

A DIAGRAMMATIC CATEGORIFICATION OF THE AFFINE q -SCHUR ALGEBRA $\widehat{\mathbf{S}}(n, n)$ FOR $n \geq 3$

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ABSTRACT. This paper is a follow-up to [MT13]. In that paper we categorified the affine q -schur algebra $\widehat{\mathbf{S}}(n, r)$ for $2 < r < n$, using a quotient of Khovanov and Lauda's categorification of $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ [KL09, KL11, KL10]. In this paper we categorify $\widehat{\mathbf{S}}(n, n)$ for $n \geq 3$, using an extension of the aforementioned quotient.

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1. INTRODUCTION

The affine q -Schur algebras were defined by Green [Gre99] for any $n, r \geq 3$, and are quotients of $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ if $r < n$. In [MT13] we defined a quotient of Khovanov and Lauda's categorification $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, denoted $\widehat{\mathbf{S}}(n, r)$, and showed that the Grothendieck group of its Karoubi envelope (idempotent completion) was exactly isomorphic to $\widehat{\mathbf{S}}(n, r)$ for $2 < r < n$. In order to establish the isomorphism, we used Doty and Green's [DG07] idempotent presentation of $\widehat{\mathbf{S}}(n, r)$ for $2 < r < n$.

The case addressed in this paper is slightly more complicated, because $\widehat{\mathbf{S}}(n, n)$ is not a quotient of $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ or $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ but of the strictly larger algebra $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, called the *extended* affine general linear quantum algebra and also due to Green [Gre99]. Therefore, we have to extend the Khovanov-Lauda calculus of the corresponding quotient of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ by adding certain generating 1 and 2-morphisms and relations. We denote that extended 2-category by $\widehat{\mathcal{S}}(n, n)$ and show that the Grothendieck group of its Karoubi envelope is isomorphic to $\widehat{\mathbf{S}}(n, n)$ for $n \geq 3$. For that isomorphism we use Deng, Du and Fu's [DDF12] presentation of $\widehat{\mathbf{S}}(n, n)$, which extends Doty and Green's.

A little warning should be made. The results in this paper are not sufficient to categorify $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ diagrammatically, because that would require a categorification of $\widehat{\mathbf{S}}(n, r)$ for $2 < n < r$ too. However, no Drinfeld-Jimbo type presentation of $\widehat{\mathbf{S}}(n, r)$ is known in that case, so even on the decategorified level there

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is an open question that would need to be solved first. For more information on this problem, see Question 4.3.2 in [Gre99] and Chapter 5 in [DDF12].

There is another technical detail that we should explain beforehand. In [MT13] we introduced a new degree-two variable y and a y -deformation of the relations in Khovanov and Lauda's $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, denoted $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$. The corresponding Schur quotients were denoted $\widehat{\mathcal{S}}(n, r)_{[y]}$. This y -deformation was introduced in order to establish a precise relation between $\widehat{\mathcal{S}}(n, r)_{[y]}$ and an extension of the affine singular Soergel bimodules built from Soergel's reflection faithful representation of the affine Weyl group, which were defined and studied by Williamson [Wil11]. However, we also proved that the ideals generated by y are virtually nilpotent, so that the Grothendieck groups of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n, r)_{[y]}$ are isomorphic to those of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ and $\widehat{\mathcal{S}}(n, r)$. Furthermore, for $y = 0$ the 2-representations in [MT13] give 2-functors from $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ to certain extensions of the affine Soergel bimodules built from the geometric representation of the affine Weyl group, which is not reflection faithful but still has some nice properties (for more information on this topic, see Section 3.1 in [EW13] and the results in [Lib08]). In order to keep the calculations simple in this paper, we put $y = 0$ here. It would not be hard to give the y -deformed relations in the definition of $\widehat{\mathcal{S}}(n, n)$, which would give a 2-category $\widehat{\mathcal{S}}(n, n)_{[y]}$, but some of the subsequent calculations would be much harder in the y -deformed setting, e.g. the ones in the proof of Proposition 3.5.

In general, it would be interesting to know more about the relation between $\widehat{\mathcal{S}}(n, r)$, for $n \geq r$, and its y -deformation and the 2-category of affine singular Soergel bimodules.

Knowing more about this relation might also help to establish a connection with the work by Lusztig [Lus99] and Ginzburg and Vasserot [GV93] on perverse sheaves and affine quantum \mathfrak{gl}_n .

Finally, we thank Ben Webster for sharing with us his ideas on a potentially interesting connection between our results in this paper and his work in [Web12], the precise formulation of which remains to be worked out.

2. AFFINE QUANTUM ALGEBRAS

In this section, we first recall the definition of the extended affine quantum general linear algebra $\widehat{\mathcal{U}}_q(\widehat{\mathfrak{gl}}_n)$ and its subalgebras $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$. After that, we recall the definition of the affine quantum Schur algebras $\widehat{\mathcal{S}}(n, r)$, due to Green [Gre99]. Furthermore, we recall an idempotent presentation of the affine quantum Schur algebras, due to Doty and Green [DG07] for $n > r$ and to Deng, Du and Fu [DDF12] for $n = r$.

2.1. The (extended) affine quantum general and special linear algebras. For the rest of this paper, let $n \geq 3$.

Since in this paper we are only interested in the affine quantum general and special linear algebras at level 0, i.e. the q -analogue of the loop algebras without central extension, we can work with the normal \mathfrak{gl}_n -weight lattice, which is isomorphic to \mathbb{Z}^n . Let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$, with 1 being on the i th coordinate, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^n$, for $i = 1, \dots, n$, where the subscripts have to be understood modulo n , e.g. $\alpha_n = \varepsilon_n - \varepsilon_1 = (-1, 0, \dots, 0, 1)$. We also define the Euclidean inner product on \mathbb{Z}^n by $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$.

Definition 2.1. [Gre99] The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$ -algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \dots, n$, subject to the relations

$$K_i K_j = K_j K_i \quad K_i K_i^{-1} = K_i^{-1} K_i = 1 \quad (2.1)$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}} \quad (2.2)$$

$$K_i E_{\pm j} = q^{\pm \langle \varepsilon_i, \alpha_j \rangle} E_{\pm j} K_i \quad (2.3)$$

$$E_{\pm i}^2 E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2 = 0 \quad \text{if } |i - j| = 1 \pmod n \quad (2.4)$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0 \quad \text{else} \quad (2.5)$$

$$R R^{-1} = R^{-1} R = 1 \quad (2.6)$$

$$R X_i R^{-1} = X_{i+1} \quad \text{for } X_i \in \{E_{\pm i}, K_i^{-1}\}. \quad (2.7)$$

In all equations, the subscripts have to be read modulo n .

Definition 2.2. The *affine quantum general linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$ -subalgebra generated by $E_{\pm i}$ and $K_i^{\pm 1}$, for $i = 1, \dots, n$.

The *affine quantum special linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subseteq \mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$ -subalgebra generated by $E_{\pm i}$ and $K_i K_{i+1}^{-1}$, for $i = 1, \dots, n$.

Remark 2.3. A little warning about the notation is needed here. Our notation follows that of [DG07, Gre99], which differs from that of [DDF12]. What we call $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$, Deng, Du and Fu call $U_{\Delta}(n)$. In Remark 5.3.2 [DDF12] they define $\widehat{\mathbf{U}}$, which is equal to our $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Finally, their $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is the *quantum loop algebra* (see their Definition 2.3.1), which contains $U_{\Delta}(n)$, i.e. our $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$, as a proper subalgebra. In their notation, $\widehat{\mathbf{U}}$ is not a subalgebra of $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, because $R \in \widehat{\mathbf{U}}$ would have to be equal to an infinite linear combination of generators of the latter.

We will also need the bialgebra structure on $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$.

Definition 2.4. [Gre99] $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is a bialgebra with counit $\varepsilon: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \mathbb{Q}(q)$ defined by

$$\varepsilon(E_{\pm i}) = 0, \quad \varepsilon(R^{\pm 1}) = \varepsilon(K_i^{\pm 1}) = 1$$

and coproduct $\Delta: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \otimes \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, defined by

$$\Delta(1) = 1 \otimes 1 \quad (2.8)$$

$$\Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i \quad (2.9)$$

$$\Delta(E_{-i}) = K_i^{-1} K_{i+1} \otimes E_{-i} + E_{-i} \otimes 1 \quad (2.10)$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \quad (2.11)$$

$$\Delta(R^{\pm 1}) = R^{\pm 1} \otimes R^{\pm 1}. \quad (2.12)$$

As a matter of fact, $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is even a Hopf algebra, but we do not need the antipode in this paper. Note that Δ and ε can be restricted to $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$, which are bialgebras too.

At level 0, we can also work with the $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ -weight lattice, which is isomorphic to \mathbb{Z}^{n-1} . Suppose that V is a $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ -weight representation with weights $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_{\lambda} V_{\lambda}$$

and K_i acts as multiplication by q^{λ_i} on V_λ . Then V is also a $U_q(\widehat{\mathfrak{sl}}_n)$ -weight representation with weights $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\bar{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j = 1, \dots, n-1$. Conversely, given a $U_q(\widehat{\mathfrak{sl}}_n)$ -weight representation with weights $\mu = (\mu_1, \dots, \mu_{n-1})$, there is not a unique choice of $U_q(\widehat{\mathfrak{gl}}_n)$ -action on V . We can fix this by choosing the action of $K_1 \cdots K_n$. In terms of weights, this corresponds to the observation that, for any $r \in \mathbb{Z}$ the equations

$$\lambda_i - \lambda_{i+1} = \mu_i \quad (2.13)$$

$$\sum_{i=1}^n \lambda_i = r \quad (2.14)$$

determine $\lambda = (\lambda_1, \dots, \lambda_n)$ uniquely, if there exists a solution to (2.13) and (2.14) at all. To fix notation, we define the map $\varphi_{n,r}: \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^n \cup \{*\}$ by

$$\varphi_{n,r}(\mu) = \lambda \quad (2.15)$$

if (2.13) and (2.14) have a solution, and put $\varphi_{n,r}(\mu) = *$ otherwise. This map already appeared in [MT13] and [MSV13].

As far as weight representations are concerned, we can restrict our attention to the Beilinson-Lusztig-MacPherson [BLM90] idempotent version of these quantum groups, denoted $\widehat{U}(\widehat{\mathfrak{gl}}_n)$, $\dot{U}(\widehat{\mathfrak{gl}}_n)$ and $\dot{U}(\widehat{\mathfrak{sl}}_n)$ respectively. To understand their definition, recall that K_i acts as q^{λ_i} on the λ -weight space of any weight representation. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_λ to $\widehat{U}_q(\widehat{\mathfrak{gl}}_n)$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\mu} 1_\lambda \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \alpha_i} E_{\pm i} \\ K_i 1_\lambda &= q^{\lambda_i} 1_\lambda \\ R 1_{(\lambda_1, \dots, \lambda_n)} &= 1_{(\lambda_n, \lambda_1, \dots, \lambda_{n-1})} R. \end{aligned}$$

Definition 2.5. The *idempotent extended affine quantum general linear algebra* is defined by

$$\widehat{U}(\widehat{\mathfrak{gl}}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} 1_\lambda \widehat{U}_q(\widehat{\mathfrak{gl}}_n) 1_\mu.$$

Of course one defines $\dot{U}(\widehat{\mathfrak{gl}}_n) \subset \widehat{U}(\widehat{\mathfrak{gl}}_n)$ as the idempotent subalgebra generated by 1_λ and $E_{\pm i} 1_\lambda$, for $i = 1, \dots, n$ and $\lambda \in \mathbb{Z}^n$. Similarly for $\widehat{U}_q(\widehat{\mathfrak{sl}}_n)$, adjoin an idempotent 1_λ for each $\lambda \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\mu} 1_\lambda \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \alpha_i} E_{\pm i} \\ K_i K_{i+1}^{-1} 1_\lambda &= q^{\lambda_i} 1_\lambda. \end{aligned}$$

Definition 2.6. The *idempotent quantum special linear algebra* is defined by

$$\dot{U}(\widehat{\mathfrak{sl}}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^{n-1}} 1_\lambda U_q(\widehat{\mathfrak{sl}}_n) 1_\mu.$$

Just to fix notation for future use.

Notation 2.7. For $\underline{i} = (\mu_1 i_1, \dots, \mu_m i_m)$, with $\mu_j = \pm$, define

$$E_{\underline{i}} := E_{\mu_1 i_1} \cdots E_{\mu_m i_m}$$

and define $\underline{i}_\Lambda \in \mathbb{Z}^n$ to be the n -tuple such that

$$E_{\underline{i}} 1_\lambda = 1_{\lambda + \underline{i}_\Lambda} E_{\underline{i}}.$$

Following Khovanov and Lauda [KL09, KL11, KL10], we call \underline{i} a *signed sequence* and denote the set of signed sequences by SSeq .

2.2. The affine q -Schur algebra. As we did in [MT13], we first copy some facts about the action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on tensor space from [DG07, Gre99]. After that we define the quotient $\widehat{\mathbf{S}}(n, r)$, for $n \geq r$, and give a presentation of that algebra. Note that the case $n = r$ was not considered in [MT13].

2.2.1. Tensor space. Let V be the $\mathbb{Q}(q)$ -vector space freely generated by $\{e_t \mid t \in \mathbb{Z}\}$.

Definition 2.8. [Gre99] The following defines an action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on V

$$E_i e_{t+1} = e_t \quad \text{if } i \equiv t \pmod{n} \quad (2.16)$$

$$E_i e_{t+1} = 0 \quad \text{if } i \not\equiv t \pmod{n} \quad (2.17)$$

$$E_{-i} e_t = e_{t+1} \quad \text{if } i \equiv t \pmod{n} \quad (2.18)$$

$$E_{-i} e_t = 0 \quad \text{if } i \not\equiv t \pmod{n} \quad (2.19)$$

$$K_i^{\pm 1} e_t = q^{\pm 1} e_t \quad \text{if } i \equiv t \pmod{n} \quad (2.20)$$

$$K_i^{\pm 1} e_t = e_t \quad \text{if } i \not\equiv t \pmod{n} \quad (2.21)$$

$$R^{\pm 1} e_t = e_{t \pm 1} \quad \text{for all } t \in \mathbb{Z}. \quad (2.22)$$

Note that V is clearly a weight-representation of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, with e_t having weight equal to ε_i , for $i \equiv t \pmod{n}$. Therefore V is also a representation of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$. Let $r \in \mathbb{N}_{>0}$ be arbitrary but fixed. As usual, one extends the above action to $V^{\otimes r}$, using the coproduct in $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Again, this is a weight-representation, and therefore also a representation of $\widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$. There is also a right action of the extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ on $V^{\otimes r}$, whose precise definition is not relevant here, which commutes with the left action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$.

Definition 2.9. [Gre99] The *affine q -Schur algebra* $\widehat{\mathbf{S}}(n, r)$ is by definition the centralizing algebra

$$\text{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}} (V^{\otimes r}).$$

It turns out that the image of the representation $\psi_{n,r}: \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \rightarrow \text{End}(V^{\otimes r})$ is isomorphic to $\widehat{\mathbf{S}}(n, r)$. If $n > r$, then we can even restrict to $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subset \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, i.e.

$$\psi_{n,r}(\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)) \cong \widehat{\mathbf{S}}(n, r).$$

If $n = r$, this is no longer true, as we will show below.

2.2.2. Presentation of $\widehat{\mathbf{S}}(n, r)$ for $n > r$. In this subsection, let $n > r$. As already mentioned, the map

$$\psi_{n,r}: \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \rightarrow \text{End}(V^{\otimes r}) \rightarrow \widehat{\mathbf{S}}(n, r)$$

is surjective. This observation gives rise to the following presentation of $\widehat{\mathbf{S}}(n, r)$. The proof can be found in [DG07] (Theorem 2.6.1).

Theorem 2.10. [DG07] For $n > r$, the $\mathbb{Q}(q)$ -algebra $\widehat{\mathbf{S}}(n, r)$ is isomorphic to the associative unital $\mathbb{Q}(q)$ -algebra generated by 1_λ and $E_{\pm i}$, for $\lambda \in \Lambda(n, r)$ and $i = 1, \dots, n$, subject to the relations

$$1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda \quad (2.23)$$

$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i} \quad (2.24)$$

$$(E_i E_{-j} - E_{-j} E_i) 1_\lambda = \delta_{i,j} [\lambda_i - \lambda_{i+1}] 1_\lambda \quad (2.25)$$

$$(E_{\pm i}^2 E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2) 1_\lambda = 0 \quad \text{if } |i - j| = 1 \pmod{n}, \quad (2.26)$$

$$(E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i}) 1_\lambda = 0 \quad \text{else.} \quad (2.27)$$

In all equations the subscripts i, j have to be read modulo n , and the equations hold for any $\lambda \in \Lambda(n, r)$. If $\lambda \pm \alpha_i \notin \Lambda(n, r)$, the corresponding idempotent is zero by convention.

We can restrict $\psi_{n,r}$ even further and obtain a surjection $\psi_{n,r}: \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \rightarrow \widehat{\mathbf{S}}(n, r)$, which can be given explicitly on the generators. For any $\lambda \in \mathbb{Z}^{n-1}$, we have

$$\psi_{n,r}(E_{\pm i}1_\lambda) = E_{\pm i}1_{\varphi_{n,r}(\lambda)},$$

where $\varphi_{n,r}: \mathbb{Z}^{n-1} \rightarrow \Lambda(n, r) \cup \{*\}$ is the map defined in (2.15). By convention, we put $1_* = 0$.

2.2.3. Presentation of $\widehat{\mathbf{S}}(n, n)$. A Drinfeld-Jimbo type presentation of $\widehat{\mathbf{S}}(n, n)$ is harder to get, because

$$\psi_{n,n}(\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)) = \psi_{n,n}(\mathbf{U}_q(\widehat{\mathfrak{gl}}_n))$$

is a proper subalgebra of $\widehat{\mathbf{S}}(n, n)$. Therefore Green [Gre99] introduced $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, which contains the new invertible element R , and proved that $\widehat{\mathbf{S}}(n, n)$ is a quotient of this extended algebra. As vector spaces, we get the following $\mathbb{Q}(q)$ -linear isomorphism:

$$\widehat{\mathbf{S}}(n, n) \cong \psi_{n,n}(\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)) \oplus \bigoplus_{t \neq 0} \mathbb{Q}[R^t, R^{-t}].$$

However, this is not an algebra isomorphism. In Theorem 5.3.5 in [DDF12] Deng, Du and Fu show which relations need to be added in order to get a presentation of the algebra $\widehat{\mathbf{S}}(n, n)$. Let us first recall a slightly different presentation obtained by adding two new elements, $E_{\pm\delta}$, instead of $R^{\pm 1}$. This presentation, also due to Deng, Du and Fu [DDF12], turns out to be easier to categorify. As in [MT13], we write $1_n := 1_{(1^n)}$. Recall that the *divided powers* are defined by

$$E_{\pm i}^{(a)} := \frac{E_{\pm i}^a}{[a]}, \quad \text{for } i = 1, \dots, n.$$

Theorem 2.11. [DDF12] The $\mathbb{Q}(q)$ -algebra $\widehat{\mathbf{S}}(n, n)$ is generated by $E_{\pm\delta}$, $E_{\pm i}$ and 1_λ , for $i = 1, \dots, n$ and $\lambda \in \Lambda(n, n)$, subject to the relations 2.23 through 2.27 together with

- i) $E_{\pm\delta}1_\lambda = 1_\lambda E_{\pm\delta} = 0$ for all $\lambda \neq (1^n)$;
- ii) $E_{\pm\delta}1_n = 1_n E_{\pm\delta}$;
- iii) $E_{+\delta}E_{-\delta}1_n = E_{-\delta}E_{+\delta}1_n = 1_n$;
- iv) $E_i E_{+\delta}1_n = E_i^{(2)} E_{i-1} \dots E_1 E_n \dots E_{i+1}1_n$;
- v) $1_n E_{+\delta} E_i = 1_n E_{i-1} \dots E_1 E_n \dots E_{i+1} E_i^{(2)}$;
- vi) $E_{-i} E_{+\delta}1_n = E_{i-1} \dots E_1 E_n \dots E_{i+1}1_n$;
- vii) $1_n E_{+\delta} E_{-i} = 1_n E_{i-1} \dots E_1 E_n \dots E_{i+1}$;
- viii) $E_{-i} E_{-\delta}1_n = E_{-i}^{(2)} E_{-(i+1)} \dots E_{-n} E_{-1} \dots E_{-(i-1)}1_n$;
- ix) $1_n E_{-\delta} E_{-i} = 1_n E_{-(i+1)} \dots E_{-n} E_{-1} \dots E_{-(i-1)} E_{-i}^{(2)}$;
- x) $E_i E_{-\delta}1_n = E_{-(i+1)} \dots E_{-n} E_{-1} \dots E_{-(i-1)}1_n$;
- xi) $1_n E_{-\delta} E_i = 1_n E_{-(i+1)} \dots E_{-n} E_{-1} \dots E_{-(i-1)}$,

for any $i = 1, \dots, n$.

To see that Theorem 2.11 really gives a presentation of $\widehat{\mathbf{S}}(n, n)$, recall that Deng, Du and Fu give the following definition in (5.3.1.1) and (5.3.1.2) in [DDF12] (They use the notation ρ where we use R):

Definition 2.12. Define

$$R^{-1} := E_{+\delta}1_n + \sum_{i=1}^n \sum_{\substack{(a_1, \dots, a_n) \in \Lambda(n, n) \\ a_i=0}} E_{i-1}^{(a_{i-1})} \dots E_1^{(a_1)} E_n^{(a_n)} \dots E_{i+1}^{(a_{i+1})} 1_{(a_n, a_1, \dots, a_{n-1})}$$

and

$$R := E_{-\delta}1_n + \sum_{i=1}^n \sum_{\substack{(a_1, \dots, a_n) \in \Lambda(n, n) \\ a_i=0}} E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})} 1_{(a_1, \dots, a_n)}.$$

Then note that

$$E_{i-1}^{(a_{i-1})} \cdots E_1^{(a_1)} E_n^{(a_n)} \cdots E_{i+1}^{(a_{i+1})} 1_{(a_n, a_1, \dots, a_{n-1})} = 1_{(a_1, \dots, a_n)} E_{i-1}^{(a_{i-1})} \cdots E_1^{(a_1)} E_n^{(a_n)} \cdots E_{i+1}^{(a_{i+1})}$$

and

$$E_{i-1}^{(a_{i-1})} \cdots E_1^{(a_1)} E_n^{(a_n)} \cdots E_{i+1}^{(a_{i+1})} 1_\lambda = 0$$

for all $\lambda \neq (a_n, a_1, \dots, a_{n-1})$. Likewise, we have

$$E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})} 1_{(a_1, \dots, a_n)} = 1_{(a_n, a_1, \dots, a_{n-1})} E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})}$$

and

$$E_{-(i-1)}^{(a_{i-1})} \cdots E_{-1}^{(a_1)} E_{-n}^{(a_n)} \cdots E_{-(i+1)}^{(a_{i+1})} 1_\lambda = 0$$

for all $\lambda \neq (a_1, \dots, a_n)$. These remarks show that Proposition 5.3.3 and Corollary 5.3.4 in [DDF12] imply that the presentation of $\widehat{\mathbf{S}}(n, n)$ in Theorem 5.3.5 in that paper, is equivalent to the one we have given in Theorem 2.11. In particular, the relations in Theorem 2.11 imply the following relations, which are exactly the ones in Theorem 5.3.5 [DDF12]:

Corollary 2.13. In $\widehat{\mathbf{S}}(n, n)$, we have

$$RR^{-1} = R^{-1}R = 1, \quad RE_{\pm i}R^{-1} = E_{\pm(i+1)}, \quad R1_\lambda R^{-1} = 1_{(\lambda_n, \lambda_1, \dots, \lambda_{n-1})}.$$

As usual, we read the indices modulo n .

Therefore, the surjective algebra homomorphism

$$\psi_{n,n}: \widehat{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathbf{S}}(n, n)$$

can be defined as

$$\psi_{n,n}(1_\lambda) = \begin{cases} 1_\lambda & \text{if } \lambda \in \Lambda(n, n) \\ 0 & \text{else} \end{cases}$$

and

$$\psi_{n,n}(E_{\pm i}1_\lambda) = E_{\pm i}\psi_{n,n}(1_\lambda), \quad \psi_{n,n}(R^{\pm 1}1_\lambda) = R^{\pm 1}\psi_{n,n}(1_\lambda).$$

In Lemma 3.2 and Corollary 5.6 in [DD13] Deng and Du also show that there exists an embedding

$$\iota_n: \widehat{\mathbf{S}}(n, n) \rightarrow \widehat{\mathbf{S}}(n+1, n),$$

which gives an isomorphism of algebras

$$\widehat{\mathbf{S}}(n, n) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, n)} 1_{(\lambda, 0)} \widehat{\mathbf{S}}(n+1, n) 1_{(\mu, 0)}.$$

At that point of their paper they use a different presentation of the affine q -Schur algebras, but by Proposition 7.1 [DD13] it is not hard to work out the image under ι_n of the generators of $\widehat{\mathbf{S}}(n, n)$ in Theorem 2.11. Note that we have multiplied their images of E_{+n} and E_{-n} by -1 , which is more convenient for categorification and does not invalidate their results.

Proposition 2.14. [DD13] The $\mathbb{Q}(q)$ -linear algebra homomorphism

$$\iota_n : \widehat{\mathbf{S}}(n, n) \rightarrow \widehat{\mathbf{S}}(n + 1, n)$$

defined by

$$\begin{aligned} 1_\lambda &\mapsto 1_{(\lambda,0)} \\ E_{\pm i} 1_\lambda &\mapsto E_{\pm i} 1_{(\lambda,0)} \\ E_n 1_\lambda &\mapsto E_n E_{n+1} 1_{(\lambda,0)} \\ E_{-n} 1_\lambda &\mapsto E_{-(n+1)} E_{-n} 1_{(\lambda,0)} \\ E_{+\delta} 1_n &\mapsto E_n E_{n-1} \cdots E_1 E_{n+1} 1_{(1^n,0)} \\ E_{-\delta} 1_n &\mapsto E_{-(n+1)} E_{-1} \cdots E_{-n} 1_{(1^n,0)} \end{aligned}$$

for any $1 \leq i \leq n - 1$ and $\lambda \in \Lambda(n, n)$, is an embedding and gives an isomorphism of algebras

$$\widehat{\mathbf{S}}(n, n) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, n)} 1_{(\lambda,0)} \widehat{\mathbf{S}}(n + 1, n) 1_{(\mu,0)}.$$

3. A DIAGRAMMATIC CATEGORIFICATION OF $\widehat{\mathbf{S}}(n, n)$

Definition 3.1. The 2-category $\widehat{\mathcal{S}}(n, n)$ is defined as the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all diagrams with regions whose labels are not contained in $\Lambda(n, n)$, just as in [MT13] (taking $y = 0$ in that paper), together with the generating 1-morphisms

$$1_n \mathcal{E}_{+\delta} 1_n \{t\} \quad \text{and} \quad 1_n \mathcal{E}_{-\delta} 1_n \{t\},$$

for $t \in \mathbb{Z}$, and the following generating 2-morphisms

$$\begin{array}{ccc} 1_{\mathcal{E}_{+\delta} 1_n \{t\}} & & 1_{\mathcal{E}_{-\delta} 1_n \{t\}} \\ \begin{array}{c} \delta \\ \uparrow \\ (1^n) \quad \uparrow \quad (1^n) \\ \downarrow \\ \delta \end{array} & & \begin{array}{c} \delta \\ \downarrow \\ (1^n) \quad \downarrow \quad (1^n) \\ \uparrow \\ \delta \end{array} \\ \text{deg } 0 & & \text{deg } 0 \end{array}$$

Notation:				
2-morphism:				
Degree:	0	0	0	0

Notation:				
2-morphism:				
Degree:	1	1	1	1

Relations:

$\mathcal{E}_{+\delta}1_n$ and $\mathcal{E}_{-\delta}1_n$ are biadjoint inverses of each other:

$$\begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ (1^n) \end{array} = \begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \begin{array}{c} (1^n) \\ \downarrow \\ \delta \end{array} \quad \begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ (1^n) \end{array} = \begin{array}{c} (1^n) \\ \downarrow \\ \delta \end{array} \begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \quad (3.1)$$

$$\begin{array}{c} (1^n) \\ \downarrow \\ \delta \end{array} \begin{array}{c} \uparrow \\ (1^n) \end{array} = \begin{array}{c} (1^n) \\ \downarrow \\ \delta \end{array} \begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \quad \begin{array}{c} (1^n) \\ \downarrow \\ \delta \end{array} \begin{array}{c} \uparrow \\ (1^n) \end{array} = \begin{array}{c} (1^n) \\ \uparrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ (1^n) \end{array} \quad (3.2)$$

$$\begin{array}{c} (1^n) \\ \circlearrowleft \\ \delta \end{array} = \begin{array}{c} (1^n) \\ \circlearrowright \\ \delta \end{array} = 1 \quad (3.3)$$

$$\begin{array}{c} \downarrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ (1^n) \end{array} = \begin{array}{c} \downarrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ \delta \end{array} \quad \begin{array}{c} \downarrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ (1^n) \end{array} = \begin{array}{c} \downarrow \\ \delta \end{array} \begin{array}{c} \downarrow \\ \delta \end{array} \quad (3.4)$$

We impose full cyclicity w.r.t. $\Uparrow_{\delta,i}$, $\Downarrow_{\delta,i}$, $\Uparrow^{\delta,i-1}$ and $\Downarrow^{\delta,i+1}$, e.g. by using the adequate cups and caps we can rotate $\Uparrow_{\delta,i}$ to obtain $\Downarrow^{\delta,i+1}$.

Furthermore, we impose the relations:

$$\begin{array}{c} i+1 \ i \ 1 \ n \ i+2 \\ \dots \\ \uparrow \\ \delta \end{array} (1^n) = \begin{array}{c} i+1 \ i \ 1 \ n \ i+3 \ i+2 \\ \dots \\ \uparrow \\ \delta \end{array} (1^n) - \begin{array}{c} i+1 \ i \ 1 \ n \ i+3 \ i+2 \\ \dots \\ \uparrow \\ \delta \end{array} (1^n) \quad (3.5)$$

$$(1^n) = (1^n) - (1^n) \quad (3.6)$$

$$(1^n) = (1^n) - (1^n) \quad (3.7)$$

$$(1^n) = (1^n) - (1^n) \quad (3.8)$$

$$(1^n) = (1^n) \quad (1^n) \quad (3.9)$$

$$(1^n) = (1^n) - (1^n) \quad (3.10)$$

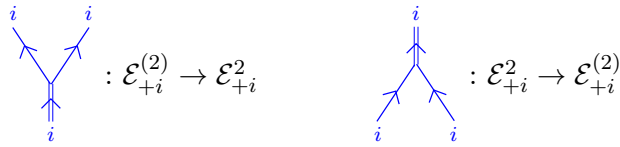
$$(1^n) = (1^n) \quad (3.11)$$

Note that cyclicity implies the analogous relations with all orientations reversed.

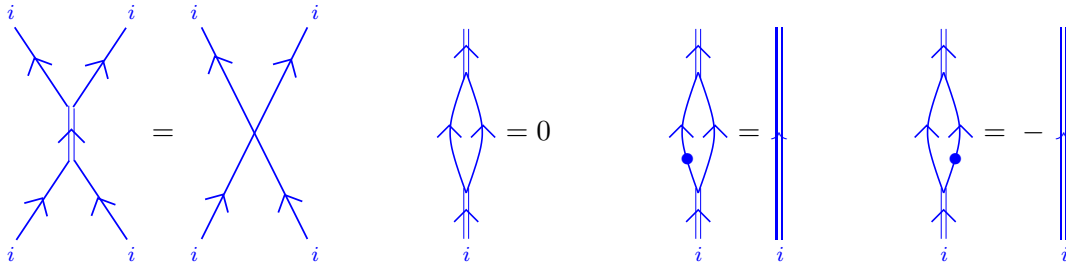
Before giving the following lemma, we recall that the Karoubi envelope (or idempotent completion) of Khovanov and Lauda's 2-categories, e.g. $\text{Kar}\mathcal{U}(\mathfrak{sl}_n)$ and $\text{Kar}\mathcal{U}(\mathfrak{gl}_n)$, contain the categorified divided powers $\mathcal{E}_{\pm i}^{(a)}$, which satisfy

$$\mathcal{E}_{\pm i}^a = \left(\mathcal{E}_{\pm i}^{(a)} \right)^{\oplus [a]}.$$

In [KLMS12] the 2-morphisms in $\text{Kar}\mathcal{U}(\mathfrak{sl}_2)$ between the divided powers were worked out explicitly. Using the fact that $\text{Kar}\mathcal{U}(\mathfrak{sl}_2)$ can be embedded into $\text{Kar}\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ for any choice of simple root, we can use the results in [KLMS12]. We do not need much of that calculus in this paper, but we do have to recall the *splitters* (see definitions below Lemma 2.2.3 and see (2.63) in [KLMS12])

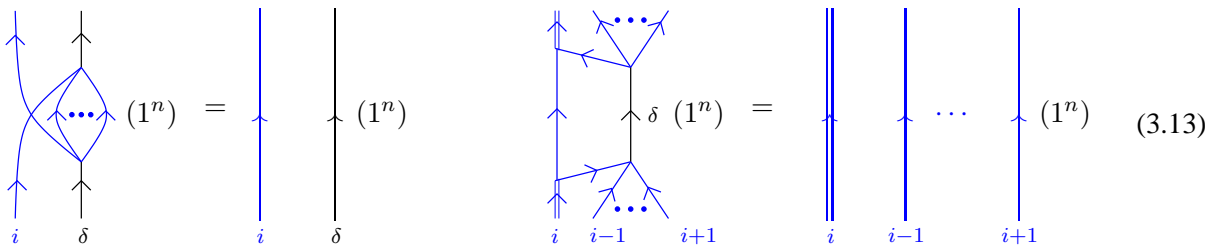
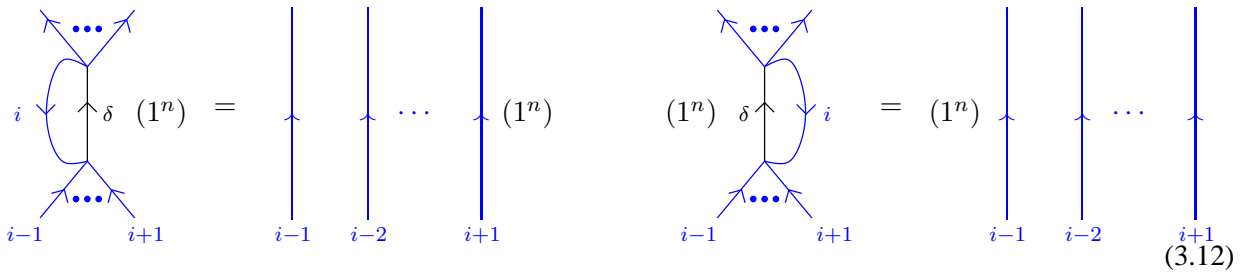


and the relations (see (2.36), (2.64) and (2.65) in [KLMS12])



for any $i = 1, \dots, n$. By cyclicity, we get similar splitters and relations for $\mathcal{E}_{-i}^{(2)}$, $i = 1, \dots, n$.

Lemma 3.2.



$$(1^n) \text{ [diagram with bubble between } \delta \text{ and } i \text{]} = (1^n) \text{ [diagram with two lines } \delta \text{ and } i \text{]} \quad (1^n) \text{ [diagram with bubble and } \delta \text{ between } i-1, i+1, i \text{]} = (1^n) \text{ [diagram with three lines } i-1, i+1, i \text{]} \quad (3.14)$$

By cyclicity, we get the analogous relations with all orientations reversed.

Proof. The equations in (3.12) follow directly from (3.10) and the bubble relations. Note that one of the terms we get by applying (3.10) has a bubble of degree -2 , which is equal to zero, and the other term has a bubble of degree 0 which is equal to -1 if it is counter-clockwise and $+1$ if it is clockwise.

We only prove the equations in (3.13). The equations in (3.14) can be proved similarly. By the second relation in (3.9), curl removal and the evaluation of degree zero bubbles, we get

$$(1^n) \text{ [diagram with bubble between } i \text{ and } \delta \text{]} = (1^n) \text{ [diagram with figure-eight bubble between } i \text{ and } \delta \text{]} = (1^n) \text{ [diagram with two lines } i \text{ and } \delta \text{]} \quad (3.13)$$

By (3.10) and the relations in (2.64) in [KLMS12], we get

$$(1^n) \text{ [diagram with } \delta \text{ and bubble]} = (1^n) \text{ [diagram with double lines and bubble]} \dots (1^n) \text{ [diagram with double line and bubble]} - (1^n) \text{ [diagram with double line and bubble]} \dots (1^n) \text{ [diagram with double lines]} \quad (3.13)$$

□

Lemma 3.3. We have

$$(1^n) \text{ [diagram with } \delta \text{ and bubble on } i \text{]} = (1^n) \text{ [diagram with bubble between } i \text{ and } i+1 \text{]} = (1^n) \text{ [diagram with } \delta \text{ and bubble on } i+1 \text{]} \quad (3.13)$$

Proof. The first equality is a direct consequence of the first relation in (3.9).

The second is a consequence of the first relation in (3.9) and the fact that

$$(1^n) \text{ [diagram with clockwise bubble on } i+1 \text{]} = (1^n) \text{ [diagram with counter-clockwise bubble on } i+1 \text{]},$$

which follows from the infinite Grassmannian relation for bubbles. □

In order to formulate the following results, define

$$\boxed{z_m} (1^n) := - \left(\text{bubble}_{i-1}^{i-1} + \text{bubble}_{i-2}^{i-2} + \cdots + \text{bubble}_m^m \right) (1^n)$$

The sum of the bubbles is over the colors $i-1, i-2, \dots, m$, if $1 \leq m \leq i-1$, and over the colors $i-1, i-2, \dots, 1, n, n-1, \dots, m$, if $m \geq i+1$. These are exactly the colors of all the strands in the diagram on the left-hand side of Lemma 3.4 between the strands $i-1$ and m . By definition we take $\boxed{z_i} = 0$ and use the convention that $0^0 = 1$.

Similarly, we define

$$\boxed{y_m} (1^n) := - \left(\text{bubble}_m^m + \text{bubble}_{m-1}^{m-1} + \cdots + \text{bubble}_{i+2}^{i+2} \right) (1^n).$$

The sum of the bubbles is over the colors $m, m-1, \dots, i+2$, if $i+2 \leq m \leq n$, and over the colors $m, m-1, \dots, 1, n, n-1, \dots, i+2$, if $m \leq i+1$. These are exactly the colors of all the strands in the diagram on the left-hand side of Lemma 3.4 between the strands m and $i+2$. By definition we take $\boxed{y_{i+1}} = 0$ and use the convention that $0^0 = 1$.

Note that

$$\boxed{y_{i-1}} = \boxed{z_{i+2}}$$

by the infinite Grassmannian relation.

Lemma 3.4. For any $1 \leq m \leq n$ and $s, t \in \mathbb{N}$, we have

$$\begin{array}{c} \text{Diagram with } t \text{ dots on strand } i \text{ and } s \text{ dots on strand } m \\ \delta \end{array} (1^n) = \sum_{j=0}^s \binom{s}{j} \text{bubble}_{s+t-j}^i \boxed{z_m^j} \begin{array}{c} \delta \\ \delta \end{array} (1^n). \quad (3.15)$$

On the left-hand side of Equation (3.15), the t dots are on the i -th strand and the s dots are on the m -th strand. Similarly, we have

$$\begin{array}{c} \text{Diagram with } t \text{ dots on strand } m \text{ and } s \text{ dots on strand } i+1 \\ \delta \end{array} (1^n) = \sum_{j=0}^t \binom{t}{j} \text{bubble}_{s+t-j}^{i+1} \boxed{y_m^j} (1^n). \quad (3.16)$$

On the left-hand side of Equation (3.15), the t dots are on the m -th strand and the s dots are on the $i+1$ -st strand.

Proof. We only prove the first equation. The second can be proved in a similar way. The proof is by induction w.r.t. s . For $s = 0$ and any $1 \leq m \leq n$ and $t \in \mathbb{N}$, the result follows from (3.9).

Suppose $s > 0$, $t \in \mathbb{N}$ and $m \neq i + 1$. The case $m = i$ follows from (3.9), so we can assume that $m \neq i$. First note the following:

$$0 = \text{Diagram 1} (1^n) = - \text{Diagram 2} (1^n) + \text{Diagram 3} (1^n). \quad (3.17)$$

The first equality holds, because the label of the region inside the curl does not belong to $\Lambda(n, n)$; its $m + 1$ -st entry equals -1 . The second equality follows from resolving the curl. The minus sign is a consequence of our normalization of degree zero bubbles in [MT13], because the label λ of the region just outside the bubble satisfies $\lambda_{m+1} = 0$. Note that the bubble in the second term has degree two, since $\lambda_m - \lambda_{m+1} = 1$, for any $m \neq i, i + 1$.

Equation (3.17) implies

$$\text{Diagram 4} (1^n) = \text{Diagram 5} (1^n). \quad (3.18)$$

Now slide the m -bubble to the left. Note that the strand directly to the left of the bubble has color $m + 1$ (we keep considering the colors modulo n). Therefore, by the bubble slide relations and the degree zero bubble relations in [MT13], we get

$$\text{Diagram 6} (1^n) = \text{Diagram 7} (1^n) - \text{Diagram 8} (1^n) \quad (3.19)$$

The new bubble, in the second diagram on the right-hand side of Equation (3.19), still has color m of course. But now it is in between the strands colored $m + 2$ and $m + 1$, reading from left to right. The label, λ , of the region between these two strands satisfies $\lambda_{m+1} = 1$. Thus, by the degree zero bubble relations in [MT13], the counter-clockwise degree zero m -bubble in that region is equal to one, which explains the positive sign of the first term on the right-hand side in (3.19). Note that the label of the region containing the m -bubble in the second term satisfies $\lambda_m - \lambda_{m+1} = 0$, so the dotless m -bubble has degree 2, as it should.

Note that the m -bubble in the second term in (3.19) can be slid completely to the left-hand side. After that, we can use (3.18) to eliminate the dot on the $m + 1$ th strand and slide the $m + 1$ -bubble completely to

the left-hand side. Repeating this for all strands between $i - 1$ and m , we get the following result

$$\text{Diagram}_1 (1^n) = \text{Diagram}_2 (1^n) - \left(\text{Circle}_{i-1} + \text{Circle}_{i-2} + \dots + \text{Circle}_m \right) \text{Diagram}_3 (1^n) \quad (3.20)$$

Induction then proves the result for $m \neq i + 1$.

For $m = i + 1$ we have to adapt our reasoning above, because the region between the $i + 2$ -th and the $i + 1$ -th strands has label $\lambda = (1^i, 2, 0, 1^{n-(i+2)})$. In particular $\lambda_{i+1} = 2$, so the left $i + 1$ -curl has degree four this time, which prevents us from using induction. Therefore, we use a slightly different argument involving a right curl.

We still assume that $s > 0$ holds. First note that, by the resolution of the curl and the degree zero bubble relations in [MT13], we have

$$0 = \text{Diagram}_1 (1^n) = \text{Diagram}_2 (1^n) - \text{Diagram}_3 (1^n) \quad (3.21)$$

because the region between the $i + 2$ -th and the $i + 1$ -th strands is labeled $\lambda = (1^i, 2, 0, 1^{n-(i+2)})$. In particular, we have $\lambda_{i+2} - \lambda_{i+3} = -1$ and $\lambda_{i+3} = 1$, which explains the signs of the terms on the right-hand side of (3.21).

We now slide the $i + 2$ -bubble in the second term on the right-hand side of (3.21) to the right:

$$\text{Diagram}_1 (1^n) = \text{Diagram}_2 (1^n) + \text{Diagram}_3 (1^n) \quad (3.22)$$

The sign of the first term on the right-hand side of (3.22) follows from the degree zero bubble relations in [MT13].

Putting (3.21) and (3.22) together, we get

$$\begin{array}{c} \uparrow \\ \text{Diagram 1} \\ \delta \end{array} (1^n) = \begin{array}{c} \uparrow \\ \text{Diagram 2} \\ \delta \end{array} (1^n) - \begin{array}{c} \uparrow \\ \text{Diagram 3} \\ \delta \end{array} (1^n). \quad (3.23)$$

The diagrams are:
 Diagram 1: A diamond shape with vertices t (top-left), s (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 2: A diamond shape with vertices t (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 3: A diamond shape with vertices t (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i , with a small circle on its boundary.

We can exchange the $i + 2$ -bubble on the right-hand side for an $i + 1$ -bubble on the left-hand side by Lemma 3.3, and invert its orientation by the infinite Grassmannian relation.

By the same reasoning as above, we get

$$\begin{array}{c} \uparrow \\ \text{Diagram 4} \\ \delta \end{array} (1^n) = \begin{array}{c} \uparrow \\ \text{Diagram 5} \\ \delta \end{array} (1^n) - \left(\text{Diagram 6} + \text{Diagram 7} + \dots + \text{Diagram 8} \right) \begin{array}{c} \uparrow \\ \text{Diagram 9} \\ \delta \end{array}. \quad (3.24)$$

The diagrams are:
 Diagram 4: A diamond shape with vertices t (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 5: A diamond shape with vertices $t+1$ (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 6-8: A series of circles of sizes $i-1$, $i-2$, ..., $i+2$.
 Diagram 9: A diamond shape with vertices t (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+1$ is attached to the bottom vertex i .

Putting (3.23) and (3.24) together, we obtain

$$\begin{array}{c} \uparrow \\ \text{Diagram 10} \\ \delta \end{array} (1^n) = \begin{array}{c} \uparrow \\ \text{Diagram 11} \\ \delta \end{array} (1^n) - \left(\text{Diagram 12} + \text{Diagram 13} + \dots + \text{Diagram 14} \right) \begin{array}{c} \uparrow \\ \text{Diagram 15} \\ \delta \end{array}. \quad (3.25)$$

The diagrams are:
 Diagram 10: A diamond shape with vertices t (top-left), s (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 11: A diamond shape with vertices $t+1$ (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+2$ is attached to the bottom vertex i .
 Diagram 12-14: A series of circles of sizes $i-1$, $i-2$, ..., $i+1$.
 Diagram 15: A diamond shape with vertices t (top-left), $s-1$ (top-right), and i (bottom). A bubble of size $i+1$ is attached to the bottom vertex i .

As before, the result follows by induction. □

Proposition 3.5.

$$\begin{array}{c}
 \begin{array}{c} \uparrow \\ s_{i-1} \quad s_{i+2} \\ \bullet \quad \bullet \\ s_i \quad s_{i+1} \\ \vdots \\ i \\ \downarrow \\ \delta \end{array} (1^n) = \\
 \\
 \sum_{j_{i-1}=0}^{s_{i-1}} \sum_{j_{i-2}=0}^{s_{i-2}} \cdots \sum_{j_{i+1}=0}^{s_{i+1}} \binom{s_{i-1}}{j_{i-1}} \cdots \binom{s_{i+1}}{j_{i+1}} \boxed{z_{i-1}^{j_{i-1}} \cdots z_{i+1}^{j_{i+1}}} \begin{array}{c} i \\ \circlearrowleft \\ s_i + \cdots + s_{i+1} - j_{i-1} - \cdots - j_{i+1} \end{array} \Bigg|_{\delta} (1^n) = \\
 \\
 \sum_{j_i=0}^{s_i} \sum_{j_{i-1}=0}^{s_{i-1}} \cdots \sum_{j_{i+2}=0}^{s_{i+2}} \binom{s_i}{j_i} \cdots \binom{s_{i+2}}{j_{i+2}} \boxed{y_i^{j_i} \cdots y_{i+2}^{j_{i+2}}} \begin{array}{c} i+1 \\ \circlearrowleft \\ s_i + \cdots + s_{i+1} - j_i - \cdots - j_{i+2} \end{array} \Bigg|_{\delta} (1^n)
 \end{array}$$

Proof. We only prove the first equation. The second can be proved by similar arguments.

We use induction w.r.t. the reverse lexicographical ordering of the dot sequences (s_i, \dots, s_{i+1}) . The base of the induction, $s_i = \dots = s_{i+1} = 0$, has been dealt with in Lemma 3.3.

The case $s_{i-1} = \dots = s_{i+1} = 0$ has been dealt with in Lemma 3.4. Suppose there exists a $j \in \{i-1, \dots, i+1\}$ with $s_j > 0$. The argument below works for arbitrary j , but let us assume that $j = i-1$ for simplicity.

By the same arguments as used in the proof of Lemma 3.4, we get

$$\begin{array}{c} \uparrow \\ s_{i-1} \quad s_{i+2} \\ \bullet \quad \bullet \\ s_i \quad s_{i+1} \\ \vdots \\ i \\ \downarrow \\ \delta \end{array} (1^n) = \begin{array}{c} \uparrow \\ s_{i-1}-1 \quad s_{i+2} \\ \bullet \quad \bullet \\ s_i+1 \quad s_{i+1} \\ \vdots \\ i \\ \downarrow \\ \delta \end{array} (1^n) - \begin{array}{c} \uparrow \\ s_{i-1}-1 \quad s_{i+2} \\ \bullet \quad \bullet \\ s_i \quad s_{i+1} \\ \vdots \\ i \\ \downarrow \\ \delta \end{array} (1^n) - \begin{array}{c} i-1 \\ \circlearrowleft \\ s_{i-1}-1 \quad s_{i+2} \\ \bullet \quad \bullet \\ s_i \quad s_{i+1} \\ \vdots \\ i \\ \downarrow \\ \delta \end{array} (1^n) \tag{3.26}$$

Induction on both terms on the right-hand side of (3.26) proves the proposition. □

Proposition 3.5 also allows us to derive two bubble slide formulas. The other two, for bubbles with the opposite orientation, can be obtained using the infinite Grassmannian relation and induction. Since we do not need them in this paper, we omit them.

Corollary 3.6. We have

$$\sum_{j=0}^s \binom{s}{j} \left(\begin{array}{c} i \\ \text{bubble} \\ s-j \end{array} \right) \boxed{z_{i+1}^j} \left(\begin{array}{c} \delta \\ (1^n) \end{array} \right) = \left(\begin{array}{c} \delta \\ (1^n) \end{array} \right) \left(\begin{array}{c} i+1 \\ \text{bubble} \\ s \end{array} \right) \quad (3.27)$$

and

$$\left(\begin{array}{c} \delta \\ (1^n) \end{array} \right) \sum_{j=0}^s \binom{s}{j} \left(\begin{array}{c} i+1 \\ \text{bubble} \\ s-j \end{array} \right) \boxed{y_i^j} = \left(\begin{array}{c} \delta \\ (1^n) \end{array} \right) \left(\begin{array}{c} i \\ \text{bubble} \\ s \end{array} \right) \quad (3.28)$$

Proof. These two bubble slide relations follow immediately from Lemma 3.4. For (3.27), apply (3.15) and (3.16) with $t = 0, m = i + 1$. For (3.28), apply (3.15) and (3.16) with $s = 0, m = i$. \square

4. TWO USEFUL 2-FUNCTORS

Definition 4.1. Let the 2-functor $\Psi_{n,n}: \mathcal{U}(\widehat{\mathfrak{sl}}_n)^* \rightarrow \widehat{\mathcal{S}}(n, n)^*$ be defined just as $\Psi_{n,r}$ in Section 3.5.3 in [MT13], i.e. on objects and 1-morphisms it is determined by

$$\begin{aligned} \mu &\mapsto \varphi_{n,n}(\mu) =: \lambda \\ \mathcal{E}_{\underline{i}} \mathbf{1}_{\mu} &\mapsto \mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda}. \end{aligned}$$

By convention, we put $1_* := 0$. On 2-morphisms it is determined by sending any diagram in $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ which is not a left cap or cup to the same diagram in $\widehat{\mathcal{S}}(n, n)$ and applying $\varphi_{n,n}$ to the labels of the regions in the diagram. The images of the left caps and cups also have to be multiplied by certain signs. To be more precise, define

$$\curvearrowright_{i,\mu} \mapsto (-1)^{\lambda_{i+1}+1} \curvearrowright_{i,\lambda} \quad \text{and} \quad \curvearrowleft_{i,\mu} \mapsto (-1)^{\lambda_{i+1}} \curvearrowleft_{i,\lambda}. \quad (4.1)$$

We define any diagram in $\widehat{\mathcal{S}}(n, n)$ to be equal to zero if it contains regions labeled $*$.

Note that, unlike $\Psi_{n,r}$ for $n > r$, $\Psi_{n,n}$ is not essentially surjective. However, it still has the following useful property.

Lemma 4.2. The 2-functor $\Psi_{n,n}$ is full.

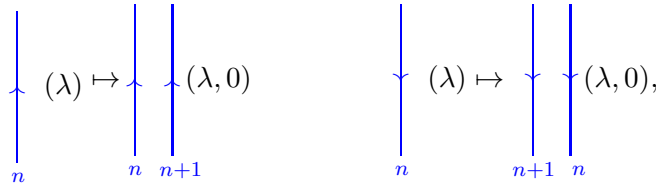
Proof. The proof follows from the following two observations, which show how to remove δ -strands from diagrams in $\text{HOM}_{\widehat{\mathcal{S}}(n,n)}(\mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda}, \mathcal{E}_{\underline{j}} \mathbf{1}_{\lambda})$, for any signed sequences \underline{i} and \underline{j} :

- Closed δ -diagrams always consist of disjoint δ -circles. By Corollary 3.6 we can move any closed i -diagram, which is always equivalent to a linear combination of disjoint i -circles, from the interior to the exterior of a δ -circle. By (3.3), we can then remove the δ -circles with empty interior.
- Any δ -strand which is not part of a δ -circle has to be part of a diagram obtained by glueing $\curvearrowright_{\delta,j}$ on top of $\curvearrowleft_{\delta,i}$ or $\curvearrowleft_{\delta,i}$ on top of $\curvearrowright_{\delta,j}$, for certain $1 \leq i, j \leq n$. In both cases we can remove the δ -strand by applying (3.10) or (3.11).

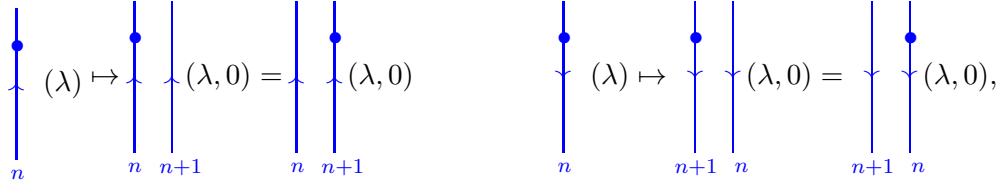
\square

Definition 4.3. We define the 2-functor $\mathcal{I}_n: \widehat{\mathcal{S}}(n, n) \rightarrow \widehat{\mathcal{S}}(n+1, n)$ as follows:

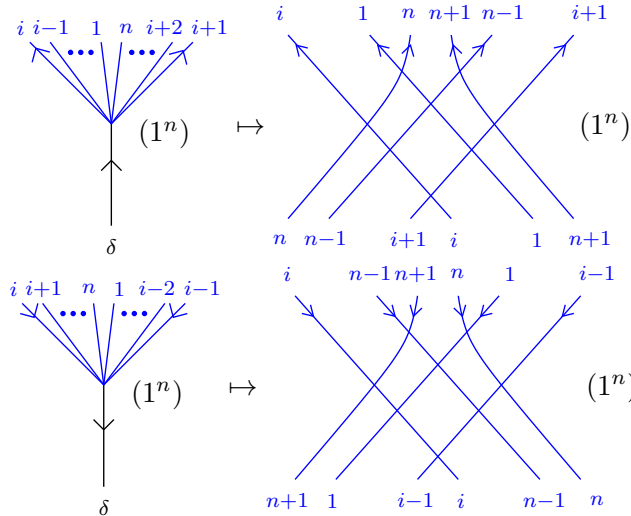
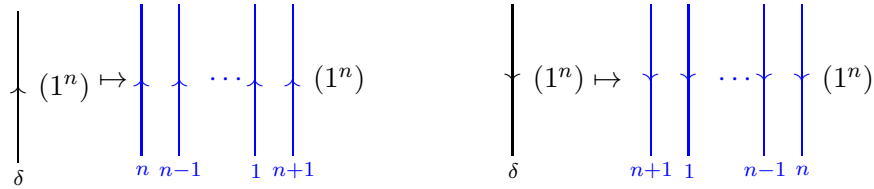
- on objects and 1-morphisms use the map in Proposition 2.14;
- on 2-morphisms take the identity on all i -strands, for $1 \leq i \leq n - 1$, map all n -strands to two parallel strands labeled n and $n + 1$, e.g.



map dots on n -strands to dots on the corresponding pairs of parallel strands as follows



and send the generators involving δ -strands to



with the image of the other two δ -splitters being defined likewise using cyclicity.

Note that the two images of the dotted n -strands which are shown, are indeed equal in $\widehat{\mathcal{S}}(n + 1, n)$. This follows from the relevant Reidemeister 2 relations, because the diagrams with the crossings in those relations are equal to zero (the last entry of the labels of their middle regions is equal to -1).

Lemma 4.4. For any $n \geq 3$, \mathcal{I}_n is well-defined.

Proof. We only have to prove that \mathcal{I}_n preserves the relations involving n and δ -strands, because \mathcal{I}_n clearly preserves all other relations.

First consider the nilHecke relations which only involve n -strands. By cyclicity, we can assume that all strands are oriented upward. We give the proof of well-definedness w.r.t. one nilHecke relation in detail.

The image of the left-hand side of

$$\begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \lambda - \begin{array}{c} \nearrow \\ \searrow \\ \bullet \end{array} \lambda = \begin{array}{c} | \\ | \end{array} \begin{array}{c} | \\ | \end{array} \lambda \quad (4.2)$$

is given by

$$\begin{array}{c} \nearrow \nearrow \\ \bullet \nearrow \\ \searrow \searrow \end{array} (\lambda, 0) - \begin{array}{c} \nearrow \nearrow \\ \nearrow \searrow \\ \bullet \searrow \end{array} (\lambda, 0).$$

By the nilHecke relation for the n -strands, this is equal to

$$\begin{array}{c} \nearrow \nearrow \\ \searrow \nearrow \\ \searrow \searrow \end{array} (\lambda, 0) = \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \end{array} (\lambda, 0),$$

which is equal to the image of the right-hand side of (4.2). Note that in the last equality we have omitted one term, which is equal to zero because it contains a region whose label has a negative entry.

Well-definedness w.r.t. the other two nilHecke relations for n -strands can be proved by similar arguments.

As for the other relations involving only n -strands, the first one we should have a look at is the infinite Grassmannian relation. The image of the n -bubbles is given by

$$\begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + a \end{array} \lambda \mapsto \begin{array}{c} \circlearrowleft \\ \circlearrowright \\ \bullet \\ \spadesuit + a \end{array} (\lambda, 0)$$

$$\begin{array}{c} \circlearrowright \\ \bullet \\ \spadesuit + a \end{array} \lambda \mapsto \begin{array}{c} \circlearrowleft \\ \circlearrowright \\ \bullet \\ \spadesuit + a \end{array} (\lambda, 0)$$

for any $a \in \mathbb{N}$ and $\lambda \in \Lambda(n, n)$. The notation \spadesuit is defined by

$$\begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + b \end{array} \lambda := \begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + b \end{array} \lambda - (\lambda_i - \lambda_{i+1}) - 1 + b$$

$$\begin{array}{c} \circlearrowright \\ \bullet \\ \spadesuit + b \end{array} \lambda := \begin{array}{c} \circlearrowright \\ \bullet \\ \spadesuit + b \end{array} \lambda - (\lambda_i - \lambda_{i+1}) - 1 + b,$$

for any $b \in \mathbb{N}$.

For $\spadesuit + a < 0$, the image of the fake n -bubbles above is a definition. For $\spadesuit + a \geq 0$, we have to prove that the image of the n -bubbles above is equal to the image assigned to them by \mathcal{I}_n . This is immediate if the two nested bubbles in the image are real (since the numbers of dots match), but one of them could be fake, in which case a proof is required. Let us give this proof for the counter-clockwise n -bubbles. Note that

$$\begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + a \end{array} \lambda = \begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + a \end{array} \lambda - (\lambda_n - \lambda_1) - 1 + a \quad (4.3)$$

By the definition above, the image of the l.h.s. of (4.3) is given by

$$\begin{array}{c} \circlearrowleft \\ \circlearrowright \\ \bullet \\ \spadesuit + a \end{array} (\lambda, 0) = - \sum_{b+c=a} \begin{array}{c} \circlearrowleft \\ \bullet \\ \spadesuit + b \end{array} \begin{array}{c} \circlearrowright \\ \bullet \\ \spadesuit + c \end{array} (\lambda, 0).$$

The equality is obtained by applying a bubble-slide relation. By the definition of \mathcal{I}_n , the image of the r.h.s. of (4.3) is given by

$$\begin{aligned}
 \begin{array}{c} n+1 \\ \circlearrowleft \\ n \\ \circlearrowleft \\ a' \end{array} (\lambda, 0) &= - \sum_{b'+c=a'+\lambda_n} \begin{array}{c} n+1 \\ \circlearrowleft \\ b' \end{array} \begin{array}{c} n \\ \circlearrowleft \\ +c \end{array} (\lambda, 0) \\
 &= - \sum_{b'+c=a'+\lambda_n} \begin{array}{c} n+1 \\ \circlearrowleft \\ +b'-\lambda_1+1 \end{array} \begin{array}{c} n \\ \circlearrowleft \\ +c \end{array} (\lambda, 0) \\
 &= - \sum_{b+c=a} \begin{array}{c} n+1 \\ \circlearrowleft \\ +b \end{array} \begin{array}{c} n \\ \circlearrowleft \\ +c \end{array} (\lambda, 0)
 \end{aligned}$$

with $a' = -(\lambda_n - \lambda_1) - 1 + a$. The first equality is obtained by applying a bubble-slide relation, the other equalities are obtained by reindexing. This finishes the proof that both definitions of the image of the counter-clockwise non-fake n -bubbles are equal. The proof for the clockwise n -bubbles is similar and is left to the reader.

We now show that with the definitions above, the images of the bubbles satisfy the infinite Grassmannian relation. To be more precise we have to show that the relation

$$\sum_{a=0}^b \begin{array}{c} n \\ \circlearrowleft \\ +b-a \end{array} \begin{array}{c} n \\ \circlearrowleft \\ +a \end{array} \lambda = -\delta_{b,0} \tag{4.4}$$

is preserved, for any $b \in \mathbb{N}$. For $b = 0$, the image of (4.4) is given by

$$\begin{array}{c} n+1 \\ \circlearrowleft \\ n \\ \circlearrowleft \\ + \end{array} \begin{array}{c} n \\ \circlearrowleft \\ + \end{array} (\lambda, 0) = -1.$$

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The equality follows immediately from the degree-zero bubble relations. For $b > 0$, the image of (4.4) is given by

$$\begin{aligned}
\sum_{a=0}^b \left(\text{bubble}_{n+1}^n \right) (\lambda, 0) &= \sum_{a=0}^b \sum_{k=0}^a \left(\text{bubble}_{n+1}^n \right) (\lambda, 0) \\
&= - \sum_{a=0}^b \sum_{k=0}^a \sum_{\ell=0}^{a-k} \left(\text{bubble}_{n+1}^n \right) (\lambda, 0) \\
&= - \sum_{a=0}^b \sum_{c=0}^a \sum_{k=0}^c \left(\text{bubble}_{n+1}^n \right) (\lambda, 0) \\
&= - \sum_{c=0}^b \sum_{k=0}^c \sum_{m=0}^{b-c} \left(\text{bubble}_{n+1}^n \right) (\lambda, 0) \\
&= 0.
\end{aligned}$$

The first two equalities follow from bubble-slide relations. The next two equalities follow from reindexing, as indicated. The last equality follows from the infinite Grassmannian relation: for the n -bubbles if $b > c$ (with c fixed), and for the $n + 1$ -bubbles if $b = c$.

Knowing the images of the fake bubbles allows us to prove the other relations involving only n -strands very easily. Let us do just one example, the other relations can be proved in a similar fashion. We show that \mathcal{I}_n preserves the relation

$$\left(\text{strand}_n \right) \lambda = - \sum_{f=0}^{\lambda_1 - \lambda_n} \left(\text{strand}_n \right) \bullet_{\lambda_1 - \lambda_n - f} \left(\text{bubble}_n \right) \lambda. \tag{4.5}$$

The image of the l.h.s. of (4.5) is given by

$$\left(\text{strand}_n \text{ strand}_{n+1} \right) (\lambda, 0),$$

which is equal to

$$-\sum_{f=0}^{\lambda_1-1} \begin{array}{c} \uparrow \uparrow \\ \text{bubble} \\ \uparrow \uparrow \\ n \quad n+1 \end{array} (\lambda, 0) = -\sum_{f=0}^{\lambda_1-1} \begin{array}{c} \uparrow \uparrow \\ \text{bubble} \\ \uparrow \uparrow \\ n \quad n+1 \end{array} (\lambda, 0) = -\sum_{f=0}^{\lambda_1-\lambda_n} \begin{array}{c} \uparrow \uparrow \\ \text{bubble} \\ \uparrow \uparrow \\ n \quad n+1 \end{array} (\lambda, 0).$$

The first summation is obtained by resolving the $n+1$ -curl. The second summation can then be obtained by applying a Reidemeister 3 relation to the strands colored $n, n+1$ and n . Note that only the terms which are shown survive, the other ones are zero because they are given by diagrams which contain a region whose label has a negative entry. The last summation is obtained by first reindexing. Then an argument similar to the one we used below (4.3) ensures that the nested bubbles, before and after the equality, match and that the first $\lambda_n - 1$ terms of the reindexed summation vanish (indeed in those terms, bubbles of negative degree appear, and those are always zero). This last expression is equal to the image of the r.h.s. of (4.5), which finishes our proof that \mathcal{I}_n preserves (4.5).

Next let us have a look at the relations involving i -strands of more than one color. We just do one example in detail, the other relations can be proved in a similar fashion. Consider the relation

$$\begin{array}{c} \uparrow \uparrow \\ \text{cross} \\ \uparrow \uparrow \\ n \quad 1 \end{array} \lambda = - \begin{array}{c} \uparrow \uparrow \\ \bullet \\ \uparrow \uparrow \\ n \quad 1 \end{array} \lambda + \begin{array}{c} \uparrow \uparrow \\ \bullet \\ \uparrow \uparrow \\ n \quad 1 \end{array} \lambda \quad (4.6)$$

in $\widehat{\mathcal{S}}(n, n)$. The image of the term on the l.h.s. is given by

$$\begin{array}{c} \uparrow \uparrow \\ \text{cross} \\ \uparrow \uparrow \\ n \quad n+1 \quad 1 \end{array} (\lambda, 0) = \begin{array}{c} \uparrow \uparrow \\ \text{cross} \\ \uparrow \uparrow \\ n \quad n+1 \quad 1 \end{array} (\lambda, 0) = - \begin{array}{c} \uparrow \uparrow \\ \bullet \\ \uparrow \uparrow \\ n \quad n+1 \quad 1 \end{array} (\lambda, 0) + \begin{array}{c} \uparrow \uparrow \\ \bullet \\ \uparrow \uparrow \\ n \quad n+1 \quad 1 \end{array} (\lambda, 0).$$

The first and the second equality follow from the Reidemeister 2 relations in $\widehat{\mathcal{S}}(n+1, n)$. The linear combination at the end is exactly the image of the r.h.s. in (4.6), which proves that (4.6) is preserved by \mathcal{I}_n .

Remains to be proved that \mathcal{I}_n preserves the relations involving δ -strands. For the relations (3.1) and (3.2) the proof follows immediately from the zig-zag relations for i -strands with $i = 1, \dots, n+1$. For the relations in (3.3) the proof follows immediately from the degree-zero i -bubble relations for $i = 1, \dots, n+1$. Let us explain the first relation in (3.4) in more detail, the second being similar. The image of

$$\begin{array}{c} \downarrow \downarrow \\ \delta \\ \downarrow \downarrow \end{array} \quad \begin{array}{c} \downarrow \downarrow \\ \delta \\ \downarrow \downarrow \end{array} (1^n)$$

is given by

$$\begin{array}{c} \downarrow \downarrow \dots \downarrow \downarrow \\ n+1 \quad 1 \quad \dots \quad n \quad n \\ \uparrow \uparrow \dots \uparrow \uparrow \\ 1 \quad n+1 \quad \dots \quad 1 \quad n+1 \end{array} (1^n, 0) = \begin{array}{c} \downarrow \downarrow \dots \downarrow \downarrow \\ n+1 \quad 1 \quad \dots \quad n \quad n \\ \uparrow \uparrow \dots \uparrow \uparrow \\ n \quad n \quad \dots \quad 1 \quad n+1 \end{array} (1^n, 0) = \dots = \begin{array}{c} \downarrow \downarrow \dots \downarrow \downarrow \\ n+1 \quad 1 \quad \dots \quad n \quad n \\ \uparrow \uparrow \dots \uparrow \uparrow \\ n+1 \quad 1 \quad \dots \quad 1 \quad n+1 \end{array} (1^n, 0),$$

which is indeed equal to the image of

$$\begin{array}{c} \downarrow \downarrow \\ \delta \\ \downarrow \downarrow \end{array} (1^n).$$

The equalities above are obtained by repeatedly applying Reidemeister 2 relations on the pairs of i -strands with $\lambda_i - \lambda_{i+1} = -1$ for all $i = 1, \dots, n + 1$. Note that the terms with two i -crossings are all equal to zero, because they contain a region whose label has one negative entry, and that all bubbles in the other terms are of degree zero and equal to -1 .

The fact that relations (3.5), (3.6), (3.7) and (3.8) are preserved follows easily from applying Reidemeister 2 and 3 relations to the images of the terms on their left-hand side. The dots appear after applying the Reidemeister 2 relation involving the i and $i + 1$ -strands.

We prove the left relation in (3.9) for $1 \leq i < n$. The proof for $i = n$ and the proof of the right relation in (3.9) are similar and are left to the reader. The image on the l.h.s. of the first relation in (3.9) is given by

$$(1^n, 0) \quad \begin{array}{c} \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \\ \begin{array}{c} \text{Diagram with crossings between strands } i \text{ and } i+1 \end{array} \\ \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \end{array} \quad . \quad (4.7)$$

We claim that this is equal to

$$(1^n, 0) \quad \begin{array}{c} \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \\ \begin{array}{c} \text{Diagram with vertical strands and a dot on the } i\text{-strand} \end{array} \\ \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \end{array} ,$$

which is indeed the image of the r.h.s. of (3.9). This follows from first applying Reidemeister 2 relations to (4.7) in order to straighten all j -strands for $j \neq i$:

$$(1^n, 0) \quad \begin{array}{c} \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \\ \begin{array}{c} \text{Diagram with vertical strands and crossings between } i \text{ and } i+1 \end{array} \\ \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \end{array} ;$$

then a Reidemeister 2 relation to the i -strands in the middle (note that the region at the top and the bottom between the i and the $i - 1$ -strand is labeled $(1, \dots, 1, 0, 1, \dots, 1)$ with 0 on the i -th position):

$$(1^n, 0) \quad \begin{array}{c} \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \\ \begin{array}{c} \text{Diagram with vertical strands and a crossing between } i \text{ and } i-1 \end{array} \\ \begin{array}{cccccccc} n & n-1 & \dots & i+1 & i & i-1 & \dots & 1 & n+1 & i \end{array} \end{array} ;$$

and finally Reidemeister 2 relations in order to straighten the downward i -strand.

Finally, the fact that \mathcal{I}_n preserves the relations (3.10) and (3.11) can be easily proved by applying Reidemeister 2 and 3 relations to the images of the diagrams on the l.h.s. of those two relations. \square

5. THE GROTHENDIECK GROUP

In this section we prove that $\widehat{\mathcal{S}}(n, n)$ categorifies $\widehat{\mathbf{S}}(n, n)$ (Theorem 5.4). All the hard work has been done already, we just have to put everything together.

Lemma 5.1. In $\widehat{\mathcal{S}}(n, n)$, we have

- i) $\mathcal{E}_{\pm\delta}\mathbf{1}_\lambda \cong \mathbf{1}_\lambda\mathcal{E}_{\pm\delta} \cong 0$ for all $\lambda \neq (1^n)$;
- ii) $\mathcal{E}_{\pm\delta}\mathbf{1}_n \cong \mathbf{1}_n\mathcal{E}_{\pm\delta}$;
- iii) $\mathcal{E}_{+\delta}\mathcal{E}_{-\delta}\mathbf{1}_n \cong \mathcal{E}_{-\delta}\mathcal{E}_{+\delta}\mathbf{1}_n \cong \mathbf{1}_n$;
- iv) $\mathcal{E}_i\mathcal{E}_{+\delta}\mathbf{1}_n \cong \mathcal{E}_i^{(2)}\mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}\mathbf{1}_n$;
- v) $\mathbf{1}_n\mathcal{E}_{+\delta}\mathcal{E}_i \cong \mathbf{1}_n\mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}\mathcal{E}_i^{(2)}$;
- vi) $\mathcal{E}_{-i}\mathcal{E}_{+\delta}\mathbf{1}_n \cong \mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}\mathbf{1}_n$;
- vii) $\mathbf{1}_n\mathcal{E}_{+\delta}\mathcal{E}_{-i} \cong \mathbf{1}_n\mathcal{E}_{i-1}\cdots\mathcal{E}_1\mathcal{E}_n\cdots\mathcal{E}_{i+1}$;
- viii) $\mathcal{E}_{-i}\mathcal{E}_{-\delta}\mathbf{1}_n \cong \mathcal{E}_{-i}^{(2)}\mathcal{E}_{-(i+1)}\cdots\mathcal{E}_{-n}\mathcal{E}_{-1}\cdots\mathcal{E}_{-(i-1)}\mathbf{1}_n$;
- ix) $\mathbf{1}_n\mathcal{E}_{-\delta}\mathcal{E}_{-i} \cong \mathbf{1}_n\mathcal{E}_{-(i+1)}\cdots\mathcal{E}_{-n}\mathcal{E}_{-1}\cdots\mathcal{E}_{-(i-1)}\mathcal{E}_{-i}^{(2)}$;
- x) $\mathcal{E}_i\mathcal{E}_{-\delta}\mathbf{1}_n \cong \mathcal{E}_{-(i+1)}\cdots\mathcal{E}_{-n}\mathcal{E}_{-1}\cdots\mathcal{E}_{-(i-1)}\mathbf{1}_n$;
- xi) $\mathbf{1}_n\mathcal{E}_{-\delta}\mathcal{E}_i \cong \mathbf{1}_n\mathcal{E}_{-(i+1)}\cdots\mathcal{E}_{-n}\mathcal{E}_{-1}\cdots\mathcal{E}_{-(i-1)}$,

for any $i = 1, \dots, n$.

Proof. The isomorphisms in (i) and (ii) are immediate.

For (iii), consider the 2-morphisms

$$\begin{array}{l}
 \begin{array}{c} \curvearrowright \\ \delta \end{array} (1^n): \quad \mathbf{1}_n \rightarrow \mathcal{E}_{-\delta}\mathcal{E}_{+\delta}\mathbf{1}_n \\
 \begin{array}{c} \curvearrowleft \\ \delta \end{array} (1^n): \quad \mathcal{E}_{-\delta}\mathcal{E}_{+\delta}\mathbf{1}_n \rightarrow \mathbf{1}_n \\
 \begin{array}{c} \curvearrowleft \\ \delta \end{array} (1^n): \quad \mathbf{1}_n \rightarrow \mathcal{E}_{+\delta}\mathcal{E}_{-\delta}\mathbf{1}_n \\
 \begin{array}{c} \curvearrowright \\ \delta \end{array} (1^n): \quad \mathcal{E}_{+\delta}\mathcal{E}_{-\delta}\mathbf{1}_n \rightarrow \mathbf{1}_n.
 \end{array}$$

Relations (3.3) and (3.4) show that these 2-morphisms are 2-isomorphisms.

Similarly, the isomorphisms in (iv) and (v) follow from the relations in (3.13) and (3.14), and the isomorphisms in (vi) and (vii) follow from the relations in (3.9) and (3.12).

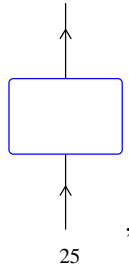
The isomorphisms in (viii)–(xi) follow from the ones above by biadjointness. \square

Recall that $\text{END}(X)$ denotes the ring generated by all homogeneous 2-endomorphisms of a given 1-morphism X , whereas $\text{End}(X) \subset \text{END}(X)$ only contains the ones of degree zero.

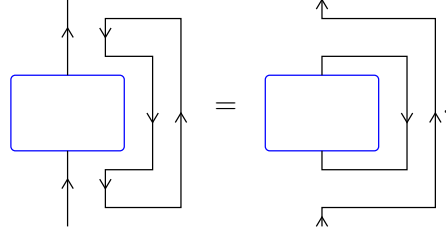
Lemma 5.2. For any $t \in \mathbb{Z}$,

$$\text{END}(\mathcal{E}_{+\delta}^t\mathbf{1}_n) \cong \mathbf{1}_{\mathcal{E}_{+\delta}^t}\text{END}(\mathbf{1}_n) \cong \text{END}(\mathbf{1}_n)\mathbf{1}_{\mathcal{E}_{+\delta}^t}$$

Proof. Note that for $t = 0$ there is nothing to prove. Let us now explain the proof for $t = 1$. Given a diagram of the form



we can create a δ -bubble by (3.3) and apply (3.4) to obtain



This proves the lemma for $t = 1$. For $t > 1$ use the same trick repeatedly until you are left with a closed diagram and t upward δ -strands. For $t < 0$, a similar trick can be applied using the opposite orientation on the δ -strands. \square

Let $K_0(\text{Kar}\widehat{\mathcal{S}}(n, n))$ be the split Grothendieck group of $\text{Kar}\widehat{\mathcal{S}}(n, n)$. This is a $\mathbb{Z}[q, q^{-1}]$ -module, where the action of q is defined by

$$q[X] := [X\{1\}].$$

Furthermore, let

$$K_0^{\mathbb{Q}(q)}(\text{Kar}\widehat{\mathcal{S}}(n, n)) := K_0(\text{Kar}\widehat{\mathcal{S}}(n, n)) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q).$$

Definition 5.3. Define the $\mathbb{Q}(q)$ -linear algebra homomorphism $\gamma_n: \widehat{\mathcal{S}}(n, n) \rightarrow K_0^{\mathbb{Q}(q)}(\text{Kar}\widehat{\mathcal{S}}(n, n))$ by

$$\gamma_n(E_{\mathbf{i}}\mathbf{1}_\lambda) := [\mathcal{E}_{\mathbf{i}}\mathbf{1}_\lambda] \otimes 1 \quad \text{and} \quad \gamma_n(E_{+\delta}^t\mathbf{1}_n) := [\mathcal{E}_{+\delta}^t\mathbf{1}_n] \otimes 1$$

for any signed sequence \mathbf{i} , $\lambda \in \Lambda(n, n)$ and $t \in \mathbb{Z}$.

Theorem 5.4. The homomorphism γ_n is well-defined and bijective.

Proof. Well-definedness follows from the corresponding statement for $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ by Khovanov and Lauda in [KL10] and from Lemma 5.1.

Let us now show surjectivity. By Lemma 5.1, any indecomposable object in $\text{Kar}\widehat{\mathcal{S}}(n, n)$ is isomorphic to an object of the form (X, e) , where X is either of the form $\mathcal{E}_{+\delta}^t$ for some $t \in \mathbb{Z}$ or of the form $\mathcal{E}_{\mathbf{i}}$ for some signed sequence \mathbf{i} , and e is some idempotent in $\text{End}(X)$. By Lemmas 4.2 and 5.2, we see that $\text{End}(\mathcal{E}_{+\delta}^t) \cong \mathbb{Q}1_{\mathcal{E}_{+\delta}^t}$. Therefore $\mathcal{E}_{+\delta}^t$ is indecomposable in $\text{Kar}\widehat{\mathcal{S}}(n, n)$. Note that its Grothendieck class lies indeed in the image of γ_n . By Lemma 4.2 we know that $\text{End}_{\widehat{\mathcal{S}}(n, n)}(\mathcal{E}_{\mathbf{i}})$ is the surjective image of the analogous endomorphism ring in $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, for any signed sequence \mathbf{i} . By Khovanov and Lauda's Theorem 1.1 [KL10] and some general arguments which were explained in detail in [MSV13], and also used in [MT13], this implies that the Grothendieck classes of all direct summands of $\mathcal{E}_{\mathbf{i}}$ in $\text{Kar}\widehat{\mathcal{S}}(n, n)$ are contained in the image of γ_n . This concludes the proof that γ_n is surjective.

For injectivity, consider the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{S}}(n, n) & \xrightarrow{\iota_n} & \widehat{\mathcal{S}}(n+1, n) \\ \gamma_n \downarrow & & \downarrow \gamma_{n+1} \\ K_0^{\mathbb{Q}(q)}(\text{Kar}\widehat{\mathcal{S}}(n, n)) & \xrightarrow{K_0(\mathcal{I}_n) \otimes 1} & K_0^{\mathbb{Q}(q)}(\text{Kar}\widehat{\mathcal{S}}(n+1, n)) \end{array},$$

where γ_{n+1} is the isomorphism from Theorem 6.4 in [MT13]. Since ι_n and γ_{n+1} are both injective, their composite is also injective. The commutativity of the diagram above then implies that γ_n is injective too. \square

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