

CATEGORIFICATIONS OF THE EXTENDED AFFINE HECKE ALGEBRA AND THE AFFINE q -SCHUR ALGEBRA $\widehat{\mathbf{S}}(n, r)$ FOR $3 \leq r < n$

MARCO MACKAAY AND ANNE-LAURE THIEL

ABSTRACT. We categorify the extended affine Hecke algebra and the affine quantum Schur algebra $\widehat{\mathbf{S}}(n, r)$ for $3 \leq r < n$, using results on diagrammatic categorification in affine type A by Elias-Williamson (extension of Elias-Khovanov for finite type A) and Khovanov-Lauda respectively. We also define 2-representations of these categorifications on an extension of the 2-category of affine (singular) Soergel bimodules. These results are the affine analogue of the results in [MSV13].

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INTRODUCTION

Khovanov and Lauda [KL10], and Rouquier [Rou08] following a slightly different approach, defined a graded additive 2-category $\mathcal{U}(\mathfrak{g})$ with “nice properties” for any root datum. The 2-morphisms are defined by string diagrams with regions labeled by \mathfrak{g} -weights. They are generated by a finite set of elementary diagrams which obey a finite set of relations. The split Grothendieck group of the Karoubi envelope of $\mathcal{U}(\mathfrak{g})$ is isomorphic to the idempotented version of the corresponding quantum group $\dot{\mathbf{U}}(\mathfrak{g})$. In Crane and Frenkel’s [CF94] terminology, we say that $\mathcal{U}(\mathfrak{g})$ *categorifies* $\dot{\mathbf{U}}(\mathfrak{g})$.

Khovanov and Lauda only proved this *categorification theorem* for $\mathfrak{g} = \mathfrak{sl}_n$. A key ingredient of that proof was a 2-representation of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ on a 2-category build out of the cohomology rings of partial flag varieties. The equivariant cohomology rings of these varieties, which also give rise to a 2-representation, are equivalent to the singular Soergel bimodules of type A , introduced and studied by Williamson in his PhD thesis in 2008 and published in [Wil11]. The general categorification theorem was proved by Webster [Web10], by geometric techniques well beyond our understanding.

In [MSV13], Mackaay, Stošić and Vaz defined a quotient of $\mathcal{U}(\mathfrak{sl}_n)$, denoted $\mathcal{S}(n, r)$, and proved that it categorifies the quantum Schur algebra $\mathbf{S}(n, r)$, for any $r \in \mathbb{Z}_{>0}$. If $n \geq r$, then $\mathcal{S}(n, r)$ contains a full sub-2-category which categorifies the Hecke algebra $\mathcal{H}_{A_{r-1}}$. This sub-2-category is equivalent to the 2-category of (ordinary) Soergel bimodules of type A_{r-1} , as was proved in [MSV13] using Elias and Khovanov’s diagrammatic presentation of the Soergel 2-category [EK10].

In the same paper, Mackaay, Stošić and Vaz also showed that Khovanov and Lauda’s 2-representation of $\mathcal{U}(\mathfrak{sl}_n)$ on the singular Soergel bimodules descends to $\mathbf{S}(n, r)$. Its restriction to the aforementioned sub-2-category of $\mathbf{S}(n, r)$ is exactly Elias and Khovanov’s 2-equivalence of their diagrammatic 2-category and the 2-category of (ordinary) Soergel bimodules.

Naturally the question arises whether the results in [MSV13] extend to affine type A . In this paper, we show that this is indeed the case for $3 \leq r < n$:

- As Libedinsky explained in [Lib08a], one can define Soergel bimodules using the geometric representation of the affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}}$ or Soergel’s extension of that representation [Soe07]. The geometric representation is not faithful, whereas Soergel’s representation is reflection faithful. Both representations give rise to categories of Soergel bimodules which categorify the affine Hecke algebra $\mathcal{H}_{\widehat{A}_{r-1}}$, as shown in [Här99, Lib08b, Lib08a, Soe07]. For more information on this topic, see also [EW13].

However, the extension of the geometric representation to the *extended* affine Weyl group $\widehat{W}_{\widehat{A}_{r-1}}$ is “too degenerate” and cannot be

used to categorify the *extended* affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$. Soergel's representation extends nicely to $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ and we show that there is a corresponding category of bimodules, denoted $\mathcal{EBim}_{\widehat{A}_{r-1}}$, which categorifies $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$.

Since Soergel's representation of $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ has dimension $r + 1$, the corresponding bimodules are defined over $\mathbb{Q}[y, x_1, \dots, x_r]$ where $\deg(y) = \deg(x_1) = \dots = \deg(x_r) = 2$. We will show that multiplication by y is always a bimodule map of degree two.

- We also define a diagrammatic 2-category $\mathcal{DEBim}_{\widehat{A}_{r-1}}$, similar to the ones in [EK10, EW13], and show that it is 2-equivalent to $\mathcal{EBim}_{\widehat{A}_{r-1}}$ (actually they are equivalent as monoidal categories, i.e. 2-categories with one object). Here we use the corresponding result for the non-extended category of affine bimodules and its diagrammatic analogue due to Elias and Williamson [EW13].
- We define a y -deformation of the level-zero 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$. In order to do that, the homogeneous 2-morphisms are defined over $\mathbb{Q}[y]$ instead of \mathbb{Q} , with y a formal variable of degree two. We denote this 2-category by $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and prove that its Karoubi envelope is Krull-Schmidt.

We recover $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ when we divide by the ideal generated by y . This ideal is virtually nilpotent, so the Grothendieck groups of the Karoubi envelopes of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ are isomorphic.

- We define a quotient of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, which we denote $\widehat{\mathcal{S}}(n, r)_{[y]}$. We prove that $\widehat{\mathcal{S}}(n, r)_{[y]}$ categorifies the affine quantum Schur algebra $\widehat{\mathbf{S}}(n, r)$. Again, the ideal generated by y is virtually nilpotent, so the quotient of $\widehat{\mathcal{S}}(n, r)_{[y]}$ by this ideal also categorifies $\widehat{\mathbf{S}}(n, r)$. We denote this quotient by $\widehat{\mathcal{S}}(n, r)$, which can also be obtained as a quotient of Khovanov and Lauda's original $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$.
- We define a 2-functor

$$\Sigma_{n,r}: \mathcal{DEBim}_{\widehat{A}_{r-1}}^* \rightarrow \widehat{\mathcal{S}}^*(n, r)_{[y]},$$

prove it to be faithful and conjecture it to be full.¹

- We define the 2-category of extended affine singular Soergel bimodules $\mathcal{ESBim}_{\widehat{A}_{r-1}}$ and give the affine analogue (and y -deformation) of Khovanov and Lauda's 2-representation, i.e. a 2-functor

$$\mathcal{F}': \widehat{\mathcal{S}}(n, r)_{[y]}^* \rightarrow \mathcal{ESBim}_{\widehat{A}_{r-1}}^*.$$

Remark 0.1. The case $n = r$ is different, because $\widehat{\mathbf{S}}(n, n)$ is not a quotient of $\widehat{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ but only of a strictly larger algebra. Therefore, one has to extend the Khovanov-Lauda affine calculus in order to define $\widehat{\mathcal{S}}(n, n)$. This case is dealt with in a follow-up paper [MT13].

¹We will explain the $*$ notation in Section 2. It basically allows us to consider 2-morphisms of arbitrary degree.

The case $n < r$ cannot be dealt with at present, because even the decategorified story has not been worked out (see Problem 2.4.5 in [DDF12]).

Remark 0.2. There is a technical detail, which will be fully explained in Section 3 but should be mentioned here already. Just as in [MSV13], we actually define $\widehat{\mathcal{S}}(n, r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, which is a 2-category obtained from $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ by switching to degenerate $\widehat{\mathfrak{gl}}_n$ -weights of level zero for the labels of the regions in the string diagrams. We conjecture that $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ categorifies the level-zero $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, but do not need that fact for the rest of this paper.

The results in this paper have several points of interest. The categories $\mathcal{EBim}_{\widehat{A}_{r-1}}$ and $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ contain the new objects B_{ρ^\pm} and \pm , respectively, and the corresponding morphisms. As our results show, these objects and morphisms also show up naturally in $\widehat{\mathcal{S}}(n, r)_{[y]}$ as 1 and 2-morphisms.

The y -deformation $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and its Schur quotient $\widehat{\mathcal{S}}(n, r)_{[y]}$ are new. As the results in this paper show, they both show up naturally when considering Soergel bimodules over $\mathbb{Q}[y, x_1, \dots, x_r]$, which is the ring of polynomial functions on Soergel’s reflection faithful representation of the affine Weyl group.

Furthermore, there are interesting (possible) links with other categorifications of the (extended) affine Hecke algebra and the quantum affine Schur algebra. Lusztig [Lus93, Lus99] and Ginzburg and Vasserot [GV93] gave a categorification of $\widehat{\mathbf{S}}(n, r)$ using perverse sheaves, extending Grojnowski and Lusztig’s approach to the categorification of $\mathbf{S}(n, r)$. It would be interesting to find the precise relation with the categorification presented here and in our follow-up paper [MT13].

In this paper we also define an extended version of the affine singular Soergel bimodules. Williamson introduced and studied the 2-category of singular Soergel bimodules for any Coxeter group in his PhD thesis in 2008, the results of which were published in [Wil11], and proved that it categorifies a certain “new” algebra, which he called the *Schur algebroid*. In finite type A the Schur algebroid is isomorphic to the quantum Schur algebra. Williamson’s affine type A Schur algebroid should also be closely related to the affine quantum Schur algebra. Whatever the precise relation turns out to be, the 2-representation of $\widehat{\mathcal{S}}(n, r)_{[y]}$ on the extended affine singular Soergel bimodules establishes an interesting relation between Khovanov and Lauda’s work and Williamson’s.

Another point of interest is related to the possibility of categorifying the so called *Kirillov-Reshetikhin modules* of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$. These level zero modules can be defined for any affine quantum group and have been intensively studied (see [CP94, DDF12, DF12, Kas02] for more information and references).

In affine type A (and only in that type), they are special examples of *evaluation modules* $V_{\lambda, a}$, where λ is a dominant weight and $a \in \mathbb{C}^*$. If λ is an n -part partition of r , then $V_{\lambda, a}$ descends to a representation of $\widehat{\mathbf{S}}(n, r)$. More precisely, $V_{\lambda, a}$ is defined by pulling back (the technical term is *inflating*) the action of $\mathbf{S}(n, r)$ on the irrep V_λ via the so called *evaluation map*

$$\mathrm{ev}_a: \widehat{\mathbf{S}}(n, r) \rightarrow \mathbf{S}(n, r).$$

If $\lambda = (m^i)$, i.e. m times the i -th fundamental \mathfrak{gl}_n -weight, then it is known that $V_{\lambda, q^{i-m+2}}$ is isomorphic to a Kirillov-Reshetikhin module and has a canonical basis. It seems likely that one can categorify the evaluation map

$$\mathrm{ev}_{q^{i-m+2}} : \widehat{\mathbf{S}}(n, mi) \rightarrow \mathbf{S}(n, mi)$$

and therefore $V_{\lambda, q^{i-m+2}}$, but such a categorification is beyond the scope of this paper.

This paper is organized as follows:

- In Section 1, we recall the definition of and some basic results on affine roots and weights, the (extended) affine Weyl group and the (extended) affine Hecke algebra.
- In Section 2, we define $\mathcal{EBim}_{\widehat{A}_{r-1}}$ and recall Härterich's categorification result. We also use Elias-Khovanov type diagrams in order to define $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ and show that it gives a diagrammatic presentation of $\mathcal{EBim}_{\widehat{A}_{r-1}}$, using Elias and Williamson's results in [EW13].
- In Section 3, we first recall the definition of $\widehat{\mathbf{S}}(n, r)$ and its relation to the extended affine Hecke algebra. After that, we give the definitions of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}$.
- In Section 4, we give the 2-functor

$$\Sigma_{n,r} : \mathcal{DEBim}_{\widehat{A}_{r-1}}^* \rightarrow \widehat{\mathcal{S}}^*(n, r)_{[y]}$$

and prove that it is well-defined.

- In Section 5, we define the affine 2-representation

$$\mathcal{F}' : \widehat{\mathcal{S}}^*(n, r)_{[y]} \rightarrow \mathcal{ESBim}_{\widehat{A}_{r-1}}^*.$$

The fact that it is well-defined follows essentially from the well-definedness of the analogous 2-functor for finite type A and a “conjugation trick”, which we will explain.

- In Section 6, we use the results in the previous sections to prove that $\widehat{\mathcal{S}}(n, r)_{[y]}$ categorifies $\widehat{\mathbf{S}}(n, r)$.

1. THE AFFINE SETTING

1.1. Affine roots of level zero. We use the well-known realization of $\widehat{\mathfrak{sl}}_r$ as the central extension of the loop algebra of \mathfrak{sl}_r together with a derivation (see for example [PS86, Kac90, Fuc95]), i.e. the underlying vector space is isomorphic to

$$\widehat{\mathfrak{sl}}_r = \mathcal{L}(\mathfrak{sl}_r) \oplus \mathbb{Q}\langle c \rangle \oplus \mathbb{Q}\langle d \rangle.$$

In order to express its root system, consider the Cartan subalgebra

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{Q}\langle c \rangle \oplus \mathbb{Q}\langle d \rangle = \mathbb{Q}\langle h_i, c, d \mid i = 1, \dots, r-1 \rangle$$

with \mathfrak{h} being the Cartan subalgebra of \mathfrak{sl}_r . The roots with respect to $\widehat{\mathfrak{h}}$ are

$$\alpha = (\bar{\alpha}, 0, m), \quad \bar{\alpha} \in \Phi(\mathfrak{sl}_r), \quad m \in \mathbb{Z}$$

and

$$\alpha = (0, 0, m), \quad m \in \mathbb{Z} \setminus \{0\}.$$

The roots of the first family are called the *real roots* and the ones of the second family are called the *imaginary roots*.

The simple roots are

$$\alpha_i = (\bar{\alpha}_i, 0, 0) \quad i = 1, \dots, r-1$$

and

$$\alpha_r = (-\bar{\theta}, 0, 1) = \delta - \bar{\theta}$$

where $\bar{\alpha}_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, r-1$ are the simple roots of \mathfrak{sl}_r , $\bar{\theta} = \bar{\alpha}_1 + \dots + \bar{\alpha}_{r-1} = \varepsilon_1 - \varepsilon_r$ is the highest root and δ is the dual element of d . The elements ε_i for $i = 1, \dots, r$ are the canonical basis vectors in \mathbb{Z}^r .

Weights are triples of the form

$$\kappa = (\bar{\kappa}, k, m),$$

where $\bar{\kappa}$ is an \mathfrak{sl}_r -weight and k and m are integers. The integer k is called the *level* of κ . The inner product between two weights $\kappa = (\bar{\kappa}, k, m)$ and $\kappa' = (\bar{\kappa}', k', m')$ is given by

$$\langle \kappa, \kappa' \rangle = \langle \bar{\kappa}, \bar{\kappa}' \rangle + km' + k'm,$$

where $\langle \bar{\kappa}, \bar{\kappa}' \rangle$ is the usual inner product of \mathfrak{sl}_r -weights. In particular, we have

$$\langle \alpha_i, \alpha_i \rangle = 2 \quad \text{for all } i = 1, \dots, r$$

and

$$\langle \alpha_r, \alpha_1 \rangle = \langle \alpha_i, \alpha_{i+1} \rangle = -1 \quad \text{for all } i = 1, \dots, r-1.$$

The simple coroots are $\alpha_i^\vee = \alpha_i$ for all $i = 1, \dots, r$.

In the following sections, we will also use $\widehat{\mathfrak{gl}}_r$ -weights

$$\kappa = (\bar{\kappa}, k, m),$$

where $\bar{\kappa}$ denotes a non-affine \mathfrak{gl}_r -weight.

Remark 1.1. In this paper, we will only consider $\widehat{\mathfrak{sl}}_r$ and $\widehat{\mathfrak{gl}}_r$ -weights of level zero.

1.2. The Weyl group action. For any root $\alpha \in \Phi(\widehat{\mathfrak{sl}}_r)$ and $\widehat{\mathfrak{sl}}_r$ -weight κ , the Weyl reflection σ_α is defined by $\sigma_\alpha(\kappa) = \kappa - \langle \kappa, \alpha^\vee \rangle \alpha$. So if $\alpha = (\bar{\alpha}, 0, n)$ and $\kappa = (\bar{\kappa}, k, m)$, we can express $\sigma_\alpha(\kappa)$ as follows

$$\sigma_\alpha(\kappa) = \left(\bar{\kappa} - \langle \bar{\kappa}, \bar{\alpha}^\vee \rangle \bar{\alpha} - kn\bar{\alpha}^\vee, k, m - \langle \bar{\kappa}, \bar{\alpha}^\vee \rangle n - \frac{2kn^2}{\langle \alpha, \alpha \rangle} \right). \quad (1.1)$$

The *affine Weyl group* $\mathcal{W}_{\widehat{A}_{r-1}}$ is the group generated by all these reflections. For any simple root α_i , we write $\sigma_i := \sigma_{\alpha_i}$.

Similarly, one can consider the action of σ_i on a level zero $\widehat{\mathfrak{gl}}_r$ -weight of the form $(\varepsilon_j, 0, m)$ with $j = 1, \dots, r$. If $i \neq r$, one gets

$$\sigma_i(\varepsilon_j, 0, m) = \begin{cases} (\varepsilon_{i+1}, 0, m) & \text{if } j = i \\ (\varepsilon_i, 0, m) & \text{if } j = i+1 \\ (\varepsilon_j, 0, m) & \text{otherwise.} \end{cases} \quad (1.2)$$

The action of σ_r is given by

$$\sigma_r(\varepsilon_j, 0, m) = \begin{cases} (\varepsilon_r, 0, m+1) & \text{if } j = 1 \\ (\varepsilon_1, 0, m-1) & \text{if } j = r \\ (\varepsilon_j, 0, m) & \text{otherwise.} \end{cases} \quad (1.3)$$

For each simple \mathfrak{sl}_r -root $\bar{\alpha}_i$, $i = 1, \dots, r-1$, there also exists a translation $t_{\bar{\alpha}_i}$, which acts on the level zero $\widehat{\mathfrak{gl}}_r$ -weights as follows

$$t_{\bar{\alpha}_i}(\bar{\kappa}, 0, m) = (\bar{\kappa}, 0, m - (\kappa_i - \kappa_{i+1})) \quad (1.4)$$

where $\bar{\kappa} = (\kappa_1, \dots, \kappa_r)$ and the indices are taken to be modulo r , e.g. $\kappa_{r+1} = \kappa_1$ by definition.

One can prove that $\mathcal{W}_{\widehat{A}_{r-1}}$ is the semidirect product of the finite Weyl group $\mathcal{W}_{A_{r-1}}$, generated by the reflections σ_i for $i = 1, \dots, r-1$, and of the abelian group $\langle t_{\bar{\alpha}_1^\vee}, \dots, t_{\bar{\alpha}_{r-1}^\vee} \rangle$ of translations along the coroot lattice of \mathfrak{sl}_r .

1.3. The extended affine Weyl group. See [Lus89], [DG07] or [DDF12] for more details about the extended affine Weyl group. For example in [DG07], there is a definition of this group different from the following, it is described as a subgroup of permutations of \mathbb{Z} .

Let us now consider the translations t_{ε_i} along the simple \mathfrak{gl}_r -roots ε_i , $i = 1, \dots, r$. Their action on a level zero $\widehat{\mathfrak{gl}}_r$ -weight κ is given by

$$t_{\varepsilon_i}(\bar{\kappa}, 0, m) = (\bar{\kappa}, 0, m - \kappa_i). \quad (1.5)$$

The *extended affine Weyl group* $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is defined as the semidirect product of the finite Weyl group $\mathcal{W}_{A_{r-1}}$ and the abelian group $\langle t_{\varepsilon_1}, \dots, t_{\varepsilon_r} \rangle$ of translations along the coroot lattice of \mathfrak{gl}_r . It contains the affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}}$ as a normal subgroup.

The group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by $\sigma_1, \dots, \sigma_{r-1}$ and $t_{\varepsilon_1}, \dots, t_{\varepsilon_r}$, which satisfy the following relations:

$$\sigma_i t_{\varepsilon_j} \sigma_i = t_{\sigma_i(\varepsilon_j)} \quad \text{for } i = 1, \dots, r-1 \text{ and } j = 1, \dots, r. \quad (1.6)$$

Hence the set of generators is not minimal, e.g. one can obtain any t_{ε_j} for $j = 2, \dots, r$ by conjugating t_{ε_1} by certain reflections.

There is another presentation of $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, which is important for this paper. It involves the following specific element

$$\rho = t_{\varepsilon_1} \sigma_1 \dots \sigma_{r-1},$$

which acts on a level zero $\widehat{\mathfrak{gl}}_r$ -weight $\kappa = (\bar{\kappa}, 0, m)$ by

$$\rho(\bar{\kappa}, 0, m) = ((\kappa_r, \kappa_1, \dots, \kappa_{r-1}), 0, m - \kappa_r). \quad (1.7)$$

The action of its inverse $\rho^{-1} = \sigma_{r-1}^{-1} \dots \sigma_1^{-1} t_{-\varepsilon_1}$ is given by

$$\rho^{-1}(\bar{\kappa}, 0, m) = ((\kappa_2, \dots, \kappa_r, \kappa_1), 0, m + \kappa_1). \quad (1.8)$$

One then sees that $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by

$$\sigma_1, \dots, \sigma_r, \rho,$$

subject to the relations

$$\sigma_i^2 = 1 \quad \text{for } i = 1, \dots, r \quad (1.9)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for distant } i, j = 1, \dots, r \quad (1.10)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \dots, r \quad (1.11)$$

$$\rho \sigma_i \rho^{-1} = \sigma_{i+1} \quad \text{for } i = 1, \dots, r \quad (1.12)$$

where the indices have to be understood modulo r , as before. We say that i and j are *distant* if $j \not\equiv i \pm 1 \pmod{r}$. Using this set of generators, any element $w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ can be written in the following way

$$w = \rho^k w' = \rho^k \sigma_{i_1} \cdots \sigma_{i_l} \quad (1.13)$$

where $k \in \mathbb{Z}$ is unique and $\sigma_{i_1} \cdots \sigma_{i_l}$ is a reduced expression of the element $w' \in \mathcal{W}_{\widehat{A}_{r-1}}$.

Note that the conventions here are opposite to the ones chosen by Doty and Green [DG07].

1.4. The extended affine braid group and Hecke algebra. One can form the *extended affine braid group* $\widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ associated to $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$. It admits the same presentation as $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ except that one omits the involutivity relations of the generators σ_i for $i = 1, \dots, r$.

One can also define the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$, which is the quotient of the $\mathbb{Q}(q)$ -group algebra of $\widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ by the relations

$$T_{\sigma_i}^2 = (q^2 - 1)T_{\sigma_i} + q^2 \quad \text{for all } i = 1, \dots, r$$

with q being a formal parameter. For more details about this algebra, see [Opd03, Opd04, DO08, OS09, DDF12].

A $\mathbb{Q}(q)$ -basis of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is given by the set

$$\{T_w, w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}\}$$

where

$$T_w = T_{\rho}^k T_{w'} = T_{\rho}^k \sigma_{i_1} \cdots \sigma_{i_l}$$

with w, k, w' and σ_{i_j} as in (1.13). See [Gre99] and [DG07].

The above shows that $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is generated by

$$\{T_{\rho}, T_{\rho^{-1}}, T_{\sigma_i}, i = 1, \dots, r\}$$

as an algebra. An alternative set of algebraic generators of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ is given by

$$\{T_{\rho}, T_{\rho}^{-1}, b_i, i = 1, \dots, r\} \quad (1.14)$$

where the $b_i := C'_{\sigma_i} = q^{-1}(1 + T_{\sigma_i})$ are the Kazhdan-Lusztig generators. The relations satisfied by these generators are the following:

$$b_i^2 = (q + q^{-1})b_i \quad \text{for } i = 1, \dots, r \quad (1.15)$$

$$b_i b_j = b_j b_i \quad \text{for distant } i, j = 1, \dots, r \quad (1.16)$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i \quad \text{for } i = 1, \dots, r \quad (1.17)$$

$$T_{\rho} b_i T_{\rho}^{-1} = b_{i+1} \quad \text{for } i = 1, \dots, r. \quad (1.18)$$

In [GH07], Grojnowski and Haiman show that $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ has the following Kazhdan-Lusztig basis

$$\{T_{\rho}^k C'_w, k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\widehat{A}_{r-1}}\}, \quad (1.19)$$

with the usual positive integrality property.

2. A CATEGORIFICATION OF THE EXTENDED AFFINE HECKE ALGEBRA

Notation 2.1. Let \mathcal{C} be a \mathbb{Q} -linear \mathbb{Z} -graded additive category (resp. 2-category) with translation (see Section 5.1 in [Lau10] for the technical definitions). In all examples in this paper, the vector space of morphisms (resp. 2-morphisms) of any fixed degree is finite-dimensional.

The Karoubi envelope of \mathcal{C} is denoted by $\text{Kar}\mathcal{C}$.

By \mathcal{C}^* we denote the category (resp. 2-category) with the same objects (resp. same objects and 1-morphisms) as \mathcal{C} , but whose hom-spaces (resp. 2-hom-spaces) are defined by

$$\mathcal{C}^*(x, y) = \oplus_{t \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(x\{t\}, y).$$

A degree preserving functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ between two such categories \mathcal{C} and \mathcal{D} lifts to a functor between the enriched categories $\mathcal{C}^* \rightarrow \mathcal{D}^*$ and to a functor between the Karoubi envelopes $\text{Kar}\mathcal{C} \rightarrow \text{Kar}\mathcal{D}$, which are both also denoted \mathcal{F} .

2.1. An extension of Soergel's categorification.

2.1.1. Action on polynomial rings. Consider the polynomial ring $R = \mathbb{Q}[y][x_1, \dots, x_r]$. The extended affine Weyl group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ acts faithfully on R as follows:

$$\begin{aligned} \rho(x_i) &= \begin{cases} x_{i+1} & \text{for } i = 1, \dots, r-1 \\ x_1 - y & \text{for } i = r \end{cases} \\ \rho^{-1}(x_i) &= \begin{cases} x_{i-1} & \text{for } i = 2, \dots, r \\ x_r + y & \text{for } i = 1 \end{cases} \\ t_{\varepsilon_j}(x_i) &= \begin{cases} x_j - y & \text{for } i = j \\ x_i & \text{otherwise} \end{cases} \\ \sigma_j(x_i) &= \begin{cases} x_{j+1} & \text{for } i = j \\ x_j & \text{for } i = j+1 \\ x_i & \text{otherwise} \end{cases} \quad \text{for } j = 1, \dots, r-1 \\ \sigma_r(x_i) &= \begin{cases} x_r + y & \text{for } i = 1 \\ x_1 - y & \text{for } i = r \\ x_i & \text{otherwise} \end{cases} \end{aligned}$$

Remark 2.2. The action above naturally extends the action of the (non-extended) affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}}$. We are working here with Soergel's original reflection faithful realization of the affine type A Coxeter system $\mathcal{W}_{\widehat{A}_{r-1}}$ considered in [Soe07] and [Här99]. Indeed, when generalized to the extended affine Weyl group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, this representation remains faithful which explains why we choose to use this precise realization to achieve a categorification of the extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ in the present paper.

Let us set $X_i = x_{i+1} - x_i$ for $i = 1, \dots, r-1$ and $X_r = x_1 - x_r - y$.

2.1.2. *Extended Soergel bimodules.* For any $i = 1, \dots, r$, we define the R -bimodule

$$B_i = R \otimes_{R^{\sigma_i}} R \quad (2.1)$$

where R^{σ_i} is the subalgebra of elements of R fixed by the reflection $\sigma_i \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$:

$$\begin{aligned} R^{\sigma_i} &= \mathbb{Q}[y][x_1, \dots, x_i + x_{i+1}, x_i x_{i+1}, \dots, x_r] \quad \text{for } i = 1, \dots, r-1 \\ R^{\sigma_r} &= \mathbb{Q}[y][x_2, \dots, x_{r-1}, x_r + x_1, (x_r + y/2)(x_1 - y/2)] \end{aligned}$$

We also define the *twisted R -bimodule* B_ρ (resp. $B_{\rho^{-1}}$), which coincides with R as a left R -module but is twisted by ρ (resp. by ρ^{-1}) as a right R -module, i.e. any $a \in R$ acts on B_ρ on the right by multiplication by $\rho(a)$ (resp. $\rho^{-1}(a)$).

Arkhipov introduced twisted bimodules associated to simple reflections and the twisted functors obtained by tensoring with them. These were used by several people in their work on category \mathcal{O} ; for some history and references see [Maz12]. Here we (only) consider twisted bimodules associated to powers of ρ .

We introduce a grading on R , R^{σ_i} , B_i and $B_{\rho^{\pm 1}}$ by setting

$$\deg(y) = \deg(x_k) = 2$$

for all $k = 1, \dots, r$. Curly brackets will indicate a shift of the grading: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a \mathbb{Z} -graded bimodule and p an integer, then the \mathbb{Z} -graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$.

Form now the category monoidally generated by the graded R -bimodules defined above. Then allow direct sums and grading shifts of these objects and consider only morphisms which are degree-preserving morphisms of R -bimodules, and denote this category $\mathcal{EBim}_{\widehat{A}_{r-1}}$. Its Karoubi envelope $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$ is a \mathbb{Q} -linear graded additive monoidal category with translation, which we call *the category of extended Soergel bimodules of type \widehat{A}_{r-1}* .

As mentioned in Remark 2.2, we are precisely using Soergel's original realization, so the bimodules B_i considered here are the ones constructed by Härterich [Här99] and Soergel [Soe07]. Therefore Soergel's category $\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}}$ of affine type A is equivalent to the full subcategory of $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$ generated by the B_i , for $i = 1, \dots, r$. Let us recall Härterich's categorification result [Här99]:

Theorem 2.3 (Härterich). *We have*

$$\mathcal{H}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}})$$

where

$$K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}}) := K_0(\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q).$$

For each $w \in \mathcal{W}_{\widehat{A}_{r-1}}$, there exists a unique indecomposable bimodule B_w in $\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}}$. Conversely, any indecomposable bimodule in this category is isomorphic to $B_w\{t\}$, for a certain $w \in \mathcal{W}_{\widehat{A}_{r-1}}$ and a certain grading shift $t \in \mathbb{Z}$.

Under the isomorphism above the Kazhdan-Lusztig basis element C'_w is mapped to $[B_w\{-1\}]$, for any $w \in \mathcal{W}_{\hat{A}_{r-1}}$.

In the rest of the paper, we keep the convention that subscripts are considered to be modulo r , e.g. the bimodule B_{r+1} is by definition equal to B_1 . We will also use the notation $B_\rho^{\otimes k}$ for any $k \in \mathbb{Z}$, where this bimodule is defined to be the tensor product of $|k|$ copies of B_ρ if $k \geq 0$ and of $B_{\rho^{-1}}$ if $k \leq 0$. In both cases $B_\rho^{\otimes k}$ is isomorphic to B_{ρ^k} .

2.1.3. Categorification of $\hat{\mathcal{H}}_{\hat{A}_{r-1}}$.

Lemma 2.4. *For any $i = 1, \dots, r$, there exists an R -bimodule isomorphism*

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho. \quad (2.2)$$

Applying these isomorphisms r times gives an isomorphism

$$B_\rho^{\otimes r} \otimes_R B_i \cong B_i \otimes_R B_\rho^{\otimes r}, \quad (2.3)$$

for any $i = 1, \dots, r$.

Proof.

- *Isomorphism (2.2)*

First note that there exist natural isomorphisms of R -bimodules:

$$B_\rho \otimes_R B_i \cong B_\rho \otimes_{R^{\sigma_i}} R \quad \text{and} \quad B_{i+1} \otimes_R B_\rho \cong R \otimes_{R^{\sigma_{i+1}}} B_\rho.$$

Define the isomorphism of R -bimodules $\psi : B_\rho \otimes_{R^{\sigma_i}} R \rightarrow R \otimes_{R^{\sigma_{i+1}}} B_\rho$ by

$$\psi(a \otimes b) = a \otimes \rho(b).$$

This isomorphism is well-defined, because ρ defines an isomorphism between R^{σ_i} and $R^{\sigma_{i+1}}$.

- *Isomorphism (2.3)*

Note that $B_\rho^{\otimes r} \cong B_{\rho^r}$ and that ρ^r leaves the ring R^{σ_i} invariant. \square

Theorem 2.5. *The category $\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}$ categorifies the extended affine Hecke algebra, i.e.*

$$\hat{\mathcal{H}}_{\hat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}),$$

where

$$K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}) := K_0(\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q).$$

Under this isomorphism, the indecomposables in $\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}$ correspond exactly to the Kazhdan-Lusztig basis of $\hat{\mathcal{H}}_{\hat{A}_{r-1}}$. In particular, $[B_i\{-1\}]$ corresponds to b_i and $[B_{\rho^{\pm 1}}]$ corresponds to $T_\rho^{\pm 1}$.

Proof. Recall that

$$\mathcal{H}_{\hat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{Bim}_{\hat{A}_{r-1}}),$$

by Theorem 2.3. The indecomposables in $\text{Kar}\mathcal{Bim}_{\hat{A}_{r-1}}$, which are denoted by $B_w\{-1\}$ for $w \in \mathcal{W}_{\hat{A}_{r-1}}$, correspond exactly to the Kazhdan-Lusztig basis elements $\{C'_w, w \in \mathcal{W}_{\hat{A}_{r-1}}\}$ of $\mathcal{H}_{\hat{A}_{r-1}}$. Moreover, B_w appears as a direct summand of the tensor product $B_{i_1} \otimes_R \dots \otimes_R B_{i_k}$ where $\sigma_{i_1} \dots \sigma_{i_k}$ is a reduced expression of w .

We define the homomorphism of algebras

$$\widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \rightarrow K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}})$$

by

$$b_i \mapsto [B_i\{-1\}] \quad \text{and} \quad T_\rho^{\pm 1} \mapsto [B_{\rho^{\pm 1}}].$$

The homomorphism is well-defined, as follows from the following isomorphisms in $\mathcal{EBim}_{\widehat{A}_{r-1}}$:

$$B_i \otimes_R B_i \cong B_i \oplus B_i\{2\} \quad (2.4)$$

$$B_i \otimes_R B_j \cong B_j \otimes_R B_i \text{ for distant } i, j \quad (2.5)$$

$$B_i \otimes_R B_{i+1} \otimes_R B_i \oplus B_{i+1}\{2\} \cong B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \oplus B_i\{2\} \quad (2.6)$$

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho, \quad (2.7)$$

for $i, j = 1, \dots, r$.

Note that any tensor product of $B_{\rho^{\pm 1}}$'s and B_i 's can be rewritten in the following way

$$B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \dots \otimes_R B_{i_l},$$

by sliding all the $B_{\rho^{\pm 1}}$'s to the left using the isomorphism (2.7).

Let us now look at the indecomposables of the category $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$. First observe that if the bimodule M is indecomposable in $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$ then, for any $k \in \mathbb{Z}$, the tensor product $B_\rho^{\otimes k} \otimes_R M$ is indecomposable as well. Indeed assume that

$$B_\rho^{\otimes k} \otimes_R M \cong P \oplus Q,$$

then tensoring on the left by $B_\rho^{\otimes -k}$ gives

$$M \cong B_\rho^{\otimes -k} \otimes_R P \oplus B_\rho^{\otimes -k} \otimes_R Q,$$

which contradicts the fact that M is supposed to be indecomposable.

So let M be an indecomposable of $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$. It is a direct summand of some tensor product $B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \dots \otimes_R B_{i_l}$. Then the indecomposable bimodule $B_\rho^{\otimes -k} \otimes_R M$ is a direct summand of $B_{i_1} \otimes_R \dots \otimes_R B_{i_l}$. The latter tensor product belongs to the subcategory $\text{Kar}\mathcal{Bim}_{\widehat{A}_{r-1}}$ of $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$. Thus $B_\rho^{\otimes -k} \otimes_R M$ is of the form B_w for some $w \in \mathcal{W}_{\widehat{A}_{r-1}}$. We can conclude that the indecomposables of the category $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$ are all of the form

$$B_\rho^{\otimes k} \otimes_R B_w \quad \text{for } k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\widehat{A}_{r-1}}.$$

Their Grothendieck classes correspond bijectively to the elements of

$$\{T_\rho^k C'_w, k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\widehat{A}_{r-1}}\},$$

which is precisely the Kazhdan-Lusztig basis of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ in (1.19). \square

Remark 2.6. In the category $\mathcal{EBim}_{\widehat{A}_{r-1}}$, the B_i are self-adjoint and B_ρ and B_ρ^{-1} form a biadjoint pair. Therefore, there exist isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{EBim}_{\widehat{A}_{r-1}}}(B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \dots \otimes_R B_{i_m}, B_\rho^{\otimes l} \otimes_R B_{j_1} \otimes_R \dots \otimes_R B_{j_n}) &\cong \\ \text{Hom}_{\mathcal{EBim}_{\widehat{A}_{r-1}}}(R, B_\rho^{\otimes l-k} \otimes_R B_{j_1} \otimes_R \dots \otimes_R B_{j_n} \otimes_R B_{i_m} \otimes_R \dots \otimes_R B_{i_1}). \end{aligned}$$

The latter hom-space is equal to zero except when $k = l$, in which case it is isomorphic to the corresponding hom-space in $\mathcal{B}im_{\widehat{A}_{r-1}}$.

Indeed any morphism

$$f \in \text{Hom}_{\mathcal{B}im_{\widehat{A}_{r-1}}}(R, B_\rho^{\otimes l-k} \otimes_R B_{j_1} \otimes_R \dots \otimes_R B_{j_n} \otimes_R B_{i_m} \otimes_R \dots \otimes_R B_{i_1})$$

is completely determined by the image p of 1. Since f is a morphism of R -bimodules, we have $pa = ap$ for any $a \in R$. In particular, for $a = \sum_{i=1}^r x_i$ we have

$$p \left(\sum_{i=1}^r x_i \right) = \left(\sum_{i=1}^r x_i \right) p.$$

Since $\sum_{i=1}^r x_i$ is invariant under all the reflections σ_j for $j = 1, \dots, r$, we also have

$$\begin{aligned} p \left(\sum_{i=1}^r x_i \right) &= \rho^{l-k} \left(\sum_{i=1}^r x_i \right) p \\ &= \left(\sum_{i=1}^r x_i - (l-k)y \right) p. \end{aligned}$$

This implies that p , and therefore the morphism f , has to be zero unless $k = l$.

2.1.4. *Representation of $\widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ in $\mathcal{K}(\text{Kar}\mathcal{E}Bim_{\widehat{A}_{r-1}})$.* Let $\mathcal{K}(\text{Kar}\mathcal{E}Bim_{\widehat{A}_{r-1}})$ be the homotopy category of bounded complexes in $\text{Kar}\mathcal{E}Bim_{\widehat{A}_{r-1}}$.

Rouquier [Rou06] obtained a representation of $\mathcal{B}_{\widehat{A}_{r-1}}$ in $\mathcal{K}(\text{Kar}\mathcal{B}im_{\widehat{A}_{r-1}})$. We generalize his result in the extended affine setting.

Remark 2.7. In what follows, we use by matter of convenience the notation $X_i = x_{i+1} - x_i$, for $i = 1, \dots, r-1$, and $X_r = x_1 - x_r - y$, as defined in Remark 2.2.

To each braid generator $\sigma_i \in \widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ we assign the cochain complex $F(\sigma_i)$ of graded R -bimodules

$$F(\sigma_i) : 0 \longrightarrow R\{2\} \xrightarrow{\text{rb}_i} B_i \longrightarrow 0 \quad (2.8)$$

where B_i sits in cohomological degree 0, and the R -bimodule morphism rb_i maps 1 to $\frac{1}{2}(X_i \otimes 1 + 1 \otimes X_i)$, for $i = 1, \dots, r$.

To σ_i^{-1} we assign the cochain complex $F(\sigma_i^{-1})$ of graded R -bimodules

$$F(\sigma_i^{-1}) : 0 \longrightarrow B_i\{-2\} \xrightarrow{\text{br}_i} R\{-2\} \longrightarrow 0 \quad (2.9)$$

where $B_i\{-2\}$ sits in cohomological degree 0 and the R -bimodule morphism br_i is just the multiplication.

To ρ (resp. ρ^{-1}) we assign the cochain complex of graded R -bimodules

$$F(\rho) : 0 \longrightarrow B_\rho \longrightarrow 0 \quad (2.10)$$

$$(\text{resp. } F(\rho^{-1}) : 0 \longrightarrow B_{\rho^{-1}} \longrightarrow 0) \quad (2.11)$$

where B_ρ (resp. $B_{\rho^{-1}}$) sits in cohomological degree 0.

To the unit element $1 \in \widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ we assign the complex of graded R -bimodules

$$F(1) : 0 \longrightarrow R \longrightarrow 0 \quad (2.12)$$

where R sits in cohomological degree 0; the complex $F(1)$ is a unit for the tensor product of complexes. Finally, to any extended affine braid word, we assign the tensor product over R of the complexes associated to the generators appearing in the expression of the word.

Proposition 2.8. *The above defines a categorical representation of $\widehat{\mathcal{B}}_{\widehat{A}_{r-1}}$ in $\mathcal{K}(\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}})$.*

2.2. The diagrammatic version. The category of Soergel bimodules of finite type A is described via planar diagrams by Elias and Khovanov in [EK10]. They associate planar diagrams to certain generating bimodule maps and give a complete set of relations on them. Elias and Williamson [EW13] worked out the generalization of the diagrammatic approach to Soergel bimodules for any Coxeter group which does not contain a standard parabolic subgroup isomorphic to H_3 .

Our aim is to define and study an extension of Elias and Williamson's diagrammatic category for extended affine type A [EW13]. In affine type A Elias and Williamson's diagrammatic category is a straightforward generalization of Elias and Khovanov's original category, which was defined for finite type A . In addition to Elias and Williamson's diagrams, we also have to introduce a new type of strand. These new strands are oriented and their endpoints are labeled $+$ or $-$ depending on orientations. The Karoubi envelope of the diagrammatic category $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ obtained in this way is equivalent to the category of extended Soergel bimodules $\text{Kar}\mathcal{EBim}_{\widehat{A}_{r-1}}$ of affine type A , as we will show.

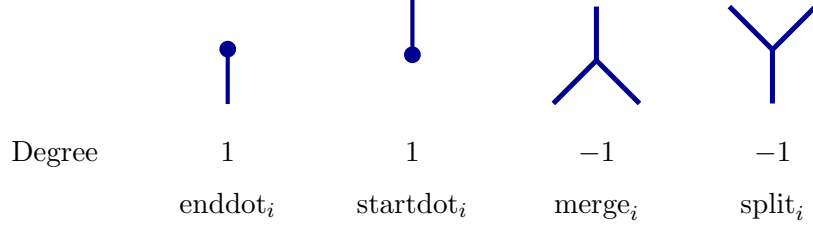
2.2.1. Definition of $\mathcal{DEBim}_{\widehat{A}_{r-1}}$. First start with the category whose objects are graded finite sequences of integers belonging to $\{1, \dots, r\}$ and the symbols $+$ and $-$. Graphically we represent these sequences by sequences of colored points (read from left to right) of the x -axis of the real plane \mathbb{R}^2 . The morphisms are then equivalence classes of \mathbb{Q} -linear combinations of graded planar diagrams in $\mathbb{R} \times [0, 1]$ (read from bottom to top) and composition is defined by vertically glueing the diagrams and rescaling the vertical coordinate. These morphisms are defined by generators and relations listed below. This category possesses a monoidal structure given by stacking sequences and diagrams next to each other.

Let $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ be the category containing all direct sums and grading shifts of these objects and let its morphisms be the degree-preserving diagrams. The diagrammatic extended Soergel category is by definition its Karoubi envelope $\text{Kar}\mathcal{DEBim}_{\widehat{A}_{r-1}}$.

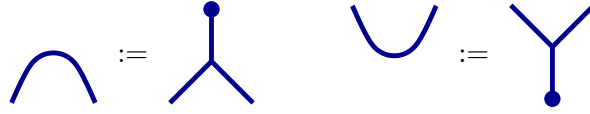
In the diagrams, the strands whose endpoints are $+$ or $-$ signs are oriented and the other strands are non-oriented. The non-oriented strands can be colored with integers belonging to $\{1, \dots, r\}$. Two colors i and j are called *adjacent* (resp. *distant*) if $i \equiv j \pm 1 \pmod r$ (resp. $i \not\equiv j \pm 1 \pmod r$). By convention, no label means that the equation holds for any color $i \in \{1, \dots, r\}$.

The morphisms of $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ are built out of the following generating diagrams. The non-oriented diagrams are the affine analogues of Elias and Khovanov's diagrams, the ones involving oriented strands are new.

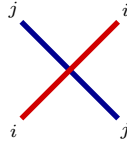
- Generators involving only one color:



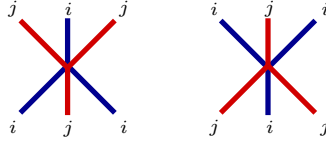
It is useful to define the cap and cup as follows



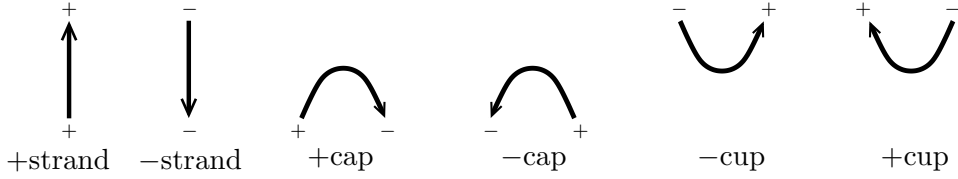
- Generators involving two colors:
 - the 4-valent vertex with distant colors, of degree 0, denoted $4\text{vert}_{i,j}$



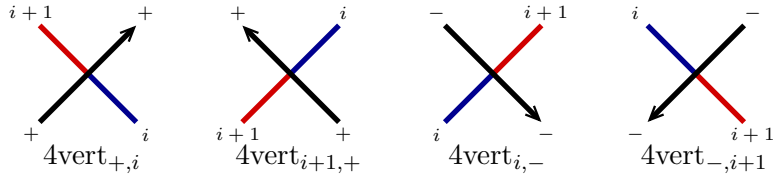
- and the 6-valent vertices with adjacent colors i and j , of degree 0, denoted $6\text{vert}_{i,j}$ and $6\text{vert}_{j,i}$



- Generators involving only oriented strands, of degree 0:



- Generators involving oriented strands and adjacent colored strands.
The mixed 4-valent vertex of degree 0:



- Generators involving boxes, of degree 2, denoted box_i



for all $i = 1, \dots, r$, and denoted box_y



The generating diagrams are subject to the following relations. The relations involving only non-oriented diagrams are the obvious affine analogues of Elias and Khovanov's relations, the ones with diagrams involving oriented strands are new.

- Isotopy relations:

$$\text{cap} = \text{straight} = \text{cup} \quad (2.13)$$

$$\text{dot-cap} = \text{dot-straight} = \text{dot-cup} \quad (2.14)$$

$$\text{Y-junction} = \text{X-junction} = \text{Z-junction} \quad (2.15)$$

$$\text{blue-red crossing} = \text{red-blue crossing} = \text{red-blue crossing} \quad (2.16)$$

$$\text{blue-red crossing} = \text{red-blue crossing} = \text{red-blue crossing} \quad (2.17)$$

$$\text{cap} = \text{straight} = \text{cup} \quad (2.18)$$

$$\text{cap} = \text{straight} = \text{cup} \quad (2.19)$$

$$\text{blue-red crossing} = \text{red-blue crossing} = \text{red-blue crossing} \quad (2.20)$$

$$\text{blue-red crossing} = \text{red-blue crossing} = \text{red-blue crossing} \quad (2.21)$$

- Relations involving one color:

$$\text{Y-junction} = \text{X-junction} \quad (2.22)$$

$$\text{loop} = 0 \quad (2.23)$$

$$\text{dot-straight} + \text{dot-straight} = 2 \text{ dot-straight} \quad (2.24)$$

- Relations involving two distant colors:

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with the blue line looping around the red line.} \end{array} = \begin{array}{c} \text{Diagram: Two parallel vertical lines, one blue and one red.} \end{array} \quad (2.25)$$

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the red line.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the blue line.} \end{array} \quad (2.26)$$

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red arc connecting them.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue arc connecting them.} \end{array} \quad (2.27)$$

- Relations involving two adjacent colors:

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red dot on the blue line.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red dot on the red line.} \end{array} + \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the blue line.} \end{array} \quad (2.28)$$

$$\begin{array}{c} \text{Diagram: Two parallel vertical lines, one blue and one red.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the red line.} \end{array} - \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the blue line.} \end{array} \quad (2.29)$$

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red dot on the blue line.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the red line.} \end{array} \quad (2.30)$$

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red dot on the blue line.} \end{array} - \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a red dot on the red line.} \end{array} = \frac{1}{2} \left(\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the blue line.} \end{array} - \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a blue dot on the red line.} \end{array} \right) \quad (2.31)$$

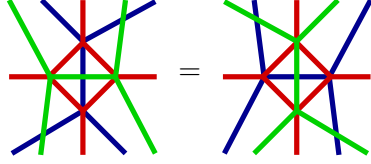
- Relation involving three distant colors:

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a green line passing through them.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a green line passing through them.} \end{array} \quad (2.32)$$

- Relation involving two adjacent colors and one distant from the other two:

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a green line passing through them.} \end{array} = \begin{array}{c} \text{Diagram: A blue line and a red line crossing, with a green line passing through them.} \end{array} \quad (2.33)$$

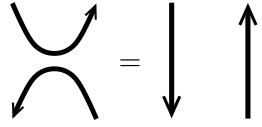
- Relation involving three adjacent colors:


(2.34)

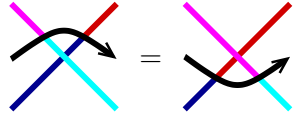
- Relations involving only oriented strands:


(2.35)

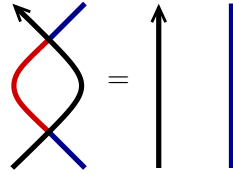

(2.36)

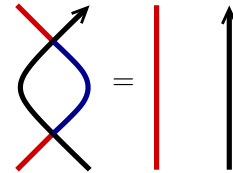

(2.37)

- Relations involving oriented strands and distant colored strands:


(2.38)

- Relations involving oriented strands and two adjacent colored strands:


(2.39)


(2.40)


(2.41)


(2.42)


(2.43)

- Relations involving oriented strands and three adjacent colored strands:

$$(2.44)$$

$$(2.45)$$

- Relations involving boxes:

$$(2.46)$$

$$(2.47)$$

$$(2.48)$$

$$(2.49)$$

$$(2.50)$$

$$(2.51)$$

$$(2.52)$$

$$(2.53)$$

$$(2.54)$$

$$\boxed{i+1} \uparrow = \uparrow \boxed{i} \quad \text{for } i \neq r \quad (2.55)$$

$$\left(\boxed{1} - \boxed{y} \right) \uparrow = \uparrow \boxed{r} \quad (2.56)$$

$$\boxed{i-1} \downarrow = \downarrow \boxed{i} \quad \text{for } i \neq 1 \quad (2.57)$$

$$\left(\boxed{r} + \boxed{y} \right) \downarrow = \downarrow \boxed{1} \quad (2.58)$$

Remark 2.9. Note that the Relations (2.38)–(2.45) also hold when the oriented strand has the opposite orientation. This follows from Relations (2.38)–(2.45) and Relations of isotopy.

Note also that the generators and relations listed above are redundant. This redundancy helps us to simplify some of the proofs later on. More specifically, let us list some of these unnecessary generators and relations:

- (i) Relation (2.37) follows from Relation (2.36) and isotopy invariance,
- (ii) Relation (2.57) follows from Relation (2.55) and isotopy invariance,
- (iii) Relation (2.56) follows from Relations (2.55) for $i = r - 1$, (2.46) for $i = r - 1$, (2.52), (2.41) and (2.42),
- (iv) Relation (2.58) follows from Relations (2.54), (2.56) and isotopy invariance,
- (v) If we sum Relation (2.46) for all $i = 1, \dots, r - 1$ and Relation (2.47), we obtain that

$$\boxed{y} = - \sum_{k=1}^r \text{blue dot at } k$$

- (vi) If we sum Relation (2.31) for $j = i - 1$ and for $j = i + 1$, we obtain that

$$\text{green dot at } i-1 + \text{blue dot at } i + \text{red dot at } i+1 \uparrow = \uparrow \text{green dot at } i-1 + \text{blue dot at } i + \text{red dot at } i+1$$

- (vii) Relation (2.52) follows from Relation (2.26), the two relations exhibited just above in (v) and (vi) and isotopy invariance,
- (viii) Relation (2.53) follows from Relations (2.41) and (2.42) and the relation exhibited in (v),
- (ix) Relation (2.54) follows from Relation (2.53) and isotopy invariance.

2.2.2. *Functor from $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$ to $\mathcal{EBim}_{\hat{A}_{r-1}}^*$.* Let us construct a degree preserving functor

$$\mathcal{F}: \mathcal{DEBim}_{\hat{A}_{r-1}}^* \rightarrow \mathcal{EBim}_{\hat{A}_{r-1}}^*$$

which extends the one by Elias and Khovanov. On objects, it is defined as follows: it maps each integer $i \in \{1, \dots, r\}$ to $B_i\{-1\}$, and the symbols $+$ and $-$ to B_ρ and $B_{\rho^{-1}}$ respectively. Sequences of these are mapped to tensor products. The empty sequence is sent to R .

For the morphisms one only needs to specify \mathcal{F} on the generators. A sequence of vertical strands is mapped to the identity of the corresponding bimodule and box_i (resp. box_y) is mapped to multiplication by x_i (resp. y). Furthermore, we define (recall that $X_i = x_{i+1} - x_i$, for $i = 1, \dots, r-1$, and $X_r = x_1 - x_r - y$)

$$\begin{aligned}
\mathcal{F}(\text{enddot}_i) &= \text{br}_i : a \otimes b \mapsto ab \\
\mathcal{F}(\text{startdot}_i) &= \text{rb}_i : a \mapsto \frac{a}{2} (X_i \otimes 1 + 1 \otimes X_i) \\
\mathcal{F}(\text{merge}_i) &= \text{pr}_i : \begin{cases} a \otimes 1 \otimes b & \mapsto 0 \\ a \otimes X_i \otimes b & \mapsto 2a \otimes b \end{cases} \\
\mathcal{F}(\text{split}_i) &= \text{inj}_i : a \otimes b \mapsto a \otimes 1 \otimes b \\
\mathcal{F}(4\text{vert}_{i,j}) &= \text{f}_{i,j} : a \otimes 1 \otimes b \mapsto a \otimes 1 \otimes b \\
\mathcal{F}(6\text{vert}_{i,i+1}) &= \text{f}_{i,i+1} : \begin{cases} a \otimes 1 \otimes 1 \otimes b & \mapsto a \otimes 1 \otimes 1 \otimes b \\ a \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes b & \mapsto 0 \end{cases} \\
\mathcal{F}(6\text{vert}_{i+1,i}) &= \text{f}_{i+1,i} : \begin{cases} a \otimes 1 \otimes 1 \otimes b & \mapsto a \otimes 1 \otimes 1 \otimes b \\ a \otimes (X_i \otimes 1 + 1 \otimes X_i) \otimes b & \mapsto 0 \end{cases} \\
\mathcal{F}(+\text{cap}) &= +, -r : a \otimes b \mapsto a\rho(b) \\
\mathcal{F}(-\text{cap}) &= -, +r : a \otimes b \mapsto a\rho^{-1}(b) \\
\mathcal{F}(-\text{cup}) &= r-, + : a \mapsto a \otimes 1 \\
\mathcal{F}(+\text{cup}) &= r+, - : a \mapsto a \otimes 1 \\
\mathcal{F}(4\text{vert}_{+,i}) &= \text{fl}_{+,i} : a \otimes b \mapsto a \otimes \rho(b) \\
\mathcal{F}(4\text{vert}_{i+1,+}) &= \text{fl}_{i+1,+} : a \otimes b \mapsto a \otimes \rho^{-1}(b) \\
\mathcal{F}(4\text{vert}_{i,-}) &= \text{fl}_{i,-} : a \otimes b \mapsto a \otimes \rho(b) \\
\mathcal{F}(4\text{vert}_{-,i+1}) &= \text{fl}_{-,i+1} : a \otimes b \mapsto a \otimes \rho^{-1}(b) \\
\mathcal{F}(\text{box}_i) &= \text{m}_i : a \mapsto ax_i \\
\mathcal{F}(\text{box}_y) &= \text{m}_y : a \mapsto ay
\end{aligned}$$

Proposition 2.10. *The functor \mathcal{F} is well-defined, degree preserving and essentially surjective.*

Proof. The fact that the functor \mathcal{F} is well-defined and degree preserving amounts to a straightforward verification that it preserves the relations (2.13)–(2.58) and the degrees of the morphisms. For the relations involving only non-oriented strands, this is completely analogous to Elias and Khovanov’s case. For the relations involving oriented strands, the calculations are new but easy.

Furthermore, in view of the definitions of the objects of $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$ and $\mathcal{EBim}_{\hat{A}_{r-1}}^*$, the functor \mathcal{F} is clearly essentially surjective. \square

Proposition 2.11. *The functor \mathcal{F} is full.*

Proof. Libedinsky has proved in [Lib08b] that all the morphisms of the category $\mathcal{Bim}_{\hat{A}_{r-1}}^*$ are generated by the following ones:

- $\text{br}_i, \text{rb}_i, \text{pr}_i$ and inj_i for all $i = 1, \dots, r$
- $\text{f}_{i,j}$ for all $i, j = 1, \dots, r$ with $i \neq j$.

In view of Remark 2.6, this implies that all the morphisms of the category $\mathcal{EBim}_{\hat{A}_{r-1}}^*$ are generated by the ones listed by Libedinsky and copied above, together with $\text{fl}_{+,i}$ and $\text{fl}_{i+1,+}$, giving

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho \quad \text{for } i = 1, \dots, r,$$

and $+, -r, -, +r, r-, +$ and $r+, -$, giving

$$B_\rho \otimes_R B_{\rho-1} \cong R \cong B_{\rho-1} \otimes_R B_\rho.$$

Thus the functor \mathcal{F} is full, since all the morphisms generating $\mathcal{EBim}_{\hat{A}_{r-1}}^*$ are in the image of \mathcal{F} . \square

Theorem 2.12. *The categories $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$ and $\mathcal{EBim}_{\hat{A}_{r-1}}^*$ are equivalent and so are their Karoubi envelopes $\text{Kar}\mathcal{DEBim}_{\hat{A}_{r-1}}^*$ and $\text{Kar}\mathcal{EBim}_{\hat{A}_{r-1}}^*$.*

Proof. Only the faithfulness of \mathcal{F} remains to be proved.

For a given object X in $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$, let k_X denote the total sum of plus and minus signs in X . Given two objects X, Y in $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$, the hom-space between X and Y is non-zero only if $k_X = k_Y$ (see Remark 2.6). Any object X in $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$ is isomorphic to the object $(\text{sign}(k_X)^{|k_X|}, X')$, where X' does not have any signs and is obtained from X by applying the commutation isomorphisms $(\pm, i) \cong (i \pm 1, \pm)$ and the isomorphisms $(\pm, \mp) \cong \emptyset$.

Let D be a diagram representing a morphism from X to Y , such that $k_X = k_Y$. Since

$$\text{End}(\text{sign}(k_X)^{|k_X|}) \cong \mathbb{Q},$$

it follows that

$$\text{Hom}_{\mathcal{DEBim}_{\hat{A}_{r-1}}^*}(X, Y) \cong \text{Hom}_{\mathcal{DEBim}_{\hat{A}_{r-1}}^*}(X', Y').$$

Let us illustrate this by the example in Figure 1.

Since

$$\text{Hom}_{\mathcal{DEBim}_{\hat{A}_{r-1}}^*}(X', Y') \cong \text{Hom}_{\mathcal{EBim}_{\hat{A}_{r-1}}^*}(X', Y'),$$

the faithfulness of \mathcal{F} follows from the faithfulness of Elias and Williamson's analogous functor for non-extended affine type A , which they proved in Theorem 6.28 of [EW13]. \square

3. THE AFFINE SCHUR QUOTIENT

3.1. Notations. Let n, r be integers, with $n > r$ and $r \geq 3$, and let q be a formal parameter.

A generic \mathfrak{gl}_n -weight will be denoted $\lambda = (\lambda_1, \dots, \lambda_n)$, and we set $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$. Define

$$\Lambda(n, r) = \{\lambda \in \mathbb{N}^n : \sum_{i=1}^n \lambda_i = r\}$$

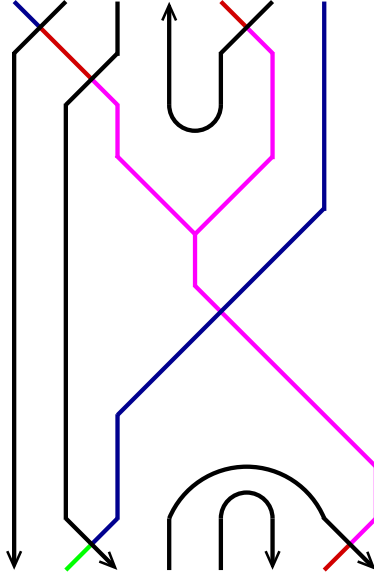


FIGURE 1. Example of a decomposition of a diagram D

and

$$\Lambda^+(n, r) = \{\lambda \in \Lambda(n, r) : r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

We will denote by (1^r) the weight $\varepsilon_1 + \cdots + \varepsilon_r$.

Let us formally denote by $\bar{\alpha}_n$ the opposite of the highest root of \mathfrak{sl}_n , i.e. $\bar{\alpha}_n = -\bar{\theta} = -\bar{\alpha}_1 - \cdots - \bar{\alpha}_{n-1} = \varepsilon_n - \varepsilon_1$.

As always in this paper, we use the convention that the indices appearing in the relations are considered modulo n .

3.2. The (extended) affine algebras. In the following definitions we do not need to consider the derivation, which we used above.

Definition 3.1. The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$ -algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \dots, n$, subject to the relations

$$K_i K_j = K_j K_i \quad K_i K_i^{-1} = K_i^{-1} K_i = 1 \quad (3.1)$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}} \quad (3.2)$$

$$K_i E_{\pm j} = q^{\pm \langle \varepsilon_i, \bar{\alpha}_j \rangle} E_{\pm j} K_i \quad (3.3)$$

$$E_{\pm i}^2 E_{\pm(i\pm 1)} - (q + q^{-1}) E_{\pm i} E_{\pm(i\pm 1)} E_{\pm i} + E_{\pm(i\pm 1)} E_{\pm i}^2 = 0 \quad (3.4)$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0 \quad \text{for distant } i, j \quad (3.5)$$

$$R R^{-1} = R^{-1} R = 1 \quad (3.6)$$

$$R X_i R^{-1} = X_{i+1} \quad \text{for } X_i \in \{E_{\pm i}, K_i^{\pm 1}\}. \quad (3.7)$$

Definition 3.2. The *affine quantum general linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$ -subalgebra generated by $E_{\pm i}$ and $K_i^{\pm 1}$, for $i = 1, \dots, n$.

The *affine quantum special linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subseteq \mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$ -subalgebra generated by $E_{\pm i}$ and $K_i K_{i+1}^{-1}$, for $i = 1, \dots, n$.

We will also need the bialgebra structure on $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$.

Definition 3.3. $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is a bialgebra with counit $\varepsilon: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \mathbb{Q}(q)$ defined by

$$\varepsilon(E_{\pm i}) = 0, \quad \varepsilon(R^{\pm 1}) = \varepsilon(K_i^{\pm 1}) = 1$$

and coproduct $\Delta: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \otimes \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, defined by

$$\Delta(1) = 1 \otimes 1 \tag{3.8}$$

$$\Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i \tag{3.9}$$

$$\Delta(E_{-i}) = K_i^{-1} K_{i+1} \otimes E_{-i} + E_{-i} \otimes 1 \tag{3.10}$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \tag{3.11}$$

$$\Delta(R^{\pm 1}) = R^{\pm 1} \otimes R^{\pm 1}. \tag{3.12}$$

As a matter of fact, $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is even a Hopf algebra, but we do not need the antipode in this paper. Note that Δ and ε can be restricted to $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$, which are bialgebras too.

At level 0 and forgetting the derivation, we can work with the $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ -weight lattice, when considering $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ -weight representations. Omitting the extra entry corresponding to the derivation makes this weight-lattice degenerate, because $\alpha_1 + \alpha_2 + \dots + \alpha_n = 0$, but that does not matter in this section. Similarly, we can work with the $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ -weight lattice, when considering $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ -weight representations.

Suppose that V is a $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ -weight representation with weights $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_{\lambda} V_{\lambda}$$

and K_i acts as multiplication by q^{λ_i} on V_{λ} . Then V is also a $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ -weight representation with weights $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\bar{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j = 1, \dots, n-1$.

Conversely, there is not a unique choice of $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ -action on a given $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ -weight representation with weights $\mu = (\mu_1, \dots, \mu_{n-1})$. We first have to fix the action of $K_1 \cdots K_n$. In terms of weights, this corresponds to the observation that, for any given $r \in \mathbb{Z}$ the equations

$$\lambda_i - \lambda_{i+1} = \mu_i \tag{3.13}$$

$$\sum_{i=1}^n \lambda_i = r \tag{3.14}$$

determine $\lambda = (\lambda_1, \dots, \lambda_n)$ uniquely, if there exists a solution to (3.13) and (3.14) at all. We therefore define the map $\varphi_{n,r}: \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^n \cup \{*\}$ by

$$\varphi_{n,r}(\mu) = \lambda$$

if (3.13) and (3.14) have a solution, and put $\varphi_{n,r}(\mu) = *$ otherwise.

As far as weight representations are concerned, we can restrict our attention to the Beilinson-Lusztig-MacPherson idempotent version of these

quantum groups, denoted $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ and $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ respectively. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_λ to $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\lambda \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \alpha_i} E_{\pm i} \\ K_i 1_\lambda &= q^{\lambda_i} 1_\lambda \\ R 1_{(\lambda_1, \dots, \lambda_n)} &= 1_{(\lambda_n, \lambda_1, \dots, \lambda_{n-1})} R. \end{aligned}$$

Definition 3.4. The *idempotent extended affine quantum general linear algebra* is defined by

$$\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} 1_\lambda \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) 1_\mu.$$

Of course one defines $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ as the idempotent subalgebra generated by 1_λ and $E_{\pm i} 1_\lambda$, for $i = 1, \dots, n$ and $\lambda \in \mathbb{Z}^n$.

Similarly for $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$, adjoin an idempotent 1_λ for each $\lambda \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\lambda \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \alpha_i} E_{\pm i} \\ K_i K_{i+1}^{-1} 1_\lambda &= q^{\lambda_i} 1_\lambda. \end{aligned}$$

Definition 3.5. The *idempotent quantum special linear algebra* is defined by

$$\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^{n-1}} 1_\lambda \mathbf{U}_q(\widehat{\mathfrak{sl}}_n) 1_\mu.$$

Any weight-representation of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ or $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ is also a representation of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ or $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, respectively. This is not true for non-weight representations, of which there are many. There are also other differences of course, e.g. $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ and $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ are not unital, because they have infinitely many idempotents. For that same reason, they are not bialgebras, although their action on tensor products of weight representations is well-defined.

3.3. The affine q -Schur algebra. Let us first recall Green's [Gre99, DG07] tensor space and the action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on it. We will also recall some basic results about this action and add some of our own. Wherever we omit a proof in this section, the corresponding result was taken from [DG07]. When we give a proof, it is because the corresponding result cannot be found in the literature and we had to prove it ourselves, e.g. the inner product on tensor space is probably known to experts, but there seems to be no written reference.

Let V be the $\mathbb{Q}(q)$ -vector space freely generated by $\{e_t \mid t \in \mathbb{Z}\}$.

Definition 3.6. The following defines an action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on V

$$E_i e_{t+1} = e_t \quad \text{if } i \equiv t \pmod{n} \quad (3.15)$$

$$E_i e_{t+1} = 0 \quad \text{if } i \not\equiv t \pmod{n} \quad (3.16)$$

$$E_{-i} e_t = e_{t+1} \quad \text{if } i \equiv t \pmod{n} \quad (3.17)$$

$$E_{-i} e_t = 0 \quad \text{if } i \not\equiv t \pmod{n} \quad (3.18)$$

$$K_i^{\pm 1} e_t = q^{\pm 1} e_t \quad \text{if } i \equiv t \pmod{n} \quad (3.19)$$

$$K_i^{\pm 1} e_t = e_t \quad \text{if } i \not\equiv t \pmod{n} \quad (3.20)$$

$$R^{\pm 1} e_t = e_{t \pm 1} \quad \text{for all } t \in \mathbb{Z}. \quad (3.21)$$

Note that V is clearly a weight representation of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, with e_t having weight ε_i , for $i \equiv t \pmod{n}$. Therefore V is also a representation of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$.

From now on, let $r \in \mathbb{N}_{>0}$ be arbitrary but fixed. As usual, one extends the above action to $V^{\otimes r}$, using the coproduct in $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Again, this is a weight representation and therefore a representation of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, which we call Green's *tensor space*.

We also define a $\mathbb{Q}(q)$ -bilinear form on V by $\langle e_s, e_t \rangle = \delta_{st}$, which extends to $V^{\otimes r}$ factorwise, i.e.

$$\langle v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r \rangle := \langle v_1, w_1 \rangle \cdots \langle v_r, w_r \rangle.$$

Lemma 3.7. For any $v \in V^{\otimes r}$, we have

$$\langle v, v \rangle \neq 0.$$

Proof. We can write v uniquely as $\sum_{\underline{t} \in T} v_{\underline{t}} e_{\underline{t}}$, such that T is a finite subset of \mathbb{Z}^r and for any $\underline{t} = (t_1, \dots, t_r) \in T$ we have

$$v_{\underline{t}} \in \mathbb{Q}(q) \quad \text{and} \quad e_{\underline{t}} = e_{t_1} \otimes \cdots \otimes e_{t_r}.$$

Thus we get

$$\langle v, v \rangle = \sum_{\underline{t} \in T} v_{\underline{t}}^2.$$

For each $\underline{t} \in T$, write $v_{\underline{t}} = f_{\underline{t}}(q)/g_{\underline{t}}(q)$, where $f_{\underline{t}}(q), g_{\underline{t}}(q) \in \mathbb{Q}[q]$ have g.c.d. equal to 1. Choose $q_0 \in \mathbb{Q}$ such that $f_{\underline{t}}(q_0) \neq 0, g_{\underline{t}}(q_0) \neq 0$ for all $\underline{t} \in T$ (such a number exists, because T is finite and each polynomial has only finitely many roots). Then $v_{\underline{t}}(q_0) = f_{\underline{t}}(q_0)/g_{\underline{t}}(q_0) \in \mathbb{Q}^*$ and we have

$$\sum_{\underline{t} \in T} v_{\underline{t}}(q_0)^2 > 0.$$

This implies that

$$\sum_{\underline{t} \in T} v_{\underline{t}}^2 \neq 0.$$

□

There is a right action of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ on $V^{\otimes r}$ which commutes with the left action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Its precise definition, which can be found in [Gre99, DG07], is not relevant here.

Definition 3.8. The affine q -Schur algebra $\widehat{\mathbf{S}}(n, r)$ is by definition the centralizing algebra

$$\text{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}}(V^{\otimes r}).$$

By affine Schur-Weyl duality, the image of $\psi_{n,r}: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \text{End}(V^{\otimes r})$ is always isomorphic to $\widehat{\mathbf{S}}(n, r)$. If $n > r$, we can even restrict to $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subset \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, i.e.

$$\psi_{n,r}(\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)) \cong \widehat{\mathbf{S}}(n, r).$$

For $n = r$, this is no longer true.

Definition 3.9. Let $\rho: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ be the $\mathbb{Q}(q)$ -linear algebra anti-involution defined by

$$\rho(E_i) = qK_i K_{i+1}^{-1} E_{-i} \quad \rho(E_{-i}) = qK_i^{-1} K_{i+1} E_i \quad \rho(K_i) = K_i, \quad \rho(R) = R^{-1},$$

for $1 \leq i \leq n$.

The proof of the following lemma is a straightforward check, which we leave to the reader.

Lemma 3.10. *We have*

$$\Delta \rho = (\rho \otimes \rho) \Delta.$$

Lemma 3.11. *For any $X \in \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and any $v, w \in V^{\otimes r}$, we have*

$$\langle Xv, w \rangle = \langle v, \rho(X)w \rangle.$$

Proof. By Lemma 3.10, it suffices to check the above for $r = 1$ and $v = e_i$ and $w = e_j$, for any $i, j \in \mathbb{Z}$. This is straightforward and left to the reader. \square

Note that ρ can also be defined on $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, such that $\rho(1_\lambda) = 1_\lambda$ for any $\lambda \in \mathbb{Z}^r$, and that it descends to $\widehat{\mathbf{S}}(n, r)$.

3.4. A presentation of $\widehat{\mathbf{S}}(n, r)$ for $n > r$. In this subsection, let $n > r$. Recall that in this case

$$\psi_{n,r}: \dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \rightarrow \text{End}(V^{\otimes r}) \rightarrow \widehat{\mathbf{S}}(n, r)$$

is surjective. This gives rise to the following presentation of $\widehat{\mathbf{S}}(n, r)$. The proof can be found in [DG07].

Theorem 3.12. [DG07] *$\widehat{\mathbf{S}}(n, r)$ is isomorphic to the associative unital $\mathbb{Q}(q)$ -algebra generated by 1_λ , for $\lambda \in \Lambda(n, r)$, and $E_{\pm i}$, for $i = 1, \dots, n$, subject to the relations*

$$1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda \tag{3.22}$$

$$\sum_{\lambda \in \Lambda(n, r)} 1_\lambda = 1 \tag{3.23}$$

$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \bar{\alpha}_i} E_{\pm i} \tag{3.24}$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, r)} [\lambda_i - \lambda_{i+1}] 1_\lambda \tag{3.25}$$

$$E_{\pm i}^2 E_{\pm(i \pm 1)} - (q + q^{-1}) E_{\pm i} E_{\pm(i \pm 1)} E_{\pm i} + E_{\pm(i \pm 1)} E_{\pm i}^2 = 0 \tag{3.26}$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0 \quad \text{for distant } i, j. \tag{3.27}$$

We use the convention that $1_\mu X 1_\lambda = 0$ whenever μ or λ is not contained in $\Lambda(n, r)$. Recall that $[a]$ is the q -integer $(q^a - q^{-a})/(q - q^{-1})$.

We will use signed sequences $\underline{i} = (\mu_1 i_1, \dots, \mu_m i_m)$, with $m \in \mathbb{N}$, $\mu_j \in \{\pm 1\}$ and $i_j \in \{1, \dots, n\}$. The set of signed sequences we denote SSeq . For a signed sequence $\underline{i} = (\mu_1 i_1, \dots, \mu_m i_m)$ we define

$$\underline{i}_\Lambda := \mu_1 \bar{\alpha}_{i_1} + \dots + \mu_m \bar{\alpha}_{i_m}.$$

We write $E_{\underline{i}}$ for the product $E_{\mu_1 i_1} \dots E_{\mu_m i_m}$. For any $\lambda \in \mathbb{Z}^n$ and $\underline{i} \in \text{SSeq}$, we have

$$E_{\underline{i}} 1_\lambda = 1_{\lambda + \underline{i}_\Lambda} E_{\underline{i}}.$$

The surjection $\psi_{n,r}: \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \rightarrow \widehat{\mathbf{S}}(n, r)$ can also be given explicitly in terms of the generators in Theorem 3.12. For any $\lambda \in \mathbb{Z}^{n-1}$, we have

$$\psi_{n,r}(E_{\pm i} 1_\lambda) = E_{\pm i} 1_{\varphi_{n,r}(\lambda)}, \quad (3.28)$$

where $\varphi_{n,r}: \mathbb{Z}^{n-1} \rightarrow \Lambda(n, r) \cup \{*\}$ is the map defined in 3.2. By convention, we put $1_* = 0$.

Recall the definition of ρ in 3.9. Since ρ is an algebra anti-homomorphism, we get

$$\rho: 1_\lambda \widehat{\mathbf{S}}(n, r) 1_\mu \rightarrow 1_\mu \widehat{\mathbf{S}}(n, r) 1_\lambda,$$

for any $\lambda, \mu \in \Lambda(n, r)$.

Lemma 3.13. *For any $\lambda, \mu \in \Lambda(n, r)$ and any non-zero element $1_\lambda X 1_\mu \in \widehat{\mathbf{S}}(n, r)$, we have*

$$1_\lambda X \rho(X) 1_\lambda \neq 0 \quad \text{and} \quad 1_\mu \rho(X) X 1_\mu \neq 0.$$

Proof. By the surjectivity of $\psi_{n,r}$, we know that there exist $e_{t_1}, \dots, e_{t_r} \in V$ such that

$$X 1_\mu (e_{t_1} \otimes \dots \otimes e_{t_r}) \neq 0.$$

By Lemmas 3.11 and 3.7, we get

$$\begin{aligned} \langle e_{t_1} \otimes \dots \otimes e_{t_r}, 1_\mu \rho(X) X 1_\mu (e_{t_1} \otimes \dots \otimes e_{t_r}) \rangle &= \\ \langle X 1_\mu (e_{t_1} \otimes \dots \otimes e_{t_r}), X 1_\mu (e_{t_1} \otimes \dots \otimes e_{t_r}) \rangle &\neq 0. \end{aligned}$$

Therefore, we see that

$$\psi_{n,r}(1_\mu \rho(X) X 1_\mu) \neq 0.$$

The other case follows automatically, because $X \rho(X) = \rho^2(X) \rho(X) = \rho(Y) Y$ for $Y = \rho(X)$. \square

We can also give an explicit formula for the well-known embedding (see [DG07]) of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ into $\widehat{\mathbf{S}}(n, r)$. Let $1_r = 1_{(1^r)}$. We define the following map

$$\sigma_{n,r}: \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \rightarrow 1_r \widehat{\mathbf{S}}(n, r) 1_r$$

by

$$\sigma_{n,r}(b_i) = 1_r E_{-i} E_i 1_r = 1_r E_i E_{-i} 1_r,$$

for $i = 1, \dots, r-1$,

$$\sigma_{n,r}(b_r) = 1_r E_{-n} \dots E_{-r} E_r \dots E_n 1_r,$$

$$\begin{aligned}\sigma_{n,r}(T_\rho) &= 1_r E_{-n} \dots E_{-r-1} E_{-1} \dots E_{-r} 1_r \\ & (= 1_r E_{-n} E_{-1} \dots E_{-r+1} E_{-n+1} \dots E_{-r} 1_r)\end{aligned}$$

and

$$\begin{aligned}\sigma_{n,r}(T_{\rho^{-1}}) &= 1_r E_r \dots E_1 E_{r+1} \dots E_n 1_r \\ & (= 1_r E_r \dots E_{n-1} E_{r-1} \dots E_1 E_n 1_r).\end{aligned}$$

It is easy to check that $\sigma_{n,r}$ is well-defined. It turns out that $\sigma_{n,r}$ is actually an isomorphism, which induces the affine q -Schur functor

$$\widehat{\mathbf{S}}(n, r) - \text{mod} \rightarrow \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} - \text{mod}$$

between the categories of finite-dimensional modules of the extended affine Hecke algebra and of the affine q -Schur algebra. This functor is an equivalence (see Theorem 4.1.3 in [DDF12], for example).

The following result will be needed in Section 6.

Proposition 3.14. *Let $n > r$. Suppose that A is a $\mathbb{Q}(q)$ -algebra and*

$$f: \widehat{\mathbf{S}}(n, r) \rightarrow A$$

is a surjective $\mathbb{Q}(q)$ -algebra homomorphism which is an embedding when restricted to $1_r \widehat{\mathbf{S}}(n, r) 1_r \cong \widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$. Then f is a $\mathbb{Q}(q)$ -algebra isomorphism

$$A \cong \widehat{\mathbf{S}}(n, r).$$

Proof. We first prove that f is an embedding when restricted to $1_r \widehat{\mathbf{S}}(n, r) 1_\lambda$ and $1_\lambda \widehat{\mathbf{S}}(n, r) 1_r$, for any $\lambda \in \Lambda(n, r)$. Suppose that this is not true in the first case, then there exists a non-zero element $1_r X 1_\lambda \in 1_r \widehat{\mathbf{S}}(n, r) 1_\lambda$ in the kernel of f . By Lemma 3.13, we have $1_r X \rho(X) 1_r \neq 0$. However, we have

$$f(1_r X \rho(X) 1_r) = f(1_r X 1_\lambda) f(1_\lambda \rho(X) 1_r) = 0,$$

which leads to a contradiction, because by hypothesis f is an embedding when restricted to $1_r \widehat{\mathbf{S}}(n, r) 1_r$. The second case can be proved similarly.

Now, for any $\lambda, \mu \in \Lambda(n, r)$, let $1_\lambda X 1_\mu \in \widehat{\mathbf{S}}(n, r)$ be an arbitrary non-zero element. By Schur-Weyl duality, we have

$$1_\lambda \widehat{\mathbf{S}}(n, r) 1_\mu \cong \text{Hom}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}} (1_\mu \widehat{\mathbf{S}}(n, r) 1_r, 1_\lambda \widehat{\mathbf{S}}(n, r) 1_r),$$

where the isomorphism is induced by left composition. Therefore, there exists an element $1_\mu Y 1_r$ such that

$$1_\lambda X Y 1_r \neq 0.$$

By the above, we have

$$f(1_\lambda X 1_\mu) f(1_\mu Y 1_r) = f(1_\lambda X Y 1_r) \neq 0,$$

so

$$f(1_\lambda X 1_\mu) \neq 0.$$

This shows that f is an embedding when restricted to $1_\lambda \widehat{\mathbf{S}}(n, r) 1_\mu$. Since λ and μ were arbitrary, this shows that f is an isomorphism. \square

Let us end this section giving an embedding between affine q -Schur algebras which we will use in Section 5.

Proposition 3.15. *The $\mathbb{Q}(q)$ -linear algebra homomorphism*

$$\iota_n: \widehat{\mathbf{S}}(n, r) \rightarrow \widehat{\mathbf{S}}(n+1, r)$$

defined by

$$1_\lambda \mapsto 1_{(\lambda, 0)} \quad (3.29)$$

$$E_{\pm i} 1_\lambda \mapsto E_{\pm i} 1_{(\lambda, 0)} \quad (3.30)$$

$$E_n 1_\lambda \mapsto E_n E_{n+1} 1_{(\lambda, 0)} \quad (3.31)$$

$$E_{-n} 1_\lambda \mapsto E_{-(n+1)} E_{-n} 1_{(\lambda, 0)}, \quad (3.32)$$

for any $1 \leq i \leq n-1$ and $\lambda \in \Lambda(n, r)$, is an embedding and gives an isomorphism of algebras

$$\widehat{\mathbf{S}}(n, r) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, r)} 1_{(\lambda, 0)} \widehat{\mathbf{S}}(n+1, r) 1_{(\mu, 0)}.$$

3.5. The 2-categories $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}$. In this section we define three 2-categories, $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}$, using a graphical calculus analogous to Khovanov and Lauda's in [KL10].

3.5.1. The 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. The 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined just as $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ in [KL10], but the 2-HOM-spaces are tensored with $\mathbb{Q}[y]$, where y is a formal variable of degree two, and some of the KL-relations are y -deformed, as shown below. In order to define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, change the weights in the definition of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ into (degenerate) level-zero $\widehat{\mathfrak{gl}}_n$ -weights (i.e. \mathfrak{gl}_n -weights). The 2-category $\widehat{\mathcal{S}}(n, r)_{[y]}$ is then defined as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. This is precisely the affine analogue of what was done in [MSV13].

Remark 3.16. We use the sign conventions from [MSV13] in the relations on 2-morphisms, which differ from Khovanov and Lauda's sign conventions. For more details on this change of convention, see below.

Remark 3.17. We do not prove that the 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ provides a categorification of $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$, although we conjecture that it does. We will prove that $\widehat{\mathcal{S}}(n, r)_{[y]}$ categorifies $\widehat{\mathbf{S}}(n, r)$.

In order to avoid giving several long definitions which are very similar, we only define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ in detail. The 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined exactly as $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, but using degenerate level-zero $\widehat{\mathfrak{sl}}_n$ -weights. The 2-category $\widehat{\mathcal{S}}(n, r)_{[y]}$ is defined as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. We will show that $\widehat{\mathcal{S}}(n, r)_{[y]}$ can also be obtained as a quotient of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$.

To be more precise, we first define the $\mathbb{Q}[y]$ -linear graded 2-category with translation $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$, whose 2-morphisms are $\mathbb{Q}[y]$ -linear combinations of homogeneous 2-morphisms of various degrees. The \mathbb{Q} -linear 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ is then obtained by restricting to the degree-zero 2-morphisms.

Definition 3.18. The additive $\mathbb{Q}[y]$ -linear graded 2-category with translation $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$ consists of

- objects: $\lambda \in \mathbb{Z}^n$.

The hom-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$ between two objects λ, λ' is an additive $\mathbb{Q}[y]$ -linear graded category with translation defined by:

- objects² of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$. A 1-morphism in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$ from λ to λ' is a formal finite direct sum of 1-morphisms

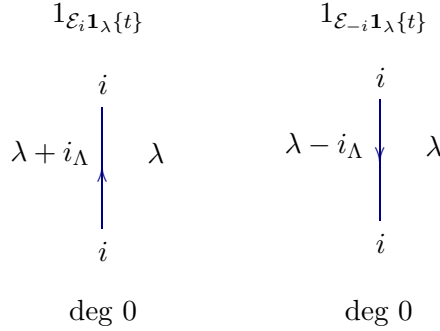
$$\mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda} \{t\} = \mathbf{1}_{\lambda'} \mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda} \{t\} := \mathcal{E}_{\mu_1 i_1} \cdots \mathcal{E}_{\mu_m i_m} \mathbf{1}_{\lambda} \{t\}$$

for any $t \in \mathbb{Z}$ and signed sequence $\underline{i} \in \text{SSeq}$ such that $\lambda' = \lambda + \underline{i}_{\Lambda}$ and $\lambda' \in \mathbb{Z}^n$.

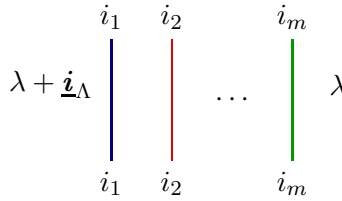
The morphisms of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$ are presented by generators and relations. Multiplication by y is indicated graphically by \boxed{y} in the diagrams below.

- generators. For 1-morphisms $\mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda} \{t\}$ and $\mathcal{E}_{\underline{j}} \mathbf{1}_{\lambda} \{t'\}$ in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$, the hom-sets $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda} \{t\}, \mathcal{E}_{\underline{j}} \mathbf{1}_{\lambda} \{t'\})$ of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$ are graded $\mathbb{Q}[y]$ -vector spaces given by linear combinations of diagrams of homogeneous degrees, modulo certain relations, built from composites of:

- Degree-zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$; in particular for any $i \in \{1, \dots, n\}$, $t \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}^n$, the identity 2-morphisms $1_{\mathcal{E}_i \mathbf{1}_{\lambda}} \{t\}$ and $1_{\mathcal{E}_{-i} \mathbf{1}_{\lambda}} \{t\}$ are represented graphically by





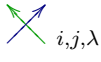
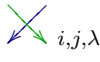
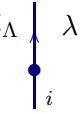
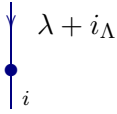
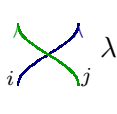
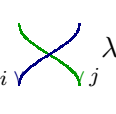
More generally, for a signed sequence $\underline{i} = (\mu_1 i_1, \mu_2 i_2, \dots, \mu_m i_m)$, the identity $1_{\mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda}} \{t\}$ 2-morphism is represented as






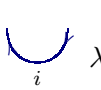
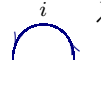
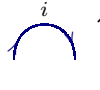


where the strand labeled i_k is oriented up if $\mu_k = +$ and oriented down if $\mu_k = -$. We will often place labels on the side of a strand and omit the labels at the top and bottom. The signed sequence can be recovered from the labels and the orientations on the strands. We might also forget the object on the left of the diagram which can be recovered from the object on the right and the signed sequence corresponding to the diagram.

²We refer to objects of the category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$ as 1-morphisms of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$. Likewise, the morphisms of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*(\lambda, \lambda')$ are called 2-morphisms in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$.

ii) For any $\lambda \in \mathbb{Z}^n$ the 2-morphisms

Notation:				
2-morphism:				
Degree:	$i \cdot i$	$i \cdot i$	$-i \cdot j$	$-i \cdot j$

Notation:				
2-morphism:				
Degree:	$1 + \bar{\lambda}_i$	$1 - \bar{\lambda}_i$	$1 + \bar{\lambda}_i$	$1 - \bar{\lambda}_i$

where the degrees are given by the symmetric \mathbb{Z} -valued bilinear form on $\mathbb{C}\{1, \dots, n\}$

$$i \cdot j = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i \equiv j \pm 1 \pmod{n} \\ 0 & \text{if } i \not\equiv j \pm 1 \pmod{n}. \end{cases}$$

• relations:

★ Biadjointness and cyclicity:

i) $\mathbf{1}_{\lambda+i_\Lambda} \mathcal{E}_i \mathbf{1}_\lambda$ and $\mathbf{1}_\lambda \mathcal{E}_{-i} \mathbf{1}_{\lambda+i_\Lambda}$ are biadjoint, up to grading shifts:

$$\begin{array}{c} \lambda + i_\Lambda \\ \downarrow \\ i \end{array} \begin{array}{c} \uparrow \\ \lambda \end{array} = \begin{array}{c} \lambda + i_\Lambda \\ \downarrow \\ i \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ \lambda + i_\Lambda \end{array} \begin{array}{c} \uparrow \\ i \end{array} \quad (3.33)$$

$$\begin{array}{c} \lambda \\ \downarrow \\ i \end{array} \begin{array}{c} \uparrow \\ \lambda + i_\Lambda \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ i \end{array} = \begin{array}{c} \lambda + i_\Lambda \\ \downarrow \\ \lambda \end{array} \begin{array}{c} \uparrow \\ i \end{array} \quad (3.34)$$

ii) As well for dotted lines:

$$\begin{array}{c} \lambda \\ \downarrow \\ i \end{array} \begin{array}{c} \uparrow \\ \lambda + i_\Lambda \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ i \end{array} = \begin{array}{c} \lambda + i_\Lambda \\ \downarrow \\ \lambda \end{array} \begin{array}{c} \uparrow \\ i \end{array} \quad (3.35)$$

- iii) All 2-morphisms are cyclic with respect to the above biadjoint structure. This is ensured by the relations (3.33)–(3.35), and the relations

$$(3.36)$$

The cyclic condition on 2-morphisms expressed by (3.33)–(3.36) ensures that diagrams related by isotopy represent the same 2-morphism in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$.

It will be convenient to introduce degree zero 2-morphisms:

$$(3.37)$$

$$(3.38)$$

where the second equality in (3.37) and (3.38) follow from (3.36).

★ Bubble relations:

- i) All dotted bubbles of negative degree are zero. That is,

$$(3.39)$$

for all $m \in \mathbb{Z}_+$, where a dot carrying a label m denotes the m -fold iterated vertical composite of $\uparrow_{i,\lambda}$ or $\downarrow_{i,\lambda}$ depending on the orientation.

- ii) A dotted bubble of degree zero equals ± 1 :

$$(3.40)$$

- iii) For the following relations we employ the convention that all summations are increasing, so that a summation of the form $\sum_{f=0}^m$ is zero if $m < 0$.

$$(3.41)$$

$$\begin{array}{c} \lambda \\ \uparrow \\ i \end{array} \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} = \begin{array}{c} \lambda \\ \text{X} \\ i \end{array} - \sum_{f=0}^{\bar{\lambda}_i-1} \sum_{g=0}^f \begin{array}{c} \lambda \\ \text{bubble} \\ i \end{array} \quad (3.42)$$

$$\begin{array}{c} \lambda \\ \downarrow \\ i \end{array} \begin{array}{c} \lambda \\ \downarrow \\ i \end{array} = \begin{array}{c} \lambda \\ \text{X} \\ i \end{array} - \sum_{f=0}^{-\bar{\lambda}_i-1} \sum_{g=0}^f \begin{array}{c} \lambda \\ \text{bubble} \\ i \end{array} \quad (3.43)$$

iv) Fake bubbles:

Notice that for some values of λ the dotted bubbles appearing above have negative labels. A composite of $\begin{array}{c} \uparrow \\ i, \lambda \end{array}$ or $\begin{array}{c} \downarrow \\ i, \lambda \end{array}$ with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called fake bubbles, are formal symbols inductively defined by the equation

$$\left(\begin{array}{c} \text{bubble} \\ i, -\bar{\lambda}_i-1 \end{array} + \begin{array}{c} \text{bubble} \\ i, -\bar{\lambda}_i-1+1 \end{array} t + \cdots + \begin{array}{c} \text{bubble} \\ i, -\bar{\lambda}_i-1+r \end{array} t^r + \cdots \right) \times \\
\left(\begin{array}{c} \text{bubble} \\ i, \bar{\lambda}_i-1 \end{array} + \begin{array}{c} \text{bubble} \\ i, \bar{\lambda}_i-1+1 \end{array} t + \cdots + \begin{array}{c} \text{bubble} \\ i, \bar{\lambda}_i-1+r \end{array} t^r + \cdots \right) = -1 \quad (3.44)$$

and the additional condition

$$\begin{array}{c} \text{bubble} \\ i, -1 \end{array} = (-1)^{\lambda_{i+1}}, \quad \begin{array}{c} \text{bubble} \\ i, -1 \end{array} = (-1)^{\lambda_{i+1}-1} \quad \text{if } \bar{\lambda}_i = 0.$$

Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles.

★ NilHecke relations:

$$\begin{array}{c} \text{X} \\ i \end{array} = 0, \quad \begin{array}{c} \text{X} \\ i \end{array} \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} = \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} \begin{array}{c} \text{X} \\ i \end{array} \quad (3.45)$$

$$\begin{array}{c} \uparrow \\ i \end{array} \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} = \begin{array}{c} \text{X} \\ i \end{array} \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} - \begin{array}{c} \text{X} \\ i \end{array} \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} = \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} \begin{array}{c} \text{X} \\ i \end{array} - \begin{array}{c} \lambda \\ \uparrow \\ i \end{array} \begin{array}{c} \text{X} \\ i \end{array} \quad (3.46)$$

★ For $i \neq j$

$$\begin{array}{c} \text{X} \\ i, j \end{array} = \begin{array}{c} \uparrow \\ i \end{array} \begin{array}{c} \uparrow \\ j \end{array} \begin{array}{c} \lambda \end{array} \quad \begin{array}{c} \text{X} \\ i, j \end{array} = \begin{array}{c} \uparrow \\ i \end{array} \begin{array}{c} \uparrow \\ j \end{array} \begin{array}{c} \lambda \end{array} \quad (3.47)$$

★ The analogue of the $R(\nu)$ -relations:

i) For $i \neq j$

$$\begin{aligned}
 & \text{Diagram: } \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} = \quad (3.48) \\
 & \left\{ \begin{array}{ll} \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ parallel.} \\ \text{Label } \lambda \text{ above.} \end{array} & \text{if } i \cdot j = 0 \\
 \varepsilon(i, j) \left(\begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} - \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} \right) & \text{if } i \cdot j = -1 \\
 & \quad \text{and } \{i, j\} \neq \{1, n\} \\
 \varepsilon(i, j) \left(\begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} - \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} \right) - \boxed{y} \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ parallel.} \\ \text{Label } \lambda \text{ above.} \end{array} & \text{if } \{i, j\} = \{1, n\}
 \end{array} \right.
 \end{aligned}$$

For $i \cdot j = -1$, we define

$$\varepsilon(i, j) := \begin{cases} 1 & \text{if } i \equiv j + 1 \pmod{n} \\ -1 & \text{if } i \equiv j - 1 \pmod{n} \end{cases}$$

Note that this sign takes into account the standard orientation of the Dynkin diagram.

$$\begin{array}{ccc} \text{Diagram: } \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} & = & \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} \end{array} \quad \begin{array}{ccc} \text{Diagram: } \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} & = & \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} \end{array} \quad (3.49)$$

ii) Unless $i = k$ and $i \cdot j = -1$

$$\begin{array}{ccc} \text{Diagram: } \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} & = & \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} \end{array} \quad (3.50)$$

For $i \cdot j = -1$

$$\begin{array}{ccc} \text{Diagram: } \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} & - & \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ cross.} \\ \text{Label } \lambda \text{ above.} \end{array} = \varepsilon(i, j) \begin{array}{c} \text{Blue line } i \text{ and red line } j \text{ parallel.} \\ \text{Label } \lambda \text{ above.} \end{array} \quad (3.51)
 \end{array}$$

- Composition of 1-morphisms is defined by multiplication and horizontal and vertical composition of 2-morphisms are defined by juxtaposition and glueing, as usual.

Definition 3.19. The 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ is the \mathbb{Q} -linear 2-subcategory of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}^*$ given by the degree-zero 2-morphisms.

Remark 3.20. Note that the 2-hom-spaces in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ are no longer $\mathbb{Q}[y]$ -linear, because $\deg(y) = 2$. They are finite-dimensional as \mathbb{Q} -vector spaces, because the original KL 2-HOM-spaces are finite-dimensional in each degree and their grading is bounded below.

As already remarked, $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined similarly. Only some signs in the relations involving right cups and caps have to be changed, so that all relations really depend on $\widehat{\mathfrak{sl}}_n$ -weights and not on $\widehat{\mathfrak{gl}}_n$ -weights. We use the convention which is the affine analogue of the signed-version in [KL10]. For more information on this change of signs, see (3.67).

Definition 3.21. Let $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ denote the \mathbb{Q} -linear 2-categories obtained by modding out $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by y .

Note that $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the original KL 2-category and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ isomorphic to its $\widehat{\mathfrak{gl}}_n$ -analogue.

We do not know if $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is isomorphic to $\mathcal{U}(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$, but it does not seem so.

3.5.2. *Further relations in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$.* The other $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ -relations expressed below follow from the relations in Definition 3.18 and are going to be used in the sequel.

★ Bubble slides:

If $\{i, j\} \neq \{1, n\}$, we have

$$\begin{array}{c} \text{green arrow } j \uparrow \text{ circle } \lambda \text{ with } i \text{ and } -\bar{\lambda}_i - 1 + m \end{array} = \begin{cases} \sum_{f=0}^m (f - m - 1) \begin{array}{c} \text{green arrow } m-f \uparrow \text{ circle } \lambda + j_{\Lambda} \text{ with } i \text{ and } -(\lambda + j_{\Lambda})_i - 1 + f \\ \text{green arrow } j \uparrow \end{array} & \text{if } i = j \\ \\ \begin{array}{c} \text{green arrow } j \uparrow \text{ circle } \lambda + j_{\Lambda} \text{ with } i \text{ and } -(\lambda + j_{\Lambda})_i - 1 + m \end{array} & \text{if } i \cdot j = 0 \end{cases} \quad (3.52)$$

$$\begin{array}{c} \text{red arrow } i+1 \uparrow \text{ circle } \lambda \text{ with } i \text{ and } -\bar{\lambda}_i - 1 + m \end{array} = \begin{array}{c} \text{red arrow } i+1 \uparrow \text{ circle } \lambda + (i+1)_{\Lambda} \text{ with } i \text{ and } -(\lambda + (i+1)_{\Lambda})_i - 2 + m \end{array} - \begin{array}{c} \text{red arrow } i+1 \uparrow \text{ circle } \lambda + (i+1)_{\Lambda} \text{ with } i \text{ and } -(\lambda + (i+1)_{\Lambda})_i - 1 + m \end{array} \quad (3.53)$$

$$\begin{array}{c} \text{red arrow } i+1 \uparrow \text{ circle } \lambda \text{ with } i \text{ and } -\bar{\lambda}_i - 1 + m \end{array} = - \sum_{f+g=m} \begin{array}{c} \text{red arrow } f \uparrow \text{ circle } \lambda - (i+1)_{\Lambda} \text{ with } i \text{ and } -(\lambda - (i+1)_{\Lambda})_i - 1 + g \\ \text{red arrow } i+1 \uparrow \end{array} \quad (3.54)$$

$$\begin{array}{c} \lambda \\ i \\ \text{bubble} \\ \overline{\lambda}_i - 1 + m \\ i+1 \end{array} = - \sum_{f+g=m} \begin{array}{c} \lambda + (i+1)_\Lambda \\ i \\ \text{bubble} \\ (\overline{\lambda + (i+1)_\Lambda})_i - 1 + g \\ i+1 \end{array} \begin{array}{c} f \\ i+1 \end{array} \quad (3.55)$$

$$\begin{array}{c} \lambda \\ i \\ \text{bubble} \\ \overline{\lambda}_i - 1 + m \\ i+1 \end{array} = \begin{array}{c} \lambda - (i+1)_\Lambda \\ i \\ \text{bubble} \\ (\overline{\lambda - (i+1)_\Lambda})_i - 2 + m \\ i+1 \end{array} - \begin{array}{c} \lambda - (i+1)_\Lambda \\ i \\ \text{bubble} \\ (\overline{\lambda - (i+1)_\Lambda})_i - 1 + m \\ i+1 \end{array} \quad (3.56)$$

If we switch labels i and $i+1$, then the r.h.s. of the above equations gets a minus sign. Bubble slides with the vertical strand oriented downwards can easily be obtained from the ones above by rotating the diagrams 180 degrees.

If $\{i, j\} = \{1, n\}$, we get

$$\begin{array}{c} \lambda \\ n \\ \text{bubble} \\ -\overline{\lambda}_n - 1 + m \\ 1 \end{array} = \begin{array}{c} \lambda + (1)_\Lambda \\ n \\ \text{bubble} \\ -(\overline{\lambda + (1)_\Lambda})_n - 2 + m \\ 1 \end{array} - \begin{array}{c} \lambda + (1)_\Lambda \\ n \\ \text{bubble} \\ -(\overline{\lambda + (1)_\Lambda})_n - 1 + m \\ 1 \end{array} - \boxed{y} \begin{array}{c} \lambda + (1)_\Lambda \\ n \\ \text{bubble} \\ -(\overline{\lambda + (1)_\Lambda})_n - 2 + m \\ 1 \end{array} \quad (3.57)$$

$$\begin{array}{c} \lambda \\ 1 \\ \text{bubble} \\ -\overline{\lambda}_1 - 1 + m \\ n \end{array} = - \begin{array}{c} \lambda + (n)_\Lambda \\ 1 \\ \text{bubble} \\ -(\overline{\lambda + (n)_\Lambda})_1 - 2 + m \\ n \end{array} + \begin{array}{c} \lambda + (n)_\Lambda \\ 1 \\ \text{bubble} \\ -(\overline{\lambda + (n)_\Lambda})_1 - 1 + m \\ n \end{array} - \boxed{y} \begin{array}{c} \lambda + (n)_\Lambda \\ 1 \\ \text{bubble} \\ -(\overline{\lambda + (n)_\Lambda})_1 - 2 + m \\ n \end{array} \quad (3.58)$$

$$\begin{array}{c} \lambda \\ n \\ \text{bubble} \\ -\overline{\lambda}_n - 1 + m \\ 1 \end{array} = - \sum_{f+g=m} \sum_{p=0}^f \binom{f}{p} (-\boxed{y})^{f-p} \begin{array}{c} \lambda - (1)_\Lambda \\ n \\ \text{bubble} \\ -(\overline{\lambda - (1)_\Lambda})_n - 1 + g \\ 1 \end{array} \begin{array}{c} p \\ 1 \end{array} \quad (3.59)$$

$$\begin{array}{c} \lambda \\ \text{1} \circlearrowleft \\ \text{---}\overline{\lambda}_1-1+m \\ \text{---} \\ n \end{array} = \sum_{f+g=m} \sum_{p=0}^f \binom{f}{p} \boxed{y}^{f-p} \begin{array}{c} \lambda - (n)_\Lambda \\ \text{1} \circlearrowleft \\ \text{---}\overline{(\lambda-(n)_\Lambda)}_1-1+g \\ \text{---} \\ n \end{array} \begin{array}{c} p \\ \bullet \\ \text{---} \\ 1 \end{array} \quad (3.60)$$

$$\begin{array}{c} \lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{\lambda}_n-1+m \end{array} = - \sum_{f+g=m} \sum_{p=0}^f \binom{f}{p} (-\boxed{y})^{f-p} \begin{array}{c} \lambda + (1)_\Lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{(\lambda+(1)_\Lambda)}_n-1+g \end{array} \begin{array}{c} p \\ \bullet \\ \text{---} \\ 1 \end{array} \quad (3.61)$$

$$\begin{array}{c} \lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{\lambda}_1-1+m \end{array} = \sum_{f+g=m} \sum_{p=0}^f \binom{f}{p} \boxed{y}^{f-p} \begin{array}{c} \lambda + (n)_\Lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{(\lambda+(n)_\Lambda)}_1-1+g \end{array} \begin{array}{c} p \\ \bullet \\ \text{---} \\ n \end{array} \quad (3.62)$$

$$\begin{array}{c} \lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{\lambda}_n-1+m \end{array} = \begin{array}{c} \lambda - (1)_\Lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{(\lambda-(1)_\Lambda)}_n-2+m \end{array} - \begin{array}{c} \lambda - (1)_\Lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{(\lambda-(1)_\Lambda)}_n-1+m \end{array} \\ - \boxed{y} \begin{array}{c} \lambda - (1)_\Lambda \\ \text{---} \\ 1 \end{array} \begin{array}{c} n \circlearrowleft \\ \text{---}\overline{(\lambda-(1)_\Lambda)}_n-2+m \end{array} \quad (3.63)$$

$$\begin{array}{c} \lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{\lambda}_1-1+m \end{array} = - \begin{array}{c} \lambda - (n)_\Lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{(\lambda-(n)_\Lambda)}_1-2+m \end{array} + \begin{array}{c} \lambda - (n)_\Lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{(\lambda-(n)_\Lambda)}_1-1+m \end{array} \\ - \boxed{y} \begin{array}{c} \lambda - (n)_\Lambda \\ \text{---} \\ n \end{array} \begin{array}{c} 1 \circlearrowleft \\ \text{---}\overline{(\lambda-(n)_\Lambda)}_1-2+m \end{array} \quad (3.64)$$

★ More Reidemeister 3 like relations:

Unless $i = k = j$ we have

$$\begin{array}{c} \lambda \\ \text{---} \\ i \end{array} \begin{array}{c} \text{---} \\ j \end{array} \begin{array}{c} \text{---} \\ k \end{array} = \begin{array}{c} \lambda \\ \text{---} \\ i \end{array} \begin{array}{c} \text{---} \\ j \end{array} \begin{array}{c} \text{---} \\ k \end{array} \quad (3.65)$$

and when $i = j = k$ we have

$$\begin{array}{c} \text{Diagram 1} \end{array} \lambda - \begin{array}{c} \text{Diagram 2} \end{array} \lambda = \sum \begin{array}{c} \text{Diagram 3} \end{array} \lambda + \sum \begin{array}{c} \text{Diagram 4} \end{array} \lambda \quad (3.66)$$

The diagrams are as follows:
 Diagram 1: A vertical line with a crossing, labeled i at the bottom and i at the top.
 Diagram 2: A vertical line with a crossing, labeled i at the bottom and i at the top.
 Diagram 3: A vertical line with a crossing, labeled i at the bottom and i at the top, with a bubble labeled f_1 and f_2 on the right, and a bubble labeled f_3 and f_4 on the left.
 Diagram 4: A vertical line with a crossing, labeled i at the bottom and i at the top, with a bubble labeled g_1 and g_2 on the right, and a bubble labeled g_3 and g_4 on the left.

where the first sum is over all $f_1, f_2, f_3, f_4 \geq 0$ with $f_1 + f_2 + f_3 + f_4 = \bar{\lambda}_i$ and the second sum is over all $g_1, g_2, g_3, g_4 \geq 0$ with $g_1 + g_2 + g_3 + g_4 = \bar{\lambda}_i - 2$. Note that the first summation is zero if $\bar{\lambda}_i < 0$ and the second is zero when $\bar{\lambda}_i < 2$.

Reidemeister 3 like relations for all other orientations are determined from (3.50), (3.51), and the above relations using duality.

3.5.3. The 2-category $\widehat{\mathcal{S}}(n, r)_{[y]}$. As explained in Section 3.4, the q -Schur algebra $\widehat{\mathbf{S}}(n, r)$ can be seen as a quotient of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all idempotents corresponding to the weights that do not belong to $\Lambda(n, r)$.

It is then natural to define the 2-category $\widehat{\mathcal{S}}(n, r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ as follows.

Definition 3.22. The 2-category $\widehat{\mathcal{S}}(n, r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n, r)$.

We remark that we only put real bubbles, whose interior has a label outside $\Lambda(n, r)$, equal to zero. To see what happens to a fake bubble, one first has to write it in terms of real bubbles with the opposite orientation using the infinite Grassmannian relation (3.44).

As in [MSV13], we define $\widehat{\mathcal{S}}(n, r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, rather than $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$. Therefore, just as in [MSV13] (see the introduction of Sections 3 and 4.3 in that paper), we have to show that there exists a full and essentially surjective 2-functor

$$\Psi_{n,r}: \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \rightarrow \widehat{\mathcal{S}}(n, r)_{[y]},$$

which categorifies the surjective homomorphism

$$\psi_{n,r}: \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \rightarrow \widehat{\mathbf{S}}(n, r)$$

defined in (3.28).

On objects $\Psi_{n,r}$ maps μ to $\lambda := \varphi_{n,r}(\mu)$, which was defined in Section 3.2. On 1 and 2-morphisms $\Psi_{n,r}$ is defined to be the identity except for the left cups and caps, on which it is given by

$$\bigcap_{i,\mu} \mapsto (-1)^{\lambda_{i+1}+1} \bigcap_{i,\lambda} \quad \text{and} \quad \bigcup_{i,\mu} \mapsto (-1)^{\lambda_{i+1}} \bigcup_{i,\lambda}. \quad (3.67)$$

Note that here we are simply extending the 2-functor used in [MSV13].

Just for completeness, we now state the following result without proof.

Proposition 3.23. *The 2-functor*

$$\Psi_{n,r}: \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \rightarrow \widehat{\mathcal{S}}(n, r)_{[y]}$$

is well-defined, full and essentially surjective.

Just as for $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, we can put $y = 0$.

Definition 3.24. Let $\widehat{\mathcal{S}}(n, r)$ be the quotient of $\widehat{\mathcal{S}}(n, r)_{[y]}$ by the ideal generated by y .

Of course there also exists a full and essentially surjective 2-functor

$$\Psi_{n,r}: \mathcal{U}(\widehat{\mathfrak{sl}}_n) \rightarrow \widehat{\mathcal{S}}(n, r),$$

which is defined and denoted just as above.

4. A FUNCTOR FROM $\mathcal{DEBim}_{\widehat{A}_{r-1}}^*$ TO $\widehat{\mathcal{S}}(n, r)_{[y]}^*$

In this section, we define a functor

$$\Sigma_{n,r}: \mathcal{DEBim}_{\widehat{A}_{r-1}}^* \rightarrow \widehat{\mathcal{S}}(n, r)_{[y]}^*,$$

which categorifies the embedding

$$\sigma_{n,r}: \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \rightarrow 1_r \widehat{\mathcal{S}}(n, r) 1_r.$$

Actually its target will be the one-object sub-2-category $\widehat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r))$ of $\widehat{\mathcal{S}}(n, r)_{[y]}^*$. Since $\widehat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r))$ has only one object, it can be seen as a monoidal category.

This functor is the affine analogue of $\Sigma_{n,d}$ in Section 6.5 in [MSV13]. For diagrams with only unoriented i -colored strands for $i = 1, \dots, r-1$ the definitions in that paper and in this one coincide, for diagrams with unoriented r -colored strands or oriented strands the definition here is new.

In Section 6, we will prove that

$$\Sigma_{n,r}: \mathcal{DEBim}_{\widehat{A}_{r-1}}^* \rightarrow \widehat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r))$$

is faithful. We conjecture that $\Sigma_{n,r}$ is also full and, therefore, that the two categories $\mathcal{DEBim}_{\widehat{A}_{r-1}}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r))$ are equivalent. The latter equivalence would be the affine analogue of the one proved in Proposition 6.9 in [MSV13] for finite type A .

4.1. The definition of the functor. The functor $\Sigma_{n,r}$ is \mathbb{Q} -linear and monoidal, so it is sufficient to specify the image of the generating objects and morphisms.

The functor $\Sigma_{n,r}$ is defined on objects by

$$\begin{aligned} \emptyset &\mapsto \mathbf{1}_r = \mathbf{1}_{(1^r)} \\ i &\mapsto \mathcal{E}_{-i} \mathcal{E}_i \mathbf{1}_r \\ r &\mapsto \mathcal{E}_{-n} \dots \mathcal{E}_{-r} \mathcal{E}_r \dots \mathcal{E}_n \mathbf{1}_r \\ + &\mapsto \mathcal{E}_{-n} \dots \mathcal{E}_{-r-1} \mathcal{E}_{-1} \dots \mathcal{E}_{-r} \mathbf{1}_r \\ - &\mapsto \mathcal{E}_r \dots \mathcal{E}_1 \mathcal{E}_{r+1} \dots \mathcal{E}_n \mathbf{1}_r. \end{aligned}$$

The functor $\Sigma_{n,r}$ is defined on morphisms as follows:

- The empty diagram is sent to the empty diagram in the region labeled (1^r)

- The vertical line colored i is sent to the identity 2-morphism on $\mathcal{E}_{-i}\mathcal{E}_i\mathbf{1}_r$:

$$i \mapsto \begin{array}{c} \downarrow \\ \uparrow \end{array} \quad (1^r)$$

- The vertical line colored r is sent to the identity 2-morphism on $\mathcal{E}_{-n}\dots\mathcal{E}_{-r}\mathcal{E}_r\dots\mathcal{E}_n\mathbf{1}_r$:

$$r \mapsto \begin{array}{c} \downarrow \dots \downarrow \uparrow \uparrow \end{array} \quad (1^r)$$

- The vertical line colored $+$ is sent to the identity 2-morphism on $\mathcal{E}_{-n}\dots\mathcal{E}_{-r-1}\mathcal{E}_{-1}\dots\mathcal{E}_{-r}\mathbf{1}_r$:

$$\uparrow \mapsto \begin{array}{c} \downarrow \dots \downarrow \uparrow \uparrow \end{array} \quad (1^r)$$

- The vertical line colored $-$ is sent to the identity 2-morphism on $\mathcal{E}_r\dots\mathcal{E}_1\mathcal{E}_{r+1}\dots\mathcal{E}_n\mathbf{1}_r$:

$$\downarrow \mapsto \begin{array}{c} \uparrow \dots \uparrow \downarrow \downarrow \end{array} \quad (1^r)$$

- The images of the startdot_i and enddot_i morphisms for $i \neq r$:

$$\text{startdot}_i \mapsto \text{cup}_i \quad (1^r) \qquad \text{enddot}_i \mapsto \text{cap}_i \quad (1^r)$$

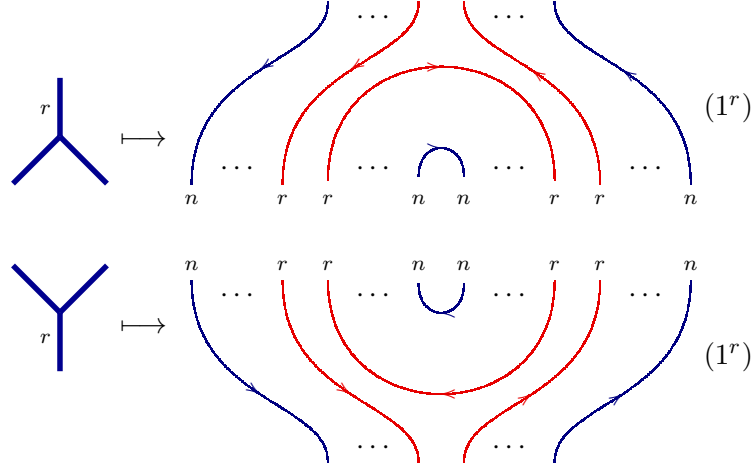
- The images of the startdot_r and enddot_r morphisms:

$$\text{startdot}_r \mapsto \text{cup}_r \quad (1^r) \qquad \text{enddot}_r \mapsto \text{cap}_r \quad (1^r)$$

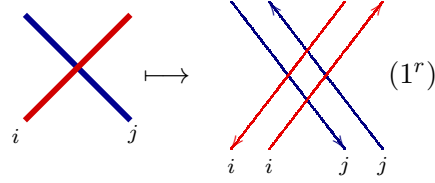
- The images of the merge_i and split_i morphisms for $i \neq r$:

$$\text{merge}_i \mapsto \text{cup}_i \quad (1^r) \qquad \text{split}_i \mapsto \text{cap}_i \quad (1^r)$$

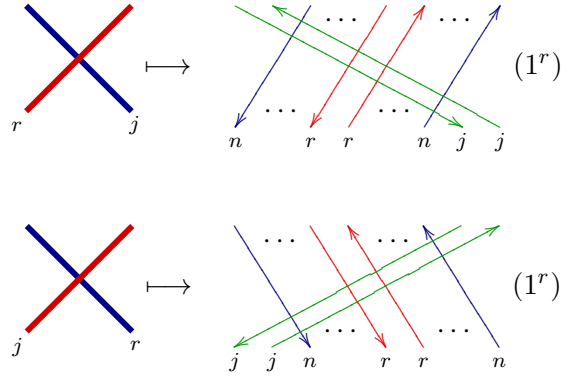
- The images of the merge_r and split_r morphisms:



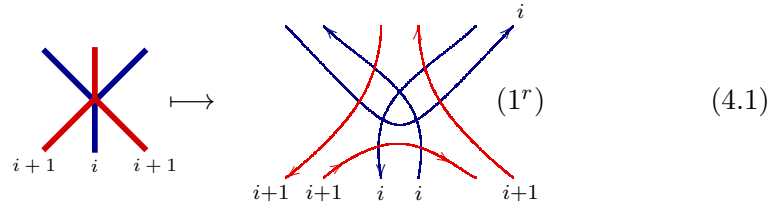
- The image of the 4-valent vertex morphism $4\text{vert}_{i,j}$ with distant colors i and j different from r :

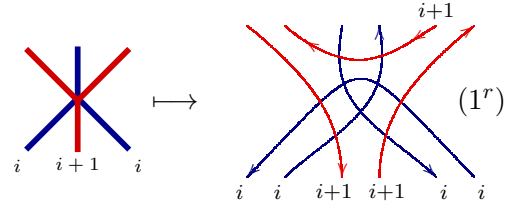


- The images of the 4-valent vertex morphisms $4\text{vert}_{r,j}$ and $4\text{vert}_{j,r}$ with distant colors r and j :

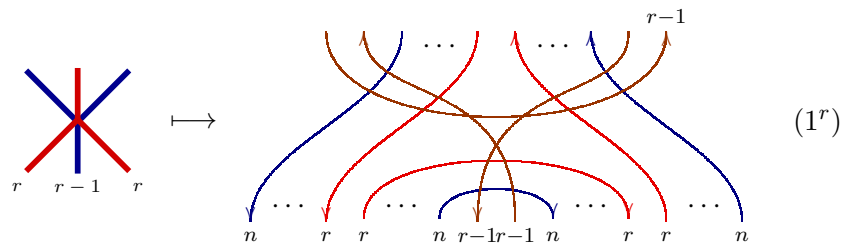
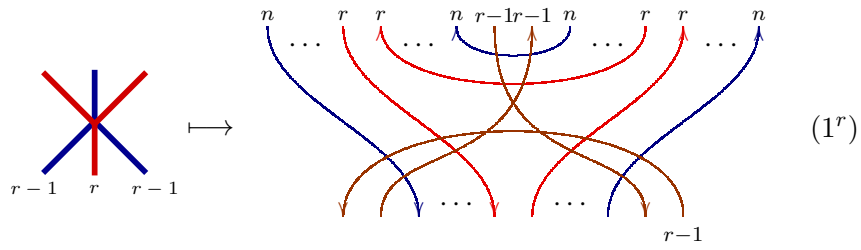
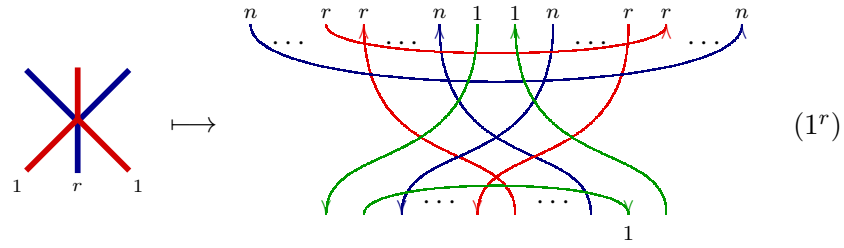
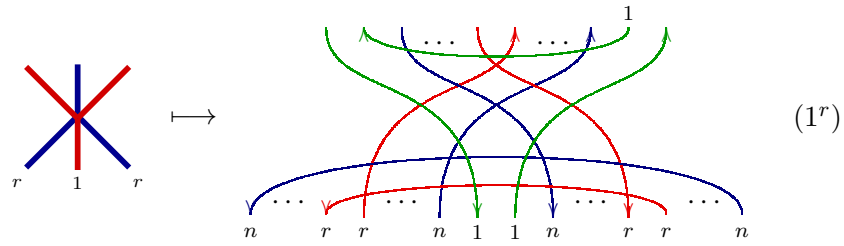


- The images of the 6-valent vertex morphisms $6\text{vert}_{i+1,i}$ and $6\text{vert}_{i,i+1}$ with colors i and $i+1$ different from r :

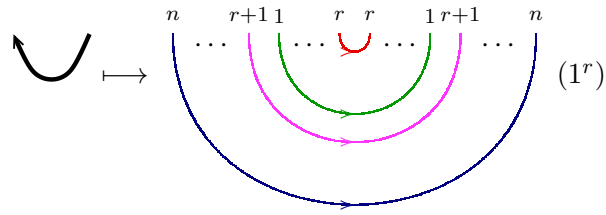
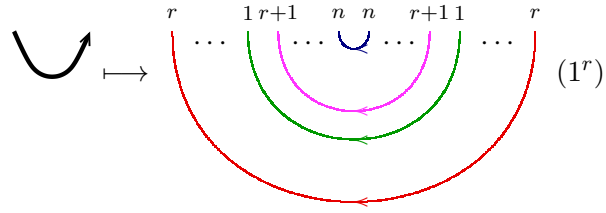
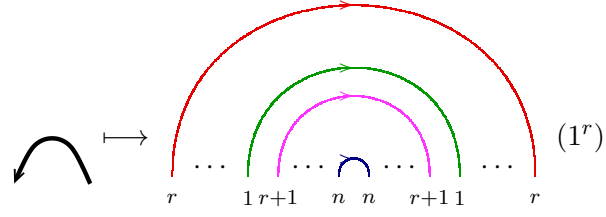
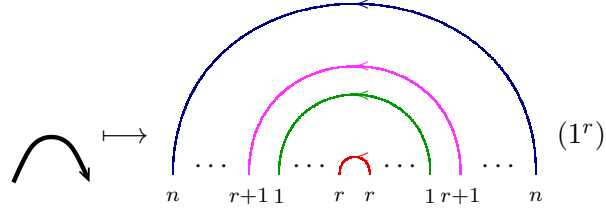




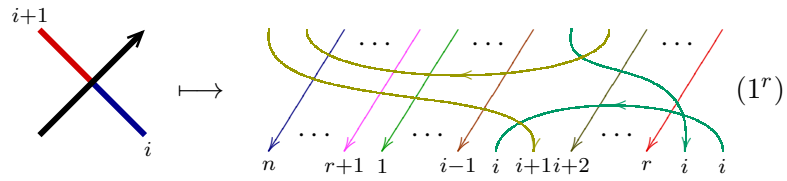
- The images of the 6-valent vertex morphisms $6\text{vert}_{r,1}$, $6\text{vert}_{1,r}$, $6\text{vert}_{r-1,r}$ and $6\text{vert}_{r,r-1}$:

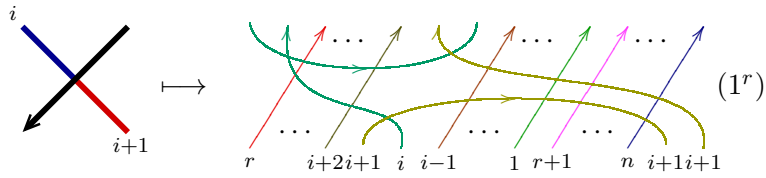
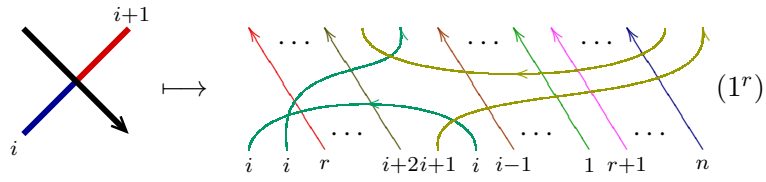
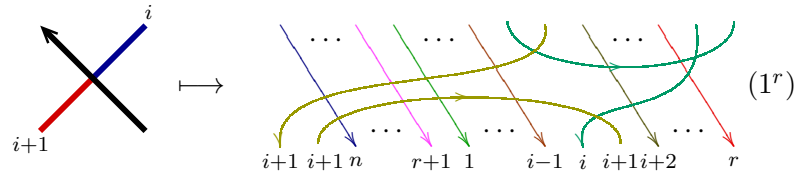


- The images of the oriented caps and cups +cap, -cap, +cup and -cup:

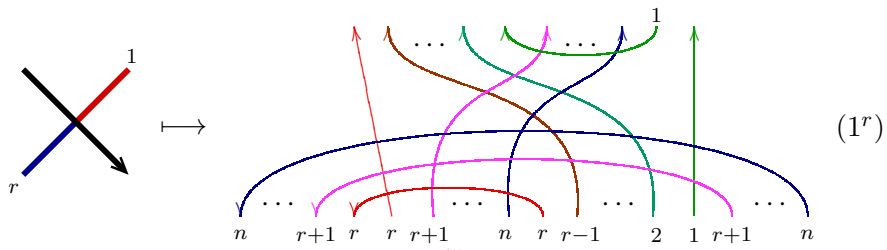
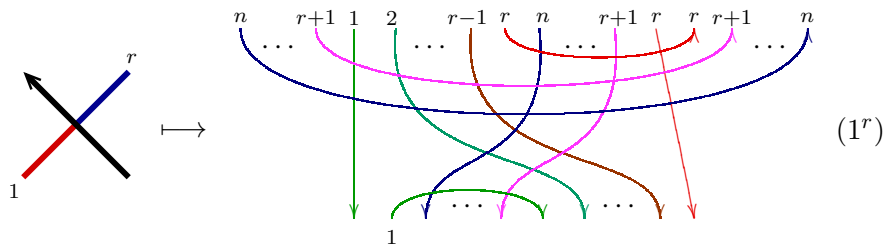
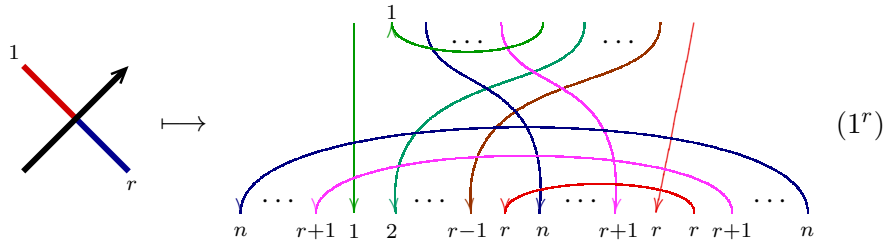


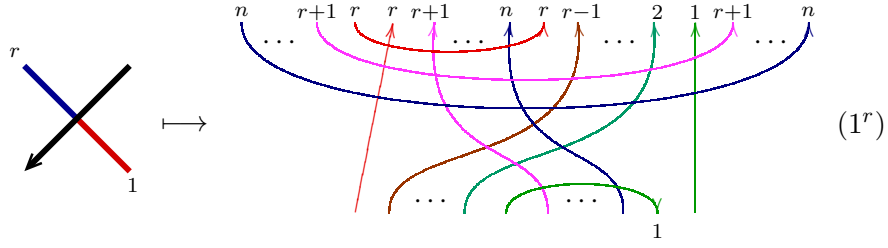
- The images of the 4-valent vertex morphisms $4\text{vert}_{+,i}$, $4\text{vert}_{i+1,+}$, $4\text{vert}_{i,-}$ and $4\text{vert}_{-,i+1}$ with colors i different from $r-1$ and r and $i+1$ different from r and 1 :



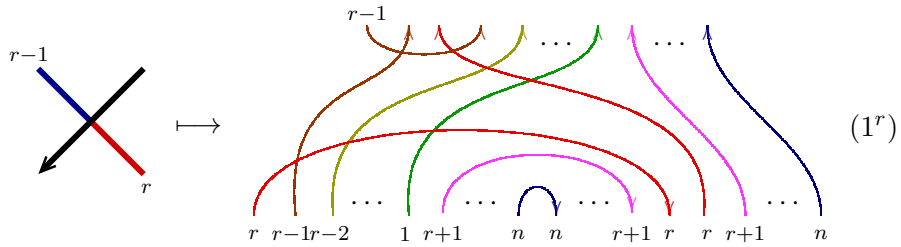
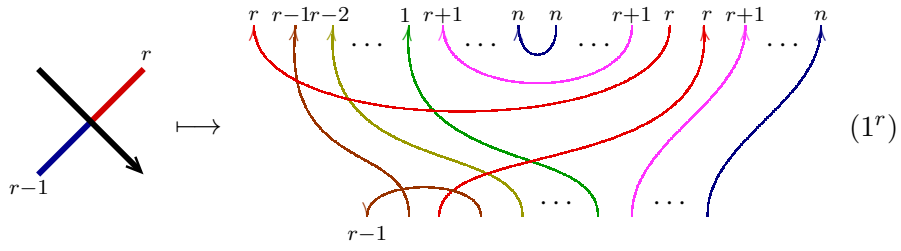
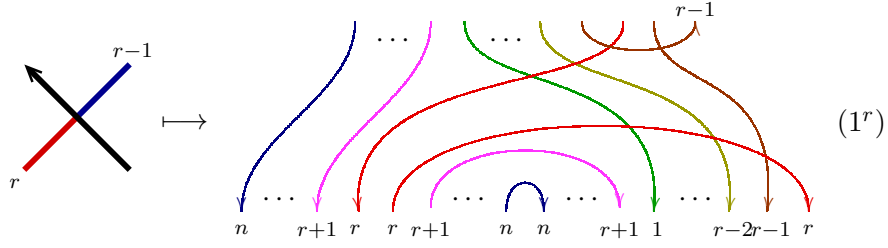
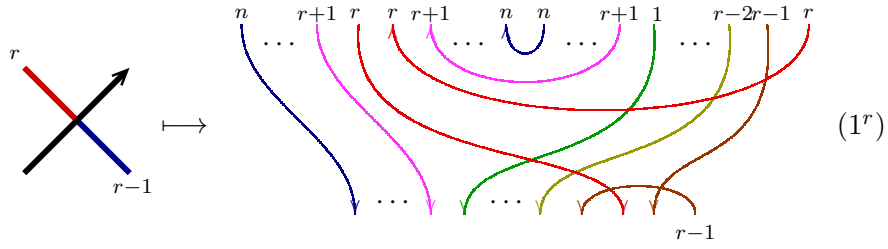


- The images of the 4-valent vertex morphisms $4\text{vert}_{+,r}$, $4\text{vert}_{1,+}$, $4\text{vert}_{r,-}$ and $4\text{vert}_{-,1}$:





- The images of the 4-valent vertex morphisms $4\text{vert}_{+,r-1}$, $4\text{vert}_{r,+}$, $4\text{vert}_{r-1,-}$ and $4\text{vert}_{-,r}$:



- The images of the box morphisms box_i for $i = 1, \dots, r$:

$$\boxed{i} \mapsto - \sum_{j=i}^{r-1} \bigcirc_j^{(1^r)} + \bigcirc_{-1}^{(1^r)}$$

- The image of the box morphism box_y :

$$\boxed{y} \mapsto \boxed{y}^{(1^r)}$$

It is easy to check that $\Sigma_{n,r}$ is degree preserving and monoidal.

Lemma 4.1. $\Sigma_{n,r}$ is well-defined.

Proof. We check that $\Sigma_{n,r}$ preserves all relations in $\mathcal{DEBim}_{\hat{A}_{r-1}}^*$.

First of all, note that all “finite type A” relations, i.e. the relations between diagrams without r -colored or oriented strands, are preserved by precisely the same arguments as in [MSV13].

Let us also remark that, except in the checks of the box relations, we will use neither Relation (3.48) nor the bubble slides Relations (3.57)–(3.64) for $\{i, j\} = \{1, n\}$.

Let us now go through the list of the remaining relations and explain why they are preserved:

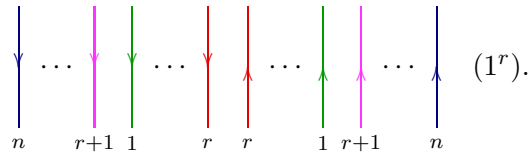
- The Isotopy Relations (2.13)–(2.21) are straightforward.
- Relations (2.22)–(2.34) with at least one r -colored strand follow from the same relations without any r -colored strand together with Relations (2.35)–(2.45). So it suffices to prove the latter relations.

Indeed, for any of these relations with an r -colored strand, one can add an oriented bubble (or if necessary two nested ones) to the diagram on the left-hand side of the equation. By Relation (2.35) the oriented bubbles are equal to one and we will show that that relation is preserved by $\Sigma_{n,r}$ below. Using the appropriate relations involving colored and oriented strands (which are checked below), slide that bubble across the diagram until it encloses that part of the diagram which differs from the diagram on the right-hand side of the equation. It is always possible to choose the orientation of the bubble so that the diagram in its interior does not have any r -colored strands. Then apply the relevant relation for colors different from r to the part of the diagram contained inside the bubble. Finally, slide the bubble aside again. This works since all these operations are proved to be preserved by $\Sigma_{n,r}$ below.

- Relation (2.35) follows directly from the fact that in $\hat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r))$ all dotted bubbles of degree zero are equal to ± 1 . Indeed we can apply successively Relations (3.40) to the nested bubbles in the image of \bigcirc and \bigcirc .

- For Relations (2.36) and (2.37) use repeatedly Relations (3.43) and (3.42). We only give the details for Relation (2.36), the other relation being completely analogous.

The diagram $\Sigma_{n,r}(\uparrow \downarrow)$ is as follows



Apply successively Relation (3.43) to each pair of the form

$$\begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \lambda.$$

For $i = r + 1, \dots, n$, we have $\lambda = (0, 1, \dots, 1, 0, \dots, 0, 1, 0, \dots, 0)$, where the entries which are equal to one are the 2nd until the r th and the $i + 1$ st (mod n). For $i = 1, \dots, r$, we have $\lambda = (1, \dots, 1, 0, 1, \dots, 1, 0, \dots, 0)$ where the entries which are equal to zero are the $r + 2$ nd until the n th and the i th.

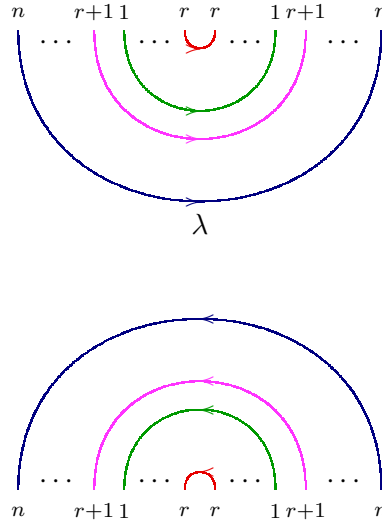
For these λ , Relation (3.43) becomes

$$\begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \lambda = \begin{array}{c} \downarrow \uparrow \\ \uparrow \downarrow \end{array} \lambda - \begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \lambda.$$

The first term on the right-hand side of this relation is equal to zero because the label appearing in interior region contains a negative entry. The bubble appearing in the second term on the right-hand side is equal to -1 . Thus, for all i , we are left with

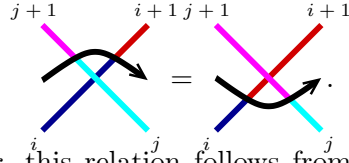
$$\begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \lambda = \begin{array}{c} \downarrow \\ \uparrow \end{array} \begin{array}{c} \downarrow \\ \uparrow \end{array} \lambda. \quad (4.2)$$

Taking all strands together, we get the following nested cups and caps:

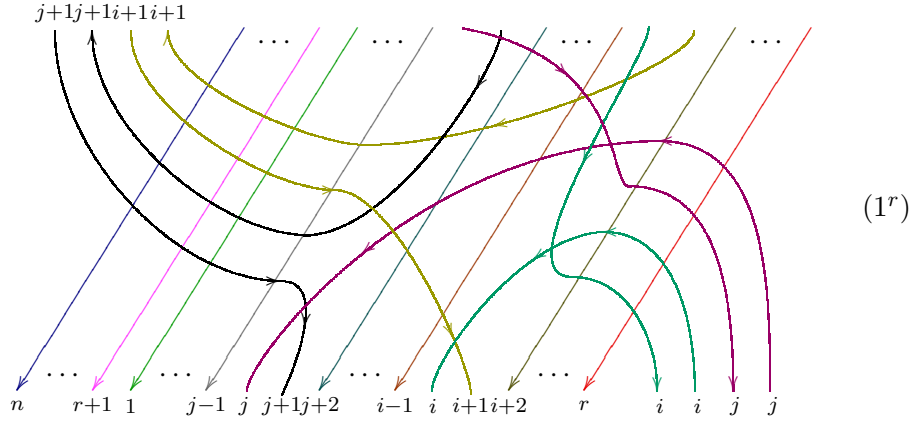


which is equal to $\Sigma_{n,r}(\bowtie)$.

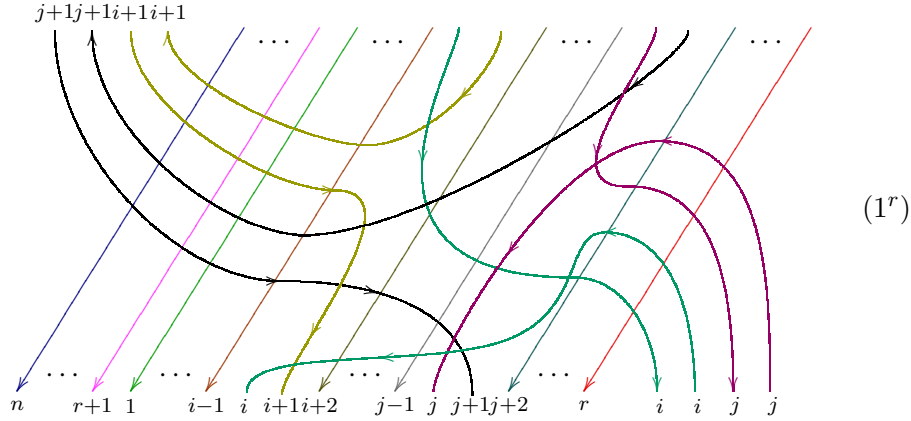
- Relation (2.38)



For $j < i$ and $i, j \neq r$, this relation follows from the fact that, using repeatedly Relations (3.47), (3.48) for distant i, j , (3.50) and (3.65), one can reduce both $\Sigma_{n,r}(\text{diagram})$ and $\Sigma_{n,r}(\text{diagram})$ to the following diagram



Similarly, for $j > i$ and $i, j \neq r$, Relation (2.38) follows from the fact that, using repeatedly Relations (3.47), (3.48) for distant i, j , (3.50) and (3.65), one can reduce both $\Sigma_{n,r}(\text{diagram})$ and $\Sigma_{n,r}(\text{diagram})$ to the following diagram

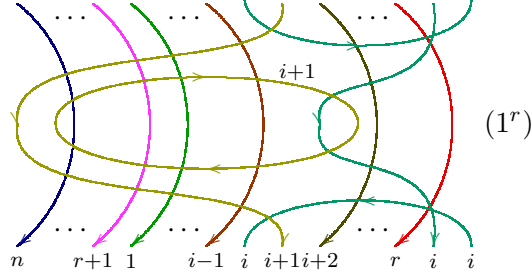


One can prove the remaining cases, in which one of the integers i or j is equal to $r - 1$ or r , in exactly the same way. Just use repeatedly Relations (3.47), (3.48) for $i \cdot j = 0$, (3.50) and (3.65).

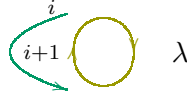
- Relations (2.39) and (2.40).

First when the colors $(i, i + 1)$ differ from $(r, 1)$ and $(r - 1, r)$. We only give the details for Relation (2.39), because the proof of the other relation is very similar.

The image under $\Sigma_{n,r}$ of is equal to

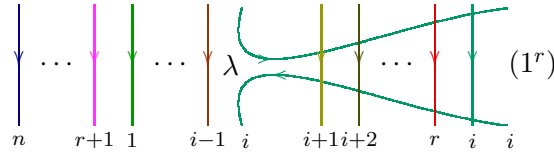


Thanks to Relation (3.47), the left part of the central bubble can be slid to the right until we end up in the position

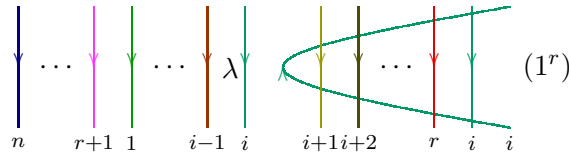


with $\lambda = (1, \dots, 1, 0, 1, \dots, 1, 0, \dots, 0)$ where the first 0 is in $i+2$ nd position and the last 1 in $r+1$ st position. That degree-zero bubble is equal to one, by Relation (3.40).

After removing the bubble, we can slide the i -colored strand over the $r-i-1$ rightmost strands and slide the $i+1$ -colored strand over the $n-r+i-1$ leftmost strands, using Relation (3.48). We get



with $\lambda = (1, \dots, 1, 0, 1, \dots, 1, 0, \dots, 0)$ where the first 0 is in i th position and the last 1 in $r+1$ st position. Apply Relation (4.2), which has only one non-zero term as before. We get



Then the strand with the rightmost endpoints can be slid all the way to the right, first using Relation (3.47) and then Relation (3.43), which again is simply equal to

$$\begin{array}{c} \downarrow \\ \downarrow \end{array} \lambda = \begin{array}{c} \downarrow \uparrow \\ \downarrow \uparrow \end{array} \lambda. \quad (4.3)$$

because $\lambda = (1^r)$. Finally, we end up with a diagram which is indeed equal to $\Sigma_{n,r} \left(\left| \begin{array}{c} \downarrow \\ \downarrow \end{array} \right| \right)$.

We can prove Relations (2.39) and (2.40) with colors $(r, 1)$ and $(r-1, r)$ in almost the same way. We can slide bubbles and strands using Relations


(3.47) and (3.48), evaluate bubbles using Relation (3.40) and use Relations (3.42) and (3.43), which again are particularly simple for the relevant labels:

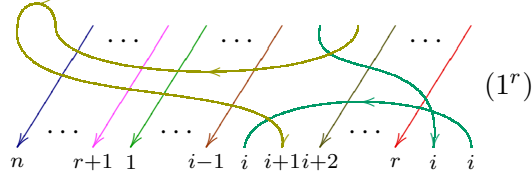
$$\begin{array}{c} \downarrow \\ i \end{array} \begin{array}{c} \downarrow \\ i \end{array} \lambda = \begin{array}{c} \text{bubble} \\ i \quad i \end{array} \lambda \quad \text{or} \quad \begin{array}{c} \downarrow \\ i \end{array} \begin{array}{c} \downarrow \\ i \end{array} \lambda = \begin{array}{c} \text{bubble} \\ i \quad i \end{array} \lambda \quad \text{or} \quad \begin{array}{c} \downarrow \\ i \end{array} \begin{array}{c} \downarrow \\ i \end{array} \lambda = \begin{array}{c} \text{bubble} \\ i \quad i \end{array} \lambda$$

depending on the value of λ . The difference here is that we have to iterate some of the steps that we used above, e.g. because we get nested bubbles or various pairs of strands to which we can apply Relations (3.42) and (3.43).

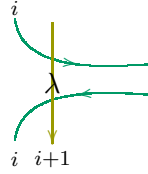
• Relations (2.41) and (2.42).

Again, first when $(i, i+1)$ differ from $(r, 1)$ and $(r-1, r)$. The proofs for these two relations are very similar. We only give the proof for Relation (2.42).

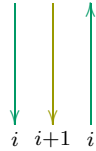
The image under $\Sigma_{n,r}$ of  is equal to



We can slide the $i+1$ -colored strand over the $n-r+i-1$ leftmost strands using Relation (3.48), so that we end up with the following picture in the central part of the diagram:



with $\lambda = (1, \dots, 1, 0, 1, \dots, 1, 0, \dots, 0)$ where the first 0 is in i th position and the last 1 in $r+1$ st position. We now use Relation (3.43), which again reduces to (4.2). Then we apply Relation (3.47) to the two rightmost strands. After doing all this, the central part of the diagram becomes equal to



Then, in the full diagram, the rightmost i -colored strand above can be slid over the $r-i-1$ rightmost parallel strands using Relation (3.47), so that we end up with

$$\begin{array}{c} \downarrow \\ n \end{array} \dots \begin{array}{c} \downarrow \\ r+1 \end{array} \begin{array}{c} \downarrow \\ 1 \end{array} \dots \begin{array}{c} \downarrow \\ r \end{array} \begin{array}{c} \text{bubble} \\ i \quad i \end{array} \quad (1^r) \quad (4.4)$$

We apply Relation (3.41) to the curl. For $\lambda = (1^r)$, this relation becomes

$$\begin{array}{c} \text{Diagram 1: A vertical strand with a curl (loop) on the right side, labeled } i \text{ at the bottom and } (1^r) \text{ at the top.} \end{array} = - \begin{array}{c} \text{Diagram 2: A vertical strand with a dot at the top labeled } 0, \text{ labeled } i \text{ at the bottom, followed by a bubble (loop) labeled } i \text{ at the top and } -1 \text{ at the bottom, with } (1^r) \text{ at the top right.} \end{array}$$

Since the bubble is equal to -1 , the diagram in (4.4) becomes equal to

$$\begin{array}{c} \text{Diagram: A sequence of vertical strands of different colors (blue, magenta, green, red) labeled } n, r+1, 1, r \text{ at the bottom. To the right is a small bubble labeled } i \text{ at the top and } i \text{ at the bottom, with } (1^r) \text{ at the top right.} \end{array}$$

which is indeed the image under $\Sigma_{n,r}$ of .

To prove Relations (2.41) and (2.42) with colors $(r, 1)$ and $(r - 1, r)$, we use exactly the same ingredients as above: sliding strands using Relations (3.47) and (3.48), removing curls using

$$\begin{array}{c} \text{Diagram 1: A vertical strand with a curl on the left side, labeled } \lambda \text{ at the top and } i \text{ at the bottom.} \end{array} = \begin{array}{c} \text{Diagram 2: A straight vertical strand labeled } \lambda \text{ at the top and } i \text{ at the bottom.} \end{array} \quad \text{or} \quad \begin{array}{c} \text{Diagram 3: A vertical strand with a curl on the right side, labeled } \lambda \text{ at the top and } i \text{ at the bottom.} \end{array} = \begin{array}{c} \text{Diagram 4: A straight vertical strand labeled } \lambda \text{ at the top and } i \text{ at the bottom.} \end{array}$$

and applying Relation (3.43), which is equal to

$$\begin{array}{c} \text{Diagram: Two strands, one labeled } i \text{ at the top and } i \text{ at the bottom, with a crossing.} \end{array} \lambda = \begin{array}{c} \text{Diagram: Two parallel vertical strands, both labeled } i \text{ at the bottom, with } \lambda \text{ at the top right.} \end{array}$$

for the relevant value of λ .

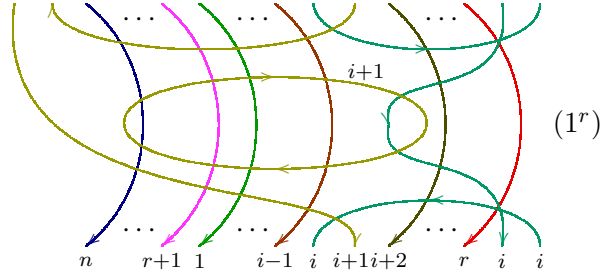
• Relation (2.43). Actually, we will prove

$$\begin{array}{c} \text{Diagram 1: A red strand and a blue strand crossing, with a black strand passing through the crossing.} \end{array} = \begin{array}{c} \text{Diagram 2: A red strand and a blue strand crossing, with a black strand passing through the crossing in a different configuration.} \end{array} \quad (4.5)$$

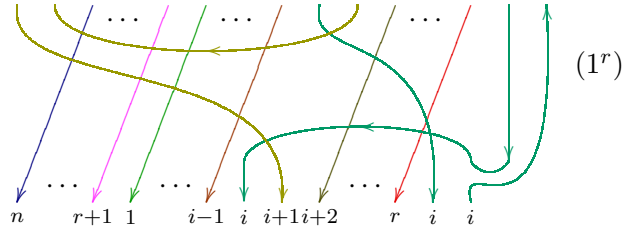
which is equivalent.

Let us start with colors $(i, i + 1)$ different from $(r, 1)$ and $(r - 1, r)$.

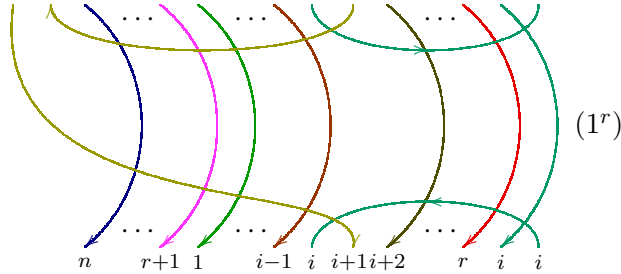
The image under $\Sigma_{n,r}$ of  is



The image under $\Sigma_{n,r}$ of  is



We can apply to the first of these two diagrams the same arguments as in the beginning of the proof of Relation (2.39), in order to simplify it to



In order to prove that this diagram is equal to the second one, it only remains to check that

$$\begin{array}{c} \text{Diagram 1} \end{array} \lambda = \begin{array}{c} \text{Diagram 2} \end{array} \lambda \quad (4.6)$$

where $\lambda = (1^r)$. Because Relation (3.42) reduces to

$$\begin{array}{c} \text{Diagram 3} \end{array} \lambda = \begin{array}{c} \text{Diagram 4} \end{array} \lambda \quad (4.7)$$

when $\lambda = (1^r)$, the left hand side of (4.6) is equal to

$$\text{Diagram 1} = \text{Diagram 2}$$

So proving (4.6) is equivalent to proving that:

$$\text{Diagram 1} = \text{Diagram 2} \quad (4.8)$$

Let us apply twice Relation (4.3) to the right hand side. This gives us

$$\text{Diagram 1} = \text{Diagram 2}$$

Then we apply Relation (3.43), with $\lambda = (1, \dots, 1, 0, 2, 1, \dots, 1, 0, \dots, 0)$ where 2 is in $i + 1$ st position and the last 1 in r th position, which gives us

$$- \text{Diagram 1} - \text{Diagram 2} - \text{Diagram 3} \quad (4.9)$$

Here the first and last term are equal to zero, due to Relation (3.45). To the second term we can apply Relation (3.46), so that we get

$$\text{Diagram 1} = - \text{Diagram 2} + \text{Diagram 3}$$

Here again the first term on the right-hand side is killed by Relation (3.45), so Relation (4.8) is satisfied and $\Sigma_{n,r} \left(\text{Diagram 1} \right)$ and $\Sigma_{n,r} \left(\text{Diagram 2} \right)$ are equal.

Proving Relation (4.5) for colors $(r-1, r)$ is not much more complicated and is left to the reader.

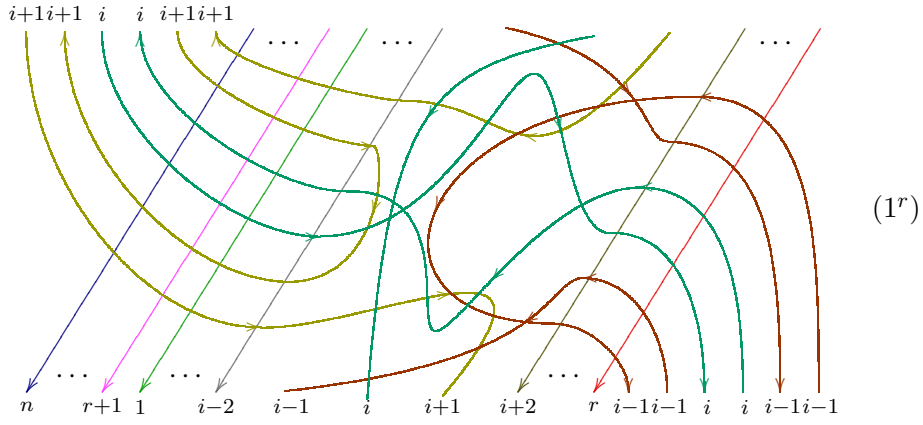
• Relations (2.44) and (2.45) involving oriented strands and three adjacent colored strands.

In order to prove these relations we bend the left end of the oriented strand downward and the right end upward in the diagrams.

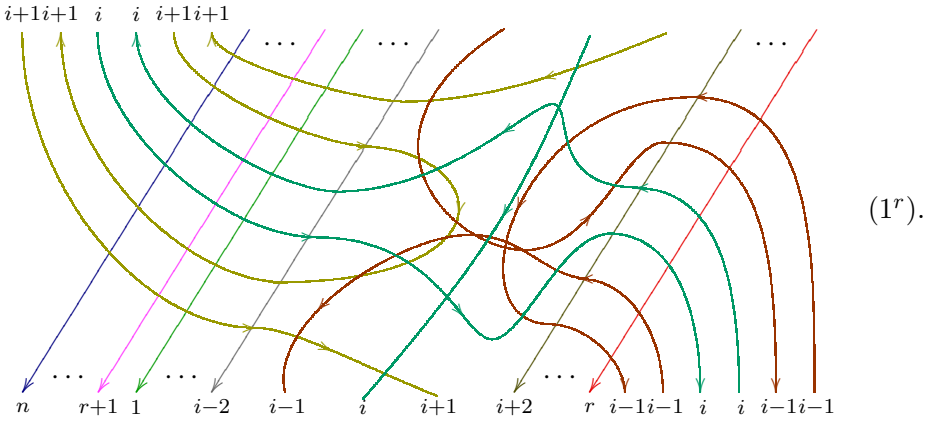
Let us start with the case when there are no r -colored strands. We prove Relation (2.44) for the case in which the bottom strands are colored $(i-1, i, i-1)$. Relation (2.45) for the case in which the bottom strands are colored $(i, i-1, i)$ can be proved similarly.

By applying Relations (3.47), (3.48), (3.50) and (3.65) to the diagrams $\Sigma_{n,r}(\text{diagram})$ and $\Sigma_{n,r}(\text{diagram})$, we can slide the entangled parts of the strands colored $i-1, i$ and $i+1$ to the middle of the diagrams. In this way, we can turn $\Sigma_{n,r}(\text{diagram})$ and $\Sigma_{n,r}(\text{diagram})$ into the following two diagrams:

the diagram $\Sigma_{n,r}(\text{diagram})$ becomes



and $\Sigma_{n,r}(\text{diagram})$ becomes



We have to prove that these two diagrams are equivalent.

First consider the diagram above which is equivalent to $\Sigma_{n,r}(\text{diagram})$. Apply Relation (3.66) to the triangle in the center formed by three i -colored strands. Since $\lambda = (1, \dots, 1, 0, 1, 2, 0, 1, \dots, 1, 0, \dots, 0)$, where 2 is in the $i+1$ st position and the last 1 in the $r+1$ st position, that relation is equal

to

$$\begin{array}{c} \text{Diagram 1} \end{array} \lambda = \begin{array}{c} \text{Diagram 2} \end{array} \lambda \quad (4.10)$$

Then apply Relation (3.50) to the two triangles formed by strands colored $(i-1, i, i+1)$. Both triangles are slightly to the right of the center and one is higher and the other is lower than the center. Sliding the i -colored strands to the left using this relation creates four bigons, two between strands colored i and $i+1$ and the other two between strands colored $i-1$ and $i+1$. The first two bigons can be solved using Relation (3.47), the other two using Relation (3.48). Finally, we apply Relation (3.50) to the top and bottom central triangles of the diagram. This proves that $\Sigma_{n,r}(\text{Diagram})$ is equivalent to

$$\begin{array}{c} \text{Diagram (1}^r\text{)} \end{array}$$

Now apply Relation (3.51)

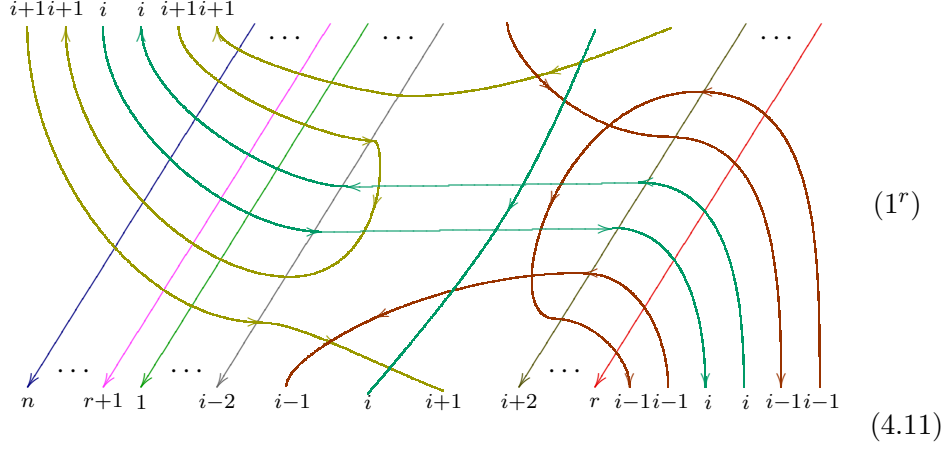
$$\begin{array}{c} \text{Diagram 1} \end{array} \lambda = \begin{array}{c} \text{Diagram 2} \end{array} \lambda - \begin{array}{c} \text{Diagram 3} \end{array} \lambda$$

to the triangle in the central right part of the diagram. This gives us two terms. The second term is killed because it contains a bigon between two i -colored strands with the same orientation, which is zero by Relation (3.45). In the remaining term we can slide the vertical i -colored strand to the left using Relation (3.66), as we did in (4.10). This leaves us with a bigon between two i -colored strands with opposite orientations. Use Relation (3.43) in order to remove this bigon. Note that this relation is equal to

$$\begin{array}{c} \text{Diagram 1} \end{array} \lambda = \begin{array}{c} \text{Diagram 2} \end{array} \lambda,$$

since $\lambda = (1, \dots, 1, 0, 2, 1, 0, 1, \dots, 1, 0, \dots, 0)$ with 2 in the i th position and the last 1 in the $r+1$ st position.

This tells us that $\Sigma_{n,r}(\text{diagram})$ is equivalent to

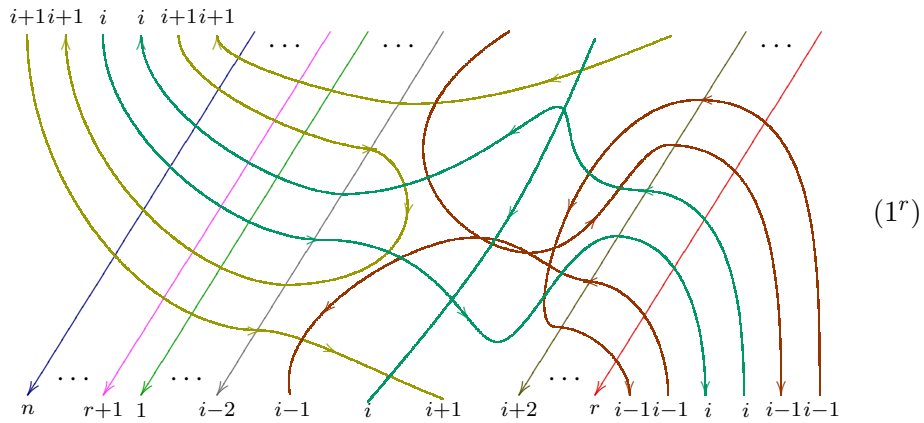


Let us now prove that $\Sigma_{n,r}(\text{diagram})$ is equivalent to this same diagram. First apply Relation (3.66) to the triangle formed by the three $i-1$ -colored strands just right of the center. Since $\lambda = (1, \dots, 1, 2, 0, 0, 1, \dots, 1, 0, \dots, 0)$ with 2 in the i th position and the last 1 in the $r+1$ st position, this relation becomes

$$\begin{array}{c} \text{diagram} \\ i-1 \quad i-1 \quad i-1 \end{array} \lambda = \begin{array}{c} \text{diagram} \\ i-1 \quad i-1 \quad i-1 \end{array} \lambda$$

Then apply Relation (3.50) to the two central triangles formed by strands colored $(i-1, i, i+1)$. Sliding the $i+1$ -colored strands to the left using this relation creates two bigons between strands colored $i-1$ and $i+1$ respectively. These bigons can be removed using Relation (3.48).

In this way, we have proved that $\Sigma_{n,r}(\text{diagram})$ is equivalent to



Next, apply Relation (3.51) to the central left part of the diagram. Locally we end up with a sum of two terms:

$$\begin{array}{c} \text{crossing of } i-1 \text{ and } i \text{ strands} \end{array} \lambda = \begin{array}{c} \text{crossing of } i-1 \text{ and } i \text{ strands with different orientation} \end{array} \lambda - \begin{array}{c} \text{three parallel strands (two } i-1, \text{ one } i) \end{array} \lambda$$

The second term is killed because it contains an $i - 1$ -colored curl which is zero by Relation (3.45). Note that $\lambda = (1, \dots, 1, 2, 0, 0, 1, \dots, 1, 0, \dots, 0)$ where 2 is in the i th position and the last 1 is in the $r + 1$ st position, so $\bar{\lambda}_{i-1} = -1$. The first term contains a bigon between two $i - 1$ -colored strands with opposite orientations. Remove this bigon using Relation (3.42), which in this case equals

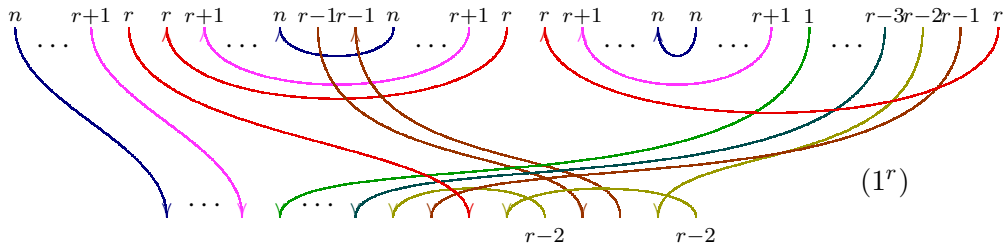
$$\begin{array}{c} \text{two parallel } i-1 \text{ strands} \end{array} \lambda = \begin{array}{c} \text{crossing of } i-1 \text{ strands} \end{array} \lambda.$$

Now apply Relation (3.50) to the top central and bottom central parts of the diagram which we have obtained so far. Apply Relation (3.65) to the top right and bottom right parts of the diagram. In this way we get two more bigons between strands colored $i - 1$ and i with opposite orientations, which we remove using Relation (3.47). This finishes our proof that $\Sigma_{n,r} \left(\begin{array}{c} \text{diagram with crossing and curls} \end{array} \right)$ is equivalent to the same diagram in (4.11).

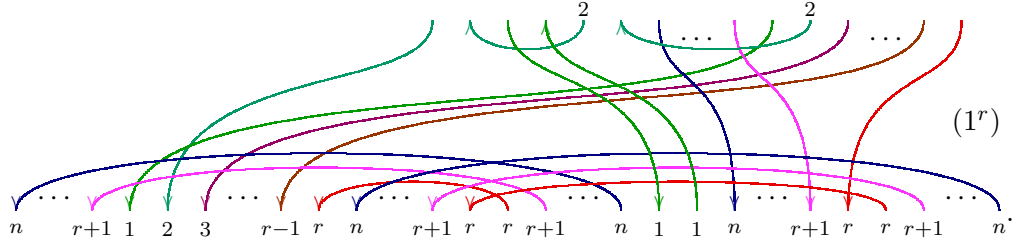
Let us now consider Relation (2.44) when one of the strands has color r (Relation (2.45) can be dealt with in a similar way). Three cases have to be considered: when the bottom strands are colored $(r - 2, r - 1, r - 2)$, $(r, 1, r)$ or $(r - 1, r, r - 1)$.

Using the same sort of reasoning as above, one can show that $\Sigma_{n,r} \left(\begin{array}{c} \text{diagram with crossing and curls} \end{array} \right)$ and $\Sigma_{n,r} \left(\begin{array}{c} \text{diagram with crossing and curls} \end{array} \right)$ both reduce to the same diagram. We omit the precise computations, but just give the diagram to which both images can be reduced.

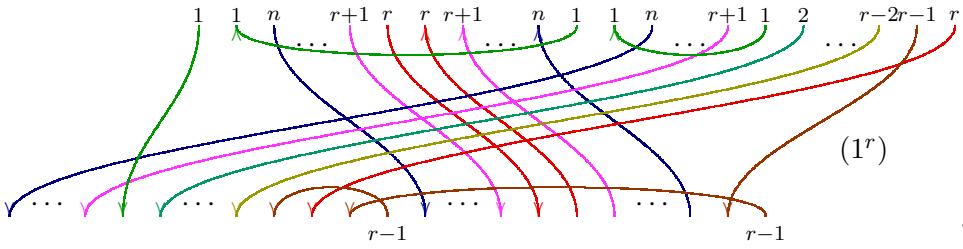
In the case of the colors $(r - 2, r - 1, r - 2)$, one can reduce both images to the following diagram



In the case of colors $(r, 1, r)$, we get



Finally, in the case of colors $(r-1, r, r-1)$, we get



• **Box relations.** Just as for Relations (2.22)–(2.34), some of the box relations with $i = r$ or $j = r$ follow from the same box relations for $i, j \neq r$ together with some other box relations. Taking into account the observations in Remark 2.9 too, we see that it suffices to prove Relations (2.47) and (2.55) here.

Let us start with Relation (2.55). It is sufficient to prove this relation for $i = r-1$, because we can write

$$\boxed{i+1} = \boxed{i+1} - \boxed{i+2} + \boxed{i+2} - \boxed{i+3} + \dots + \boxed{r-1} - \boxed{r} + \boxed{r}$$

and

$$\boxed{i} = \boxed{i} - \boxed{i+1} + \boxed{i+1} - \boxed{i+2} + \dots + \boxed{r-2} - \boxed{r-1} + \boxed{r-1}$$

and use Relations (2.41), (2.42) and (2.46).

Let us prove Relation (2.55) for $i = r-1$, i.e.

$$\begin{array}{c} \text{red circle with } r \text{ and } -1 \\ \vdots \\ \text{blue strand } n \\ \vdots \\ \text{magenta strand } r+1 \\ \vdots \\ \text{green strand } 1 \\ \vdots \\ \text{red strand } r \end{array} = \begin{array}{c} \text{blue strand } n \\ \vdots \\ \text{magenta strand } r+1 \\ \vdots \\ \text{green strand } 1 \\ \vdots \\ \text{red strand } r \end{array} - \text{blue circle with } r-1 + \begin{array}{c} \text{red circle with } r \text{ and } -1 \end{array}.$$

By Relations (2.41) and (2.42), we have

$$\begin{array}{c} \text{blue circle with } r-1 \\ \vdots \\ \text{red circle with } r \text{ and } -1 \\ \vdots \\ \text{blue strand } n \\ \vdots \\ \text{magenta strand } r+1 \\ \vdots \\ \text{green strand } 1 \\ \vdots \\ \text{red strand } r \end{array} = \begin{array}{c} \text{blue strand } n \\ \vdots \\ \text{magenta strand } r+1 \\ \vdots \\ \text{green strand } 1 \\ \vdots \\ \text{red strand } r \end{array} - \text{blue circle with } r-1.$$

Therefore, it suffices to prove

$$\left(\cdots \bigcirc_r \cdots \right)_n + \left(\bigcirc_{-1}^r \downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) = \left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) \bigcirc_{-1}^r.$$

On the one hand, we observe that

$$\left(\bigcirc_{-1}^r \downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) = \left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) - \left(\downarrow_n \cdots \downarrow_{r+1} \bigcirc_r \downarrow_1 \cdots \downarrow_r \right)$$

since the bubble can be slid through the first $n - r - 1$ left strands and then bubble slide Relation (3.53) can be applied to the bubble and the strand colored $r + 1$. On the other hand, we have

$$\left(\cdots \bigcirc_r \cdots \right)_n \downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r = \left(\downarrow_n \cdots \downarrow_{r+1} \bigcirc_r \downarrow_1 \cdots \downarrow_r \right)$$

which is obtained by using repeatedly Relation (3.42), which for the relevant labels λ reduces to

$$\downarrow_i \downarrow_i \lambda = \begin{array}{c} \text{bigon} \\ \text{with } i \text{ strands} \end{array} \lambda.$$

So it only remains to prove that

$$\left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) = \left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) \bigcirc_{-1}^r.$$

Notice that

$$\left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) \bigcirc_{-1}^r = \left(\downarrow_n \cdots \downarrow_{r+1} \downarrow_1 \cdots \downarrow_r \right) \bigcirc_1^r,$$

by the infinite Grassmannian relation. We can apply Relation (3.42) to the bubble and the strand colored r . The first term that appears is killed, because the weight inside the bigon has a negative entry. The bubble appearing

in the second term satisfies

$$\begin{array}{c} \lambda \\ \textcircled{r} \\ -2 \end{array} = -1,$$

since $\lambda = (1, \dots, 1, 0, 1, 0, \dots, 0)$ with the last one in $r + 1$ st position. Thus we get

$$\begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} \begin{array}{c} \textcircled{r} \\ -1 \end{array} = \begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array}$$

Finally, we have to verify that

$$\begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} = \begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} \quad (4.12)$$

which is true: just slide the left dotted strand over all the strands colored $1, \dots, r-1$, using the first case of Relation (3.48), and then apply the second case of Relation (3.48). Observe that the term with the bigon is killed, because the weight inside the bigon has a negative entry. Note that this argument is not valid if $r = n - 1$. Indeed in this case, $r + 1$ and 1 are adjacent colors, hence the left dotted strand cannot be simply slid over the strand colored 1 . In order to prove this remaining case together with Relation (2.47), let us remark that the right hand side of Equation (4.12) can also be expressed as follows:

$$\begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} = \begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} + \left(\sum_{j=1}^{r-1} \begin{array}{c} \textcircled{j} \\ (1^r) \end{array} - \begin{array}{c} \textcircled{r} \\ -1 \end{array} + \begin{array}{c} \textcircled{n} \\ 1 \end{array} + \boxed{y} \right) \begin{array}{c} \downarrow \dots \downarrow \downarrow \downarrow \\ n \quad r+1 \quad 1 \quad r \end{array} \quad (4.13)$$

This expression is obtained using repeatedly kink resolutions and bubble slides. Indeed the kink on the left hand side of Relation (3.41) for $i = r$ is equal to zero here since the label inside the kink possesses a negative entry. Hence one can express the dotted r strand as a non-dotted strand times a bubble colored r on the left. This bubble can then be slid through the $r - 1$ strand using Relation (3.56), creating two terms: one is a dotted $r - 1$ strand while the other is a non-dotted $r - 1$ strand times a bubble colored r on the left. This bubble can be slid all the way to the left using (3.56) making appear an extra term which is the dotted $r + 1$ strand on the right hand side of (4.13). One applies this same trick successively to the dotted strands for colors $r - 1$ to 1 . The only exception is that, in the end, to slide the bubble colored 1 through the strand colored n , we have to use the deformed Relation (3.64), which brings out the y term in (4.13). Finally the dotted n

strand that thus appears can also be expressed as a non-dotted strand times a n -colored bubble on the left using Relation (3.41).

Therefore, by (4.13), it suffices to prove

$$\sum_{j=1}^{r-1} \text{bubble}_j^{(1^r)} - \text{bubble}_{-1}^{(1^r)} + \text{bubble}_1^{(1^r)} + \boxed{y} = 0 \quad (4.14)$$

in order to show (4.12). In Section 5 we define a 2-representation of $\widehat{\mathcal{S}}(n, r)_{[y]}^*$. Its restriction to $\text{END}(\mathbf{1}_r)$ gives an algebra homomorphism

$$\mathcal{F}': \text{END}(\mathbf{1}_r) \rightarrow \mathbb{Q}[y, x_1, \dots, x_r]$$

which on the elements of degree two is determined by

$$\begin{aligned} \boxed{y}^{(1^r)} &\mapsto y \\ \text{bubble}_i^{(1^r)} &\mapsto \begin{cases} x_{i+1} - x_i & 1 \leq i \leq r-1 \\ 0 & r+1 \leq i \leq n-1 \end{cases} \\ \text{bubble}_{-1}^{(1^r)} &\mapsto x_r \\ \text{bubble}_1^{(1^r)} &\mapsto x_1 - y. \end{aligned}$$

Note that \mathcal{F}' maps the l.h.s. of (4.14) to zero. We are going to show that this implies (4.14) by showing that \mathcal{F}' is an isomorphism. From the definition it is clear that \mathcal{F}' is surjective. Injectivity follows if we can prove that $\text{END}(\mathbf{1}_r)$ is generated by

$$\boxed{y}^{(1^r)}, \text{bubble}_i^{(1^r)}, \text{bubble}_{-1}^{(1^r)}, \text{bubble}_1^{(1^r)} \quad (4.15)$$

for $i = 1, \dots, r-1$, because that implies that $\text{END}(\mathbf{1}_r)$ is isomorphic to a quotient of $\mathbb{Q}[y, x_1, \dots, x_r]$ by the surjectivity of \mathcal{F}' , which means that the two algebras have to be isomorphic.

In order to prove that $\text{END}(\mathbf{1}_r)$ is indeed generated by the 2-morphisms in (4.15), first note that $\text{END}(\mathbf{1}_r)$ is generated by all counter-clockwise bubbles of arbitrary degree and \boxed{y} . It therefore suffices to prove that any counter-clockwise bubble is in the span of the 2-morphisms in (4.15).

We first prove this fact for counter-clockwise i -bubbles with $1 \leq i \leq r-1$. The result follows from the recursive formula

$$\text{bubble}_{t+1}^{(1^r)} = - \text{bubble}_t^{(1^r)} \left(\text{bubble}_{i+1}^{(1^r)} + \dots + \text{bubble}_{r-1}^{(1^r)} - \text{bubble}_{-1}^{(1^r)} \right) \quad (4.16)$$

for $t \geq 0$. The equation in (4.16) can be obtained by unnesting the l.h.s. of

$$\begin{array}{c} \text{\tiny $(1^r$)} \\ \cdots \quad \circlearrowleft_r \quad \cdots \end{array} \quad i = 0 \tag{4.17}$$

using bubble slides from the inside to the outside. Equation (4.17) holds, because the region in the center has label λ with $\lambda_{r+1} = -1$.

For $i = r$ the argument is simpler, because

$$\int_{-2+t}^{1^r} r = 0$$

for any $t \geq 2$. This holds because the inner region has label λ with $\lambda_{r+1} = -1$.

Similarly, for $i = n$ we have

$$n^{(1^r)}_{-2+t} = 0$$

for any $t \geq 2$. By the infinite Grassmannian relation, this implies that

$$n^{(1^r)}_t = \left(n^{(1^r)}_1 \right)^t$$

for any $t \geq 2$.

For $r + 1 \leq i \leq n - 1$ there is nothing to prove. In that case, the counter-clockwise i -bubbles of positive degree are all zero, because their interior is labeled by λ with $\lambda_{i+1} = -1$. This finishes the proof that

$$\text{END}(\mathbf{1}_r) \cong \mathbb{Q}[y, x_1, \dots, x_r],$$

which implies (4.14).

Now let us consider Relation (2.47). The image under $\Sigma_{n,r}$ of \bullet_r is

$$\dots \text{ (large blue circle } (1^r) \text{ containing a small red circle } r \text{)} \dots = - \text{ (red circle } r \text{ with dot at } -1 \text{)} + \text{ (blue circle } 1 \text{ with dot at } 1 \text{)}$$

This expression can be obtained using repeatedly bubble slide Relation (3.54). Observe that, at each step but the first one, only one term survives, the second being systematically zero since it includes a real bubble whose inside is a label with a negative entry.

If one replaces, in this expression, the bubble colored n using Equation (4.14), one recognizes precisely the image under $\Sigma_{n,r}$ of $\boxed{1} - \boxed{r} - \boxed{y}$.

We have checked that $\Sigma_{n,r}$ preserves all the relations of $\mathcal{DEBim}_{\widehat{A}_{r-1}}^*$, so this ends the proof. \square

5. A 2-REPRESENTATION OF $\widehat{\mathcal{S}}(n, r)_{[y]}^*$

In this section we define a 2-category $\mathcal{ESBim}_{\widehat{A}_{r-1}}$ and a 2-functor

$$\mathcal{F}' : \widehat{\mathcal{S}}(n, r)_{[y]}^* \rightarrow \mathcal{ESBim}_{\widehat{A}_{r-1}}^*.$$

The 2-category $\mathcal{ESBim}_{\widehat{A}_{r-1}}$ is an extension of the category of singular Soergel bimodules in affine type A considered by Williamson in [Wil11] (see also [MSV11, MSV13]). The 2-functor \mathcal{F}' is a generalization of Khovanov and Lauda's 2-representation Γ_r^G defined in [KL10, KL11].

5.1. Extended singular bimodules. Let $R = \mathbb{Q}[y][x_1, \dots, x_r]$. As in Section 2.1.2, there is a grading on R defined by $\deg y = \deg x_i = 2$, for $i = 1, \dots, r$, and a degree preserving action of $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ on R . For any partition (i_1, \dots, i_k) of r , let

$$S_{i_1} \times \dots \times S_{i_k} \subseteq W_{A_{r-1}} \subset \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$$

be the parabolic subgroup which is contained in the finite Weyl group. let $R_{i_1 \dots i_k} \subseteq R$ denote the subring of $S_{i_1} \times \dots \times S_{i_k}$ -invariant polynomials.

Using these rings of partially symmetric polynomials, we can construct bimodules by induction and restriction. Induction is defined as follows: suppose i_j splits into i_j^0 plus i_j^1 , then we define the *induction functor* by

$$\text{Ind}_{i_j}^{i_j^0, i_j^1}(R_{i_1 \dots i_k}) := R_{i_1 \dots i_j^0 i_j^1 \dots i_k} \otimes_{i_1 \dots i_k} R_{i_1 \dots i_k}.$$

The subscript of the tensor product means that it is taken over $R_{i_1 \dots i_k}$. The *restriction functor* is defined by

$$\text{Res}_{i_j, i_{j+1}}^{i_j + i_{j+1}}(R_{i_1 \dots i_k}) := R_{i_1 \dots i_j + i_{j+1} \dots i_k} | R_{i_1 \dots i_k}.$$

Definition 5.1. Let $\mathcal{ESBim}_{\widehat{A}_{r-1}}$ be the 2-category whose objects are the rings $R_{i_1 \dots i_k}$, for all partitions (i_1, \dots, i_k) of r .

For any two partitions (i_1, \dots, i_k) and (j_1, \dots, j_l) of r , the 1-morphisms between $R_{i_1 \dots i_k}$ and $R_{j_1 \dots j_l}$ are by definition the direct sums and shifts of tensor products of $R_{i_1 \dots i_k} - R_{j_1 \dots j_l}$ -bimodules obtained by induction and restriction and by tensoring with the twisted bimodules $R_{i_k i_1 \dots i_{k-1}, \rho}$ and $R_{i_2 \dots i_k i_1, \rho^{-1}}$, which we will define below.

The 2-morphisms are the degree-preserving bimodule maps.

The twisted bimodules are defined like the bimodules $B_{\rho^{\pm 1}}$ of section 2.1.2: $R_{i_k i_1 \dots i_{k-1}, \rho}$ (resp. $R_{i_2 \dots i_k i_1, \rho^{-1}}$) is equal to $R_{i_k i_1 \dots i_{k-1}}$ (resp. $R_{i_2 \dots i_k i_1}$) as a left $R_{i_k i_1 \dots i_{k-1}}$ -module (resp. as a left $R_{i_2 \dots i_k i_1}$ -module) whereas the action on the right is twisted. The right action of any $a \in R_{i_1 \dots i_k}$ on $R_{i_k i_1 \dots i_{k-1}, \rho}$ and $R_{i_2 \dots i_k i_1, \rho^{-1}}$ is given by multiplication by $\rho^{i_k}(a)$ and $\rho^{-i_1}(a)$ respectively.

Recall that the action of $\rho^{\pm 1}$ was defined in section 2.1.1. The functors defined by tensoring with twisted bimodules are denote by

$$R_{\rho^{i_k}}(R_{i_1 \dots i_k}) := R_{i_k i_1 \dots i_{k-1}, \rho} \otimes_{i_1 \dots i_k} R_{i_1 \dots i_k}$$

and

$$R_{\rho^{-i_1}}(R_{i_1 \dots i_k}) := R_{i_2 \dots i_k i_1, \rho^{-1}} \otimes_{i_1 \dots i_k} R_{i_1 \dots i_k}$$

respectively.

Lemma 5.2. *The map ρ^{i_k} gives an isomorphism between $R_{i_1 \dots i_k}$ and $R_{i_k i_1 \dots i_{k-1}}$, while ρ^{-i_1} gives an isomorphism between $R_{i_1 \dots i_k}$ and $R_{i_2 \dots i_k i_1}$.*

This implies that the twisted bimodules $R_{i_k i_1 \dots i_{k-1}, \rho}$ and $R_{i_2 \dots i_k i_1, \rho^{-1}}$ are well-defined.

Proof. We only prove the lemma for ρ^{i_k} . The proof for ρ^{-i_1} is similar and is left to the reader.

It is clear that ρ^{i_k} is a bijection. What remains to be shown is that the image of $R_{i_1 \dots i_k}$ is indeed $R_{i_k i_1 \dots i_{k-1}}$.

The ring $R_{i_1 \dots i_k}$ is generated by the following elementary symmetric polynomials:

$$\varepsilon_{p_j}(x_{i_1+\dots+i_{j-1}+1}, \dots, x_{i_1+\dots+i_{j-1}+i_j}) = \sum_{\substack{q_1 < \dots < q_{p_j} \\ q_l \in \{1, \dots, i_j\}}} x_{i_1+\dots+i_{j-1}+q_1} \dots x_{i_1+\dots+i_{j-1}+q_{p_j}}$$

for all $j = 1, \dots, k$ and $p_j = 1, \dots, i_j$.

Similarly, the ring $R_{i_k i_1 \dots i_{k-1}}$ is generated by the elementary symmetric polynomials

$$\varepsilon_{p_j}(x_{i_k+i_1+\dots+i_{j-1}+1}, \dots, x_{i_k+i_1+\dots+i_{j-1}+i_j}) = \sum_{\substack{q_1 < \dots < q_{p_j} \\ q_l \in \{1, \dots, i_j\}}} x_{i_1+\dots+i_{j-1}+q_1+i_k} \dots x_{i_1+\dots+i_{j-1}+q_{p_j}+i_k}$$

and

$$\varepsilon_{p_k}(x_1, \dots, x_{i_k}) = \sum_{\substack{q_1 < \dots < q_{p_k} \\ q_l \in \{1, \dots, i_k\}}} x_{q_1} \dots x_{q_{p_k}}$$

for all $j = 1, \dots, k-1$, $p_j = 1, \dots, i_j$ and $p_k = 1, \dots, i_k$.

Note that

$$\rho^{i_k}(x_q) = \begin{cases} x_{q+i_k} & \text{for } q = 1, \dots, i_1 + \dots + i_{k-1} \\ x_{q-i_1-\dots-i_{k-1}} - y & \text{for } q = i_1 + \dots + i_{k-1} + 1, \dots, i_1 + \dots + i_k \end{cases}.$$

Let us look at the image under ρ^{i_k} of the elementary symmetric polynomials generating $R_{i_1 \dots i_k}$. For $j = 1, \dots, k-1$, we have

$$\rho^{i_k}(\varepsilon_{p_j}(x_{i_1+\dots+i_{j-1}+1}, \dots, x_{i_1+\dots+i_{j-1}+i_j})) = \varepsilon_{p_j}(x_{i_1+\dots+i_{j-1}+1+i_k}, \dots, x_{i_1+\dots+i_{j-1}+i_j+i_k}),$$

which indeed belongs to $R_{i_k i_1 \dots i_{k-1}}$.

For $j = k$ and $p_k = 1, \dots, i_k$, we get

$$\rho^{i_k}(\varepsilon_{p_k}(x_{i_1+\dots+i_{k-1}+1}, \dots, x_{i_1+\dots+i_k})) = \sum_{\substack{q_1 < \dots < q_{p_k} \\ q_l \in \{1, \dots, i_k\}}} (x_{q_1} - y) \dots (x_{q_{p_k}} - y).$$

This product is equal to

$$\begin{aligned} & \sum_{\substack{q_1 < \dots < q_{p_k} \\ q_l \in \{1, \dots, i_k\}}} x_{q_1} \dots x_{q_{p_k}} - y \sum_{\substack{q_1 < \dots < q_{p_k-1} \\ q_l \in \{1, \dots, i_k\}}} x_{q_1} \dots x_{q_{p_k-1}} \\ & + y^2 \sum_{\substack{q_1 < \dots < q_{p_k-2} \\ q_l \in \{1, \dots, i_k\}}} x_{q_1} \dots x_{q_{p_k-2}} + \dots + (-1)^{p_k-1} y^{p_k-1} (x_1 + \dots + x_{i_k}) + (-1)^{p_k} y^{p_k} = \\ & \varepsilon_{p_k}(x_1, \dots, x_{i_k}) - y \varepsilon_{p_k-1}(x_1, \dots, x_{i_k}) + y^2 \varepsilon_{p_k-2}(x_1, \dots, x_{i_k}) \\ & + \dots + (-1)^{p_k-1} y^{p_k-1} \varepsilon_{p_k-2}(x_1, \dots, x_{i_k}) + (-1)^{p_k} y^{p_k}, \end{aligned}$$

which again belongs to $R_{i_k i_1 \dots i_{k-1}}$.

This shows that ρ^{i_k} sends the ring $R_{i_1 \dots i_k}$ isomorphically to the ring $R_{i_k i_1 \dots i_{k-1}}$. \square

The proof of the following lemma is straightforward and is left to the reader.

Lemma 5.3. *We have the following isomorphisms of bimodules relating twisting, induction and restriction:*

$$\begin{aligned} R_{\rho^{-i_k}} R_{\rho^{i_k}}(R_{i_1 \dots i_k}) &\cong R_{\rho^{i_1}} R_{\rho^{-i_1}}(R_{i_1 \dots i_k}) \cong R_{i_1 \dots i_k}, \\ \text{Ind}_{i_j}^{j^0, i_j^1} R_{\rho^{i_k}}(R_{i_1 \dots i_k}) &\cong R_{\rho^{i_k}} \text{Ind}_{i_j}^{j^0, i_j^1}(R_{i_1 \dots i_k}) \quad \text{for } j \neq k, \\ \text{Ind}_{i_k}^{j^0, i_k^1} R_{\rho^{i_k}}(R_{i_1 \dots i_k}) &\cong R_{\rho^{i_k}} R_{\rho^{i_k}^1} \text{Ind}_{i_k}^{j^0, i_k^1}(R_{i_1 \dots i_k}), \\ \text{Res}_{i_j, i_{j+1}}^{i_j+i_{j+1}} R_{\rho^{i_k}}(R_{i_1 \dots i_k}) &\cong R_{\rho^{i_k}} \text{Res}_{i_j, i_{j+1}}^{i_j+i_{j+1}}(R_{i_1 \dots i_k}) \quad \text{for } j \neq k-1, \\ \text{Res}_{i_{k-1}, i_k}^{i_{k-1}+i_k} R_{\rho^{i_{k-1}}} R_{\rho^{i_k}}(R_{i_1 \dots i_k}) &\cong R_{\rho^{i_{k-1}+i_k}} \text{Res}_{i_{k-1}, i_k}^{i_{k-1}+i_k}(R_{i_1 \dots i_k}). \end{aligned}$$

There exist analogous isomorphisms for the negative twists.

Lemma 5.4. *The category $\mathcal{EBim}_{\hat{A}_{r-1}}$ is a full subcategory of $\mathcal{ESBim}_{\hat{A}_{r-1}}$.*

Proof. For $i = 1, \dots, r-1$, the full embedding of $\mathcal{EBim}_{\hat{A}_{r-1}}$ into $\mathcal{ESBim}_{\hat{A}_{r-1}}$ sends B_i to B_i and $B_{\rho^{\pm 1}}$ to $B_{\rho^{\pm 1}} = R_{(1^r), \rho^{\pm 1}}$. For $i = r$, the bimodule B_r is sent to

$$B_{\rho} \otimes_R B_{r-1} \otimes_R B_{\rho^{-1}} \in \mathcal{ESBim}_{\hat{A}_{r-1}}.$$

The fact that the isomorphism in $\mathcal{EBim}_{\hat{A}_{r-1}}$ between B_r and $B_{\rho} \otimes_R B_{r-1} \otimes_R B_{\rho^{-1}}$ is unique ensures that $\mathcal{EBim}_{\hat{A}_{r-1}}$ is a full subcategory of $\mathcal{ESBim}_{\hat{A}_{r-1}}$. \square

5.2. The 2-representation.

We will mostly refer the reader to [KL10, KL11, MSV13] for the definition of

$$\mathcal{F}' : \widehat{\mathcal{S}}(n, r)_{[y]}^* \rightarrow \mathcal{ESBim}_{\widehat{A}_{r-1}}^*,$$

since \mathcal{F}' is a straightforward generalization of the equivariant Khovanov-Lauda 2-representations discussed in those papers.

Remark 5.5. Khovanov and Lauda used the equivariant cohomology rings of the varieties of partial flags in \mathbb{C}^r for the definition of their equivariant 2-representations. These cohomology rings are isomorphic to the finite type A singular Soergel bimodules which were used in [MSV13].

We do not know if the 2-representation in this paper, which we define using the extended affine singular Soergel bimodules, can be defined in terms of equivariant cohomology rings of the varieties of cyclic partial flags (or periodic lattices) in $\mathbb{C}[\varepsilon, \varepsilon^{-1}]^r$ defined in [Lus99] and [GV93].

5.2.1. *Definition of \mathcal{F}' .* Note that for $y = 0$ the restriction of $\mathcal{F}' \circ \Psi_{n,r}$ to $\mathcal{U}(\mathfrak{sl}_n)$ is simply equal to Γ_r^G , where

$$\Psi_{n,r} : \mathcal{U}(\widehat{\mathfrak{sl}_n})_{[y]}^* \rightarrow \widehat{\mathcal{S}}(n, r)_{[y]}^*$$

was defined just before Proposition 3.23.

We will define the 2-functor \mathcal{F}' on all objects and 1-morphisms of $\widehat{\mathcal{S}}(n, r)_{[y]}^*$, give explicitly the images of the 2-morphisms for which the color n appears and explain how they are related to Khovanov and Lauda's 2-representation Γ_r^G . Here we are using their notation $k_i = \lambda_1 + \dots + \lambda_i$, for $i = 1, \dots, n$, with the convention that $k_0 = 0$.

- On objects $\lambda \in \Lambda(n, r)$, the 2-functor \mathcal{F}' is given by:

$$\lambda = (\lambda_1, \dots, \lambda_n) \mapsto R_{\lambda_1 \dots \lambda_n}.$$

- On 1-morphisms we define \mathcal{F}' as follows:

$$\mathbf{1}_\lambda \{t\} \mapsto R_{\lambda_1 \dots \lambda_n} \{t\}.$$

For $i = 1, \dots, n-1$ and $t \in \mathbb{Z}$ we define:

$$\mathcal{E}_i \mathbf{1}_\lambda \{t\} \mapsto \text{Res}_{\lambda_i, 1}^{\lambda_i+1} \text{Ind}_{\lambda_{i+1}}^{1, \lambda_{i+1}-1} (R_{\lambda_1 \dots \lambda_n} \{t+1+k_{i-1}+k_i-k_{i+1}\})$$

and

$$\mathcal{E}_{-i} \mathbf{1}_\lambda \{t\} \mapsto \text{Res}_{1, \lambda_{i+1}}^{\lambda_{i+1}+1} \text{Ind}_{\lambda_i}^{\lambda_i-1, 1} (R_{\lambda_1 \dots \lambda_n} \{t+1-k_i\}).$$

For $i = n$ and $t \in \mathbb{Z}$ we define:

$$\begin{aligned} \mathcal{E}_n \mathbf{1}_\lambda \{t\} \mapsto \\ \text{Res}_{\lambda_n, 1}^{\lambda_n+1} R_{\rho-1} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} (R_{\lambda_1 \dots \lambda_n} \{t+n-(r+k_1)-(k_1+\dots+k_{n-2})\}) \end{aligned}$$

and

$$\mathcal{E}_{-n} \mathbf{1}_\lambda \{t\} \mapsto \text{Res}_{1, \lambda_1}^{\lambda_1+1} R_\rho \text{Ind}_{\lambda_n}^{\lambda_n-1, 1} (R_{\lambda_1 \dots \lambda_n} \{t+k_1+\dots+k_{n-1}\}).$$

Remark 5.6. Let us explain where the shifts in the image of the new 1-morphisms come from. We will denote by $A_n(\lambda)$ the shift appearing in $\mathcal{F}'(\mathcal{E}_n \mathbf{1}_\lambda)$ and by $B_n(\lambda)$ the one appearing in $\mathcal{F}'(\mathcal{E}_{-n} \mathbf{1}_\lambda)$. To understand their origin, let us go back to the decategorified level (see Section 3.4), where the embedding of $\widehat{\mathbf{S}}(n, r)$ into $\widehat{\mathbf{S}}(n+1, r)$ sends $E_n \mathbf{1}_\lambda$ to $E_n E_{n+1} \mathbf{1}_{(\lambda, 0)}$ and $E_{-n} \mathbf{1}_\lambda$ to $E_{-(n+1)} E_{-n} \mathbf{1}_{(\lambda, 0)}$. We want this embedding to have a categorical analogue. Although we will not work out the details of the corresponding functor in this paper, a necessary condition for the existence of such a functor is that the shifts satisfy the following recurrence relations:

$$\begin{aligned} A_n(\lambda) &= k_{n-1} - 1 + A_{n+1}(\lambda) \\ B_n(\lambda) &= B_{n+1}(\lambda') + 1 - k_n \end{aligned}$$

where $\lambda' = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n - 1, 1)$. This determines the value of $A_n(\lambda)$ and $B_n(\lambda)$ up to a constant which does not depend on n . To fix these constants, we use the fact that we want the triangle of lemma 6.1 to be commutative and the 2-functor \mathcal{F}' to be degree preserving. Note that $\mathcal{F}(-)$ has a shift equal to zero, so $\mathcal{F}'(\mathcal{E}_r \dots \mathcal{E}_1 \mathcal{E}_{r+1} \dots \mathcal{E}_n \mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(-)$ should have a shift equal to zero too. In this way we obtain a constraint on $A_n((1^r))$ and we deduce that the aforementioned overall constant is equal to $-(r + k_1)$. Similarly $\mathcal{F}(+)$ has a shift equal to zero, so $\mathcal{F}'(\mathcal{E}_{-n} \dots \mathcal{E}_{-r-1} \mathcal{E}_{-1} \dots \mathcal{E}_{-r} \mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(+)$ has to have a zero-shift too. This gives us a constraint on $B_n((1^r) + \bar{\alpha}_n)$ and allows us to deduce that the aforementioned overall constant is equal to zero. The reader can verify that this choice of overall constants also fits with the -1 -shift of $\mathcal{F}'(\mathcal{E}_{-n} \dots \mathcal{E}_{-r} \mathcal{E}_r \dots \mathcal{E}_n \mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(r)$.

• To define \mathcal{F}' on 2-morphisms, we have to give the bimodule maps which correspond to the generating 2-morphisms of $\widehat{\mathcal{S}}(n, r)_{[y]}^*$. The ones not involving color n have the same image as under Khovanov and Lauda's 2-representation [KL10, KL11, MSV13].

When the color n occurs in a generating 2-morphism, the 2-functor \mathcal{F}' is as follows. Here ε_α and η_α denote the elementary and the complete symmetric polynomials respectively.

$$\mathcal{F}' \left(\begin{array}{c} \text{cup} \\ n \end{array} \lambda \right) : 1 \mapsto$$

$$\begin{aligned} &\sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} x_1^f \otimes 1 \otimes 1 \otimes \varepsilon_{\lambda_n-f}(x_{r-\lambda_n+1} + y, \dots, x_r + y) = \\ &\sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} (x_1 - y)^f \otimes 1 \otimes 1 \otimes \varepsilon_{\lambda_n-f}(x_{r-\lambda_n+1}, \dots, x_r) \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{cup} \\ n \end{array} \lambda \right) : 1 \mapsto$$

$$\begin{aligned} &(-1)^{\lambda_1} \sum_{f=0}^{\lambda_1} (-1)^{\lambda_1-f} (x_r + y)^f \otimes 1 \otimes 1 \otimes \varepsilon_{\lambda_1-f}(x_1, \dots, x_{\lambda_1}) = \\ &(-1)^{\lambda_1} \sum_{f=0}^{\lambda_1} (-1)^{\lambda_1-f} x_r^f \otimes 1 \otimes 1 \otimes \varepsilon_{\lambda_1-f}(x_1 - y, \dots, x_{\lambda_1} - y) \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} n \\ \text{blue arc} \\ \lambda \end{array} \right) : x_1^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2} \mapsto$$

$$\begin{aligned} & (-1)^{\lambda_1+1} \eta_{\alpha_1+\alpha_2+1-\lambda_1}(x_1, \dots, x_{\lambda_1}) = \\ & (-1)^{\lambda_1+1} \sum_{p=0}^{\alpha_1} \sum_{q=0}^{\alpha_2} \binom{\alpha_1}{p} \binom{\alpha_2}{q} y^{\alpha_1+\alpha_2-p-q} \eta_{p+q+1-\lambda_1}(x_1-y, \dots, x_{\lambda_1}-y) \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} n \\ \text{blue arc} \\ \lambda \end{array} \right) : x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_r^{\alpha_2} \mapsto$$

$$\begin{aligned} & \sum_{p=0}^{\alpha_1} \sum_{q=0}^{\alpha_2} \binom{\alpha_1}{p} \binom{\alpha_2}{q} (-y)^{\alpha_1+\alpha_2-p-q} \eta_{\alpha_1+\alpha_2+1-\lambda_n}(x_{r-\lambda_n+1}+y, \dots, x_r+y) = \\ & \eta_{\alpha_1+\alpha_2+1-\lambda_n}(x_{r-\lambda_n+1}, \dots, x_r) \end{aligned}$$

If $|n-j| > 1$,

$$\mathcal{F}' \left(\begin{array}{c} \text{blue and magenta crossing} \\ n \text{ blue, } j \text{ magenta} \\ \lambda \end{array} \right) : x_r^{\alpha_1} \otimes 1 \otimes x_{k_j+1}^{\alpha_2} \mapsto x_{k_j}^{\alpha_2} \otimes 1 \otimes (x_1-y)^{\alpha_1}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{blue and magenta crossing} \\ j \text{ magenta, } n \text{ blue} \\ \lambda \end{array} \right) : x_{k_j}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2} \mapsto (x_r+y)^{\alpha_2} \otimes 1 \otimes x_{k_j+1}^{\alpha_1}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{blue and magenta crossing} \\ n \text{ blue, } j \text{ magenta} \\ \lambda \end{array} \right) : x_1^{\alpha_1} \otimes 1 \otimes x_{k_j}^{\alpha_2} \mapsto x_{k_j+1}^{\alpha_2} \otimes 1 \otimes (x_r+y)^{\alpha_1}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{blue and magenta crossing} \\ j \text{ magenta, } n \text{ blue} \\ \lambda \end{array} \right) : x_{k_j+1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \mapsto (x_1-y)^{\alpha_2} \otimes 1 \otimes x_{k_j}^{\alpha_1}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{blue crossing} \\ n \text{ blue, } n \text{ blue} \\ \lambda \end{array} \right) : x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2} \mapsto$$

$$\begin{aligned} & (x_r+y)^{\alpha_2} \sum_{p=0}^{\alpha_1} \sum_{f=0}^{p-1} \binom{\alpha_1}{p} (-y)^{\alpha_1-p} (x_r+y)^{p-1-f} \otimes 1 \otimes 1 \otimes x_1^f \\ & - x_r^{\alpha_1} \sum_{g=0}^{\alpha_2-1} (x_r+y)^{\alpha_2-1-g} \otimes 1 \otimes 1 \otimes x_1^g = \\ & (x_r+y)^{\alpha_2} \sum_{f=0}^{\alpha_1-1} x_r^{\alpha_1-1-f} \otimes 1 \otimes 1 \otimes (x_1-y)^f \\ & - x_r^{\alpha_1} \sum_{q=0}^{\alpha_2} \sum_{g=0}^{q-1} \binom{\alpha_2}{q} y^{\alpha_2-q} x_r^{q-1-g} \otimes 1 \otimes 1 \otimes (x_1-y)^g \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two blue lines crossing, top-left to bottom-right and top-right to bottom-left.} \\ n \quad n \end{array} \lambda \right) : x_1^{\alpha_1} \otimes 1 \otimes 1 \otimes x_r^{\alpha_2} \mapsto$$

$$\begin{aligned} & x_1^{\alpha_1} \sum_{p=0}^{\alpha_2} \sum_{f=0}^{p-1} \binom{\alpha_2}{p} (-y)^{\alpha_2-p} x_1^{p-1-f} \otimes 1 \otimes 1 \otimes (x_r + y)^f \\ & - (x_1 - y)^{\alpha_2} \sum_{g=0}^{\alpha_1-1} x_1^{\alpha_1-1-g} \otimes 1 \otimes 1 \otimes (x_r + y)^g = \\ & x_1^{\alpha_1} \sum_{f=0}^{\alpha_2-1} (x_1 - y)^{\alpha_2-1-f} \otimes 1 \otimes 1 \otimes x_r^f \\ & - (x_1 - y)^{\alpha_2} \sum_{q=0}^{\alpha_1} \sum_{g=0}^{q-1} \binom{\alpha_1}{q} y^{\alpha_1-q} (x_1 - y)^{q-1-g} \otimes 1 \otimes 1 \otimes x_r^g \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left blue to bottom-right green, top-right green to bottom-left blue.} \\ 1 \quad n \end{array} \lambda \right) : x_{\lambda_1}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2} \mapsto$$

$$\begin{aligned} & \left((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1+1} - (x_r + y)^{\alpha_2+1} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1} \right) \{-1\} = \\ & \left((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1} (x_{\lambda_1+1} - y) - x_r (x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1} \right) \{-1\} \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left red to bottom-right blue, top-right blue to bottom-left red.} \\ n \quad n-1 \end{array} \lambda \right) : x_r^{\alpha_1} \otimes 1 \otimes x_{r-\lambda_n+1}^{\alpha_2} \mapsto$$

$$\begin{aligned} & \left((x_{r-\lambda_n} + y)^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1} x_1 - (x_{r-\lambda_n} + y) x_{r-\lambda_n}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1} \right) \{-1\} = \\ & \left((x_{r-\lambda_n} + y)^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1+1} - x_{r-\lambda_n}^{\alpha_2+1} \otimes 1 \otimes (x_1 - y)^{\alpha_1} \right) \{-1\} \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left blue to bottom-right green, top-right green to bottom-left blue.} \\ 1 \quad n \end{array} \lambda \right) : x_{\lambda_1+1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \mapsto \left((x_1 - y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1}^{\alpha_1} \right) \{-1\}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left red to bottom-right blue, top-right blue to bottom-left red.} \\ n \quad n-1 \end{array} \lambda \right) : x_1^{\alpha_1} \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_2} \mapsto \left(x_{r-\lambda_n+1}^{\alpha_2} \otimes 1 \otimes (x_r + y)^{\alpha_1} \right) \{-1\}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left green to bottom-right blue, top-right blue to bottom-left green.} \\ n \quad 1 \end{array} \lambda \right) : x_r^{\alpha_1} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_2} \mapsto \left(x_{\lambda_1}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1} \right) \{1\}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left blue to bottom-right red, top-right red to bottom-left blue.} \\ n-1 \quad n \end{array} \lambda \right) : x_{r-\lambda_n}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2} \mapsto \left((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{r-\lambda_n+1}^{\alpha_1} \right) \{1\}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left green to bottom-right blue, top-right blue to bottom-left green.} \\ n \quad 1 \end{array} \lambda \right) : x_1^{\alpha_1} \otimes 1 \otimes x_{\lambda_1}^{\alpha_2} \mapsto$$

$$\begin{aligned} & \left(x_{\lambda_1+1}^{\alpha_2+1} \otimes 1 \otimes (x_r + y)^{\alpha_1} - x_{\lambda_1+1}^{\alpha_2} \otimes 1 \otimes (x_r + y)^{\alpha_1+1} \right) \{1\} = \\ & \left(x_{\lambda_1+1}^{\alpha_2} (x_{\lambda_1+1} - y) \otimes 1 \otimes (x_r + y)^{\alpha_1} - x_{\lambda_1+1}^{\alpha_2} \otimes 1 \otimes x_r (x_r + y)^{\alpha_1} \right) \{1\} \end{aligned}$$

$$\mathcal{F}' \left(\begin{array}{c} \text{Diagram: Two lines crossing, top-left red to bottom-right blue, top-right blue to bottom-left red.} \\ n-1 \quad n \end{array} \lambda \right) : x_{r-\lambda_n+1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \mapsto$$

$$\begin{aligned} & \left((x_1 - y)^{\alpha_2} x_1 \otimes 1 \otimes (x_{r-\lambda_n} + y)^{\alpha_1} - (x_1 - y)^{\alpha_2} \otimes 1 \otimes (x_{r-\lambda_n} + y) x_{r-\lambda_n}^{\alpha_1} \right) \{1\} = \\ & \left((x_1 - y)^{\alpha_2+1} \otimes 1 \otimes (x_{r-\lambda_n} + y)^{\alpha_1} - (x_1 - y)^{\alpha_2} \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_1+1} \right) \{1\} \end{aligned}$$

$$\begin{aligned}
\mathcal{F}' \left(\begin{array}{c} \text{blue strand with dot} \\ \uparrow \\ n \end{array} \right) & : 1 \otimes 1 \mapsto x_r \otimes 1 = 1 \otimes (x_1 - y) \\
\mathcal{F}' \left(\begin{array}{c} \text{blue strand with dot} \\ \downarrow \\ n \end{array} \right) & : 1 \otimes 1 \mapsto (x_1 - y) \otimes 1 = 1 \otimes x_r
\end{aligned}$$

Remark 5.7. It is natural to wonder how the previous images relate to Khovanov and Lauda's 2-representation Γ_r^G . Indeed, take a generating 2-morphism with n -strands. By Lemma 5.3, the images of its source and target 1-morphisms are isomorphic, up to a same shift, to conjugates of the images of 1-morphisms which do not contain factors $\mathcal{E}_{\pm n}$. Here conjugation means conjugation by certain invertible twisted bimodules. One can thus ask if the image of the 2-morphism can be obtained by conjugating a 2-morphism which does not contain strands of color n . Before answering this question, let us do an example to make things more concrete (we omit the shifts here). Consider the 2-morphism



with $|n - j| > 1$. The image of its source and target 1-morphisms are

$$\text{Res}_{\lambda_n, 1}^{\lambda_n+1} R_{\rho^{-1}} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} \text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} (R_{\lambda_1, \dots, \lambda_n})$$

and

$$\text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} \text{Res}_{\lambda_n, 1}^{\lambda_n+1} R_{\rho^{-1}} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} (R_{\lambda_1, \dots, \lambda_n}).$$

These are isomorphic to

$$R_{\rho^{-(\lambda_n+1)}} \text{Res}_{\lambda_n, 1}^{\lambda_n+1} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} \text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} R_{\rho^{\lambda_n}} (R_{\lambda_1, \dots, \lambda_n}) \quad (5.1)$$

and

$$R_{\rho^{-(\lambda_n+1)}} \text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} \text{Res}_{\lambda_n, 1}^{\lambda_n+1} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} R_{\rho^{\lambda_n}} (R_{\lambda_1, \dots, \lambda_n}) \quad (5.2)$$

respectively, where in both cases the isomorphism is given by

$$a \otimes 1 \otimes b \mapsto a \otimes 1 \otimes 1 \otimes \rho^{\lambda_n}(b) = 1 \otimes \rho^{\lambda_n+1}(a) \otimes \rho^{\lambda_n}(b) \otimes 1. \quad (5.3)$$

The inverse is given by

$$1 \otimes a \otimes b \otimes 1 = \rho^{-(\lambda_n+1)}(a) \otimes 1 \otimes \rho^{-\lambda_n}(b) = 1 \otimes \rho^{-\lambda_n}(a) \otimes \rho^{-\lambda_n}(b). \quad (5.4)$$

Note that the tensor factors

$$\text{Res}_{\lambda_n, 1}^{\lambda_n+1} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} \text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} (R_{\lambda_n, \lambda_1, \dots, \lambda_{n-1}})$$

and

$$\text{Res}_{\lambda_j, 1}^{\lambda_j+1} \text{Ind}_{\lambda_{j+1}}^{1, \lambda_{j+1}-1} \text{Res}_{\lambda_n, 1}^{\lambda_n+1} \text{Ind}_{\lambda_1}^{1, \lambda_1-1} (R_{\lambda_n, \lambda_1, \dots, \lambda_{n-1}})$$

in the middle of (5.1) and (5.2) are, up to a same shift, the images of $\mathcal{E}_1 \mathcal{E}_{j+1} \mathbf{1}_{\lambda_n, \lambda_1, \dots, \lambda_{n-1}}$ and $\mathcal{E}_{j+1} \mathcal{E}_1 \mathbf{1}_{\lambda_n, \lambda_1, \dots, \lambda_{n-1}}$ under \mathcal{F}' .

One can see that if one applies the isomorphism (5.3) to $x_r^{\alpha_1} \otimes 1 \otimes x_{k_j+1}^{\alpha_2}$ followed by the tensor product of the identity on the two twisted bimodules

to show that

$$\begin{aligned} \sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} \otimes (x_r + y)^f \otimes 1 \otimes \varepsilon_{\lambda_n-f}(x_{r-\lambda_n+1} + y, \dots, x_r + y) = \\ \sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} \otimes x_r^f \otimes 1 \otimes \varepsilon_{\lambda_n-f}(x_{r-\lambda_n+1}, \dots, x_r). \end{aligned}$$

For a fixed power of x_r , say k , this amounts to showing that

$$\begin{aligned} 1 \otimes x_r^k \otimes 1 \otimes \sum_{i=0}^{\lambda_n-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{\lambda_n-k-i}(x_{r-\lambda_n+1} + y, \dots, x_r + y) = \\ 1 \otimes x_r^k \otimes 1 \otimes \varepsilon_{\lambda_n-k}(x_{r-\lambda_n+1}, \dots, x_r). \end{aligned}$$

This follows from the following lemma.

Lemma 5.8. *For any $0 \leq k \leq n$, we have*

$$\varepsilon_{n-k}(a_1, \dots, a_n) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_n + y).$$

Proof. By induction w.r.t. n . For $n = 0$ there is nothing to prove.

Suppose $n > 0$. Write

$$\varepsilon_{n-k}(a_1, \dots, a_n) = \varepsilon_{n-k}(a_1, \dots, a_{n-1}) + a_n \varepsilon_{n-1-k}(a_1, \dots, a_{n-1}).$$

Note that $n - k = (n - 1) - (k - 1)$, so by induction the sum above is equal to

$$\begin{aligned} \sum_{i=0}^{n-k} (-1)^i \binom{k-1+i}{i} y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y) + \\ a_n \sum_{i=0}^{n-1-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{n-1-k-i}(a_1 + y, \dots, a_{n-1} + y). \end{aligned}$$

Write $a_n = -y + a_n + y$. Then we get

$$\sum_{i=0}^{n-k} (-1)^i \binom{k-1+i}{i} y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y) - \quad (5.6)$$

$$\sum_{i=0}^{n-1-k} (-1)^i \binom{k+i}{i} y^{i+1} \varepsilon_{n-1-k-i}(a_1 + y, \dots, a_{n-1} + y) + \quad (5.7)$$

$$(a_n + y) \sum_{i=0}^{n-1-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{n-1-k-i}(a_1 + y, \dots, a_{n-1} + y). \quad (5.8)$$

After reindexing the sum in (5.7), the difference of the sums in (5.6) and (5.7) becomes

$$\begin{aligned} & \varepsilon_{n-k}(a_1 + y, \dots, a_{n-1} + y) + \\ & \sum_{i=1}^{n-k} (-1)^i \left(\binom{k-1+i}{i} + \binom{k-1+i}{i-1} \right) y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y) = \\ & \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y). \end{aligned}$$

Together with the sum in (5.8), we now get

$$\begin{aligned} & \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{i} y^i (\varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y) + \\ & (a_n + y) \varepsilon_{n-1-k-i}(a_1 + y, \dots, a_{n-1} + y)) = \\ & \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{i} y^i \varepsilon_{n-k-i}(a_1 + y, \dots, a_{n-1} + y, a_n + y). \end{aligned}$$

□

The proof for the image of the left n -cup is similar, using $x_r \otimes 1 \otimes 1 \otimes 1 = 1 \otimes (x_1 - y) \otimes 1 \otimes 1$ and the lemma above with $-y$ instead of y .

The result for the cups implies the result for the caps, because of the biadjointness relations (3.33) and (3.34). If two maps corresponding to a cap both satisfy the biadjointness relations w.r.t. one fixed map associated to the corresponding cup, then the two maps have to be equal.

The result for the upward oriented n -crossing follows from the fact that both bimodule maps satisfy (easy calculations):

- (1) $f(1 \otimes 1 \otimes 1 \otimes 1) = 0$;
- (2) for any $\alpha_1, \alpha_2 \in \mathbb{N}$,

$$(x_r \otimes 1 \otimes 1 \otimes 1) f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) - f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2} (x_1 - y)) = x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2};$$

- (3) for any $\alpha_1, \alpha_2 \in \mathbb{N}$,

$$f(x_r^{\alpha_1+1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) - (1 \otimes 1 \otimes 1 \otimes (x_1 - y)) f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) = x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}.$$

These three properties determine the maps completely by recursion, so they have to be equal.

A similar argument proves the result for downward oriented n -crossings and the remaining cases of the crossings colored $(1, n)$ and $(n, n-1)$ are easy computations.

Just to summarize, all generating 2-morphisms containing n -strands can be obtained by the conjugation trick which turns n into $n-1$, while only the ones whose n -strands have no dots can also be obtained by the conjugation trick which turns n into 1.

Proposition 5.9. $\mathcal{F}' : \widehat{\mathcal{S}}(n, r)_{[y]}^* \rightarrow \mathcal{ESBim}_{A_{r-1}}^*$ is a well-defined degree preserving 2-functor.

Proof. All relations between 2-morphisms which do not have n -colored strands are satisfied by the results in Section 6 in [KL10]. Since no relation in $\widehat{\mathcal{S}}(n, r)_{[y]}$ involves all colors at the same time, the proof that \mathcal{F}' preserves a given relation can always be reduced to the fact that \mathcal{F}' preserves the same relation with colors belonging to $\{1, \dots, n-1\}$ by using the conjugation trick which turns n into $n-1$, except in the case of relations (3.48) and (3.49) for $\{i, j\} = \{1, n\}$. The proof that \mathcal{F}' preserves these relations is straightforward and is left to the reader. \square

Remark 5.10. Note that the twisted bimodules $B_\rho = R_{(1^r), \rho}$ and $B_{\rho^{-1}} = R_{(1^r), \rho^{-1}}$ are isomorphic to the images under \mathcal{F}' of respectively

$$\mathcal{E}_{-n} \dots \mathcal{E}_{-r-1} \mathcal{E}_{-1} \dots \mathcal{E}_{-r} \mathbf{1}_r \quad \text{and} \quad \mathcal{E}_r \dots \mathcal{E}_1 \mathcal{E}_{r+1} \dots \mathcal{E}_n \mathbf{1}_r.$$

As a matter of fact, any twisted singular bimodule is isomorphic to the image under \mathcal{F}' of a certain product of categorified divided powers (see [KLMS12] and [MSV13] for more details on divided powers and extended graphical calculus). Indeed, let $n > r$ and let $\lambda \in \Lambda(n, r)$ be arbitrary. At least one entry of λ is equal to zero, let us assume it is λ_i . Then

$$R_{\lambda_n \lambda_1 \dots \lambda_{n-1}, \rho} \cong \mathcal{F}' \left(\mathcal{E}_{-i-1}^{(\lambda_{i+1})} \dots \mathcal{E}_{-n}^{(\lambda_n)} \mathcal{E}_{-1}^{(\lambda_1)} \dots \mathcal{E}_{-i+1}^{(\lambda_{i-1})} \mathbf{1}_\lambda \right)$$

and

$$R_{\lambda_2 \dots \lambda_n \lambda_1, \rho^{-1}} \cong \mathcal{F}' \left(\mathcal{E}_{i-2}^{(\lambda_{i-1})} \dots \mathcal{E}_1^{(\lambda_2)} \mathcal{E}_n^{(\lambda_1)} \dots \mathcal{E}_i^{(\lambda_{i+1})} \mathbf{1}_\lambda \right).$$

6. THE GROTHENDIECK GROUP OF $\widehat{\mathcal{S}}(n, r)_{[y]}$

The following Lemma is the affine analogue of Lemma 6.6 in [MSV13].

Lemma 6.1. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{DEBim}_{\widehat{A}_{r-1}}^* & \xrightarrow{\mathcal{F}} & \mathcal{ESBim}_{\widehat{A}_{r-1}}^* \\ & \searrow \Sigma_{n,r} & \nearrow \mathcal{F}' \\ & \widehat{\mathcal{S}}(n, r)_{[y]}^*((1^r), (1^r)) & \end{array}$$

Proof. The proof is straightforward and follows from checking the definitions carefully. \square

Note that the 2-hom spaces of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ are finite-dimensional \mathbb{Q} -vector spaces, because the original 2-HOM spaces in $\mathcal{U}(\widehat{\mathfrak{sl}}_n)^*$ are finite-dimensional in each degree, their grading is bounded below and $\deg(y) = 2 > 0$. Therefore, the Karoubi envelope (or idempotent completion) of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, denoted $\text{Kar}(\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]})$, is Krull-Schmidt. The same holds for $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}$, of course.

By the same arguments, we see that the ideal generated by y is virtually nilpotent (for virtually nilpotent ideals and basic facts about them, see Section 3.8.1 and 3.8.2 in [KL10]). This proves that

$$K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{C}_{[y]}) \cong K_0^{\mathbb{Q}(q)}(\text{Kar}\mathcal{C}), \quad (6.1)$$

where \mathcal{C} is $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ or $\widehat{\mathcal{S}}(n, r)$.

Corollary 6.2. *The algebra homomorphism*

$$K_0^{\mathbb{Q}(q)}(\Sigma_{n,r}) : \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \rightarrow K_0^{\mathbb{Q}(q)}(\mathrm{Kar} \widehat{\mathcal{S}}(n,r)_{[y]})$$

is an embedding.

Proof. We already know that $K_0^{\mathbb{Q}(q)}(\mathcal{F})$ is injective, by Theorem 2.12 and the fact that $\mathcal{EBim}_{\widehat{A}_{r-1}}$ is a full sub-2-category of $\mathcal{ESBim}_{\widehat{A}_{r-1}}$. The result now follows from the commutativity of the diagram in Lemma 6.1. \square

Theorem 2.12 and Lemma 6.1 also imply that $\Sigma_{n,r}$ is faithful. We do not know if it is full, as in the finite type A case (Proposition 6.9 in [MSV13]), but we conjecture that to be true.

Conjecture 6.3. *The functor*

$$\Sigma_{n,r} : \mathcal{DEBim}_{\widehat{A}_{r-1}} \rightarrow \widehat{\mathcal{S}}(n,r)_{[y]}((1^r), (1^r))$$

is an equivalence of 2-categories.

Theorem 6.4. *The algebra homomorphism*

$$\gamma : \widehat{\mathcal{S}}(n,r) \rightarrow K_0^{\mathbb{Q}(q)}(\mathrm{Kar} \widehat{\mathcal{S}}(n,r)_{[y]})$$

defined by

$$\gamma(E_{\pm i} 1_{\lambda}) := [\mathcal{E}_{\pm i} \mathbf{1}_{\lambda}]$$

is an isomorphism.

Proof. Khovanov and Lauda proved surjectivity of the homomorphism

$$\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \rightarrow K_0^{\mathbb{Q}(q)}(\mathcal{U}(\widehat{\mathfrak{sl}}_n))$$

in Theorem 1.1 in [KL10]. The same arguments which proved Lemma 7.7 in [MSV13] can thus be used to prove that

$$\gamma : \widehat{\mathcal{S}}(n,r) \rightarrow K_0^{\mathbb{Q}(q)}(\mathrm{Kar} \widehat{\mathcal{S}}(n,r))$$

is surjective. By (6.1) this implies that the analogous homomorphism

$$\gamma : \widehat{\mathcal{S}}(n,r) \rightarrow K_0^{\mathbb{Q}(q)}(\mathrm{Kar} \widehat{\mathcal{S}}(n,r)_{[y]})$$

is surjective.

The rest of the proof follows from Lemma 3.14 and Corollary 6.2 with $A = K_0^{\mathbb{Q}(q)}(\mathrm{Kar} \widehat{\mathcal{S}}(n,r)_{[y]})$. \square

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M.M.: *Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal; Departamento de Matemática, FCT, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal*

email: mmackaay@ualg.pt

A.-L.T.: *Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal; Matematiska Institutionen, Uppsala Universitet, 75106 Uppsala, Sverige*

email: anne-laure.thiel@math.uu.se