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Weighted Hardy and potential operators in the generalized Morrey spaces

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ABSTRACT

We study the weighted $p \to q$-boundedness of the multi-dimensional Hardy type operators in the generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\mathbb{R}^n, w)$ defined by an almost increasing function $\varphi(r)$ and radial type weight $w(|x|)$. We obtain sufficient conditions, in terms of some integral inequalities imposed on φ and w, for such a $p \to q$-boundedness. In some cases the obtained conditions are also necessary. These results are applied to derive a similar weighted $p \to q$-boundedness of the Riesz potential operator.

1. Introduction

The well known Morrey spaces $\mathcal{L}^{p,\lambda}$ were introduced in [27] in relation to the study of partial differential equations, and presented in various books, see e.g. [13,21,44]. They were widely investigated during the last decades, including the study of classical operators of harmonic analysis – maximal, singular and potential operators on Morrey spaces and there generalizations were studied, including also the case of functions defined on metric measure spaces. We refer for instance to papers [1–4,8,7,9–11,28–34,41–43]. Surprisingly, weighted estimates of these classical operators, in fact, were not studied. Just recently, in [38] we proved weighted $p \to p$-estimates in Morrey spaces $\mathcal{L}^{p,\lambda}$ for Hardy operators on $\mathbb{R}^1_+$ and one-dimensional singular operators (on $\mathbb{R}^1$ or on Carleson curves in the complex plane). In paper [39] we gave the conditions for the $p \to q$-boundedness of multidimensional Hardy and potential operators within the frameworks of the Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$.

In this paper we make use of another approach which allows us to obtain weighted estimations for such operators in the generalized Morrey spaces $\mathcal{L}^{p,\varphi}$ obtained from Morrey spaces when we replace r^λ by a function $\varphi(r)$. The admitted weights $w(|x - x_0|)$ are generated by functions $w(r)$ from the Bary–Stechkin-type class; they may be characterized as weights continuous and positive for $r \in (0, \infty)$, with possible decay or growth at $r = 0$ and $r = \infty$, which become almost increasing or almost decreasing after the multiplication by a power function. Such weights are oscillating between two powers at the origin and infinity (with different exponents for the origin and infinity).

We obtain conditions for the weighted boundedness of the Hardy operators for local and global generalized Morrey spaces (see definitions in Section 3.1). These conditions are necessary and sufficient, in the case of local spaces, and sufficient in the case of global ones; in the latter case they are also necessary in some cases, see Theorems 4.2 and 4.4.

The paper is organized as follows. In Section 2 we give necessary preliminaries on some classes of weight functions. In Section 3, which plays a crucial role in the preparation of the proofs of the main results, we prove some important lemmas concerning belongness of the generalized Morrey spaces of some classes of radial functions. In Section 4 we prove

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theorems on the weighted $p \to q$-boundedness of Hardy operators in Morrey spaces. Finally, in Section 5 we apply the results of Section 4 to a similar weighted boundedness of potential operators in the global generalized Morrey spaces. The main results are given in Theorems 4.2, 4.4 and 5.5.

2. Preliminaries on weight functions

2.1. Zygmund–Bary–Stechkin (ZBS) classes and Matuszewska–Orlicz (MO) type indices

2.1.1. On classes of the type W_0 and W_∞

In the sequel, a non-negative function f on $[0,\ell]$, $0<\ell\leqslant\infty$, is called almost increasing (almost decreasing), if there exists a constant C ($\geqslant 1$) such that $f(x)\leqslant Cf(y)$ for all $x\leqslant y$ ($x\geqslant y$, respectively). Equivalently, a function f is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function g, i.e. $c_1 f(x)\leqslant g(x)\leqslant c_2 f(x)$, $c_1>0$, $c_2>0$.

Definition 2.1. Let $0<\ell<\infty$.

1) By $W=W([0,\ell])$ we denote the class of continuous and positive functions φ on $(0,\ell]$ such that the limit $\lim_{x\to 0}\varphi(x)$ exists and is finite;
2) by $W_0=W_0([0,\ell])$ we denote the class of almost increasing functions $\varphi\in W$ on $(0,\ell)$;
3) by $\overline{W}=\overline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $x^a\varphi(x)\in W_0$ for some $a=a(\varphi)\in\mathbb{R}^1$;
4) by $\underline{W}=\underline{W}([0,\ell])$ we denote the class of functions $\varphi\in W$ such that $\frac{\varphi(t)}{t^b}$ is almost decreasing for some $b\in\mathbb{R}^1$.

Definition 2.2. Let $0<\ell<\infty$.

1) By $W_\infty=W_\infty([\ell,\infty])$ we denote the class of functions φ which are continuous and positive and almost increasing on $[\ell,\infty)$ and which have the finite limit $\lim_{x\to\infty}\varphi(x)$,
2) by $\overline{W}_\infty=\overline{W}_\infty([\ell,\infty))$ we denote the class of functions $\varphi\in W_\infty$ such $x^a\varphi(x)\in W_\infty$ for some $a=a(\varphi)\in\mathbb{R}^1$.

Finally, by $\overline{W}(\mathbb{R}^1_+)$ we denote the set of functions on $\mathbb{R}^1_+$ whose restrictions onto $(0,1)$ are in $\overline{W}([0,1])$ and restrictions onto $[1,\infty)$ are in $\overline{W}_\infty([1,\infty))$. Similarly, the set $\underline{W}(\mathbb{R}^1_+)$ is defined.

2.1.2. ZBS-classes and MO-indices of weights at the origin

In this subsection we assume that $\ell<\infty$.

Definition 2.3. We say that a function $\varphi\in W_0$ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta\in\mathbb{R}^1$, if

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}}\,dt\leqslant c\frac{\varphi(x)}{x^\beta},\quad x\in(0,\ell),$$

and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma\in\mathbb{R}^1$, if

$$\int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}}\,dt\leqslant c\frac{\varphi(x)}{x^\gamma},\quad x\in(0,\ell).$$

We also denote

$$\Phi^\beta_\gamma:=\mathbb{Z}^\beta\cap\mathbb{Z}_\gamma,$$

the latter class being also known as Bary–Stechkin–Zygmund class [5].

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska–Orlicz indices. We refer to [18,20], [24, p. 20], [25,26,36,37] for the properties of the indices of such a type. For a function $\varphi\in\overline{W}$, the numbers

$$m(\varphi)=\sup_{0<x<1}\frac{\ln(\limsup_{h\to 0}\frac{\varphi(hx)}{\varphi(h)})}{\ln x}=\lim_{x\to 0}\frac{\ln(\limsup_{h\to 0}\frac{\varphi(hx)}{\varphi(h)})}{\ln x} \tag{2.1}$$

and

$$M(\varphi)=\sup_{x>1}\frac{\ln(\limsup_{h\to 0}\frac{\varphi(hx)}{\varphi(h)})}{\ln x}=\lim_{x\to\infty}\frac{\ln(\limsup_{h\to 0}\frac{\varphi(hx)}{\varphi(h)})}{\ln x} \tag{2.2}$$

are known as *the Matuszewska–Orlicz type lower and upper indices* of the function $\varphi(r)$. Note that in this definition $\varphi(x)$ need not to be an N-function: only its behavior at the origin is of importance. Observe that $0 \leqslant m(\varphi) \leqslant M(\varphi) \leqslant \infty$ for $\varphi \in W_0$, and $-\infty < m(\varphi) \leqslant M(\varphi) \leqslant \infty$ for $\varphi \in \overline{W}$, and the following formulas are valid:

$$m\big[x^a\varphi(x)\big] = a + m(\varphi), \qquad M\big[x^a\varphi(x)\big] = a + M(\varphi), \quad a \in \mathbb{R}^1, \tag{2.3}$$

$$m\big(\big[\varphi(x)\big]^a\big) = am(\varphi), \qquad M\big(\big[\varphi(x)\big]^a\big) = aM(\varphi), \quad a \geqslant 0, \tag{2.4}$$

$$m\left(\frac{1}{\varphi}\right) = -M(\varphi), \qquad M\left(\frac{1}{\varphi}\right) = -m(\varphi), \tag{2.5}$$

$$m(uv) \geqslant m(u) + m(v), \qquad M(uv) \leqslant M(u) + M(v) \tag{2.6}$$

for $\varphi, u, v \in \overline{W}$.

The following statement is known, see [18, Theorems 3.1, 3.2 and 3.5]. (In the formulation of Theorem 2.4 in [18] it was supposed that $\beta \geqslant 0$, $\gamma > 0$ and $\varphi \in W_0$. It is evidently true also for $\varphi \in \overline{W}$ and all $\beta, \gamma \in \mathbb{R}^1$, in view of formulas (2.3).)

Theorem 2.4. *Let $\varphi \in \overline{W}$ and $\beta, \gamma \in \mathbb{R}^1$. Then*

$$\varphi \in \mathbb{Z}^\beta \quad \Longleftrightarrow \quad m(\varphi) > \beta \quad \text{and} \quad \varphi \in \mathbb{Z}_\gamma \quad \Longleftrightarrow \quad M(\varphi) < \gamma.$$

Besides this

$$m(\varphi) = \sup\left\{\mu > 0\colon \frac{\varphi(x)}{x^\mu} \text{ is almost increasing}\right\}, \tag{2.7}$$

$$M(\varphi) = \inf\left\{\nu > 0\colon \frac{\varphi(x)}{x^\nu} \text{ is almost decreasing}\right\} \tag{2.8}$$

and for $\varphi \in \Phi_\gamma^\beta$ the inequalities

$$c_1 x^{M(\varphi)+\varepsilon} \leqslant \varphi(x) \leqslant c_2 x^{m(\varphi)-\varepsilon} \tag{2.9}$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$.

The following simple lemma is also useful. For its formulation and also for other goals in the sequel we find it convenient to introduce the following notation for a subclass in $\overline{W}_0$:

$$\overline{W}_{0,b} = \left\{\varphi \in \overline{W}_0\colon \frac{\varphi(t)}{t^b} \text{ is almost increasing}\right\}, \quad b \in \mathbb{R}^1.$$

Lemma 2.5. *Let $\varphi \in W_{0,b}([0,\ell])$, $0 < \ell \leqslant \infty$. Then $\varphi \in \mathbb{Z}^\beta([0,\ell])$ for any $\beta < b$. If there exists a ν $(> b)$ such that $\frac{\varphi(t)}{t^\nu}$ is almost decreasing, then there exist positive constants c_1 and c_2 such that*

$$c_1\frac{\varphi(r)}{r^\beta} \leqslant \int_0^r \frac{\varphi(t)\,dt}{t^{1+\beta}} \leqslant c_2\frac{\varphi(r)}{r^\beta}, \quad r \in (0,\ell], \tag{2.10}$$

where $\beta < b$.

The proof is simple so we leave out the details. We also mention the following obvious consequence of the lemma:

Corollary 2.6. *Let $\varphi \in \overline{W}$ with $-\infty < m(\varphi) \leqslant M(\varphi) < \infty$. Then* (2.10) *holds for every $\beta < m(\varphi)$.*

2.1.3. ZBS-classes and MO-indices of weights at infinity

Following [19, Subsection 4.1], and [35, Subsection 2.2], we introduce the following definitions:

Definition 2.7. Let $-\infty < \alpha < \beta < \infty$. We put $\Psi_\alpha^\beta := \widehat{\mathbb{Z}}^\beta \cap \widehat{\mathbb{Z}}_\alpha$, where $\widehat{\mathbb{Z}}^\beta$ is the class of functions $\varphi \in \overline{W}_\infty$ satisfying the condition

$$\int_x^\infty \left(\frac{x}{t}\right)^\beta \frac{\varphi(t)\,dt}{t} \leqslant c\varphi(x), \quad x \in (\ell,\infty), \tag{2.11}$$

and $\widehat{\mathbb{Z}}_\alpha$ is the class of functions $\varphi \in W([\ell,\infty))$ satisfying the condition

$$\int_\ell^x \left(\frac{x}{t}\right)^\alpha \frac{\varphi(t)\,dt}{t} \leqslant c\varphi(x), \quad x \in (\ell,\infty) \tag{2.12}$$

where $c = c(\varphi) > 0$ does not depend on $x \in [\ell,\infty)$.

The indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$ responsible for the behavior of functions $\varphi \in \Psi_\alpha^\beta([\ell,\infty))$ at infinity are introduced in the way similar to (2.1) and (2.2):

$$m_\infty(\varphi) = \sup_{x>1} \frac{\ln[\liminf_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}]}{\ln x}, \qquad M_\infty(\varphi) = \inf_{x>1} \frac{\ln[\limsup_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}]}{\ln x}. \tag{2.13}$$

Properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha([0,\ell])$ because of the following equivalence

$$\varphi \in \Psi_\alpha^\beta\big([\ell,\infty)\big) \iff \varphi_* \in \Phi_{-\alpha}^{-\beta}\big(\big[0,\ell^*\big]\big), \tag{2.14}$$

where $\varphi_*(t) = \varphi(\frac{1}{t})$ and $\ell_* = \frac{1}{\ell}$. Direct calculation shows that

$$m_\infty(\varphi) = -M(\varphi_*), \qquad M_\infty(\varphi) = -m(\varphi_*), \qquad \varphi_*(t) := \varphi\left(\frac{1}{t}\right). \tag{2.15}$$

Making use of (2.14) and (2.15), one can easily reformulate properties of functions of the class Φ_γ^β near the origin, given in Theorem 2.4 for the case of the corresponding behavior at infinity of functions of the class Ψ_α^β and obtain that

$$c_1 t^{m_\infty(\varphi)-\varepsilon} \leqslant \varphi(t) \leqslant c_2 t^{M_\infty(\varphi)+\varepsilon}, \quad t \geqslant \ell,\ \varphi \in \overline{W}_\infty, \tag{2.16}$$

$$m_\infty(\varphi) = \sup\big\{\mu \in \mathbb{R}^1\colon t^{-\mu}\varphi(t) \text{ is almost increasing on } [\ell,\infty)\big\}, \tag{2.17}$$

$$M_\infty(\varphi) = \inf\big\{\nu \in \mathbb{R}^1\colon t^{-\nu}\varphi(t) \text{ is almost decreasing on } [\ell,\infty)\big\}. \tag{2.18}$$

We say that a continuous function φ in $(0,\infty)$ is in the class $\overline{W}_{0,\infty}(\mathbb{R}_+^1)$, if its restriction to $(0,1)$ belongs to $\overline{W}([0,1])$ and its restriction to $(1,\infty)$ belongs to $\overline{W}_\infty([1,\infty])$. For functions in $\overline{W}_{0,\infty}(\mathbb{R}_+^1)$ the notation

$$\mathbb{Z}^{\beta_0,\beta_\infty}\big(\mathbb{R}_+^1\big) = \mathbb{Z}^{\beta_0}\big([0,1]\big) \cap \mathbb{Z}^{\beta_\infty}\big([1,\infty)\big), \qquad \mathbb{Z}_{\gamma_0,\gamma_\infty}\big(\mathbb{R}_+^1\big) = \mathbb{Z}_{\gamma_0}\big([0,1]\big) \cap \mathbb{Z}_{\gamma_\infty}\big([1,\infty)\big)$$

has an obvious meaning. In the case where the indices coincide, i.e. when $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbb{Z}^\beta(\mathbb{R}_+^1)$ and similarly for $\mathbb{Z}_\gamma(\mathbb{R}_+^1)$. We also denote

$$\Phi_\gamma^\beta\big(\mathbb{R}_+^1\big) := \mathbb{Z}^\beta\big(\mathbb{R}_+^1\big) \cap \mathbb{Z}_\gamma\big(\mathbb{R}_+^1\big). \tag{2.19}$$

Making use of Theorem 2.4 for $\Phi_\beta^\alpha([0,1])$ and relations (2.15), we easily arrive at the following statement.

Lemma 2.8. *Let $\varphi \in \overline{W}(\mathbb{R}_+^1)$. Then*

$$\varphi \in \mathbb{Z}^{\beta_0,\beta_\infty}\big(\mathbb{R}_+^1\big) \iff m(\varphi) > \beta_0,\ m_\infty(\varphi) > \beta_\infty \tag{2.20}$$

and

$$\varphi \in \mathbb{Z}_{\gamma_0,\gamma_\infty}\big(\mathbb{R}_+^1\big) \iff M(\varphi) < \gamma_0,\ M_\infty(\varphi) < \gamma_\infty. \tag{2.21}$$

2.2. On classes $\mathbf{V}_\pm^\mu$

Note that we slightly changed the notation of the class introduced in the following definition, in comparison with its notation in [37].

Definition 2.9. Let $0 < \mu \leqslant 1$. By $\mathbf{V}_\pm^\mu$, we denote the classes of functions φ which are non-negative on $[0,\ell]$ and positive on $(0,\ell]$, $0 < \ell \leqslant \infty$, defined by the following conditions:

$$\mathbf{V}_+^\mu\colon \quad \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\mu} \leqslant C\frac{\varphi(x_+)}{x_+^\mu}, \tag{2.22}$$

$$\mathbf{V}_-^\mu\colon \quad \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\mu} \leqslant C\frac{\varphi(x_-)}{x_+^\mu}, \tag{2.23}$$

where $x, y \in (0,\ell]$, $x \neq y$, and $x_+ = \max(x,y)$, $x_- = \min(x,y)$.

Lemma 2.10. *Functions $\varphi \in \mathbf{V}_+^\mu$ are almost increasing on $[0,\ell]$ and functions $\varphi \in \mathbf{V}_-^\mu$ are almost decreasing on $[0,\ell]$.*

Proof. Let $\varphi \in \mathbf{V}_+^\mu$ and $y < x$. By (2.22) we have $|\varphi(x) - \varphi(y)| \leqslant C\varphi(x)(1-\frac{y}{x})^\mu \leqslant C\varphi(x)$. Then $\varphi(y) \leqslant |\varphi(x)-\varphi(y)| + \varphi(x) \leqslant (C+1)\varphi(x)$. By instead of (2.22) using (2.23) the second statement follows in a similar way. □

Corollary 2.11. *Functions $\varphi \in \mathbf{V}_+^\mu$ have non-negative indices $0 \leqslant m(\varphi) \leqslant M(\varphi)$ and functions $\varphi \in \mathbf{V}_-^\mu$ have non-positive indices $m(\varphi) \leqslant M(\varphi) \leqslant 0$, the same being also valid with respect to the indices $m_\infty(\varphi)$, $M_\infty(\varphi)$ in the case $\ell = \infty$.*

Proof. Use (2.7) and (2.8). □

Note that

$$V_+^\mu \subset V_+^\nu, \qquad V_-^\mu \subset V_-^\nu, \quad 0 < \nu < \mu \leqslant 1, \tag{2.24}$$

the classes $V_\pm^\mu$ being trivial for $\mu > 1$. We also have that

$$x^\gamma \in \bigcup_{\mu\in[0,1]} \mathbf{V}_+^\mu \iff \gamma \geqslant 0, \qquad x^\gamma \in \bigcup_{\mu\in[0,1]} \mathbf{V}_-^\mu \iff \gamma \leqslant 0,$$

which follows from the fact that $x^\gamma \in \mathbf{V}_+^1 \iff \gamma \geqslant 0$, and $x^\gamma \in \mathbf{V}_-^1 \iff \gamma \leqslant 0$ (see [38, Subsection 2.3]), Remark 2.10 and property (2.24).

An example of a function which is in $\mathbf{V}_+^\mu$ with some $\mu > 0$, but does not belong to the total intersection $\bigcap_{\mu\in[0,1]} \mathbf{V}_+^\mu$ is given by

$$\varphi(x) = ax^\gamma + b|x-x_0|^\beta \in \bigcap_{\mu\in[0,\beta]} \mathbf{V}_+^\mu,$$

where $x_0 > 0$, $\gamma \geqslant 0$, $0 < \beta < 1$, $a > 0$, and $b > 0$.

The following lemmas (proved in [38], see Lemmas 2.10 and 2.11 there) show that conditions (2.22) and (2.23) are fulfilled with $\mu = 1$ not only for power functions, but for an essentially larger class of functions (which in particular may oscillate between two power functions with different exponents). Note that the information about this class in Lemmas 2.12 and 2.13 is given in terms of increasing or decreasing functions, without the word "almost".

Lemma 2.12. *Let $\varphi \in W$. Then*

i) *$\varphi \in \mathbf{V}_+^1$ in the case φ is increasing and the function $\frac{\varphi(x)}{x^\nu}$ is decreasing for some $\nu > 0$;*
ii) *$\varphi \in \mathbf{V}_-^1$ in the case $\varphi(x)$ is decreasing and there exists a number $\mu \geqslant 0$ such that $x^\mu\varphi(x)$ is increasing.*

Lemma 2.13. *Let $\varphi \in W \cap C^1((0,\ell])$. If there exist $\varepsilon > 0$ and $\nu \geqslant 0$ such that $0 \leqslant \frac{\varphi'(x)}{\varphi(x)} \leqslant \frac{\nu}{x}$ for $0 < x \leqslant \varepsilon$, then $\varphi \in \mathbf{V}_+^1$. If there exist $\varepsilon > 0$ and $\mu \geqslant 0$ such that $-\frac{\mu}{x} \leqslant \frac{\varphi'(x)}{\varphi(x)} \leqslant 0$ for $0 < x \leqslant \varepsilon$, then $\varphi \in \mathbf{V}_-^1$.*

3. On weighted integrability of functions in Morrey spaces

3.1. Definitions and belongness of some functions to generalized Morrey spaces

Let Ω be an open set in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ and $\ell = \mathit{diam}\,\Omega$, $0 < \ell \leqslant \infty$, $B(x,r) = \{y \in \mathbb{R}^n\colon\ |x-y| < r\}$ and $\widetilde{B}(x,r) = B(x,r) \cap \Omega$.

Definition 3.1. Let $\varphi(r)$ be a non-negative function on $[0,\ell]$, positive on $(0,\ell]$, and $1 \leqslant p < \infty$. The generalized Morrey spaces $\mathcal{L}^{p,\varphi}(\Omega)$, $\mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$, are defined as the spaces of functions $f \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\|f\|_{p,\varphi} := \sup_{x\in\Omega, r>0} \left(\frac{1}{\varphi(r)} \int_{\widetilde{B}(x,r)} |f(y)|^p\,dy\right)^{\frac{1}{p}} < \infty, \tag{3.1}$$

$$\|f\|_{p,\varphi;\mathrm{loc}} := \sup_{r>0} \left(\frac{1}{\varphi(r)} \int_{\widetilde{B}(x_0,r)} |f(y)|^p\,dy\right)^{\frac{1}{p}} < \infty, \tag{3.2}$$

respectively, where $x_0 \in \Omega$.

Obviously,

$$\mathcal{L}^{p,\varphi}(\Omega) \subset \mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega).$$

The spaces $\mathcal{L}^{p,\varphi}(\Omega)$, $\mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$ are known under the names of *global* and *local Morrey spaces*, see for instance [7,8].

Let ω denote a weight function on Ω. Then the weighted Morrey space $\mathcal{L}^{p,\varphi}(\Omega,\omega)$ is defined as follows:

$$L^{p,\varphi}(\Omega,\omega) := \{f\colon\ \omega f \in L^{p,\varphi}(\Omega)\}.$$

Everywhere in the sequel we assume that

$$\varphi(r) \geqslant cr^n \tag{3.3}$$

for $0 < r \leqslant \ell$, if $\ell < \infty$, and $0 < r \leqslant N$ with an arbitrary $N > 0$, if $\ell = \infty$, the constant c depending on N in the latter case. Condition (3.3) makes the spaces $\mathcal{L}^{p,\varphi}(\Omega)$, $\mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$ non-trivial, see Corollary 3.4. Sometimes in the case $\ell = \infty$ we will use the condition (3.3) also on the whole semiaxis $\mathbb{R}^1_+$, that is

$$\sup_{0<r<\infty} \frac{r^n}{\varphi(r)} < \infty. \tag{3.4}$$

Remark 3.2. The space $\mathcal{L}^{p,\varphi}(\Omega)$ as defined above, is not necessarily embedded into $L^p(\Omega)$, in the case when Ω is unbounded. A counterexample in the case when $\Omega = \mathbb{R}^n$ is $f(x) = (\frac{\varphi(|x|)}{|x|^n})^{\frac{1}{p}}$ which is not in $L^p(\mathbb{R}^n)$, but belongs to $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ under the conditions

1) $\varphi \in \mathbb{Z}^0$,
2) the function $\frac{\varphi(r)}{r^n}$ is almost decreasing.

Indeed, we have that

$$\|f\|_{p,\varphi} = \sup_{x\in\Omega,\, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{B(x,r)} \frac{\varphi(|y|)}{|y|^n}\, dy\right)^{\frac{1}{p}},$$

which is bounded (when $|x| \geqslant 2r$, take into account that $|y| \geqslant r$ and $\frac{\varphi(r)}{r^n}$ is almost decreasing; when $|x| \leqslant 2r$, make use of the inclusion $B(x,r) \subset B(0,3r)$ and the fact that $\varphi \in \mathbb{Z}^0$).

In the next lemma we give sufficient or necessary conditions for radial type functions $u(|x-x_0|)$, $x_0 \in \Omega$, to belong to generalized Morrey spaces. They are given in terms of the condition

$$\mathcal{M}_\varphi(u) := \frac{1}{\varphi(r)} \int\limits_0^r u^p(t) t^{n-1}\, dt \leqslant C, \quad 0 < r < \ell, \tag{3.5}$$

in the case of the local spaces $\mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$, and in terms of the condition

$$\sup_{0<r<\ell} \frac{r^n u^p(r)}{\varphi(r)} < \infty \tag{3.6}$$

in the case of the spaces $\mathcal{L}^{p,\varphi}(\Omega)$. Note that (3.5) $\Longrightarrow$ (3.6), if $r^n\varphi(r) \in Z^0$. The belongness of $u(|x|)$ to $\mathcal{L}^{p,\varphi}(\Omega)$ makes also use of the notion of the Matuszewska–Orlicz indices $m(u)$ and $M(u)$ of the function u. Our principal result in this subsection reads:

Proposition 3.3. *Let $\ell = \mathrm{diam}\,\Omega \leqslant \infty$, $\varphi(r)$ be almost increasing on $[0,\ell]$ satisfying condition (3.3), $u \in \underline{W}([0,\ell])$ and $x_0 \in \Omega$.*

I. *Condition (3.5) is necessary and sufficient for a function $f(x) := u(|x-x_0|)$ to belong to the local Morrey space $\mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$ and*

$$\|f\|_{p,\varphi} \leqslant C\big(\mathcal{M}_\varphi(u)\big)^{\frac{1}{p}}, \tag{3.7}$$

where C does not depend on u and φ. Condition (3.5) and consequently (3.6) are necessary also for $f \in \mathcal{L}^{p,\varphi}(\Omega)$.

II. *Let $u, \varphi \in \overline{W}([0,\ell])$. Condition (3.6) is sufficient for $f \in \mathcal{L}^{p,\varphi}(\Omega)$, if either*

i) *$u(r)$ is bounded and, in the case $\ell = \infty$, condition (3.4) holds, or*
ii) *$M(u) < 0$, $u(2r) \leqslant Cu(r)$ and $r^n u^p(r) \in Z^0$.*

Let $\ell < \infty$. In terms of the indices of the functions u and φ the conditions for $f \in \mathcal{L}^{p,\varphi}(\Omega)$ have the form

$$M(\varphi) - pm(u) < n \quad (\textit{sufficient condition}), \tag{3.8}$$

$$m(\varphi) - pM(u) \leqslant n \quad (\textit{necessary condition}) \tag{3.9}$$

independently of the signs of the indices. In the case of the power function, the inclusion $|x - x_0|^\gamma \in \mathcal{L}^{p,\varphi}(\Omega)$ holds, if and only if

$$n + \gamma p > 0 \quad \textit{and} \quad \sup_{r>0} \frac{r^{n+\gamma p}}{\varphi(r)} < \infty. \tag{3.10}$$

Proof. I. We first prove the necessity of (3.5) for $f \in \mathcal{L}^{p,\varphi}_{\mathrm{loc},x_0}(\Omega)$. Let $\delta(x) = dist(x, \partial\Omega)$. By passing to polar coordinates and using obvious estimates we have that

$$\|f\|_{p,\varphi} = \sup_{x\in\Omega, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{\widetilde{B}(x,r)} u^p\big(|y - x_0|\big)\, dy\right)^{\frac{1}{p}} \geqslant \sup_{\substack{x\in\Omega,\\ p0<r<\delta(x)}} \left(\frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y - x_0|\big)\, dy\right)^{\frac{1}{p}}$$

$$\geqslant \sup_{0<r<\delta(x_0)} \left(\frac{1}{\varphi(r)} \int\limits_{B(x_0,r)} u^p\big(|y - x_0|\big)\, dy\right)^{\frac{1}{p}} = C \sup_{p0<r<\delta(x_0)} \left(\frac{1}{\varphi(r)} \int\limits_0^r u^p(t) t^{n-1}\, dt\right)^{\frac{1}{p}} \tag{3.11}$$

and we arrive at the necessity of condition (3.5) for $0 < r < \delta(x_0)$ and then for all $r \in (0, \ell]$ by properties of the functions φ and u. From the above estimates the necessity of (3.5) for $f \in \mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$ is also seen.

The sufficiency of (3.5) for $f \in \mathcal{L}^{p,\varphi}_{\mathrm{loc};x_0}(\Omega)$ is a trivial fact since

$$\|f\|^p_{p,\varphi;\mathrm{loc}} \leqslant \sup_{0<r\leqslant\ell} \frac{1}{\varphi(r)} \int\limits_{B(0,r)} \big|u\big(|y|\big)\big|^p\, dy = C \sup_{0<r\leqslant\ell} \frac{1}{\varphi(r)} \int\limits_0^r u^p(t) t^{n-1}\, dt.$$

II. The case i) is easy:

$$\|f\|_{p,\varphi} \leqslant \sup_{x\in B(0,\ell),\, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy\right)^{\frac{1}{p}} \leqslant \sup_{0<r<\ell} \left(\frac{Cr^n}{\varphi(r)}\right)^{\frac{1}{p}} < \infty. \tag{3.12}$$

In the case ii) we distinguish the cases $|x| \leqslant 2r$ and $|x| > 2r$. In the first case we have $B(x, r) \subset B(0, 3r)$ and then

$$\frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy \leqslant \frac{1}{\varphi(r)} \int\limits_{B(0,3r)} u^p\big(|y|\big)\, dy \leqslant \frac{C}{\varphi(r)} \int\limits_0^{3r} u^p(t) t^{n-1}\, dt \tag{3.13}$$

where it remains to refer to the fact that $r^n \varphi(r) \in Z^0$ and observe that $u(2r) \leqslant Cu(r) \Longrightarrow u(3r) \leqslant Cu(r)$. In the case $|x| \geqslant 2r$ we have $|y| \geqslant |x| - |x - y| \geqslant r$ in the first integral in (3.13). Since $M(u) < 0$, the function $u(r)$ is almost decreasing by (2.8). Therefore,

$$\sup_{x\in\Omega, r>0} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy \leqslant C \sup_{r>0} \frac{u(r) r^n}{\varphi(r)} < \infty.$$

To cover the case of conditions (3.8)–(3.9), we prove the estimates

$$C_1 r^{n+pM(u)-m(\varphi)+\varepsilon} \leqslant \sup_{x\in B(0,\ell)} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy \leqslant C_2 r^{n+pm(u)-M(\varphi)-\varepsilon} \tag{3.14}$$

with an arbitrarily small $\varepsilon > 0$ and $C_i = C_i(\varepsilon)$, $i = 1, 2$, from which (3.8)–(3.9) follow. To this end, we make use of property (2.9) and obtain that

$$\frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy \leqslant \frac{C}{\varphi(r)} \int\limits_{B(x,r)} |y|^{pm(u)-p\varepsilon}\, dy.$$

If $m(u) > 0$, we choose $0 < \varepsilon < m(u)$ and δ small enough and then the right-hand side of the last inequality is bounded in view of (3.3). If $m(u) \leqslant 0$, as above we distinguish the cases $|x| \geqslant 2r$ and $|x| < 2r$. In the first case we have $|y| > r$ and then

$$\frac{1}{\varphi(r)} \int\limits_{B(x,r)} u^p\big(|y|\big)\, dy \leqslant \frac{r^{pm(u)-p\varepsilon}}{r^{M(\varphi)+\delta}} \int\limits_{B(x,r)} dy = C \sup_{r>0} r^{pm(u)+n-M(\varphi)-\delta-p\varepsilon}$$

and in the second case the usage of the embedding $B(x, r) \subset B(0, 3r)$ yields the same estimate.

Similarly, the left-hand side inequality in (3.14) is obtained. $\square$

Corollary 3.4. *Let φ be a non-negative measurable function. Then*

$$L^{\infty}(\Omega) \subseteq L^{p,\varphi}(\Omega), \tag{3.15}$$

if and only if (3.4) holds. In the case $\varphi \in \overline{W}([0,\ell])$, the condition $m(\varphi) \leqslant n$ is necessary for (3.15) *and $M(\varphi) < n$ is sufficient.*

Proof. Embedding (3.15) is equivalent to saying that the function $f(x)$: $u(|x|) \equiv 1$ belongs to $L^{p,\varphi}(\Omega)$. Then the equivalence of (3.15) to (3.3) follows from the definition of the space. To check the conditions in terms of the indices $m(\varphi)$ and $M(\varphi)$, note that $m(u) = M(u) = 0$ for $u \equiv 1$, and then it suffices to refer to (3.10). □

Remark 3.5. In the case of $\ell = \infty$, sufficient or necessary conditions (3.8), (3.9) must be complemented by similar conditions related to the indices $m(u), M(u), m(\varphi), M(\varphi)$ defined in Section 2.1.3.

3.2. Some weighted estimates of functions in Morrey spaces

Our first result in this subsection reads:

Proposition 3.6. *Let $1 \leqslant p < \infty$, $0 < s \leqslant p$, $v \in \overline{W}([0,\ell])$, $v(2t) \leqslant cv(t)$, $\frac{\varphi^{\frac{s}{p}}}{v} \in \underline{W}([0,\ell])$, $0 < \ell \leqslant \infty$. Then*

$$\left(\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)}\, dz \right)^{\frac{1}{s}} \leqslant C\mathcal{A}(|y|)\|f\|_{p,\varphi;\mathrm{loc}}, \quad 0 < |y| \leqslant \ell, \tag{3.16}$$

where $C > 0$ does not depend on y and f and

$$\mathcal{A}(r) = \left(\int_0^r t^{n(1-\frac{s}{p})-1} \frac{\varphi^{\frac{s}{p}}(t)}{v(t)}\, dt \right)^{\frac{1}{s}}. \tag{3.17}$$

Proof. We have

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)}\, dz = \sum_{k=0}^{\infty} \int_{B_k(y)} \frac{|f(z)|^s}{v(|z|)}\, dz, \tag{3.18}$$

where $B_k(y) = \{z\colon\ 2^{-k-1}|y| < |z| < 2^{-k}|y|\}$. Making use of the fact that there exists a β such that $t^{\beta}v(t)$ is almost increasing, we observe that

$$\frac{1}{v(|z|)} \leqslant \frac{C}{v(2^{-k-1}|y|)}$$

on $B_k(y)$. Applying this in (3.18) and making use of the Hölder inequality with the exponent $\frac{p}{s} \geqslant 1$, we obtain

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)}\, dz \leqslant C \sum_{k=0}^{\infty} \frac{(2^{-k-1}|y|)^{n(1-\frac{s}{p})}}{v(2^{-k-1}|y|)} \left(\int_{B_k(y)} \left|f(z)\right|^p dz \right)^{\frac{s}{p}}.$$

Hence

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)}\, dz \leqslant C \sum_{k=0}^{\infty} \left(2^{-k-1}|y|\right)^{n(1-\frac{s}{p})} \frac{\varphi^{\frac{s}{p}}(2^{-k}|y|)}{v(2^{-k-1}|y|)} \|f\|_{p,\varphi;\mathrm{loc}}^s.$$

It remains to prove that

$$\sum_{k=0}^{\infty} \left(2^{-k-1}|y|\right)^{n(1-\frac{s}{p})} \frac{\varphi^{\frac{s}{p}}(2^{-k}|y|)}{v(2^{-k-1}|y|)} \leqslant C\left[\mathcal{A}\left(|y|\right)\right]^s. \tag{3.19}$$

We have

$$\int_0^r t^{n(1-\frac{s}{p})-1} \frac{\varphi^{\frac{s}{p}}(t)}{v(t)}\, dt = \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} t^{n(1-\frac{s}{p})-1} \frac{\varphi^{\frac{s}{p}}(t)}{v(t)}\, dt.$$

We use the fact that $\frac{\varphi^{\frac{s}{p}}(t)}{t^b v(t)}$ is almost decreasing with some b and that $v(2t) \leqslant cv(t)$ and obtain

$$\int_0^r t^{n(1-\frac{s}{p})-1}\frac{\varphi^{\frac{s}{p}}(t)}{v(t)}\,dt \geqslant C\sum_{k=0}^{\infty}\left(2^{-k}r\right)^{n(1-\frac{s}{p})}\frac{\varphi^{\frac{s}{p}}(2^{-k}r)}{v(2^{-k}r)} \geqslant C\sum_{k=0}^{\infty}\left(2^{-k-1}r\right)^{n(1-\frac{s}{p})}\frac{\varphi^{\frac{s}{p}}(2^{-k}r)}{v(2^{-k-1}r)} \tag{3.20}$$

which proves (3.19). □

Corollary 3.7. *Let $1 < p < \infty$. Then*

$$\int_{|z|<|y|}\frac{|f(z)|}{\varphi^{\frac{1}{p}}(|z|)}\,dz \leqslant c|y|^{\frac{n}{p'}}\|f\|_{p,\varphi;\mathrm{loc}}, \quad 0 < |y| \leqslant \ell \leqslant \infty. \tag{3.21}$$

The next result is the following complement to Proposition 3.6.

Proposition 3.8. *Let $1 \leqslant p < \infty$, $0 \leqslant s \leqslant p$, φ satisfy condition (3.3) and $v \in \overline{W}(\mathbb{R}^1_+)$. Then*

$$\left(\int_{|z|>|y|} v\left(|z|\right)\left|f(z)\right|^s dz\right)^{\frac{1}{s}} \leqslant c\mathcal{B}(|y|)\|f\|_{p,\varphi;\mathrm{loc}}, \quad y \neq 0, \tag{3.22}$$

where $C > 0$ does not depend on y and f and

$$\mathcal{B}(r) = \left(\int_r^{\infty} t^{n-1}\left(\frac{\varphi(t)}{t^n}\right)^{\frac{s}{p}} v(t)\,dt\right)^{\frac{1}{s}}. \tag{3.23}$$

Proof. The proof is similar to that of Proposition 3.6. We have

$$\int_{|z|>|y|} v(t)\left|f(z)\right|^s dz = \sum_{k=0}^{\infty}\int_{B^k(y)} v(z)\left|f(z)\right|^s dz,$$

where $B^k(y) = \{z\colon\ 2^k|y| < |z| < 2^{k+1}|y|\}$. Since there exists a $\beta \in \mathbb{R}^1$ such that $t^{\beta}v(t)$ is almost increasing, we obtain

$$\sum_{k=0}^{\infty}\int_{B^k(y)} v\left(|z|\right)\left|f(z)\right|^s dz \leqslant C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\int_{B^k(y)}\left|f(z)\right|^s dz$$

where C may depend on β, but does not depend on y and f. Applying the Hölder inequality with the exponent $\frac{p}{s}$, we get

$$\int_{|z|>|y|} v\left(|z|\right)\left|f(z)\right|^s dz \leqslant C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n(1-\frac{s}{p})}\left(\int_{B^k(y)}\left|f(z)\right|^p dz\right)^{\frac{s}{p}}$$

$$\leqslant C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n(1-\frac{s}{p})}\varphi^{s/p}\left(2^{k+1}|y|\right)\|f\|^s_{p,\varphi;\mathrm{loc}}.$$

It remains to prove that

$$\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n(1-\frac{s}{p})}\varphi^{s/p}\left(2^{k+1}|y|\right) \leqslant C\left[\mathcal{B}(|y|)\right]^s.$$

We have

$$\int_r^{\infty} t^{n-1}\left(\frac{\varphi(t)}{t^n}\right)^{\frac{s}{p}} v(t)\,dt = \sum_{k=0}^{\infty}\int_{2^k r}^{2^{k+1}r} t^{n-1-\frac{ns}{p}}\varphi^{s/p}(t)v(t)\,dt \geqslant C\sum_{k=0}^{\infty} v\left(2^k r\right)\varphi^{s/p}\left(2^k r\right)\left(2^k r\right)^{n(1-s/p)}$$

$$\geqslant C\sum_{k=0}^{\infty} v\left(2^{k+1}r\right)\varphi^{s/p}\left(2^{k+1}r\right)\left(2^k r\right)^{n(1-s/p)},$$

which completes the proof. □

Remark 3.9. The analysis of the proof shows that estimate (3.22) remains in force, if the assumption $v \in \overline{W}(\mathbb{R}^1_+)$ is replaced by the condition that $\frac{1}{v} \in \overline{W}(\mathbb{R}^1_+)$ and v satisfies the doubling condition $v(2t) \leqslant cv(t)$.

Corollary 3.10. *Let* $1 \leqslant p < \infty$. *Then*

$$\int\limits_{|z|>|y|} \frac{|f(z)|}{|z|^b \varphi^{\frac{1}{p}}(|z|)}\, dz \leqslant c|y|^{\frac{n}{p'}-b} \|f\|_{p,\varphi;\mathrm{loc}}, \quad y \neq 0, \tag{3.24}$$

for every $b > \frac{n}{p'}$.

4. On weighted Hardy operators in generalized Morrey spaces

4.1. Pointwise estimations

We consider the following generalized Hardy operators

$$H^\alpha_w f(x) = |x|^{\alpha-n} w\big(|x|\big) \int\limits_{|y|<|x|} \frac{f(y)\,dy}{w(|y|)}, \qquad \mathcal{H}^\alpha_w f(x) = |x|^{\alpha} w\big(|x|\big) \int\limits_{|y|>|x|} \frac{f(y)\,dy}{|y|^n w(|y|)}, \tag{4.1}$$

and their one-dimensional versions:

$$H^\alpha_w f(x) = x^{\alpha-1} w(x) \int\limits_0^x \frac{f(t)\,dt}{w(t)}, \qquad \mathcal{H}^\alpha_w f(x) = x^{\alpha} w(x) \int\limits_x^\infty \frac{f(t)\,dt}{t w(t)}, \quad x > 0 \tag{4.2}$$

adjusted for the half-axis $\mathbb{R}^1_+$. In the sequel $\mathbb{R}^n$ with $n = 1$ may be read either as $\mathbb{R}^1$ or $\mathbb{R}^1_+$.

We also use the notation

$$H^\alpha = H^\alpha_w\big|_{w\equiv 1}.$$

The proof of our main result of this section given in Theorem 4.2 is prepared by the following theorem on the pointwise estimates of the Hardy-type operators.

Theorem 4.1. *Let* $1 \leqslant p < \infty$ *and* φ *satisfy condition* (3.3).

I. *Let* $w \in \overline{W}$, $w(2t) \leqslant Cw(t)$ *and* $\frac{\varphi^{\frac{1}{p}}}{w} \in \underline{W}([0,\ell])$. *The condition*

$$\int\limits_0^\varepsilon \frac{t^{\frac{n}{p'}-1} \varphi^{1/p}(t)}{w(t)}\, dt < \infty, \tag{4.3}$$

with $\varepsilon > 0$, *is sufficient for the Hardy operator* H^α_w *to be defined on the space* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ *or on the space* $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$. *Under this condition*

$$\big|H^\alpha_w(x)\big| \leqslant C|x|^{\alpha-n} w\big(|x|\big) \int\limits_0^{|x|} \frac{t^{\frac{n}{p'}-1} \varphi^{1/p}(t)}{w(t)}\, dt\, \|f\|_{p,\varphi;\mathrm{loc}}. \tag{4.4}$$

Condition (4.3) *is also necessary in the case of* $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$ *if also* $\int_0^h \frac{\varphi(t)}{t}\, dt \leqslant c\varphi(h)$. *In the case of* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ *it is also necessary, if either* $\varphi \in \Phi^0_n$ *or* $\varphi(r) = r^n$.

II. *Let* $\frac{1}{w} \in \overline{W}$, *or* $w \in \overline{W}$ *and* $w(2t) \leqslant Cw(t)$. *The condition*

$$\int\limits_\varepsilon^\infty \frac{t^{-\frac{n}{p}-1} \varphi^{1/p}(t)}{w(t)}\, dt < \infty \tag{4.5}$$

with $\varepsilon > 0$, *is sufficient for the Hardy operator* $\mathcal{H}^\alpha_w$ *to be defined on the space* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ *or* $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$, *and in this case*

$$\big|\mathcal{H}^\alpha_w(x)\big| \leqslant C|x|^{\alpha} w\big(|x|\big) \int\limits_{|x|}^\infty \frac{t^{-\frac{n}{p}-1} \varphi^{1/p}(t)}{w(t)}\, dt\, \|f\|_{p,\varphi;\mathrm{loc}}. \tag{4.6}$$

Condition (4.5) *is also necessary in the case of* $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$. *It is also necessary in the case of* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$, *if either* $\varphi \in \Phi^0_n$ *or* $\varphi(r) = r^n$.

Proof. I. *The "sufficiency" part*. The sufficiency of condition (4.3) and estimate (4.4) follow from (3.16) under the choice $s = 1$ and $v(t) = w(t)$.

The "necessity" part. We choose a function $f(x)$ equal to $[\frac{\varphi(|x|)}{|x|^n}]^{\frac{1}{p}}$ in a neighborhood of the origin and zero beyond this neighborhood. Then $f \in \mathcal{L}^{p,\varphi}_{\mathrm{loc};0}$ by condition (3.6), see Lemma 3.3. It is also in $\mathcal{L}^{p,\varphi}$, if either $\varphi(r) = r^n$ or $\varphi \in \Phi^0_n$, by conditions i) and ii) of Lemma 3.3. For this function f, the existence of the integral $H^\alpha_\varphi f$ coincides with condition (4.3).

II. *The "sufficiency" part*. The sufficiency of condition (4.5) and estimate (4.6) follow from (3.22) under the choice $s = 1$ and $v(t) = \frac{1}{t^n w(t)}$.

The "necessity" part. The proof is the same as in Part I, via the choice $f(x) = [\frac{\varphi(|x|)}{|x|^n}]^{\frac{1}{p}}$. □

4.2. Weighted $p \to q$ estimates for Hardy operators in the generalized Morrey spaces

The statements of Theorem 4.2 are well known in the case of Lebesgue space $\varphi = 1$ when $1 < p < \frac{n}{\alpha}$, see for instance [22, pp. 6, 54]. As can be seen from the results below, inequalities for the Hardy operators in Morrey spaces admit the case $p = 1$ when $\varphi(0) = 0$.

4.2.1. The case of local spaces $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$

Theorem 4.2. *Let $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and φ satisfy condition* (3.3).

I. *Suppose that*

$$w \in \overline{W}\left(\mathbb{R}^1_+\right), \quad w(2t) \leqslant c w(t), \qquad \frac{\varphi^{\frac{1}{p}}}{w} \in \underline{W}\left(\mathbb{R}^1_+\right). \tag{4.7}$$

The operator H^α_w is bounded from $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$, if

$$\sup_{r>0} \frac{1}{\varphi(r)} \int_0^r w^q(s) s^{q(\alpha-n)+n-1} \left(\int_0^s \frac{t^{\frac{n}{p'}-1} \varphi^{\frac{1}{p}}(t)}{w(t)} \, dt \right)^q ds < \infty. \tag{4.8}$$

If $\int_0^h \frac{\varphi(t)}{t} \, dt \leqslant c\varphi(h)$, then (4.8) *is also necessary.*

II. *Suppose that*

$$\frac{1}{w} \in \overline{W}\left(\mathbb{R}^1_+\right), \text{ or } w \in \overline{W}\left(\mathbb{R}^1_+\right) \quad \text{and} \quad w(2t) \leqslant C w(t). \tag{4.9}$$

The operator $\mathcal{H}^\alpha_w$ is bounded from $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$, if and only if

$$\sup_{r>0} \frac{1}{\varphi(r)} \int_0^r w^q(s) s^{q\alpha+n-1} \left(\int_s^\infty \frac{t^{-\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t)}{w(t)} \, dt \right)^q ds < \infty. \tag{4.10}$$

Proof. I. The "if" part follows from estimate (4.4) of Lemma 4.1 and Lemma 3.3, since condition (4.8) is nothing else but the requirement that the radial function arising on the right-hand side of (4.4) belongs to $\mathcal{L}^{q,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$ according to Lemma 3.3. To prove the "only if part", as in the proof of Lemma 4.1, we choose $f(x) = \frac{\varphi^{\frac{1}{p}}(|x|)}{|x|^{\frac{n}{p}}}$ which is in $\mathcal{L}^{p,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$ by Lemma 3.3. Then condition (4.8) is nothing else but the statement that $H^\alpha_w f$ belongs to $\mathcal{L}^{q,\varphi}_{\mathrm{loc};0}(\mathbb{R}^n)$.

II. In the case of the operator $\mathcal{H}^\alpha_w(x)$ the arguments are similar, based on part II of Lemma 4.1 and Lemma 3.3. □

Remark 4.3. Conditions (4.8) and (4.10) represent joint restriction on p, q, φ and w. In the case $w \equiv 1$ and $\varphi(r) = r^\lambda$, they recover the Sobolev and Adams [1] exponent $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

4.2.2. The case of global spaces $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$

We can formulate the statement of Theorem 4.2 for the case of the global spaces.

Theorem 4.4. *Let $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and φ satisfy condition* (3.3). *Under conditions* (4.7) *and* (4.9) *for the operators H^α_w and $\mathcal{H}^\alpha_w$, respectively, these operators are bounded from $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\varphi}(\mathbb{R}^n)$, if*

$$\sup_{x\in\Omega,\, r>0} \frac{1}{\varphi(r)} \int_{B(x,r)} w^q\big(|y|\big)|y|^{q(\alpha-n)} \left(\int_0^{|y|} \frac{t^{\frac{n}{p'}-1} \varphi^{\frac{1}{p}}(t)}{w(t)} \, dt \right)^q dy < \infty \tag{4.11}$$

and

$$\sup_{x\in\Omega,\, r>0} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} w^q(|y|)|y|^{q\alpha} \left(\int\limits_{|y|}^{\infty} \frac{t^{-\frac{n}{p}-1}\varphi^{\frac{1}{p}}(t)}{w(t)}\,dt \right)^q dy < \infty, \tag{4.12}$$

respectively. These conditions are also necessary, when either $\varphi \in \Phi_n^0$ or $\varphi(r) = r^n$.

Proof. The sufficiency part follows from estimates (4.4) and (4.6). For the necessity, just observe that (4.11) is nothing else but the statement that $H_w^\alpha f \in \mathcal{L}^{q,\varphi}(\mathbb{R}^n)$ under the choice $f(x) = [\frac{\varphi(|x|)}{|x|^n}]^{\frac{1}{p}}$, which belongs to $\mathcal{L}^{q,\varphi}(\mathbb{R}^n)$ under the assumptions of the theorem. Similarly (4.12) is interpreted. □

5. Application to potential operators

We consider the potential operator

$$I^\alpha f(x) := \int\limits_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n, \tag{5.1}$$

and show that its weighted boundedness in Morrey spaces – in the case of weights $w \in V_+^\mu \cup V_-^\mu$ with $\mu = \min\{1, n-\alpha\}$ – is a consequence of the non-weighted boundedness and the weighted boundedness of Hardy operators, the latter being given in Theorem 4.4.

5.1. Reduction to Hardy operators

The necessity of the boundedness of the Hardy operators for that of potential operators is a consequence of the following simple fact, where $X = X(\mathbb{R}^n)$ and $Y = Y(\mathbb{R}^n)$ are arbitrary Banach function spaces in the sense of Luxemburg (cf., for example, [6]).

Lemma 5.1. *(See [39].) Let $w = w(x)$ be any weight function. For the boundedness of the weighted potential operator $wI^\alpha \frac{1}{w}$ from X to Y, it is necessary that the Hardy operators H_w^α and $\mathcal{H}_{w_\alpha}^\alpha$ are bounded from X to Y, where $w_\alpha(x) = |x|^{-\alpha} w(x)$.*

The proof of the sufficiency of the obtained conditions is based on the pointwise estimate of the following Lemma 5.2 which was proved in [39].

Lemma 5.2. *(See [39].) Let $w \in \mathbf{V}_-^\mu \cup \mathbf{V}_+^\mu$ with $\mu = \min\{1, n-\alpha\}$ be a weight and f a non-negative function. Then the following pointwise estimate holds*

$$wI^\alpha \frac{1}{w} f(x) \leqslant I^\alpha f(x) + c \begin{cases} H_w^\alpha f(x) + \mathcal{H}_{-\alpha}^\alpha f(x), & \text{if } w \in \mathbf{V}_+^\mu, \\ H^\alpha f(x) + \mathcal{H}_{w_\alpha}^\alpha f(x), & \text{if } w \in \mathbf{V}_-^\mu. \end{cases} \tag{5.2}$$

In view of Lemma 5.2, for the weighted boundedness of the potential operator under the corresponding assumptions on weights it suffices to apply the results obtained in Section 4 for the Hardy operators and the results for the boundedness of the non-weighted Riesz potential.

5.2. On a non-weighted $p \to q$-boundedness of potential operators in the generalized Morrey spaces

Potential operators in the non-weighed setting were studied in [12,14,15,17,23,28,40]. We make use of the Adams-type p- to q-statement which we give in Theorem 5.4 following the approach developed in [14,15]. Note that Theorem 5.4 does not impose any monotonicity condition of the function φ. A version of Theorem 5.4 for bounded domains but in a more general setting of variable exponent Morrey spaces was proved in [16].

To prove Theorem 5.4, we first need the following statement on the boundedness of the maximal operator in generalized Morrey spaces. In such a form Theorem 5.3 was proved in fact in [14,15]. For the proof we refer also to [16] where it was extended to the case of variable $p(x)$ and $\varphi(x,r)$ (note that in [16] the maximal operator was considered over bounded domains in $\mathbb{R}^n$, but the proof presented there is valid for unbounded domains as well when p is constant).

Theorem 5.3. *Let $1 < p < \infty$ and $\varphi(r)$ be a non-negative measurable function satisfying the condition*

$$\int\limits_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}+1}}\,dt \leqslant C\frac{\varphi^{\frac{1}{p}}(r)}{r^{\frac{n}{p}}}. \tag{5.3}$$

Then the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy$$

is bounded in the space $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$.

By means of Theorem 5.3, the following theorem is proved.

Theorem 5.4. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $q > p$ *and* $\varphi(r)$ *a non-negative measurable function satisfying condition* (5.3) *and the condition*

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}-\alpha+1}}\,dt \leqslant Cr^{-\frac{\alpha p}{q-p}}. \tag{5.4}$$

Then the Riesz potential operator I^α *is bounded from* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{L}^{q,\varphi}(\mathbb{R}^n)$.

Proof. We follow the approach developed in [14–16], and as in [16] prove the following pointwise estimate

$$|I^\alpha f(x)| \leqslant Ct^\alpha Mf(x) + C\int_t^\infty r^{\alpha-\frac{n}{p}-1}\|f\|_{L^p(B(x,r))}\,dr, \quad t > 0. \tag{5.5}$$

To this end, we represent the function f in the form

$$I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x),$$

where $f_1(y) = f(y)\chi_{B(x,2t)}(y)$, $f_2(y) = f(y)\chi_{\mathbb{R}^n\setminus B(x,2t)}(y)$. For $I^\alpha f_1(x)$, by the well known Hedberg trick we have

$$|I^\alpha f_1(x)| \leqslant C_1 t^\alpha Mf(x).$$

For $I^\alpha f_2(x)$ we obtain

$$\begin{aligned}
|I^\alpha f_2(x)| &\leqslant \int_{|y-x|>2t} |x-y|^{\alpha-n}|f(y)|\,dy \\
&= (n-\alpha)\int_{|y-x|>2t} |f(y)|\,dy \int_{|x-y|}^\infty r^{\alpha-n-1}\,dr \\
&= (n-\alpha)\int_{2t}^\infty \left(\int_{2t<|x-y|<r} |f(y)|\,dy\right) r^{\alpha-n-1}\,dr \\
&\leqslant (n-\alpha)\int_t^\infty \|f\|_{L_1(B(x,r))} r^{\alpha-n-1}\,dr \\
&\leqslant C\int_{2t}^\infty \|f\|_{L_p(B(x,r))} r^{\alpha-\frac{n}{p}-1}\,dr \\
&\leqslant C\int_t^\infty \|f\|_{L^p(B(x,r))} r^{\alpha-\frac{n}{p}-1}\,dr,
\end{aligned}$$

which proves (5.5).

By (5.5) we get

$$|I^\alpha f(x)| \leqslant Cr^\alpha Mf(x) + C\|f\|_{p,\varphi}\int_r^\infty t^{\alpha-\frac{n}{p}-1}\varphi^{\frac{1}{p}}(t)\,dt.$$

In view of (5.4), we obtain

$$|I^\alpha f(x)| \leqslant Cr^\alpha Mf(x) + Cr^{-\frac{\alpha p}{q-p}}\|f\|_{p,\varphi}.$$

Now we choose $r = (\frac{\|f\|_{p,\varphi}}{Mf(x)})^{\frac{q-p}{\alpha q}}$ and arrive at

$$\left|I^{\alpha} f(x)\right| \leqslant C\left(Mf(x)\right)^{\frac{p}{q}} \|f\|_{p,\varphi}^{1-\frac{p}{q}},$$

whence the statement of the theorem follows by the boundedness of the maximal operator M in $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ provided by Theorem 5.3 in virtue of condition (5.3). □

5.3. $p \to q$-boundedness of potential operators in generalized Morrey spaces

To formulate the result, we need the following conditions derived from (4.11)–(4.12) according to the estimation in (5.2):

$$\sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} |y|^{q(\alpha-n)} \left(\int\limits_0^{|y|} t^{\frac{n}{p'}-1} \varphi^{\frac{1}{p}}(t)\, dt \right)^q dy < \infty, \tag{5.6}$$

$$\sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} \left(\int\limits_{|y|}^{\infty} t^{\alpha-\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t)\, dt \right)^q dy < \infty, \tag{5.7}$$

and

$$\sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(r)} \int\limits_{B(x,r)} w^q\left(|y|\right) \left(\int\limits_{|y|}^{\infty} \frac{t^{\alpha-\frac{n}{p}-1} \varphi^{\frac{1}{p}}(t)}{w(t)}\, dt \right)^q dy < \infty. \tag{5.8}$$

Theorem 5.5. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $q > p$ and $\varphi(r)$ a non-negative measurable function satisfying conditions* (3.3) *and* (5.3)–(5.4)*. Let the weight $w \in \overline{W}(\mathbb{R}^1_+) \cap \underline{W}(\mathbb{R}^1_+)$ satisfy the conditions*

$$w \in \mathbf{V}_-^{\mu} \cup \mathbf{V}_+^{\mu}, \quad \mu = \min\{1, n-\alpha\}$$

and $\frac{\varphi^{\frac{1}{p}}}{w} \in \underline{W}(\mathbb{R}^1_+)$. Then the weighted Riesz potential operator $wI^{\alpha}\frac{1}{w}$ is bounded from $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\varphi}(\mathbb{R}^n)$ under conditions (4.11) *and* (5.7) *in the case $w \in \mathbf{V}_+^{\mu}$ and conditions* (5.6) *and* (5.8) *in the case $w \in \mathbf{V}_-^{\mu}$.*

In the case where when either $\varphi \in \Phi_n^0$ or $\varphi(r) = r^n$, conditions (4.11)*,* (5.7) *and* (5.6)*,* (5.8) *are also necessary.*

Proof. Apply Lemmas 5.1, 5.2 and Theorems 5.4, 4.4. □

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