Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 429 (2008) 1810–1822
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On a “stability” in the linear complementarity problem

M. Pires *, V.G. Kravchenko

Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Portugal

Received 15 February 2008; accepted 20 May 2008
Available online 7 July 2008
Submitted by Queiro

Abstract

In this work we rewrote the linear complementarity problem in a formulation based on unknown projector operators. In particular, this formulation allows the introduction of a concept of “stability” that, in a certain way, might explain the way block pivotal algorithm performs.

Keywords: Linear complementarity problem; Block pivotal algorithm

1. Introduction

For a given vector $q \in \mathbb{R}^n$ and a given matrix $M \in \mathbb{R}^{n\times n}$ the linear complementarity problem (LCP) consists in finding vectors z and w in $\mathbb{R}^n$ such that

$$\begin{aligned} &w = Mz + q, \\ &w^{\mathrm{T}} z = 0, \\ &z \geqslant 0; \quad w \geqslant 0. \end{aligned} \tag{1.1}$$

The first and to our best knowledge the only monograph completely dedicated to this problem is by Murty [1]. In this fundamental work a deep analysis of the LCP has been carried out under different restrictions on the matrix M. Some applications of this problem to other problems have been given.

The most typical application of the LCP is the quadratic programming problem. The LCP can appear naturally from specific properties of a problem or as a necessary optimality condition for a

* Corresponding author.
E-mail addresses: mpires@ualg.pt (M. Pires), vkravch@ualg.pt (V.G. Kravchenko).

0024-3795/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2008.05.029

Quadratic Programming Problem or as a result of establishing the equivalence between the LCP and the Linear Variational Inequality Problem. In the present paper the main emphasis is on the case of a matrix M possessing non negative entries only. Problems of this type have applications for instance in finances (see, e.g. [2]).

In [3] Murty proposed an algorithm for solving the LCP. Murty's algorithm belongs to the class of direct methods which are based on single principal pivot operations and search for an exact solution of the problem. In the same work Murty proves the convergence of his algorithm. Some years later Kostreva in the work [4] proposed the idea that the direct methods of solution of the LCP can make use of block pivotal operations.

In the work [5] Murty's algorithm was generalized onto the case of principal block pivot operation (BPA) in the way explained below.

We need some notations that are usually used in this kind of problems. A pair of vectors (z, w) which satisfies the system $w = Mz + q$ and the condition $w_j z_j = 0$, $j = 1, \ldots, n$ is called a complementary solution. A pair of vectors (z, w) satisfying the system $w = Mz + q$, $z \geqslant 0$ and $w \geqslant 0$ is called a feasible solution.

In what follows x_J will represent the components of a vector x whose indices belong to the set J and M_{JK} is the submatrix of M whose indices of rows are in the index set J and indeces of columns are in the index set K. A principal block pivot operation with pivot M_{JJ} transforms the problem as follows (here $K = \{1, \ldots, n\} - J$)

$$\begin{bmatrix} z_J \\ w_K \end{bmatrix} = \bar{q} + \overline{M} \begin{bmatrix} w_J \\ z_K \end{bmatrix},$$

where $\bar{q} = \begin{bmatrix} -M_{JJ}^{-1} q_J \\ q_K - M_{KJ} M_{JJ}^{-1} q_J \end{bmatrix}$ and $\overline{M} = \begin{bmatrix} M_{JJ}^{-1} & -M_{JJ}^{-1} M_{JK} \\ M_{KJ} M_{JJ}^{-1} & M_{KK} - M_{KJ} M_{JJ}^{-1} M_{JK} \end{bmatrix}$.

Let us consider any complementary solution of (1.1) and define two index sets $F = \{i : z_i \text{ is basic}\}$ and $T = \{i : w_i \text{ isbasic}\}$. As we have a complementary solution, then $F \cap T = \emptyset$ and $F \cup T = \{1, \ldots, n\}$ at each iteration. If one could find a set of indices F leading to the solution of the problem, that is such that $z \geqslant 0$ and $w \geqslant 0$, then it would be possible to recover from the above formulas the values of the vectors z and w corresponding to the solution. Indeed, as the components of the vectors w_F and z_T are non-basic variables then $w_F = 0$ and $z_T = 0$, and we have $z_F = \bar{q}_F \geqslant 0$ and $w_T = \bar{q}_T \geqslant 0$. If F is not the right set of indices, at least one component of z_F or w_T is negative. The set $H = \{j : w_j < 0 \vee z_j < 0\} = \{j : \bar{q}_j < 0\}$ is called the infeasibility set. Whenever we can remove one index from this set we say that one infeasibility is removed.

Murty's algorithm, at each iteration, chooses the maximum index $j \in H$ and performs a single principal pivot operation with pivot $\bar{m}_{jj}$. This is equivalent to interchanging j from F to T or from T to F according to the circunstances.

Block pivotal algorithm (BPA), at each iteration, performs a block principal pivot operation with pivot $\overline{M}_{HH}$. This is equivalent to interchanging all the indeces of $H \cap F$ from F to T and all the indeces of $H \cap T$ from T to F.

The aim of both methods is to reduce the number of infeasibilities from iteration to iteration.

Cycling examples of the BPA with P matrices have been constructed [6,1].

A proof of convergence of BPA for diagonal dominant matrices of order 3 is presented in [6] and a proof of its convergence for Minkowski matrices is presented in [7]. In spite of the fact that other convergence conditions for BPA have not yet been obtained, the method is in use and nobody has ever reported the existence of cycling with strictly diagonal dominant matrices. As it can be seen in [5] computational experience shows the exceptional superiority of BPA when compared to single pivotal algorithms.

Single pivotal methods, such as Murty's method [3] begin with $F = \phi$, and remove one infeasibility at each iteration. Block pivotal algorithm (BPA) [5] begins with $F = \{i : q_i < 0\}$ and at each iteration tries to remove all the infeasibilities. In the present work a modification of the BPA is proposed which improves the initial set of the iterative algorithm.

In Section 2 we present a formulation of the LCP (1.1) based on the use of orthogonal complementary projection operators and in Section 3 we study the case when the matrix M does not have negative entries. Based on the analysis from Section 2 we propose a modification of BPA which we call Algorithm 1. As our computational experience shows, Algorithm 1 has generally a better performance. In the table presented in Section 3 it is possible to see that the initial set that is found using Algorithm 1, most of the times, is the solution set of the problem. But there are some cases where the number of systems to solve is the same as with the conventional BPA algorithm. Nevertheless the dimensions of the systems are smaller. Only problem P14 does not follow this pattern.

In order to clarify this behavior of Algorithm 1 in Section 4 we introduce a certain concept of stability (Definition 4.1) which helps to separate these "bad", nonstable (in the sense of our definition) cases. Moreover, this concept representing, in our opinion, independent interest and leads us naturally to Algorithm 2. With the aid of this algorithm it is possible to find an initial set closer to the right set in the sense of the symmetric difference of sets.

2. Formulation of the LCP using projection operators

Let F be a subset of the index set $N = \{1, 2, \ldots, n\}$. By P_F and Q_F we denote two projection matrices such that

$$P_F = (p_{i,j}), \quad \text{where } p_{i,j} = \begin{cases} p_{ii} = 1, & i \in F, \\ p_{ii} = 0, & i \notin F, \\ p_{ij} = 0, & i \neq j, \end{cases} \text{ and } Q_F = I - P_F.$$

With these definitions the LCP (1.1) can be written as follows:

$$MP_Fx - Q_Fx = g, \quad x \geqslant 0, \tag{2.1}$$

where in order to simplify notations, $g = -q$.

Observing that

$$\det(MP_F - Q_F) = (-1)^{(n-\#F)} \det(P_F M P_F),$$

the equation

$$MP_Fx - Q_Fx = g$$

has a unique solution for any set F.

The constrain $x \geqslant 0$ implies that the main work to be done in order to solve the LCP is to find the orthogonal complementary projection operators P_F *and* Q_F such that the problem (2.1) has a nonnegative solution.

Under the condition that M is a PD matrix, the LCP (1.1) has a unique solution and so this pair of operators P_F and Q_F exists and is also uniquely determined by the set F. Such set F that determines the solution of the problem will be called the solution set.

Let

$$M = (\Lambda + B - A), \quad \text{where } \Lambda = diag(m_{11}, m_{22}, \ldots, m_{nn}); \quad A, B \geqslant 0.$$

Due to the fact that the diagonal elements of a PD matrix are all positive, Λ^{-1} exists. As $P_F + Q_F = I$, $P_F Q_F = Q_F P_F = 0$ and Λ^{-1} commutes with P_F and Q_F, the following equality holds:

$$(\Lambda + B - A)P_F - Q_F = \Lambda[(I + \Lambda^{-1}B - \Lambda^{-1}A)P_F - Q_F][P_F + \Lambda^{-1}Q_F].$$

This shows us that (2.1) is equivalent to the following problem:

$$(I + \Lambda^{-1}B - \Lambda^{-1}A)P_F x - Q_F x = \Lambda^{-1}g, \quad x \geqslant 0. \tag{2.2}$$

Without loss of generality it is possible to suppose that $\Lambda = I$.

It is easy to verify that

$$[(I + B - A)P_F - Q_F][P_F - Q_F][I - Q_F(B - A)P_F] = I + P_F(B - A)P_F.$$

Thus, if y_* is a solution of the equation

$$y_* + P_F(B - A)P_F y_* = g,$$

then

$$\begin{aligned} x_* &= [P_F - Q_F][I - Q_F B P_F + Q_F A P_F]y_* \\ &= P_F y_* - Q_F y_* + Q_F B P_F y_* - Q_F A P_F y_* \end{aligned}$$

is a solution of the system (2.1).

It is easy to see that the vectors x_* and y_* are also related by the equation

$$y_* = (I + Q_F B P_F - Q_F A P_F)(P_F - Q_F)x_*.$$

The vector y_* must satisfy the following equalities:

$$\begin{aligned} &Q_F y_* = Q_F g, \\ &P_F y_* + P_F B P_F y_* = P_F g + P_F A P_F y_*. \end{aligned}$$

The equalities

$$\begin{aligned} &P_F x_* = P_F y_*, \\ &Q_F x_* = Q_F B P_F y_* - Q_F y_* - Q_F A P_F y_* \end{aligned} \tag{2.3}$$

imply that the constrain $x \geqslant 0$ is equivalent to the following conditions:

$$\begin{aligned} &P_F y_* \geqslant 0, \\ &Q_F B P_F y_* \geqslant Q_F y_* + Q_F A P_F y_* \end{aligned}$$

or

$$\begin{aligned} &P_F y_* \geqslant 0, \\ &Q_F B P_F y_* \geqslant Q_F g + Q_F A P_F y_*. \end{aligned}$$

3. Case $A = 0$

In this case the previous conditions can be simplified to

$$Q_F y_* = Q_F g, \tag{3.1}$$
$$P_F y_* + P_F B P_F y_* = P_F g,$$

$$\begin{aligned} &P_F y_* \geqslant 0, \\ &Q_F B P_F y_* \geqslant Q_F g. \end{aligned} \tag{3.2}$$

As $P_F B P_F y_* \geqslant 0$ we have

$$P_F y_* \leqslant P_F g. \tag{3.3}$$

Let $u \in \mathbb{R}^n$; $\pi_j(u) = u_j$, $j \in N$; $N^+ = N^+(g) = \{j \in N : \pi_j(g) > 0\}$.

According to the second equality in (3.1) $j \in F \Rightarrow \pi_j(g) \geqslant 0$. Then $F \subseteq N^+$.

As

$$\pi_j(g - BP_{N^+}g) = \pi_j(g - BP_F g) - \pi_j(BP_{N^+\setminus F}g) \leqslant \pi_j(g - BP_F g),$$

then

$$\pi_j(g - BP_{N^+}g) \geqslant 0 \Rightarrow \pi_j(g - BP_F g) \geqslant 0 \quad \forall j \in F. \tag{3.4}$$

From the second inequality in (3.2) together with the inequality (3.3) we get

$$\pi_j(g - BP_{N^+}g) \leqslant 0 \quad \forall j \notin F.$$

Denote

$$N_0^+ = \{j \in N^+ : \pi_j(g - BP_{N^+}g) > 0\}.$$

From (3.4) it is obvious that $N_0^+ \subseteq F \subseteq N^+$

Thus, it makes sense to suggest N_0^+ as a starting set for the BPA. Besides it is useful to include in the BPA a step (see the following step 1) in which the condition $Q_F BP_F g - Q_F g \geqslant 0$ is verified.

We also observe that, as the matrix M does not have negative elements, it is clear that whenever $g_i < 0$ the corresponding variable w_i must be a basic variable. So, defining $\mathscr{T} = \{i : g_i < 0\}$ at each iteration we must have $\mathscr{T} \subseteq T$.

Algorithm 1

Step 0. Define $\mathscr{T} = \{i : g_i < 0\}$;
Determine $u = (B - I)g$;
Define $F = \{i : u_i < 0\}$.
Step 1. Evaluate $\nu = Q_{F\cup\mathscr{T}}(BP_F - I)g$
If $\nu \geqslant 0$ go to step 2
Otherwise make $F = F \cup \{i : \nu_i < 0\}$ and repeat step 1
Step 2. Evaluate $x = ((I + B)P_F - Q_F)^{-1}g$.
If $x \geqslant 0$ stops. The solution is $z_F = P_F x$; $z_T = 0$; $w_F = 0$; $w_T = Q_F x$
Otherwise go to step 3
Step 3. Define $H_1 = \{i \in F : x_i < 0\}$; $H_2 = \{i \in T : x_i < 0\}$
Step 4. Make $F = (F - H_1) \cup H_2$ and go to step 1.

In Table 1 we present some computational results corresponding to randomly generated symmetric diagonal dominant matrices of dimension 700. The diagonal was defined to be equal to the sum of the absolute values of the entries of the corresponding row plus a small positive perturbation, to be sure that no singular matrices are obtained. We constructed the vector g starting from a randomly generated solution (z, w) with a given percentage of basic z-components.

In certain examples the application of step 1 leads directly to the solution of the problem as it can be seen in Table 1.

Computational experience showed that application of step 1 could reduce the number of pivot operations of BPA. In the cases of a worse behavior the number of pivot operations did not change

Table 1
A comparison between the dimensions of systems solved by the BPA and by Algorithm 1

Problem	BPA				Algorithm1		
	# Sytems	Dimensions			# Sytems	Dimensions	
P1	3	700	150	89	1		89
P2	2		700	605	1		605
P3	3	700	155	70	1		70
P4	3	700	196	142	1		142
P5	3	700	236	211	1		211
P6	3	700	281	275	1		275
P7	2		700	339	1		339
P8	2		700	408	1		408
P9	2		700	474	1		474
P10	2		700	534	2	533	534
P11	4		700	386	4	310	500
			340	339		354	339
P12	4		700	291	3	175	187
			183	178		178	
P13	4		700	224	3	68	71
			83	70		70	
P14	4		700	517	5	427	700
			503	502		517	503
						502	
P15	2		700	339	1		339
P16	2		700	178	1		178
P17	2		700	70	1		70
P18	2		700	502	1		502
P19	2		700	605	1		605

but the dimensions of the systems to solve were smaller. The only exception was problem P14. We realized that for the problems for which the algorithm had a worst performance (P11 to P14) the values of positive variables in the solution are very close to 0.

4. Algorithm 2

Let $\rho \in \mathbb{R}$ and $\Phi \subseteq N^+$. We shall denote

$$\mathcal{N}^{\rho}(\Phi) = \{j \in N^+ : \pi_j(g - BP_{\Phi}) \geqslant \rho\},$$
$$\mathcal{N}^{\rho}_{-}(\Phi) = \{j \in N^+ : \pi_j(g - BP_{\Phi}) < \rho\}$$

and consider two sets Φ and Ψ such that $\Phi \subseteq \Psi \subseteq N^+$.

We have

$$\pi_j(g - BP_{\Psi}g) = \pi_j(g - BP_{\Phi}g) - \pi_j(BP_{\Psi \setminus \Phi}g) \leqslant \pi_j(g - BP_{\Phi}g),$$

so

$$\pi_j(g - BP_{\Psi}g) \geqslant \rho \Rightarrow \pi_j(g - BP_{\Phi}g) \geqslant \rho,$$
$$\pi_j(g - BP_{\Phi}g) < \rho \Rightarrow \pi_j(g - BP_{\Psi}g) < \rho,$$

which implies the validity of the following statement.

Proposition 4.1. *If* $\Phi \subseteq \Psi \subseteq N^+$, *then*

$$\begin{aligned}&\mathscr{N}^\rho(\Psi) \subseteq \mathscr{N}^\rho(\Phi),\\&\mathscr{N}^\rho_-(\Phi) \subseteq \mathscr{N}^\rho_-(\Psi).\end{aligned} \tag{4.1}$$

Let

$$\begin{aligned}&F_1^\rho = \mathscr{N}^\rho(N^+),\\&\dots\\&F_k^\rho = \mathscr{N}^\rho(F_{k-1}^\rho),\\&\dots\end{aligned}$$

Proposition 4.2. *Both sequences* $F_2^\rho, F_4^\rho, \dots, F_{2k}^\rho, \dots$ *and* $F_1^\rho, F_3^\rho, \dots, F_{2k+1}^\rho, \dots$ *are convergent and the following equalities are valid*:

$$F_{\text{odd}}^\rho = \lim F_{2k+1}^\rho \subseteq F_{\text{even}}^\rho = \lim F_{2k}^\rho. \tag{4.2}$$

Besides, *if* $F_{2k}^\rho = F_{2k_0}\ \forall k \geqslant k_0$, *then* $F_{2k+1}^\rho = F_{2k_0+1}^\rho\ \forall k \geqslant k_0$.

Proof. As $F_m^\rho \subseteq N^+, \forall m$, we have the following inclusions:

$$F_1^\rho \subseteq F_{m+1}^\rho \Rightarrow F_{m+2}^\rho \subseteq F_2^\rho \Rightarrow F_3^\rho \subseteq F_{m+3}^\rho \Rightarrow F_{m+4}^\rho \subseteq F_4^\rho \Rightarrow F_5^\rho \subseteq F_{m+5}^\rho \Rightarrow \cdots$$

or

$$F_1^\rho \subseteq F_3^\rho \subseteq F_5^\rho \subseteq \cdots \subseteq F_{2k+1}^\rho \subseteq \cdots \subseteq F_{2k}^\rho \subseteq \cdots \subseteq F_4^\rho \subseteq F_2^\rho \subseteq N^+.$$

Thus, the sequences

$$F_2^\rho, F_4^\rho, \dots, F_{2k}^\rho, \dots$$

and

$$F_1^\rho, F_3^\rho, F_5^\rho, \dots, F_{2k+1}^\rho, \dots$$

are both convergent and we can deduce the relation (4.2). □

We also observe that

$$\lim_{\rho\to+\infty} F_{\text{odd}}^\rho = \lim_{\rho\to+\infty} F_{\text{even}}^\rho = \emptyset \quad \text{and} \quad \lim_{\rho\to-\infty} F_{\text{odd}}^\rho = \lim_{\rho\to-\infty} F_{\text{even}}^\rho = N^+.$$

Let ρ and $\Phi \subseteq N^+$ be such that $\mathscr{N}^\rho(\Phi) \neq \emptyset$ and $\Phi \subseteq \mathscr{N}^\rho(\Phi)$. Consider the sequence

$$\begin{aligned}&\Phi_1^\rho = \mathscr{N}^\rho(\Phi),\\&\dots\\&\Phi_k^\rho = \mathscr{N}^\rho(\Phi_{k-1}^\rho),\\&\dots\end{aligned}$$

According to (4.1) for any k we can establish the following properties:

$$\begin{aligned}&\Phi_{2k}^\rho \subseteq \Phi_{2k+1}^\rho,\\&\Phi_{2k}^\rho \subseteq F_{2k}^\rho,\\&F_{2k+1}^\rho \subseteq \Phi_{2k+1}^\rho.\end{aligned}$$

In fact,

$$\Phi \subseteq \Phi_1^\rho \Rightarrow \Phi_2^\rho \subseteq \Phi_1^\rho \Rightarrow \Phi_2^\rho \subseteq \Phi_3^\rho \Rightarrow \Phi_4^\rho \subseteq \Phi_3^\rho \Rightarrow \cdots$$
$$\Phi \subseteq N^+ \Rightarrow F_1^\rho \subseteq \Phi_1^\rho \Rightarrow \Phi_2^\rho \subseteq F_2^\rho \Rightarrow F_3^\rho \subseteq \Phi_3^\rho \Rightarrow \cdots$$

From the definition of the set $\mathcal{N}^\rho(\Phi)$ it is possible to conclude that if one of the sequences

$$\Phi_2^\rho, \Phi_4^\rho, \ldots, \Phi_{2k}^\rho, \ldots$$

or

$$\Phi_1^\rho, \Phi_3^\rho, \ldots, \Phi_{2k+1}^\rho, \ldots$$

is convergent, then the other converges also. Moreover, if the sequences are convergent and

$$\Phi_{\text{odd}}^\rho = \lim \Phi_{2k+1}^\rho; \quad \Phi_{\text{even}}^\rho = \lim \Phi_{2k}^\rho,$$

then

$$\Phi_{\text{even}}^\rho \subseteq \Phi_{\text{odd}}^\rho,$$
$$F_{\text{odd}}^\rho \subseteq \Phi_{\text{odd}}^\rho \quad \text{and} \quad \Phi_{\text{even}}^\rho \subseteq F_{\text{even}}^\rho.$$

Definition 4.1. The set $\Phi \subseteq \mathcal{N}^\rho(\Phi)$ is called a ρ-**stabilization set** (in relation with the pair (B, g)), if the sequences $\{\Phi_{2k}^\rho\}$ and $\{\Phi_{2k+1}^\rho\}$ are convergent and

$$\Phi_{\text{odd}}^\rho = \Phi_{\text{even}}^\rho.$$

In this case we denote

$$\Phi_s^\rho = \Phi_{\text{odd}}^\rho = \Phi_{\text{even}}^\rho.$$

The ρ-**stabilization set** $\Phi \subseteq \mathcal{N}^+(\Phi)$ is a ρ-**stable set** (in relation with the pair (B, g)), if

$$\Phi_s^\rho = \Phi.$$

As the sequences $\{\Phi_{2k}^\rho\}$ and $\{\Phi_{2k+1}^\rho\}$ are monotone, the set Φ is a ρ-stable set iff

$$\Phi = \mathcal{N}^\rho(\Phi) \tag{4.3}$$

If there is at least one ρ such that $\Phi = \mathcal{N}^\rho(\Phi)$ we shall simply say that the set Φ is a stable set. We denote

$$\rho_{\min}(\Phi) = \min_{j\in\Phi}\{\pi_j[g - BP_\Phi g]\} \quad \text{and} \quad \rho_{\max}(\Phi^c) = \max_{j\in N^+\setminus\Phi}\{\pi_j[g - BP_\Phi g]\} \tag{4.4}$$

Proposition 4.3. *There is a ρ such that Φ is a ρ-stable set iff*

$$\rho_{\max}(\Phi^c) < \rho_{\min}(\Phi). \tag{4.5}$$

Proof. If $\rho_{\max}(\Phi^c) < \rho_{\min}(\Phi)$ and $\rho \in (\rho_{\max}(\Phi^c), \rho_{\min}(\Phi))$, then

$$\mathcal{N}^\rho(\Phi) = \{j \in N^+ : \pi_j(g - BP_\Phi g) \geqslant \rho\} = \Phi$$

and the set Φ is a ρ-stable set.

If Φ is a ρ-stable set, then $\mathcal{N}^\rho(\Phi) = \Phi$. So, $\pi_j(g - BP_\Phi g) < \rho$ for all $j \in N^+\setminus\Phi$ and $\rho_{\max}(\Phi^c) < \rho_{\min}(\Phi)$ must hold. □

In the linear space of all the pairs (B, g) we define the norm

$$\|(B, g) - (C, f)\| = \|(B - C, g - f)\| = \max\{\|B - C\|, \|g - f\|\}.$$

Proposition 4.4. *If the set Φ is a stable set in relation to the pair (B, g), then there is a positive ϑ such that the set Φ is a stable set in relation to any pair (C, f) that satisfies the following inequality*:

$$\|(B, g) - (C, f)\| < \vartheta.$$

Proof. Let Φ be a ρ-stable set in relation with the pair (B, g). From the inequality (4.5) it follows that

$$\rho \in (\rho_{\max}(\Phi^c), \rho_{\min}(\Phi)).$$

If we choose ϑ such that

$$\vartheta < \frac{1}{2 + \|(B, g)\|} \min\left\{\frac{\rho_{\min}(\Phi) - \rho}{2}, \frac{\rho - \rho_{\max}(\Phi^c)}{2}\right\},$$

then, from the equality

$$\pi_j(f - CP_\Phi f) = \pi_j[g + f - g - BP_\Phi(f - g + g) + (B - C)P_\Phi f],$$

we obtain

$$\pi_j(f - CP_\Phi f) \geqslant \rho_{\min}(\Phi) - \vartheta(2 + \|(B, g)\|) \quad \forall j \in \Phi$$

and

$$\pi_j(f - CP_\Phi f) < \rho_{\max}(\Phi^c) + \vartheta(2 + \|(B, g)\|) \quad \forall j \in (N^+ \setminus \Phi)$$

As

$$\rho_{\min}(\Phi) - \vartheta(2 + \|(B, g)\|) > \rho_{\max}(\Phi^c) + \vartheta(2 + \|(B, g)\|),$$

then there is a $\tilde{\rho}$ such that

$$\pi_j(f - CP_\Phi f) \geqslant \tilde{\rho} \quad \forall j \in \Phi$$

and

$$\pi_j(f - CP_\Phi f) < \tilde{\rho} \quad \forall j \in (N^+ \setminus \Phi).$$

Thus, Φ is a $\tilde{\rho}$-stable set in relation with the pair (C, f). □

If N^+ is a ρ-stabilization set, then we denote

$$F^\rho_{\mathrm{odd}} = F^\rho_{\mathrm{even}} = F^\rho_s.$$

Using the former ideas it is possible to establish the following lemma.

Proposition 4.5. *Let ρ be such that nonempty ρ-stabilization sets exist.*

(1) *If $\Phi \subseteq \mathcal{N}^\rho(\Phi)$ is a ρ- stabilization set, then*

$$F^\rho_{\mathrm{odd}} \subseteq \Phi^\rho_s \subseteq F^\rho_{\mathrm{even}}.$$

(2) *If $\Phi \subseteq \mathcal{N}^\rho(\Phi)$ is a ρ-stable set, then*

$$F^\rho_{\mathrm{odd}} \subseteq \Phi \subseteq F^\rho_{\mathrm{even}}. \tag{4.6}$$

(3) *If N^+ is a ρ-stabilization set, $\Phi \subseteq \mathcal{N}^\rho(\Phi)$ and the sequences $\{\Phi^\rho_{2k}\}$ and $\{\Phi^\rho_{2k+1}\}$ converge, then*

$$\Phi^\rho_{\text{even}} \subseteq F^\rho_s \subseteq \Phi^\rho_{\text{odd}}$$

(4) *If N^+ is a ρ-stabilization set and $\Phi \subseteq \mathcal{N}^\rho(\Phi)$ is a ρ-stable set, then*

$$\Phi = F^\rho_s.$$

In other words, if N^+ is a ρ-stabilization set, then F^ρ_s is the unique ρ-stable set.

Conclusion 4.1. If the solution F of the problem (2.2) is a ρ-stable set, then

$$F^\rho_{\text{odd}} \subseteq F \subseteq F^\rho_{\text{even}}$$

and, besides, if N^+ is a ρ-stabilization set, then

$$F = F^\rho_s$$

Thus, in some cases, we can find the solution of the problem (2.2) without solving any system. We shall denote

$$F^+ = \{j \in F : \pi_j(g - BP_F g) > 0\}.$$

Proposition 4.6. *If F is the solution of the problem and $F = F^+$, then there exists $\rho > 0$ such that F is a ρ-stable set.*

Proof. We define $\rho = \min_F\{\pi_j(g - BP_F g)\}$. then $\rho > 0$ and

$$\forall j \notin F \Rightarrow \pi_j(g - BP_F g) \leqslant \pi_j(g - BP_F y) \leqslant 0 < \rho$$

From this relation it follows that $F = \mathcal{N}^\rho(F)$ and hence F is a ρ-stable set. □

Example 4.1. If the matrix $I + B$ is diagonal dominant, $\|B\| = \sigma$ and

$$|\pi_j(g - BP_F g)| > \frac{\sigma^2}{1-\sigma}\|g\|, \tag{4.7}$$

then $F = F^+$.

As a matter of fact, in this case we have

$$y = g - P_F BP_F g + (P_F BP_F)^2 y$$

and if the inequality (4.7) holds, the parcel $(P_F BP_F)^2 y$ does not change the sign of $\pi_j(g - BP_F y)$ which must be positive in order that $j \in F$.

We do not know in advance if the solution F of problem (2.2) is stable or not for some ρ. What we do next is to try to find a ρ such that we can expect that for that ρ the solution F of problem (2.2) is stable. If actually the solution F of problem (2.2) is a ρ-stable set, then it is reasonable to choose F^ρ_{odd} as a starting set. In fact F^ρ_{odd} must be a better approximation to the solution F of problem (2.2) than the set N^+ or the empty set as it is done usually by BPA or single pivotal algorithm, respectively.

Let x_* and F be the solution of the problem. The following lemma establishes conditions under which F can be a ρ-stable set.

Denote

$$\varepsilon = \min_{j \subseteq F}\{\pi_j[BP_F(g - x_*)]\} \quad \text{and} \quad \delta = \max_{j \subseteq N^+ \setminus F}\{\pi_j[BP_F(g - x_*)]\}.$$

We observe that $\varepsilon \geqslant 0$ and $\delta \geqslant 0$.

According to (2.3) in this definitions x_* can be replaced by y_*.

Proposition 4.7. *If $\delta < \varepsilon$, then there exists ρ such that the solution F is a ρ-stable set.*

Proof. The proof follows immediately from the conditions:

$$j \in F \Leftarrow \pi_j(g - BP_F y) > 0$$

and

$$j \notin F \Leftarrow \pi_j(g - BP_F y) < 0.$$

As a matter of fact, if $\delta < \varepsilon$ and $-\varepsilon < \rho < -\delta$, then
$j \subseteq \mathscr{N}^{\rho}(F) \Rightarrow \pi_j(g - BP_F y) = \pi_j(g - BP_F g) + \pi_j[BP_F(g - y)] \geqslant \rho + \varepsilon > 0 \Rightarrow j \in F$,
and $j \subseteq \mathscr{N}^{\rho}_{-}(F) \Rightarrow \pi_j(g - BP_F y) = \pi_j(g - BP_F g) + \pi_j[BP_F(g - y)] < \rho + \delta < 0 \Rightarrow j \notin F$.

Thus, $F = \mathscr{N}^{\rho}(F)$ and F is a ρ-stable set. □

To verify the conditions of this lemma it is necessary to know the solution F and y, but neither the set F nor the vector y are known when we begin to solve the problem.

From the condition

$$j \notin F \Rightarrow \pi_j(g - BP_F g) \leqslant \pi_j(g - BP_F y) \leqslant 0,$$

it follows that $\rho_{\max}(F^c)$ is a non positive number. Then, if $\rho_{\min}(F)$ is a non negative number, the solution F is a 0-stable set.

Denote

$$\rho_{\min}(N^+) = \min_{j \in N^+}\{\pi_j[g - BP_{N^+}(g)]\}.$$

Proposition 4.8. *If the solution F is a ρ-stable set, then there exists ρ_0 such that*

$$\rho_0 \in [\rho_{\min}(N^+), 0]$$

and F is a ρ_0-stable set.

Proof. It is easy to verify that

$$\rho_{\min}(N^+) \leqslant \pi_j(g - BP_F g) - \pi_j(BP_{N^+ \setminus F} g) \leqslant \pi_j(g - BP_F g) \quad \forall j \in F. \qquad \square$$

From the previous analysis we can suspect that in the case when $F^{\rho}_{\text{odd}} = F^{\rho}_{\text{even}}$ the solution F can be stable and $F^{\rho}_{s} \subseteq F$. In the following algorithm we begin by searching the smaller value for ρ not equal to the trivial situation $\rho = \rho_{\min}(N^+)$ such that $F^{\rho}_{\text{odd}} = F^{\rho}_{\text{even}}$. For that purpose we determine the value $\zeta = \min_{j \in N^+}\{|\pi_j[(BP_{F^0_{\text{odd}}})^2(g - BP_{F^0_{\text{odd}}} g)]|\}$ as an approximation of the smaller value that causes an alteration in the behavior of the sequences $\{F^{\rho}_{2k+1}\}$ and $\{F^{\rho}_{2k}\}$.

Algorithm 2

Step1. Evaluate

$$\rho_{\min}(N^+) = \min_{j\in N^+}\{\pi_j[g - BP_{N^+}(g)]\} \text{ and}$$

$$\zeta = \min_{j\in N^+}\{|\pi_j[(BP_{F^0_{\text{odd}}})^2(g - BP_{F^0_{\text{odd}}} g)]|\}.$$

Make $k = 1$ and $\rho = \min\{\rho_{\min}(N^+) + \zeta, 0\}$.

Step 2. Find F^ρ_{odd} and F^ρ_{even}. If $F^\rho_{\text{odd}} = F^\rho_{\text{even}}$ go to step 3, otherwise do $k = k + 1$ and $\rho = \min\{\rho_{\min}(N^+) + k\zeta, 0\}$.
If $\rho < 0$ repeat step 2, otherwise find F^ρ_{odd} and go to step 3.

Step 3. Start BPA with $F = F^\rho_{\text{odd}}$.

In Table 2, we present computational results corresponding to the same problems we considered in the case of the Algorithm 1. In addition we give the values of $\rho_{\min}(F)$ and $\rho_{\max}(F^C)$ that helps

Table 2
Behavior of Algorithm 2

Problem	$\rho_{\min}(F)$	$\rho_{\max}(F^C)$	Stable	$\rho_{\min}(N^+)$	ζ	ρ	$\#F^\rho_{\text{odd}}$	# Systems	Dimensions
P1	1.10	−0.0529	Yes	−0.479	0.0482	−0.0455	89	1	89
P2	−1.63	−2.55	Yes	−3.02	0.988	−2.03	605	1	605
P3	0.969	−0.0337	Yes	−0.369	0.0267	−0.0216	70	1	70
P4	0.853	−0.140	Yes	−0.742	0.123	−0.126	142	1	142
P5	0.657	−0.312	Yes	−1.11	0.272	−0.294	211	1	211
P6	0.432	−0.522	Yes	−1.41	0.431	−0.122	275	1	275
P7	−1.06	−1.97	Yes	−1.71	0.591	−0.532	339	1	339
P8	−0.272	−1.13	Yes	−2.03	0.723	−1.31	408	1	408
P9	−0.638	−1.54	Yes	−2.37	0.823	−1.55	475	2	475
									474
P10	−1.06	−1.97	Yes	−2.68	0.911	−1.77	534	1	534
P11	−0.240	−0.223	No	−0.491	0.153	−0.184	334	3	334
									347
									339
P12	−0.0705	−0.0622	No	−0.273	0.0563	−0.0474	176	3	176
									179
									178
P13	−0.00859	−0.00910	Yes	−0.108	0.00748	−0.00286	68	3	68
									71
									70
P14	−0.523	−0.492	No	−0.723	0.252	−0.220	439	5	439
									692
									516
									503
									502
P15	3.18	−1.68	Yes	−3.66	1.27	−1.13	339	1	339
P16	4.47	−0.462	Yes	−1.96	0.413	−0.306	178	1	178
P17	4.93	−0.0777	Yes	−0.769	0.0564	−0.0359	70	1	70
P18	1.06	−3.70	Yes	−5.40	1.76	−3.63	502	1	502
P19	−0.682	−5.49	Yes	−6.48	1.93	−4.55	605	1	605

In last column are the dimensions of the systems solved by algorithm 2.

to understand the behavior of the algorithm. According to Lemma 3 $\rho_{\max}(F^C) < \rho_{\min}(F)$ assures that there is ρ such that F is stable.

As it should be expected, the value $\rho_{\min}(N^+)$ is always less than $\rho_{\max}(F^C)$. In algorithm 2, for the sake of simplicity, we chose to stop the searching for ρ on the first where $F^{\rho}_{\text{odd}} = F^{\rho}_{\text{even}}$, this sometimes prevents the algorithm to reach the optimal value of ρ. That was precisely what happened for two F stable problems (P9 and P13), as it is shown in table 2 the value of ρ found by the Algorithm 2, in these two problems, is not inside the interval defined by $\rho_{\max}(F^C)$ and $\rho_{\min}(F)$. It is interesting to verify that the worst behavior is achieved in this case and in the case of a non-stable set F. It is interesting to note that in this cases the solutions are "almost" degenerate.

It is a reality that, whenever we could find a $\rho \in (\rho_{\max}(F^C), \rho_{\min}(F))$ the algorithm BPA had only to solve a system to verify that F^{ρ}_{even} was actually the solution set and to determine the values of the solution vectors.

References

[1] K.G. Murty, Linear Complementarity, Linear and Nonlinear Programming, Heldermann Verlag, Berlin, 1988.
[2] G. Cornuejols, R. Tutuncu, Optimization Methods in Finance, Cambridge University Press, 2007.
[3] K.G. Murty, Note on a Bard-type scheme for solving the complementarity problem, Opsearch 11 (1974) 23–130.
[4] M. Kostreva, Block pivot methods for solving the complementarity problem, Linear Algebra Appl. 21 (1978) 07–215.
[5] J.J. Júdice, F.M. Pires, Bard-type methods for the linear complementarity problem with symmetric positive definite matrices, IMA J. Math. Appl. Business Ind. 2 (1988/9) 51–68.
[6] S. Takriti, K.G. Murty, On the convergence of the block principal pivotal algorithm for the LCP, Eur. J. Oper. Res. 102 (1997) 657–666.
[7] R. Chandrasekaran, A special case of the complementarity pivot problem, Opsearch 7 (1970) 263–268.