programmer the chance to create files for an incredible wide set of applications that can be used to teach or to present any subject that involves any type of calculus.

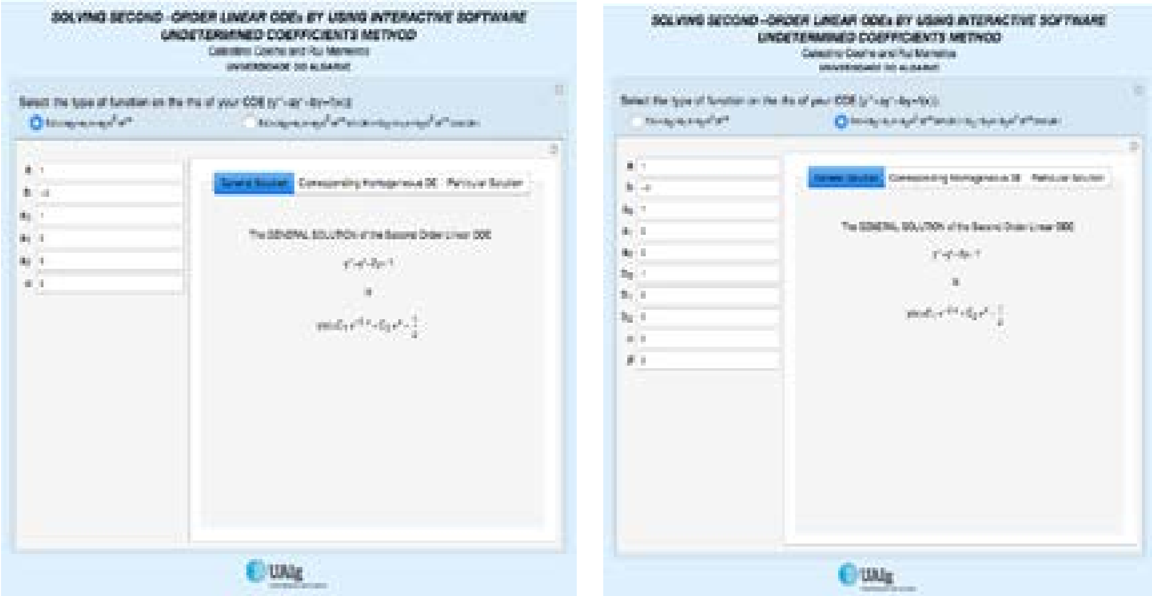
[1] Wolfram's webpage (at the bottom of the webpage look for Products/CDF Player).

Figure 1. Outlook of the software - regions of input and output



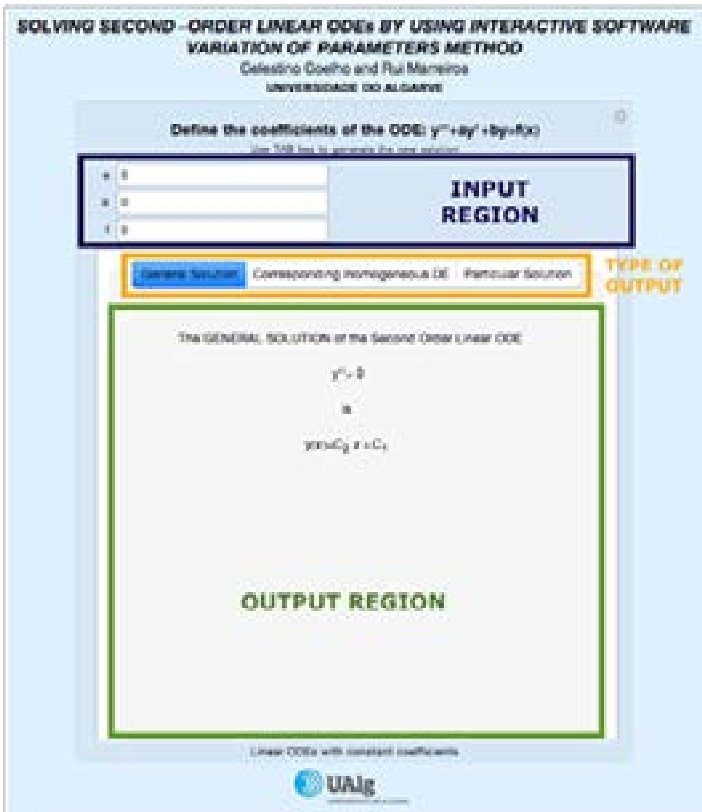
In what concerns to the outlook, we can see it as two main areas, one of input and another of output results. In figure 1 the regions that correspond to input conditions are signalized as regions 1 and 2. Region 1 is not exactly an input region but is directly related with the input region marked as region 2. Figure 2 shows the changes that occur when the user chooses one of the types presented to define the function on the rhs of the ODE. When the function on the rhs has the form of a product between a polynomial and an exponential, the user is compelled to choose the first option. This way it will be necessary to define the coefficients of the polynomial as well as the exponent constant coefficient. Since this was created to be a didactical tool we limited the degree of the polynomial to a maximum of two. So, in this case, the user has to give the program the coefficients of the ODE, the coefficients of the polynomial, and the exponent coefficient, that is, the user has to define six input values. The other option to define the function on the rhs includes trigonometric functions. This is actually the general case of the function that we can have on the rhs of the ODE when using the undetermined coefficients method to solve the type of ODEs covered in this work. Attending to figure 2 we notice a substantial increment in the number of input fields in this case. In fact, we have to define the coefficients of another polynomial, that will have, again, a maximum degree of two, and the constant coefficient in the argument of the trigonometric functions, which sums up to ten input values. To deal with the trigonometric functions the user must be a little bit more careful, because sometimes the function on the rhs of the ODE depends only on a cosine function, case that will be accomplished by setting the coefficient β equal to 0, but in other occasions the function defined on the rhs depends only on a sine function. In this situation, the manipulation is slightly different, because the choice of a coefficient β that eliminates the first part of the function is not the easiest way to solve the problem. Therefore, in this case, the user must set all the coefficients of the polynomial equal to zero. Other situation that is important to emphasize is the one that happens when the function on the rhs depends simultaneously on sine and cosine functions. In this case, additional caution must be taken, because if the constant coefficients in the arguments of the trigonometric functions are not the same the problem can't be solved by a solely application of the program. To clarify this situation, when the constant coefficients in the arguments of the trigonometric functions are exactly the same the user can define the function on the rhs in one application of the program, in any other situation it will be necessary to apply the principle of superposition described in the first section. In fact, this principle has to be applied in all situations that occur when the user is confronted with the impossibility of defining the rhs function in the way that the program exhibits. Turning to the output area we may divide it in two regions, one where the user sets the part of the solution he wants to see or analyse, if the idea is to see the process of obtaining the solution. In figure 1 this is marked as region 3. By clicking on one of the three options available, the user will see the general solution, the process to obtain the solution to the homogeneous associated ODE, or the process to obtain the particular solution to the ODE defined in the input area. The result of the choice in region 3 will be displayed in the region marked as region 4 in figure 1.

Figure 2. Outlook of the software - choices available for *f* in the ODE: *y''+ay'+by=f*.



In which concerns to the software developed to solve the problem with the method of variation of parameters, printed in figure 3, we easily see that its structure is exactly the same as the one described above, with one slight difference worth mentioning. Since this is the general method we do not have restrictions to the form of the function to be used in the rhs of the ODE, so, to define it, the user has an input field where he has to define the analytical expression of the function to use. Note that the input has to be done using the syntax of the programming language that is being used, more precisely the one used when programming with Mathematica®. The possibility of inputting analytical expressions is what implies the buying of a license to run the CDF player.

Figure 3. Outlook of the software: method of variation of parameters.



To end this section, we describe in a very briefly way the most important commands used to create the software presented here. As the list of commands used to implement the software is vast, it will be impossible point and explain all of them. The central command used to construct this type of files is the Manipulate command (see figure 4), used to enable interactive manipulation.

Figure 4. Command Manipulate

```
Manipulate[
 TabView[{
   {"op1", "General Solution" → Column[
```

Another important command used is the InputField command, presented in figure 5, which is used to represent an editable input field that, in our case, contains the values the user wants to attribute to the constant coefficients of the ODE, the polynomial and also to the trigonometric constant arguments.

Figure 5. Command InputField

```
{{alfa, 0, alfaL}, -10, 10}, Alignment → Center, ControlPlacement → Left, ControlType → InputField
], (*end manipulate*)
```

Analysing figure 4 we notice the existence of another command, also with greater importance, used to manipulate the output. The command TabView provides the possibility to choose, by clicking, the type of output.

Figure 6. Command DSolve

```
"y(x)=" FullSimplify[y[x] /. DSolve[y''[x] + a y'[x] + b y[x] == 0, y[x], x,
     GeneratedParameters → (Subscript[C, #] &)][[1]]],
```

Sometimes, due to the specificity of the problem, we also used commands that are intrinsic to Mathematica® and enable the user to perform symbolic calculus, such as DSolve (see figure 6).

EXAMPLES

The examples we decide to show here were chosen from the references (Braun, 1983), (Vrabie, 2004), and (Demidovich, 1978). Our choice was performed in a constructive way, which is building new cases upon the ones already solved by using the software. In what follows, we decided to state the problem we want to solve and give the main direction guidelines to obtain its solution. This way the user can see the correspondence between the process of obtaining the solution and the results provided by the software.

The homogeneous case

Since in all the situations we have to start the process by solving the associated homogeneous ODE we start this section by presenting the procedure when having a homogeneous ODE.

Example 1. Find the general solution of the ODE
$y''-2y'+y=0.$

Using the theory presented in the first section we solve this type of ODE by constructing and solving the characteristic equation
$r^2-2r+1=0 \Leftrightarrow (r-1)^2=0.$

Since it has only one solution with multiplicity 2, r = 1, the general solution of the ODE presented in example 1 will be
$y(x)=C_1e^x+C_2xe^x=(C_1+C_2x)\, e^x,$

with $C_1,C_2 \in \mathbb{R}$.
The results we get by applying our software are shown in figure 7. Note that in this case the particular solution has to be $y_p(x) = 0$, which means that the general solution of the ODE is equal to the solution of the homogeneous associated ODE.

Figure 7. Results obtained by using the software when solving example 1.

The nonhomogeneous case

To start the analysis of the application of our software to the nonhomogeneous case assume that the coefficients of the ODE are exactly the same as the ones used in example 1, meaning that the solution of the homogeneous associated equation is the general solution obtained to the ODE defined in example 1. First, we will turn our focus to examples that can be solved by using the undetermined coefficients method, presenting, at the end of the section, an example that can't be solved with the undetermined coefficients method, that is, that has to be solved by using the method of variation of parameters.

The method of undetermined coefficients. To start the exemplification of the application of the software developed for this method consider a function on the rhs that is the result of a product between a polynomial and an exponential.

Example 2. Find the general solution of the ODE
$y''-2y'+y=6xe^x$.

From example 1 the solution of the homogeneous associated ODE is given by
$y_h(x)=C_1e^x+C_2xe^x$,

where $C_k,k=1,2$ are arbitrary real constants. The characteristic equation, $r^2-2r+1=0$, has a unique solution with multiplicity two, r = 1. This implies that we need to look for a particular solution that has the following form,
$y_p(x)=x^2\ (A_0+A_1x)\ e^x=(A_0x^2+A_1\ x^3\ )e^x$.

In this situation, we need to multiply by x^2 because the associated homogeneous ODE has a solution that already is a linear combination of the functions in the set $\{e^x,xe^x\}$. This means that we need to look for a solution that is a linear combination of x^2e^x and x^3e^x. The application of the undetermined coefficient method to this case yields,
$y_p(x)=x^3e^x$.

Which leads us to conclude that
$y(x)=y_h(x)+y_p(x)=C_1e^x+C_2xe^x+x^3\ e^x$,

with $C_1,C_2 \in \mathbb{R}$..

In figure 8 we present the results obtained by our software when it is applied to example 2.

Figure 8. Results obtained by using the software when solving example 2.

As a third example for the application of this software we will consider a function on the rhs of the ODE that is the result of a product between a polynomial and a trigonometric function. Note that when it is mentioned a trigonometric function we are considering only the sine and/or cosine functions.

Example 3. Find the general solution of the ODE
$y''-2y'+y=4\sin x$.

In what concerns to the homogeneous associated ODE we already know its solution, since is the same of the ODE presented in example 1. According to the form of the function on the rhs of the ODE we need to search for a particular solution in the form,
$y_p(x)=A_0 \sin x+B_0 \cos x$.

Applying the method described in the first section we obtain
$2B_0 \sin x-2A_0 \cos x=4\sin x$,

which leads to the following system of linear equations,

$$\begin{cases} -2A_0 = 0 \\ 2B_0 = 4 \end{cases} \Leftrightarrow \begin{cases} A_0 = 0, \\ B_0 = 2, \end{cases}$$

and, consequently,
$y_p(x)=2\cos x$.

Therefore, the general solution of the ODE is given by
$y(x)=y_h(x)+y_p(x)=C_1e^x+C_2xe^x+2\cos x$,

with $C_1,C_2 \in \mathbb{R}$..

Figure 9 shows the results obtained with our software.

Figure 9. Results obtained by using the software when solving example 3.

The method of variation of parameters. The real importance of considering this method it's because it can be applied in all situations in particular when the function on the rhs of the ODE doesn't meet the form that can be used to apply the method of undetermined constants.

Example 4. Find the general solution of the ODE
$y''-2y'+y=e^x \ln x$,

with $x > 0$.

The major difference between the software that uses the undetermined coefficients method and the one that uses the method of variation of parameters resides in the way that user inputs the data, more precisely the form allowed to define the analytical expression of the function present in the rhs of the ODE. In this method, the choice for the analytical expression of the function to be used in the rhs is free. However, to input the analytical expression the user must follow the syntax used when programming with Mathematica®. For instance, in the example 4 the user must write Exp[x]*Log[x] in the field that corresponds to *f*. After inputting the data that defines the equation in the correspondent fields, the software will generate three tabs of results. The first one, appearing by default, is the one that corresponds to the general solution of the ODE defined, see figure 10 (left side). Clicking on the second tab the user can inspect the procedure used to obtain the general solution of the corresponding homogeneous differential equation, see figure 10 (middle). Finally, clicking on the third tab the user will find the procedure applied to obtain the particular solution using the method of variation of parameters, see figure 10 (right side). Relating the results obtained with this program with the ones given by the one that uses the method of undetermined coefficients method we discover that the only difference is the tab that shows the process of obtaining the particular solution, the other two tabs show exactly the same results.

Figure 10. Results obtained by using the software when solving example 4.

The superposition theorem

To end the section dedicated to examples let us explain how to use the software implemented for the method of the undetermined coefficients when a single application of it does not solve the problem. This situation occurs when the rhs of the equation is defined as a sum of functions that can be used with this method. For example, suppose that the rhs of the ODE corresponds to the sum of the function

$f(x)=6xe^x$

with the function

$g(x)=4\sin x$

i.e., the ODE to solve is given by

$$y'' - 2y' + y = f(x) + g(x) \Leftrightarrow y'' - 2y' + y = 6xe^x + 4\sin x. \qquad (6)$$

In this type of situations, the user must run the application twice, one to obtain the particular solution to the ODE

$$y'' - 2y' + y = f(x) + g(x) \Leftrightarrow y'' - 2y' + y = 6xe^x. \qquad (7)$$

which corresponds to the one obtained in example 4.2, and another to obtain the particular solution to the ODE

$$y'' - 2y' + y = f(x) + g(x) \Leftrightarrow y'' - 2y' + y = 4\sin x. \qquad (8)$$

which corresponds to the one obtained in example 3. Denoting the particular solutions to the ODE (7) and (8) by y_{pf} and y_{pg}, respectively, the application of the superposition principle presented at the end of the first section allow us to conclude that the particular solution to the ODE (6) is given by

$y_p(x)=y_{pf}(x)+y_{pg}(x).$

4. FINAL REMARKS

In (Coelho & Marreiros, 2013) we presented software that could be used to solve this type of ODEs, but, due to the way that we choose to create it, the user can only apply it freely when the function on the rhs of the equation is constant. In any other case the user is obliged to buy a license for Mathematica® or CDF Player. As a consequence of this fact, we can assure that one of most important conclusions of this work is that the software developed using the method of undetermined coefficients can be used without the necessity of buying any license. This feature allows us to share it with our students, so they can use it at home, or anywhere else, when studying linear second-order ODEs with constant coefficients. This may be the most important conclusion of this work, but we may draw other minor considerations about it, such as the importance it represents as a complement in studying this subject and the simplicity of its use, whether we use the program built with the undetermined coefficients method or the method of variation of parameters.

We should point out that in the last couple of years there has been a little revolution in the publication and distribution of this type of applications, as we can see in Wolfram's webpage, but in our humble opinion the one we present in this is more complete and give a more concise approach about the process of how we should obtain the particular solution.

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