

GRAND LEBESGUE SPACES WITH MIXED LOCAL AND GLOBAL AGGRANDIZATION AND THE MAXIMAL AND SINGULAR OPERATORS

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Dedicated to Professor Oleg Besov on the occasion of his 90th birthday

Abstract. The approach to “locally” aggrandize Lebesgue spaces, previously suggested by the authors and based on the notion of “aggrandizer”, is combined with the usual “global” aggrandization. We study properties of such spaces including embeddings, dependence of the choice of the aggrandizer and, in particular, we discuss the question when these spaces are not new, coinciding with globally aggrandized spaces, and when they proved to be new. We study the boundedness of the maximal, singular, and maximal singular operators in the introduced spaces.

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1. Introduction

The grand Lebesgue spaces $L^{p),\theta}(\Omega)$, over sets Ω with finite measure, were introduced in [11,12] and are defined by the norm

$$\|f\|_{L^{p),\theta}(\Omega)} = \sup_{0 < \varepsilon < p-1} \varepsilon^\theta \left(\int_{\Omega} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Grand spaces have been intensively studied during the last decades; see, for instance, [5–7,15,16,24,25]. We refer also to [18] and references therein.

An approach to aggrandize Lebesgue spaces on sets of infinite measure may be found in [24,28–30,32] and is given by

$$\|f\|_{L_a^{p),\theta}(\Omega)} = \sup_{0 < \varepsilon < p-1} \varepsilon^\theta \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}} dx \right)^{\frac{1}{p-\varepsilon}},$$

where a is a weight on Ω , called *aggrandizer*. Note that the condition $a \in L^1(\Omega)$ is necessary and sufficient for the embedding $L^p(\Omega) \hookrightarrow L_a^{p),\theta}(\Omega)$.

In recent papers [26,31] there was suggested an approach to “aggrandize” Lebesgue spaces not globally but on a given set F of measure zero, where the norm was defined by

$$(1) \quad \|f\|_{L_{a,F}^{p),\theta}(\Omega)} := \sup_{0 < \varepsilon < p-1} \varepsilon^\theta \left(\int_{\Omega} |f(x)|^p a_F(x)^{\frac{\varepsilon}{p}} dx \right)^{\frac{1}{p}},$$

with $a_F(x) = \mathbf{a}(\min\{1, \text{dist}(x, F)\}) \in L^\infty(\Omega)$ where $\mathbf{a}: (0, \mathfrak{D}) \rightarrow \mathbb{R}_+$ is continuous near the origin such that $\lim_{x \rightarrow 0^+} \mathbf{a}(x) = 0$. Note that, instead of $a(x)^{\varepsilon p}$ used in [26], in (1) we wrote $a(x)^{\frac{\varepsilon}{p}}$ similarly to notations used in a number of papers on aggrandizers, see e.g. [29], because writing ε/p , instead of εp , is more convenient for formulation of results in this paper. Boundedness of Hardy, maximal, and singular operators was studied in such spaces with local aggrandization.

In particular, the boundary of the domain Ω may be chosen as such a set F . Some application to Poisson equation in this case were considered in [26].

Our goal in this paper is to consider grand Lebesgue spaces aggrandized globally and locally at the same time. We define the norm in this case by

$$\|f\|_{L_{a,\beta}^{p),\varphi}(\Omega)} := \sup_{0 < \varepsilon < p-1} \varphi(\varepsilon) \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\beta(\varepsilon)}{p}} dx \right)^{\frac{1}{p-\varepsilon}},$$

where the local aggrandization is understood with respect to the set where the weight $a(x)$ vanishes, though it may be defined in a more general form,

see (2). Note that in this paper we consider aggrandizers $a(x)$ of a more general form, see Section 2.1. Note also that we mainly do not explicitly use the fact that $a(x)$ certainly vanishes somewhere in the proof of our main results. In particular, this means that we admit usual grand spaces, i.e. $a(x) \equiv 1$, as a particular case. Information about such vanishing is of course important when we want to ensure that the aggrandized space is larger than the usual grand space.

The principal effect we get is the following: such spaces, in general, prove to be not new in the case $\sup_{0 < \varepsilon < p-1} \frac{\beta(\varepsilon)}{\varepsilon} < \infty$, see Theorem 2.6. But they prove to be new when $\frac{\beta(\varepsilon)}{\varepsilon}$ blows up as ε goes to zero, see Section 2.3.

In the spaces $L^{p,\varphi}_{a,\beta}(\Omega)$ we prove theorems on the boundedness of the maximal, singular, and maximal singular operators.

The paper is organized as follows. In Section 2 we introduce new grand Lebesgue type spaces and study several structural properties, e.g. embeddings, dependence of the choice of the aggrandizer, and discuss when such spaces coincide with already known spaces. In Section 3 we study the boundedness of the maximal operator, singular operators with standard kernel, and maximal singular operators. In Section 4, which is the appendix, we recall the notion of Matuszewska–Orlicz indices and discuss the lower Assouad codimension and its connection with power distance functions as Muckenhoupt weights.

- Notation:*
- $L^p(\Omega, w) = \{f \in L^0(\Omega) \mid \int_{\Omega} |f(x)|^p w(x) dx < \infty\}$,
 - $L^p(w) := L^p(\mathbb{R}^n, w)$,
 - $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^n$,
 - $\int_{\Omega} f(x) dx = \frac{1}{|\Omega|} \int_{\Omega} f(x) dx$,
 - $f(x) \lesssim g(x)$ stands for $f(x) \leq Cg(x)$,
 - $f(x) \simeq g(x)$ denotes $f(x) \lesssim g(x) \lesssim f(x)$.

2. Grand spaces with general aggrandization

2.1. Grand Lebesgue spaces with mixed aggrandization. In this section we deal with general definition of grand Lebesgue spaces using aggrandizers of arbitrary form $A(x, \varepsilon)$.

In what follows, we denote by Φ the class of real-valued functions φ which are left-continuous in a neighborhood of the origin, bounded, and almost increasing on $(0, p - 1)$ with $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $\inf_{\varepsilon \in (\delta, p-1)} \varphi(\varepsilon) > 0$, for all $\delta \in (0, p - 1)$.

DEFINITION 2.1. Let $1 < p < \infty$, $\Omega \subseteq \mathbb{R}^n$ be a Lebesgue measurable set, $\varphi \in \Phi$, and $A: \Omega \times (0, p - 1) \rightarrow \mathbb{R}_+$ be a measurable function such that, for

a.e. $x \in \Omega$, $\lim_{\varepsilon \rightarrow 0} A(x, \varepsilon) = 1$. Then the space $L_{A(\cdot, \cdot)}^{p, \varphi}(\Omega)$ is defined as the set of measurable functions f such that

$$(2) \quad \|f\|_{L_{A(\cdot, \cdot)}^{p, \varphi}(\Omega)} := \sup_{0 < \varepsilon < p-1} \varphi(\varepsilon) \left(\int_{\Omega} |f(x)|^{p-\varepsilon} A(x, \varepsilon) \, dx \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

Usual grand Lebesgue spaces over bounded sets with a general function φ were studied in [3].

We write

$$L^{p, \varphi}(\Omega) := L_{A(\cdot, \cdot)}^{p, \varphi}(\Omega) \Big|_{A \equiv 1}.$$

Let us adopt the shorthand $a_{\varepsilon}(x) := A(x, \varepsilon)$.

Following [29, Theorem 3.1], we prove the following lemma.

LEMMA 2.2. *If there exists $t \in (0, 1)$ such that*

$$(3) \quad \sup_{0 < \varepsilon < p-1} \left[\frac{\varphi(\varepsilon)}{\varphi(t\varepsilon)} \right]^{\frac{(p-\varepsilon)(p-t\varepsilon)}{1-t}} \left(\int_{\Omega} a_{\varepsilon}(x)^{\frac{p-t\varepsilon}{\varepsilon(1-t)}} b_{t\varepsilon}(x)^{-\frac{p-\varepsilon}{\varepsilon(1-t)}} \, dx \right)^{\varepsilon} < \infty,$$

then $L_{B(\cdot, \cdot)}^{p, \varphi}(\Omega) \hookrightarrow L_{A(\cdot, \cdot)}^{p, \varphi}(\Omega)$.

PROOF. For $t \in (0, 1)$, by Hölder’s inequality we have

$$(4) \quad \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a_{\varepsilon}(x) \, dx \right)^{\frac{1}{p-\varepsilon}} \\ \leq \left(\int_{\Omega} |f(x)|^{p-t\varepsilon} b_{t\varepsilon}(x) \, dx \right)^{\frac{1}{p-t\varepsilon}} \left(\int_{\Omega} a_{\varepsilon}(x)^{\frac{p-t\varepsilon}{\varepsilon(1-t)}} b_{t\varepsilon}(x)^{-\frac{p-\varepsilon}{\varepsilon(1-t)}} \, dx \right)^{\frac{\varepsilon(1-t)}{(p-\varepsilon)(p-t\varepsilon)}}.$$

From (4), we obtain

$$\sup_{0 < \varepsilon < p-1} \varphi(\varepsilon) \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a_{\varepsilon}(x) \, dx \right)^{\frac{1}{p-\varepsilon}} \\ \leq \sup_{0 < \eta < p-1} \varphi(\eta) \left(\int_{\Omega} |f(x)|^{p-\eta} b_{\eta}(x) \, dx \right)^{\frac{1}{p-\eta}} \\ \times \sup_{0 < \varepsilon < p-1} \frac{\varphi(\varepsilon)}{\varphi(t\varepsilon)} \left(\int_{\Omega} a_{\varepsilon}(x)^{\frac{p-t\varepsilon}{\varepsilon(1-t)}} b_{t\varepsilon}(x)^{-\frac{p-\varepsilon}{\varepsilon(1-t)}} \, dx \right)^{\frac{\varepsilon(1-t)}{(p-\varepsilon)(p-t\varepsilon)}},$$

from which the result follows. \square

Of special interest for us is the case $A(x, \varepsilon) = a(x)^{\beta(\varepsilon)/p}$. Denote

$$(5) \quad L_{a, \beta}^{p, \varphi}(\Omega) := L_{A(\cdot, \cdot)}^{p, \varphi}(\Omega) \Big|_{A(x, \varepsilon) = a(x)^{\frac{\beta(\varepsilon)}{p}}}$$

and

$$L_a^{p),\varphi}(\Omega) := L_{a,\beta}^{p),\varphi}(\Omega) \Big|_{\beta(\varepsilon)=\varepsilon}$$

for the “usual” grand space. Everywhere in the sequel we assume that a is a weight on Ω , i.e. $a \geq 0$ and $a \in L^1(\Omega)$, and $\beta \in \mathfrak{B}$, where

$$\mathfrak{B} := \left\{ \beta : (0, p - 1) \rightarrow [0, \infty) \mid \beta \in L^\infty((0, p - 1)), \lim_{\varepsilon \rightarrow 0^+} \beta(\varepsilon) = 0, \right. \\ \left. \text{and } \beta \text{ is left-continuous in a neighborhood of the origin} \right\}.$$

Note that the condition (3), for the validity of the embedding $L_{b,\beta}^{p),\varphi}(\Omega) \hookrightarrow L_{a,\beta}^{p),\varphi}(\Omega)$, takes the form

$$(6) \quad \sup_{0 < \varepsilon < p-1} \left[\frac{\varphi(\varepsilon)}{\varphi(t\varepsilon)} \right]^{\frac{(p-\varepsilon)(p-t\varepsilon)}{1-t}} \left(\int_{\Omega} a(x)^{\frac{\beta(\varepsilon)}{p} \frac{p-t\varepsilon}{\varepsilon(1-t)}} b(x)^{-\frac{\beta(t\varepsilon)}{p} \frac{p-\varepsilon}{\varepsilon(1-t)}} dx \right)^\varepsilon < \infty.$$

REMARK 2.3. In the case $|\Omega| < \infty$, we expect that the space $L_{a,\beta}^{p),\varphi}(\Omega)$ may be larger than the classical grand space $L^{p),\varphi}(\Omega)$ when $a(x) = 0$. Of main interest, from this point of view, is the case where $a(x)$ is bounded. If so, without loss of generality, we may assume that $a(x) \leq 1$, which follows from the inequalities

$$c_1 \|f\|_{L_{a,\beta}^{p),\varphi}(\Omega)} \leq \|f\|_{L_{\lambda a,\beta}^{p),\varphi}(\Omega)} \leq c_2 \|f\|_{L_{a,\beta}^{p),\varphi}(\Omega)}, \quad \lambda > 0,$$

where $c_1 = c_1(\lambda)$ and $c_2 = c_2(\lambda)$.

LEMMA 2.4. *Let $\delta_0 \in (0, p - 1)$ be a point of left-continuity of β and φ . For $\varepsilon_0 \in [\delta_0, p - 1)$, the norm $\|f\|_{L_{a,\beta}^{p),\varphi}(\Omega)}$ is equivalent to the norm*

$$\|f\|_{L_{a,\beta;\varepsilon_0}^{p),\varphi}(\Omega)} := \sup_{0 < \varepsilon < \varepsilon_0} \varphi(\varepsilon) \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\beta(\varepsilon)}{p}} dx \right)^{\frac{1}{p-\varepsilon}},$$

if

$$(7) \quad \sup_{\delta_0 < \varepsilon < p-1} \int_{\Omega} a(x)^{\alpha(\varepsilon,\delta_0)} dx < \infty,$$

where

$$(8) \quad \alpha(\varepsilon, \delta_0) = \frac{\beta(\delta_0)}{p} + \frac{p - \delta_0}{p} \cdot \frac{\beta(\varepsilon) - \beta(\delta_0)}{\varepsilon - \delta_0} = \frac{(p - \varepsilon)(p - \delta_0)}{p(\varepsilon - \delta_0)} \left[\frac{\beta(\varepsilon)}{p - \varepsilon} - \frac{\beta(\delta_0)}{p - \delta_0} \right].$$

PROOF. It is sufficient to prove the equivalence result for δ_0 .

Since the inequality $\|f\|_{L^{p),\varphi}_{a,\beta;\delta_0}(\Omega)} \leq \|f\|_{L^{p),\varphi}_{a,\beta}(\Omega)}$ is obvious, it suffices to estimate

$$B_{\delta_0} := \sup_{\delta_0 < \varepsilon < p-1} \varphi(\varepsilon) \left(\int_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\beta(\varepsilon)}{p}} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Applying Hölder’s inequality, with the exponent $q = \frac{p-\delta_0}{p-\varepsilon}$, we obtain

$$\begin{aligned} & \int_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\beta(\varepsilon)}{p}} dx \\ & \leq \left(\int_{\Omega} |f(x)|^{p-\delta_0} a(x)^{\frac{\beta(\delta_0)}{p}} dx \right)^{1/q} \left(\int_{\Omega} a(x)^{\frac{q'}{p}(\beta(\varepsilon)-\frac{\beta(\delta_0)}{q})} dx \right)^{1/q'}. \end{aligned}$$

By a straightforward calculation and Fatou’s lemma, we get

$$\begin{aligned} B_{\delta_0} & \leq \|f\|_{L^{p),\varphi}_{a,\beta;\delta_0}(\Omega)} \frac{\sup_{\delta_0 < \varepsilon < p-1} \varphi(\varepsilon)}{\varphi(\delta_0)} \left(\int_{\Omega} a(x)^{\alpha(\varepsilon,\delta_0)} dx \right)^{\frac{\varepsilon-\delta_0}{(p-\delta_0)(p-\varepsilon)}} \\ & \leq c(\delta_0) \|f\|_{L^{p),\varphi}_{a,\beta;\varepsilon_0}(\Omega)}, \end{aligned}$$

which completes the proof. \square

COROLLARY 2.5. *Let Ω be bounded, $a \in L^\infty(\Omega)$, and $\beta(t)/(p-t)$ be increasing on $(0, p-1)$. Then, for any $\varepsilon_0 \in (0, p-1)$, $\|\cdot\|_{L^{p),\varphi}_{a,\beta}(\Omega)}$ and $\|\cdot\|_{L^{p),\varphi}_{a,\beta;\varepsilon_0}(\Omega)}$ are equivalent norms.*

PROOF. We have, from (8), that $\alpha(\varepsilon, \varepsilon_0) = \frac{\beta(\varepsilon)(p-\varepsilon_0)-\beta(\varepsilon_0)(p-\varepsilon)}{p(\varepsilon-\varepsilon_0)} \geq 0$ if and only if $\beta(\varepsilon)/(p-\varepsilon)$ is increasing on $(\varepsilon_0, p-1)$. Taking into account that, according to Remark 2.3, we can suppose that $a(x) \leq 1$, the equivalence of norms now follows from Lemma 2.4, since φ and β are left-continuous for all δ_0 in a small neighborhood of the origin. \square

2.2. On coincidence of the mixed grand spaces with the usual grand spaces. In this section we assume that $|\Omega| < \infty$.

We have the embedding

$$(9) \quad L^{p),\varphi}(\Omega) \hookrightarrow L^{p),\varphi}_{a,\beta}(\Omega)$$

whenever $a \in L^\infty(\Omega)$ and $\beta \in L^\infty(0, p-1)$. In Theorem 2.6 we show that these spaces in fact coincide for a wide class of aggrandizers, namely, apart from an integral condition on the aggrandizer a , if

$$(10) \quad \mathcal{B} := \sup_{0 < \varepsilon < p-1} \frac{\beta(\varepsilon)}{\varepsilon} < \infty.$$

Thus, an additional “local” aggrandization by means of the aggrandizer $a(x)^{\beta(\varepsilon)}$, simultaneous with the “global” aggrandization, does not provide a new type of grand spaces, if (10)–(11) are fulfilled.

We recall that φ is doubling if there exists a constant C such that $\varphi(2x) \leq C\varphi(x)$.

THEOREM 2.6. *Let φ be doubling and $a \in L^\infty(\Omega)$. The spaces $L^{p),\varphi}(\Omega)$ and $L_{a,\beta}^{p),\varphi}(\Omega)$ coincide, up to equivalence of norms, if (10) holds and there exists $\delta > 0$ such that*

$$(11) \quad \int_{\Omega} a(x)^{-\delta} dx < \infty.$$

PROOF. In view of (9), we have to prove the inverse embedding. To this end, we use the condition (6) for the embedding $L_{b,\beta}^{p),\varphi}(\Omega) \hookrightarrow L_{a,\beta}^{p),\varphi}(\Omega)$, where we replace b by a and a by 1, obtaining

$$(12) \quad \sup_{0 < \varepsilon < p-1} \int_{\Omega} \frac{dx}{a(x)^{\frac{\beta(t\varepsilon)}{\varepsilon} \frac{(p-\varepsilon)}{p(1-t)}}} < \infty,$$

where we took into account that φ is doubling, so that $\sup_{0 < \varepsilon < p-1} \frac{\varphi(\varepsilon)}{\varphi(t\varepsilon)} < \infty$. Since $a \in L^\infty(\Omega)$ we can assume that $a(x) \leq 1$, see Remark 2.3. By (10) we have

$$\frac{\beta(t\varepsilon)}{\varepsilon} \frac{(p-\varepsilon)}{p(1-t)} \leq \mathcal{B} \frac{t}{1-t}.$$

Then, in view of (11), the condition (12) is satisfied under the choice of t sufficiently small: $\frac{t}{1-t} = \frac{\delta}{\mathcal{B}}$. \square

2.3. Grand spaces $L_{a,\beta}^{p),\varphi}(\Omega)$ as new spaces when (10) is violated. As a consequence of Theorem 2.6, we can obtain the space $L_{a,\beta}^{p),\varphi}(\Omega)$ with mixed local and global aggrandization larger than the usual grand space $L^{p),\varphi}(\Omega)$ only when the condition (10) is violated, at least when the aggrandizer vanishes at a single point. As can be seen from the results below, a larger space really arises when (10) is violated.

We start with the case where $\varphi(\varepsilon) = \varepsilon^\theta$, $\theta > 0$ (the most popular case in the theory of grand spaces) and $\beta(\varepsilon) = \varepsilon^\lambda$, $0 < \lambda < 1$. Denote

$$f_\nu(x) = \frac{\left(\ln \frac{e}{|x|}\right)^{\frac{\nu}{p}}}{|x|^{\frac{\nu}{p}}}, \quad |x| < 1, \nu > -1.$$

Note that, for $\theta > 0$, we have

$$(13) \quad f_\nu \in L^{p),\varphi} \Big|_{\varphi(\varepsilon)=\varepsilon^\theta} \iff -1 < \nu \leq \theta p - 1,$$

see [2]. We extend the assertion (13) to the case of the spaces $L_{\alpha,\beta}^{p),\varphi}(\Omega)$, see Corollary 2.8.

LEMMA 2.7. *Let $a(x) = |x|$ and $\beta \in \mathfrak{B}$. Then*

$$\left(\int_{B(0,1)} f_\nu(x)^{p-\varepsilon} |x|^{\frac{\beta(\varepsilon)}{p}} dx \right)^{\frac{1}{p-\varepsilon}} \simeq \frac{1}{(\varepsilon + \beta(\varepsilon))^{\frac{\nu+1}{p}}},$$

where the constants implicit in the symbol \simeq do not depend on ε .

PROOF. We have

$$I_\varepsilon := \int_{B(0,1)} \left(\frac{(\ln \frac{e}{|x|})^{\frac{\nu}{p}}}{|x|^{\frac{n}{p}}} \right)^{p-\varepsilon} |x|^{\frac{\beta(\varepsilon)}{p}} dx = C \int_0^1 \left(\ln \frac{e}{t} \right)^{\nu(1-\frac{\varepsilon}{p})} t^{\frac{n\varepsilon}{p}-1+\frac{\beta(\varepsilon)}{p}} dt.$$

By the change of variable $\ln \frac{e}{t} =: y$, we obtain

$$I_\varepsilon \simeq \int_1^\infty y^{\nu(1-\frac{\varepsilon}{p})} e^{-y\frac{n\varepsilon+\beta(\varepsilon)}{p}} dy.$$

The change $\frac{n\varepsilon+\beta(\varepsilon)}{p}y =: s$ yields

$$\begin{aligned} I_\varepsilon &\simeq \left(\frac{n\varepsilon + \beta(\varepsilon)}{p} \right)^{\nu(\frac{\varepsilon}{p}-1)-1} \int_{\frac{n\varepsilon+\beta(\varepsilon)}{p}}^\infty s^{\nu(1-\frac{\varepsilon}{p})} e^{-s} ds \\ &\simeq (n\varepsilon + \beta(\varepsilon))^{\nu(\frac{\varepsilon}{p}-1)-1} \Gamma(\nu(1 - \varepsilon/p) + 1) \simeq (\varepsilon + \beta(\varepsilon))^{\nu(\frac{\varepsilon}{p}-1)-1}, \end{aligned}$$

hence

$$I_\varepsilon^{\frac{1}{p-\varepsilon}} \simeq \frac{1}{(\varepsilon + \beta(\varepsilon))^{\frac{\nu}{p}+\frac{1}{p-\varepsilon}}} = \frac{1}{(\varepsilon + \beta(\varepsilon))^{\frac{\nu+1}{p}}} (\varepsilon + \beta(\varepsilon))^{-\frac{\varepsilon}{p(p-\varepsilon)}}.$$

It remains to observe that $0 < c_1 \leq (\varepsilon + \beta(\varepsilon))^\varepsilon \leq c_2 < \infty$. \square

COROLLARY 2.8. *Let $\varphi \in \Phi$. Under the assumptions of Lemma 2.7,*

$$(14) \quad f_\nu \in L_{\alpha,\beta}^{p),\varphi}(B(0,1))$$

if and only if

$$(15) \quad \sup_{0<\varepsilon<\varepsilon_0} \frac{\varphi(\varepsilon)}{(\varepsilon + \beta(\varepsilon))^{\frac{\nu+1}{p}}} < \infty$$

for some $\varepsilon_0 \in (0, p - 1)$.

PROOF. Since $\varphi \in \Phi$ and $\beta \in \mathfrak{B}$, the condition (15) remains valid if we replace $\sup_{0 < \varepsilon < \varepsilon_0}$ by $\sup_{0 < \varepsilon < p-1}$. After that, it remains to use Lemma 2.7. \square

THEOREM 2.9. Let Ω be an open set in \mathbb{R}^n , $|\Omega| < \infty$, $1 < p < \infty$, $\varphi(\varepsilon) = \varepsilon^\theta$, $\theta > 0$, and $\beta(\varepsilon) = \varepsilon^\lambda$, $0 < \lambda < 1$. Moreover, let $a(x) = |x - x_0|$ if Ω is bounded, and

$$a(x) = \begin{cases} |x - x_0|, & |x - x_0| \leq 1; \\ 1, & |x - x_0| > 1 \end{cases}$$

otherwise. Then

$$(16) \quad L^{p),\varphi}(\Omega) \subsetneq L_{a,\beta}^{p),\varphi}(\Omega).$$

PROOF. We can assume that $x_0 = 0$ and Ω contains $B := B(0, 1)$. Since

$$L_{a,\beta}^{p),\varphi}(\Omega \setminus B) = L^{p),\varphi}(\Omega \setminus B),$$

up to equivalence of norms, we can consider our spaces only over B .

The embedding $L^{p),\varphi}(B) \subseteq L_{a,\beta}^{p),\varphi}(B)$ being obvious, it suffices to show, by a counterexample, the strictness of the embedding. The corresponding counterexample will be provided by the function $f = f_\nu$. By Corollary 2.8 we have $f_\nu \in L_{a,\beta}^{p),\varphi}(B)$ if and only if $\varepsilon^\theta \lesssim \varepsilon^{\lambda \frac{\nu+1}{p}}$, i.e. $\nu \leq \frac{\theta p}{\lambda} - 1$. On the other hand, $f_\nu \in L^{p),\varphi}(B)$ if and only if $\varepsilon^\theta \lesssim \varepsilon^{\frac{\nu+1}{p}}$, i.e. $\nu \leq \theta p - 1$. It remains to choose ν in the interval

$$\theta p - 1 < \nu \leq \frac{\theta p}{\lambda} - 1,$$

which completes the proof. \square

Turning to the validity of the strict embedding of the type (16) for the case of general functions $\varphi(\varepsilon)$ and $\beta(\varepsilon)$, we start with the following statement.

LEMMA 2.10. Let $\Omega = B(0, 1)$, $a(x) = |x|$, $\varphi \in \Phi$, $\beta \in \mathfrak{B}$, and $\beta(\varepsilon) \geq C\varepsilon$. If there exists $\sigma > 0$ such that

$$(17) \quad \sup_{0 < \varepsilon < \varepsilon_0} \frac{\varphi(\varepsilon)}{\beta(\varepsilon)^\sigma} < \infty \quad \text{but} \quad \sup_{0 < \varepsilon < \varepsilon_0} \frac{\varphi(\varepsilon)}{\varepsilon^\sigma} = \infty$$

for some $\varepsilon_0 \in (0, p - 1)$, then

$$(18) \quad L^{p),\varphi}(\Omega) \subsetneq L_{a,\beta}^{p),\varphi}(\Omega).$$

PROOF. We take the function f_ν and choose $(\nu + 1)/p = \sigma$. Since $\varepsilon + \beta(\varepsilon) \simeq \beta(\varepsilon)$ by Corollary 2.8 we have that $f_\nu \in L^{p, \varphi}_{a, \beta}(\Omega)$ but $f_\nu \notin L^{p, \varphi}(\Omega)$, which proves the lemma. \square

The condition (17) of Lemma 2.10 can be dressed into “numerical form”, as is shown in Theorem 2.11. To this end, keeping in mind that $\varphi(\varepsilon)$ and $\beta(\varepsilon)$ tend to zero as $\varepsilon \rightarrow 0$, by

$$(0, d)$$

we denote the interval of those ε_0 for which

$$\max \left\{ \sup_{0 < \varepsilon < \varepsilon_0} \beta(\varepsilon), \sup_{0 < \varepsilon < \varepsilon_0} \varphi(\varepsilon), \varepsilon_0 \right\} < 1.$$

THEOREM 2.11. *Let $\Omega = B(0, 1)$, $a(x) = |x|$, $\varphi \in \Phi$, $\beta \in \mathfrak{B}$, and $\beta(\varepsilon) \geq C\varepsilon$. If there exists an $\varepsilon_0 \in (0, d)$ such that*

$$(19) \quad \sup_{0 < \varepsilon < \varepsilon_0} \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\varepsilon}} < \inf_{0 < \varepsilon < \varepsilon_0} \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\beta(\varepsilon)}},$$

then the strict embedding (18) is valid.

PROOF. First we note that (19) holds for $\varphi(\varepsilon) = \varepsilon^\theta$ and $\beta(\varepsilon) = \varepsilon^\lambda$, when $\theta > 0$ and $0 < \lambda < 1$, and even for $\varepsilon = p - 1$.

To prove (18), we use Lemma 2.10. We have to verify that

$$(20) \quad \varphi(\varepsilon)\beta(\varepsilon)^{-\sigma} \leq c < \infty, \quad 0 < \varepsilon < \varepsilon_0,$$

and

$$(21) \quad \varphi(\varepsilon)\varepsilon^{-\sigma} \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0$$

for some $\sigma > 0$.

We have, for $\varepsilon \in (0, d)$, that

$$\varphi(\varepsilon)\beta(\varepsilon)^{-\sigma} \leq c \text{ with } c > 1 \iff \sigma \leq \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\beta(\varepsilon)}} + \frac{\ln c}{\ln \frac{1}{\beta(\varepsilon)}}.$$

Denote $\mathbb{A} := \inf_{0 < \varepsilon < \varepsilon_0} \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\beta(\varepsilon)}}$. From (19) it follows that $\mathbb{A} > 0$. To satisfy (20), it suffices to choose $\sigma \leq \mathbb{A}$.

To satisfy (21), it suffices to find σ such that

$$(22) \quad \varphi(\varepsilon)\varepsilon^{-\sigma} \geq \xi(\varepsilon),$$

where $\xi(\varepsilon)$ with $\lim_{\varepsilon \rightarrow 0} \xi(\varepsilon) = \infty$ will be chosen in the sequel. We have

$$(22) \iff \sigma \geq \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\varepsilon}} + \frac{\ln[\xi(\varepsilon)]}{\ln \frac{1}{\varepsilon}}.$$

Now we choose $\xi(\varepsilon)$ so that $\frac{\ln[\xi(\varepsilon)]}{\ln \frac{1}{\varepsilon}} = \delta > 0$, i.e. $\xi(\varepsilon) = \varepsilon^{-\delta}$. Then (22) is fulfilled, if $\sigma \geq \mathbb{B} + \delta$, where $\mathbb{B} = \sup_{0 < \varepsilon < \varepsilon_0} \frac{\ln \frac{1}{\varphi(\varepsilon)}}{\ln \frac{1}{\varepsilon}}$. Taking $\delta \in (0, \mathbb{A} - \mathbb{B})$, we obtain (20) and (21) under the choice $\mathbb{B} + \delta \leq \sigma \leq \mathbb{A}$. \square

We conclude the study of the possibility of strict embedding of the type (18) by considering the end-point cases $\lambda = 0$ and $\lambda = 1$ in Theorem 2.9. Correspondingly we consider the following choices of the functions $\beta(\varepsilon)$ and $\varphi(\varepsilon)$:

$$(23) \quad \beta(\varepsilon) = \left(\ln \frac{C}{\varepsilon}\right)^{-\gamma}, \quad \varphi(\varepsilon) = \left(\ln \frac{C}{\varepsilon}\right)^{-\mu}, \quad \gamma > 0, \mu > 0,$$

and

$$(24) \quad \beta(\varepsilon) = \varepsilon \left(\ln \frac{C}{\varepsilon}\right)^s, \quad \varphi(\varepsilon) = \varepsilon^\theta \left(\ln \frac{C}{\varepsilon}\right)^t, \quad s > 0, t > 0, \theta \geq 0,$$

where $C > p - 1$.

THEOREM 2.12. *The strict embedding (18) holds in the case (23), for all $\mu > 0$ and $\gamma > 0$. It also holds in the case (24) for all $t > 0$ and $s > 0$ such that $t/s \leq \theta$.*

PROOF. The proof is obtained by means of Lemma 2.10 with the choice of $\sigma \in (0, \mu/\gamma)$ in the case (23) and $\sigma = \theta$ in the case (24). \square

2.4. Grand spaces with specific aggrandizers. In this section we consider aggrandizers which are functions of the distance to a given closed set F of Lebesgue measure zero.

In what follows, we define

$$(25) \quad \delta_F(x) := \min\{1, \text{dist}(x, F)\}.$$

We consider aggrandizers of the form

$$(26) \quad a(x) = \mathbf{a}(\delta_F(x)),$$

where $\mathbf{a}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and show that for a wide class of such functions the space $L_{\mathbf{a}, \beta}^{p, \varphi}(\Omega)$ does not depend much on the choice of the function \mathbf{a} , see Theorem 2.14.

In the lemma below we impose the following condition on $\beta(\varepsilon)$:

$$(27) \quad \lim_{t \rightarrow 0} \sup_{0 < \varepsilon < p-1} \frac{\beta(t\varepsilon)}{\beta(\varepsilon)} = 0.$$

LEMMA 2.13. *Let $|\Omega| < \infty$, F be bounded, β satisfy condition (27), and φ be doubling. Then the spaces $L_{a,\beta}^{p,\varphi}(\Omega)$ and $L_{b,\beta}^{p,\varphi}(\Omega)$, with $a(x) := (\delta_F(x))^\lambda$ and $b(x) := (\delta_F(x))^\mu$, coincide, up to equivalence of norms, for all $\lambda, \mu \in \mathbb{R}_+$.*

PROOF. Denote $\Omega_F := \{x \in \Omega \mid \text{dist}(x, F) < 1\}$. Since $L_{a,\beta}^{p,\varphi}(\Omega \setminus \Omega_F) = L_{b,\beta}^{p,\varphi}(\Omega \setminus \Omega_F)$, it suffices to prove the coincidence of the spaces $L_{a,\beta}^{p,\varphi}(\Omega_F)$ and $L_{b,\beta}^{p,\varphi}(\Omega_F)$.

Let $0 < \lambda < \mu$. The embedding $L_{a,\beta}^{p,\varphi}(\Omega_F) \hookrightarrow L_{b,\beta}^{p,\varphi}(\Omega_F)$ is immediate. The validity of the converse will follow if condition (6) is satisfied for some $t \in (0, 1)$. Since φ is doubling it remains to show, after simple calculations, that

$$(28) \quad I(\lambda, \mu) := \sup_{0 < \varepsilon < p-1} \int_{\Omega_F} (\delta_F(x))^{\frac{\lambda\beta(\varepsilon) - \mu\beta(t\varepsilon)}{\varepsilon p(1-t)} + \frac{\mu\beta(t\varepsilon)}{p(1-t)} - \frac{\lambda t\beta(\varepsilon)}{p(1-t)}} dx < \infty.$$

In view of (27),

$$\frac{\beta(t\varepsilon)}{\beta(\varepsilon)} \leq \frac{\lambda}{\mu}$$

for all $\varepsilon \in (0, p-1)$ and all sufficiently small $t > 0$. Then $\lambda\beta(\varepsilon) - \mu\beta(t\varepsilon) \geq 0$ for such values of t . Hence

$$(29) \quad I(\lambda, \mu) \leq \int_{\Omega_F} (\delta_F(x))^{-\frac{\lambda t\beta(\varepsilon)}{p(1-t)}} dx.$$

It is known that the function $\delta_F(x)^\gamma$ belongs to the Muckenhoupt class A_p , if $-\mathfrak{N}_F < \gamma < \mathfrak{N}_F(p-1)$, where \mathfrak{N}_F is the so-called codimension of F , see Section 4.2. Since the set Ω_F is bounded, the integral in (29) is finite, if we choose t sufficiently small: $\frac{t}{1-t} < \frac{p\mathfrak{N}_F}{\xi}$, where $\xi = \sup_{0 < \varepsilon < p-1} \beta(\varepsilon)$. \square

In the theorem below we use the notion of quasi-monotone functions and the Matuszewska–Orlicz indices $m(\mathbf{a})$ and $M(\mathbf{a})$ of such functions as introduced in [21], see also [27] where properties of these indices are given in a form convenient for our goals. For the reader’s convenience, we provide definition and some properties of such indices in Section 4.1.

THEOREM 2.14. *Let $|\Omega| < \infty$, F be a bounded set with $|F| = 0$, β satisfy the condition (27), φ be doubling, $a(x) = \mathbf{a}(\delta_F(x))$, and $b(x) = \mathbf{b}(\delta_F(x))$, where \mathbf{a} and \mathbf{b} are quasi-monotone functions on $(0, 1)$. If $m(\mathbf{a}) > 0$ and*

$m(\mathbf{b}) > 0$, then the spaces $L_{\mathbf{a},\beta}^{p,\varphi}(\Omega)$ and $L_{\mathbf{b},\beta}^{p,\varphi}(\Omega)$ coincide, up to equivalence of norms.

PROOF. By the property (48), we have

$$(30) \quad c_1 \delta_F(x)^{M(\mathbf{a})+\xi} \leq \mathbf{a}(\delta_F(x)) \leq c_2 \delta_F(x)^{m(\mathbf{a})-\xi},$$

and

$$(31) \quad c_3 \delta_F(x)^{M(\mathbf{b})+\xi} \leq \mathbf{b}(\delta_F(x)) \leq c_4 \delta_F(x)^{m(\mathbf{b})-\xi},$$

where $\xi > 0$ may be chosen arbitrarily small, so that $m(\mathbf{a}) - \xi > 0$ and $m(\mathbf{b}) - \xi > 0$. The norms corresponding to all four power bounds in (30) and (31) are equivalent by Lemma 2.13. Then the same is true for the norms corresponding to $\mathbf{a}(\delta(x))$ and $\mathbf{b}(\delta(x))$. \square

3. Boundedness of operators

3.1. The maximal operator. Recall that the maximal operator M_Ω is defined by

$$(32) \quad M_\Omega f(x) := \sup_{r>0} \int_{B(x,r) \cap \Omega} |f(y)| \, dy,$$

and let M denote $M_{\mathbb{R}^n}$.

The Muckenhoupt class $A_p(\Omega)$, $1 < p < \infty$, is introduced as the set of all weights w on Ω , i.e. $w \geq 0$ a.e. and $w \in L^1_{\text{loc}}(\Omega)$, such that

$$(33) \quad [w]_{p,\Omega} := \sup_{B \subset \mathbb{R}^n} \int_{B \cap \Omega} w \, dx \left(\int_{B \cap \Omega} w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken with respect to all balls B in \mathbb{R}^n . For $p = \infty$, the class $A_\infty(\Omega)$ is defined by $A_\infty(\Omega) := \bigcup_{1 < p < \infty} A_p(\Omega)$. We write A_p and $[w]_p$ for $A_p(\mathbb{R}^n)$ and $[w]_{p,\mathbb{R}^n}$, respectively. Recall that M is bounded on $L^p(w)$, $1 < p < \infty$, if and only if $w \in A_p$. An excellent reference for this topic is [9, Chapter IV].

In the next lemma we single out one property of Muckenhoupt weights, due to its importance in subsequent proofs.

LEMMA 3.1. *Let $1 < q < \infty$ and $w \in A_q(\Omega)$. Then, for $1 < p < q$ and $0 < \lambda \leq \frac{p-1}{q-1}$, we have $[w^\lambda]_{p,\Omega} \leq [w]_{q,\Omega}^\lambda$.*

PROOF. For $0 < \alpha \leq 1$ and $1 < q < \infty$, by the concavity of the function $f(t) = t^\alpha$, we have $[w^\alpha]_{q,\Omega} \leq [w]_{q,\Omega}^\alpha$. For $1 < p < q$ and $\lambda_0 = \frac{p-1}{q-1}$, the estimate $[w^{\lambda_0}]_{p,\Omega} \leq [w]_{q,\Omega}^{\lambda_0}$ follows, once more, from concavity arguments. It suffices now to note that $[w^\lambda]_{p,\Omega} = [w^{\lambda_0 \frac{\lambda}{\lambda_0}}]_{p,\Omega} \leq [w^{\frac{\lambda}{\lambda_0}}]_{q,\Omega}^{\lambda_0} \leq [w]_{q,\Omega}^\lambda$. \square

By $\mathcal{G}(\Omega)$ we denote the class of weights w on Ω with an A_∞ -power, viz.

$$(34) \quad \mathcal{G}(\Omega) := \{w \mid \text{there exists } \delta > 0 \text{ such that } w^\delta \in A_\infty(\Omega)\},$$

and $\mathcal{G} := \mathcal{G}(\mathbb{R}^n)$.

REMARK 3.2. The class $\mathcal{G}(\Omega)$ coincides, by Lemma 3.1, with the class of weights

$$\tilde{\mathcal{G}}(\Omega) := \{w \mid \text{for all } p \in (1, \infty) \text{ there exists } \delta_p > 0 \text{ such that } w^{\delta_p} \in A_p(\Omega)\},$$

and we will use them interchangeably without further comment. The introduction of the class \mathcal{G} goes back to [33].

In what follows, we shall concern ourselves only with open sets Ω of type \mathcal{A} , viz. if $\Omega \subset \mathbb{R}^n$ and there exists a constant $A > 0$ such that, for every $x \in \bar{\Omega}$ and all $r \in (0, \text{diam}(\Omega))$, $|\Omega \cap B(x, r)| \geq Ar^n$.

The class of weights $\mathcal{G}(\Omega)$ is extendable to \mathcal{G} in the following sense.

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set of type \mathcal{A} . If $w \in \mathcal{G}(\Omega)$, then there exists $W \in \mathcal{G}$ such that $W|_\Omega(x) = w(x)$ almost everywhere.*

PROOF. If $w \in \mathcal{G}(\Omega)$, then for $p \in (1, \infty)$ there exists $\delta > 0$ such that $w^\delta \in A_p(\Omega)$. Therefore, for $0 < \lambda < 1$, we have $v := w^{\lambda\delta} \in A_p(\Omega)$, which yields $v^{1+\varepsilon} \in A_p(\Omega)$, when $\lambda(1+\varepsilon) = 1$. From [9, Theorem 5.6, p. 439] there exists a weight $V \in A_p$ such that $V|_\Omega(x) = v(x)$ a.e. (note that for Ω sets of type \mathcal{A} the class $A_p(\Omega)$ and the class $A_{p,\Omega}$, as defined in [9, p. 438], are equivalent). In other words, if $W(x) := V^{\frac{1}{\lambda\delta}}(x)$, then $W|_\Omega(x) = w(x)$ a.e. and $W \in \mathcal{G}$ since $W^{\lambda\delta} \in A_p$. \square

We are now in a position to state and prove our main result in this section.

THEOREM 3.4. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set of type \mathcal{A} , $a \in \mathcal{G}(\Omega) \cap L^\infty$, $\beta \in \mathfrak{B}$, $\varphi \in \Phi$, and $\beta(\varepsilon)/(p - \varepsilon)$ be increasing on $(0, p - 1)$. Then the operator M_Ω is bounded on $L_{a,\beta}^{p,\varphi}(\Omega)$.*

PROOF. We use the quantitative estimate from [1, Theorem 2.5],

$$(35) \quad \|Mf\|_{L^p(w)} \leq C_{n,p} [w]^{\frac{1}{p-1}} \|f\|_{L^p(w)},$$

where we should replace p by $p - \varepsilon$. By Corollary 2.5, due to the monotonicity of the function $\beta(\varepsilon)/(p - \varepsilon)$, we can use the norm $\| \cdot \|_{L_{a,\beta;\varepsilon_0}^{p,\varphi}(\Omega)}$, with $\varepsilon_0 < p - 1$ to be chosen latter.

Defining \tilde{f} as the zero extension of f from Ω to \mathbb{R}^n and still denoting by a the extension of $a \in \mathcal{G}(\Omega)$ to \mathcal{G} , such extension being possible by Lemma 3.3, we have

$$(36) \quad \|M_\Omega f\|_{L^{p-\varepsilon}(\Omega, a^{\frac{\beta(\varepsilon)}{p}})} \leq \|M\tilde{f}\|_{L^{p-\varepsilon}(a^{\frac{\beta(\varepsilon)}{p}})} \leq C_{n,p-\varepsilon} [a^{\frac{\beta(\varepsilon)}{p}}]_{\frac{1}{p-\varepsilon}} \|\tilde{f}\|_{L^{p-\varepsilon}(a^{\frac{\beta(\varepsilon)}{p}})},$$

assuming that $a^{\frac{\beta(\varepsilon)}{p}} \in A_{p-\varepsilon}$. The latter follows from (37) for ε sufficiently small.

There exists $\delta > 0$ such that $a^\delta \in A_p$, since $a \in \mathcal{G}$. From Lemma 3.1, it follows

$$(37) \quad [a^{\frac{\beta(\varepsilon)}{p}}]_{p-\varepsilon} = [a^{\delta \frac{\beta(\varepsilon)}{p\delta}}]_{p-\varepsilon} \leq [a^\delta]_{\frac{\beta(\varepsilon)}{p\delta}} \leq [a^\delta]_p,$$

whenever $\varrho(\varepsilon) := \frac{\beta(\varepsilon)}{p\delta} + \frac{\varepsilon}{p-1} \leq 1$. Choosing ε_0 such that $\varrho(\varepsilon) \leq 1$ for all $0 < \varepsilon < \varepsilon_0$, the result now follows by taking $\sup_{0 < \varepsilon < \varepsilon_0} \varphi(\varepsilon)$ in estimate (36) and to note that $\sup_{0 < \varepsilon < \varepsilon_0} C_{n,p-\varepsilon} < \infty$. \square

REMARK 3.5. The estimate norm in Theorem 3.4 may be given a sharp form if we slightly strengthen assumptions on the aggrandizer a . Namely, suppose that the aggrandizer a is extendable to \mathbb{R}^n so that there exists a $\delta > 0$ such that $a^\delta \in A_p$. Then the norm estimate holds in the form

$$\|M_\Omega f\|_{L_{a,\beta}^{p,\varphi}(\Omega)} \lesssim [a^\delta]_p \|f\|_{L_{a,\beta}^{p,\varphi}(\Omega)}.$$

The proof of Theorem 3.4 relies on a Buckley-type quantitative estimate. Such a quantitative estimate was already used in [18, §14.1.2, §14.1.3] to prove the boundedness of the maximal operator in weighted grand Lebesgue spaces on a set of finite measure.

In the rest of this subsection, we are interested in Muckenhoupt weights defined as the distance function to a closed null set F , which leads to the notion of lower Assouad codimension, see Section 4.2 for the definition and properties of such notion.

COROLLARY 3.6. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set of type \mathcal{A} , F be a bounded set with $|F| = 0$, $\varphi \in \Phi$ be doubling, $\beta \in \mathfrak{B}$ satisfy the condition (27), $\beta(\varepsilon)/(p - \varepsilon)$ be increasing on $(0, p - 1)$, $a(x) = \mathbf{a}(\delta_F(x))$, where \mathbf{a} is a quasi-monotone function on $(0, 1)$, and $m(\mathbf{a}) > 0$. Then the operator M_Ω is bounded on $L_{a,\beta}^{p,\varphi}(\Omega)$.*

PROOF. Since $m(\mathbf{a}) > 0$, by Theorem 2.14, we can replace $\mathbf{a}(t)$ by $\mathbf{a}(t) \equiv t$ in the definition of the space. Denoting $d_F(x) = \text{dist}(x, F)$, notice that if $d_F^\beta \in A_p$, then $\delta_F^\beta = \min\{1, d_F^\beta\} \in A_p$, with $[\delta_F^\beta]_p \leq (2^{p-2} + 1)([d_F^\beta]_p + 1)$,

see [10, Example 9.1.9]. Since $d_F \in \mathcal{G}$, by Proposition 4.1, the result now follows from Theorem 3.4. \square

3.2. Singular integral operators. We shall base ourselves on a result of the paper [14]. To this end, we recall necessary definitions.

The triplet (S, d, μ) is said to be a *space of homogeneous type* (SHT for short) if there is a constant $C \geq 1$ such that for any $x \in S$ and any $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)),$$

where $B(x, r) := \{y \in S \mid d(x, y) < r\}$. We denote by \mathcal{B} the family of all balls in (S, d) . We assume that $0 < \mu(B) < \infty$ for every ball $B \in \mathcal{B}$ (to avoid trivial measures) and that μ is a complete measure. For more on SHT, see [17, § 2.5.3].

If (S, d, μ) is a space of homogeneous type, then $K: S \times S \setminus \{x = y\} \rightarrow S$ is a *standard kernel* if there exists $\eta > 0$ such that it satisfies the *size condition*:

$$|K(x_0, y)| \lesssim \frac{1}{\mu B(x_0, d(x_0, y))},$$

for all $x_0 \neq y \in \Omega, x_0 \in \Omega, x \in \Omega$, and the *regularity conditions*:

$$(38) \quad |K(x, y) - K(x_0, y)| \lesssim \left(\frac{d(x, x_0)}{d(x_0, y)}\right)^\eta \frac{1}{\mu B(x_0, d(x_0, y))}$$

$$(39) \quad |K(y, x) - K(y, x_0)| \lesssim \left(\frac{d(x, x_0)}{d(x_0, y)}\right)^\eta \frac{1}{\mu B(x_0, d(x_0, y))}$$

for $d(x_0, x) \leq \eta d(x_0, y)$.

Let T be a singular integral operator associated with the standard kernel K . In addition, if T is bounded in $L^2(\mu)$, then T is a *Calderón–Zygmund singular operator*.

In Corollary 3.8 we provide a quantitative estimate for the operator norm of T in terms of operator norms of M , which is one of the crucial ingredients in the proof of Theorem 3.9. In order to obtain Corollary 3.8, we need the following general result, demonstrated in [14, Lemma 3.2], with the operator \mathcal{M} defined by

$$\mathcal{M}f(x) = \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{\mu B} \int_B |f| d\mu,$$

where \mathcal{B} is the set of all balls in (S, d) .

PROPOSITION 3.7. *Let (S, d, μ) be a SHT and $X \subset L^1_{\text{loc}}(S, \mu)$ be a Banach function lattice with the weak Fatou property. Suppose that the maximal*

operator \mathcal{M} is bounded in X and X' . Then the Calderón–Zygmund singular integral operator K is bounded in X , with

$$(40) \quad \|T\|_X \lesssim \|\mathcal{M}\|_X \|\mathcal{M}\|_{X'},$$

where the implicit constant in the symbol \lesssim depends on S , μ , and T .

COROLLARY 3.8. *Let $1 < p < \infty$, $w \in A_p(\Omega)$, and $\Omega \subset \mathbb{R}^n$ be a bounded open set of type \mathcal{A} . Then*

$$(41) \quad \|T\|_{L^p(\Omega, w)} \lesssim \|M_\Omega\|_{L^p(\Omega, w)} \|M_\Omega\|_{L^{p'}(\Omega, w^{1-p'})},$$

where the implicit constant in (41) does not depend on p and w .

PROOF. The operator M_Ω is point-wise equivalent to \mathcal{M} , since Ω is of type \mathcal{A} . The estimate (41) now follows from Proposition 3.7 taking into account that, under the hypothesis of the theorem, we have:

- (i) $(L^p(\Omega, w))' = L^{p'}(\Omega, w^{1-p'})$;
- (ii) $w \in A_p(\Omega)$ if and only if $w^{1-p'} \in A_{p'}(\Omega)$.

A close analysis of the proof of [14, Lemma 3.2] reveals that the implicit constant in (40) is independent of the underlying space. \square

THEOREM 3.9. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set of type \mathcal{A} , $a \in \mathcal{G}(\Omega) \cap L^\infty$, $\beta \in \mathfrak{B}$, $\varphi \in \Phi$, and $\beta(\varepsilon)/(p - \varepsilon)$ be increasing on $(0, p - 1)$. Then the operator T is bounded on $L_{a, \beta}^{p, \varphi}(\Omega)$.*

PROOF. From (41), we deduce

$$(42) \quad \|T\|_{L^{p-\varepsilon}(\Omega, a^{\frac{\beta(\varepsilon)}{p}})} \lesssim \|M_\Omega\|_{L^{p-\varepsilon}(\Omega, a^{\frac{\beta(\varepsilon)}{p}})} \|M_\Omega\|_{L^{(p-\varepsilon)'}\left(\Omega, a^{\frac{(1-(p-\varepsilon)')\beta(\varepsilon)}{p}}\right)}.$$

By (36) and (37) there exists η_1 , $0 < \eta_1 < p - 1$, such that

$$(43) \quad \sup_{0 < \varepsilon < \eta_1} \|M_\Omega\|_{L^{p-\varepsilon}(\Omega, a^{\frac{\beta(\varepsilon)}{p}})} \leq C < \infty.$$

Since $a^{-1} \in \mathcal{G}(\Omega)$, because $a \in \mathcal{G}(\Omega)$, similar arguments to those used to obtain (43) entail that there exists η_2 , $0 < \eta_2 < p - 1$, for which

$$(44) \quad \sup_{0 < \varepsilon < \eta_2} \|M_\Omega\|_{L^{(p-\varepsilon)'}\left(\Omega, a^{\frac{(1-(p-\varepsilon)')\beta(\varepsilon)}{p}}\right)} \leq C < \infty.$$

From Corollary 2.5, taking $\varepsilon_0 = \min\{\eta_1, \eta_2\}$, the boundedness of T now follows from (42), (43), (44), and taking the corresponding supremum. \square

The maximal singular operator T^* is introduced as

$$T^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| \geq \varepsilon} K(x, y) f(y) dy \right|,$$

where $K(x, y)$ is a standard kernel.

THEOREM 3.10. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set of type \mathcal{A} , $a \in \mathcal{G}(\Omega) \cap L^\infty$, $\beta \in \mathfrak{B}$, $\varphi \in \Phi$, and $\beta(\varepsilon)/(p - \varepsilon)$ be increasing on $(0, p - 1)$. Then T^* is bounded on $L_{a,\beta}^{p,\varphi}(\Omega)$.*

PROOF. Let T be the corresponding Calderón–Zygmund singular integral operator associated with T^* . Recall that Cotlar’s inequality

$$(45) \quad T^*f(x) \lesssim M(|Tf|)(x) + Mf(x),$$

is valid for $f \in L^p(w)$, $w \in A_p$, $1 < p < \infty$, see [13, p. 56]. Note that in [13] the regularity conditions on the Calderón–Zygmund kernel are given in gradient terms. The analysis of the proof in [13] shows that it remains valid for the standard kernel with regularity conditions (38) and (39), taking into account that such singular operators satisfy a weak (1,1) estimate.

The boundedness of T^* now follows from (45) and the corresponding boundedness of the maximal and singular operator. \square

4. Appendix

4.1. On Matuszewska–Orlicz indices. The notion of *Matuszewska–Orlicz indices* goes back to [20,21]. They are widely used in the literature (see for instance [17,22,23,27,34]), we refer the reader to [27] for a good presentation. These indices are closely related with the property of a function to be almost increasing or almost decreasing after multiplication (division) by a power function.

We say that the function ω is quasi-monotone at the origin, if there exist $a, b \in \mathbb{R}$ such that the function $\frac{\omega(t)}{t^a}$ is almost increasing and the function $\frac{\omega(t)}{t^b}$ is almost decreasing in a neighborhood of the origin.

The Matuszewska–Orlicz indices are defined as follows:

$$(46) \quad m(\omega) = \sup_{0 < t < 1} \frac{\ln(\overline{\lim}_{h \rightarrow 0} \frac{\omega(ht)}{\omega(h)})}{\ln t} = \lim_{t \rightarrow 0} \frac{\ln(\overline{\lim}_{h \rightarrow 0} \frac{\omega(ht)}{\omega(h)})}{\ln t},$$

and

$$(47) \quad M(\omega) = \sup_{t > 1} \frac{\ln(\overline{\lim}_{h \rightarrow 0} \frac{\omega(ht)}{\omega(h)})}{\ln t} = \lim_{t \rightarrow \infty} \frac{\ln(\overline{\lim}_{h \rightarrow 0} \frac{\omega(ht)}{\omega(h)})}{\ln t}.$$

If the function ω is quasi-monotone at the origin, then its indices $m(\omega)$, $M(\omega)$ are finite and $-\infty < m(\omega) \leq M(\omega) < +\infty$. It is known that

$$(48) \quad c_1 t^{M(\omega)+\varepsilon} \leq \omega(t) \leq c_2 t^{m(\omega)-\varepsilon}, \quad 0 < t < 1.$$

4.2. On codimension. Let $F \subset \mathbb{R}^n$ and $F_r := \{x \in \mathbb{R}^n \mid d_F(x) < r\}$, where $d_F(x) := \text{dist}(x, F)$. The *lower Assouad codimension*, denoted by \mathfrak{N}_F , is defined as the supremum of all $\nu \geq 0$ for which there exists a constant $C \geq 1$ such that

$$\frac{|F_r \cap B(x, R)|}{|B(x, R)|} \leq C \left(\frac{r}{R}\right)^\nu,$$

for every $x \in F$ and all $0 < r < R < \infty$. Observe that $\mathfrak{N}_F > 0 \Rightarrow |F| = 0$, see [4, p. 6]. More information on Assouad dimension can be found in [8, 19].

PROPOSITION 4.1 [4, Theorem 3.4]. *Let $F \subset \mathbb{R}^n$ be a closed non-empty set. Then*

$$-\mathfrak{N}_F < \beta \leq 0 \implies d_F^\beta \in A_1,$$

and

$$-\mathfrak{N}_F < \beta < (p-1)\mathfrak{N}_F \implies d_F^\beta \in A_p, \quad 1 < p < \infty,$$

here $d_F(x) = \text{dist}(x, F)$ and \mathfrak{N}_F is the lower Assouad codimension of the set F .

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