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Market configurations when marginal costs are quality-dependent

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Abstract

Most quality-then-price decision models under vertical product differentiation consider a predetermined market configuration. We endogenize market configuration considering quality-dependent marginal costs and conclude that a strictly interior full coverage duopoly holds for some parameter values, unveiling the relevance of this commonly assumed market structure. Moreover, we show that a monopoly never arises in equilibrium, and (i) there are multiple equilibria at the frontier between interior and corner full coverage duopoly, (ii) the market is fully (partially) covered when relative tastes' heterogeneity is low (high), and (iii) there is a discontinuity in the transition from partial coverage to full coverage duopoly.

1 | INTRODUCTION

Consumers have different preferences regarding quality and hence different willingness to pay for it. Some consumers are eager to have the highest quality product or the latest innovation and are willing to pay for it, whereas others do not bother having a lower quality good but paying less. As the attractiveness of firms' products depends on the quality-price mix they offer, firms spend considerable time and effort making these two strategic decisions. This applies to many decisions, as quality may stand for many different product characteristics apart from product innovation, such as reliability, trust, no risk of failure, adequate sales service, handmade, customized, social, and environmental responsibility. Vertical product differentiation (VPD) models have been used to analyze firms' quality and price decisions as an outcome of a competition game in which they try to maximize profits, considering the heterogeneity in the way consumers value quality. Two relevant issues in these models, which have a great impact on consumers' welfare, are whether a monopoly or an oligopoly arises and, in both cases, whether the whole market is served or not. An uncovered market may happen, for instance, when not the whole population has access to housing, to buy a car, to buy a new mobile phone model, to buy a holiday package, and so on. Despite the vast VPD literature, a complete analysis of these issues is yet to be done, especially when offering a higher quality implies higher variable costs, as it happens in many real-world examples where a higher-quality product can only be obtained by using better inputs and a more specialized labor force.

In fact, the vast majority of the existing studies assume either partial market coverage (Aoki & Prusa, 1996; Benassi et al., 2006; Lambertini & Tampieri, 2012; Motta, 1993; Niem, 2019) or full market coverage (Crampes & Hollander, 1995; Garella & Lambertini, 2014; Schmidt, 2006; Schubert, 2017), hence disregarding the important fact that the market configuration is determined endogenously when firms choose their quality-price combinations.[1] We use the term market configuration to capture both market structure (monopoly or duopoly) and market coverage (partial or full). Assuming a particular market configuration in a quality-then-price setup is equivalent to restricting the firms' quality choices to the interval where the assumed market configuration holds. However, as deviations to other quality choices are not checked, the equilibria found may turn out not to be equilibria at all. But this questions the relevance of the existing results, which can only be confirmed (or not) if we do not impose such restrictions and allow firms to choose among all feasible qualities, considering the corresponding market configuration, that is, a correct analysis implies endogenizing the market configuration. By endogenizing the market configuration, we mean considering all possible quality choices in the quality-then-price game and the corresponding market structures and market coverages. In addition, the analysis should not be done for a particular set of parameters. Instead, one should study how changes in the parameters affect the equilibrium market configuration.

Wauthy (1996) was the first to draw attention to the importance of endogenizing market coverage: «in models of vertical differentiation, it should not be imposed *a priori* if the market is covered or not covered. Covering the market or not is at the heart of the strategic problem for firms» (page 352). Through his complete analysis under nil costs, Wauthy is able to identify the interval where each market coverage holds (rectifying Choi & Shin, 1992 and Tirole, 1988). Liao (2008) also considers nil production costs but assumes that higher quality implies higher investment costs, amending Motta (1993) regarding the partial coverage interval. However, the analysis in Wauthy (1996) and Liao (2008) does not apply to many real-world examples where higher quality can only be achieved with higher marginal costs. A common result in these two papers, which has been overlooked, is that a full coverage duopoly only occurs in a minimal sense; that is, the lowest valuation consumer has a nil surplus. The same result holds in Chambers et al. (2006), assuming increasing marginal costs and a decreasing consumers' tastes density function defined on the interval $[1,\infty)$. So, one may question if a strictly interior full coverage duopoly will ever arise as an equilibrium when firms are free to choose any quality.

Therefore, most models of VPD that analyze quality-then-price decisions consider a predetermined market coverage, which is problematic since there is no guarantee that the assumed market coverage will arise in equilibrium when firms are free to choose any quality. Moreover, the few articles that endogenize market coverage in VPD show that a strictly interior full coverage duopoly never arises in equilibrium, questioning the relevance of this market structure. The current article aims to contribute to this puzzle by considering the case where marginal production costs vary quadratically with quality and consumers' quality valuations are uniformly distributed in a closed interval. Our assumptions are relatively common in the VPD literature (e.g., Lambertini, 1996; Moorthy, 1988; Schmidt, 2006; Schubert, 2017), but to the best of our knowledge, under these assumptions, no work has yet studied the equilibrium market configurations that emerge for different parameter values, when firms take their quality-then-price decisions without being restricted. By doing this, we aim not only to contribute to the relatively small literature that endogenizes the market configuration but also to check the robustness of previous results derived under predetermined market configurations. We obtain two novel results. First, there exists a region of parameters with multiple subgame perfect Nash equilibria that are in the frontier between the interior full coverage duopoly and the corner full coverage duopoly. Second, there is a region of parameters where a strictly interior full coverage duopoly is the market configuration that emerges in the subgame perfect Nash equilibrium (SPNE). This last result is new in the endogenous market coverage literature and is very important to the VPD literature, because it shows the relevance of this market configuration in setups where marginal costs vary in a convex way with quality.

Before presenting in more detail our model and results, we give a short overview of the VPD literature, to correctly position our paper in the literature. Following the work of Mussa and Rosen (1978), who study how a monopolist chooses its quality and price to serve a market of heterogenous consumers, and the works of Gabszewicz and Thisse (1979) and Shaked and Sutton (1982), who extend the analysis to an oligopoly setup, there is a huge VPD literature. Most of this literature assumes that consumers' utility increases linearly with quality and that firms decide the quality of their product before making their price or quantity decisions. However, the existing literature differs on assumptions such as the type of competition (Bertrand or Cournot), the timing of quality choices (sequential or simultaneous), the distribution of consumers' valuation of quality, and the type of costs of quality improvements. Regarding these costs, many of the initial VPD models assumed nil costs (Choi & Shin, 1992; Gabszewicz & Thisse, 1979; Tirole, 1988; Wauthy, 1996). Fixed or investment quality costs, such as R&D or advertising activities performed to improve quality, have been considered by authors such as Shaked and Sutton (1982), Lambertini (1999), Liao (2008), García-Gallego and Georgantzís (2009), and Niem (2019). Marginal production costs increasing with quality, which happens when higher quality requires more expensive inputs, have been assumed by Mussa and Rosen (1978), Lambertini (1996), Schmidt (2006), Schubert (2017), and Pires et al. (2022).[2] The assumptions of a uniform distribution of consumer tastes, Bertrand competition, and simultaneous quality choices are clearly dominant in the literature and are also assumed in our work. VPD models have been used to obtain insights into topics such as the impact of imposing a minimum quality standard (e.g., Crampes & Hollander, 1995; Kuhn, 2007), the impact of income inequality or consumers' heterogeneity (e.g., Benassi et al., 2006; Miao, 2019), entry and entry deterrence (e.g., Lutz, 1997; Noh & Moschini, 2006), and environmental regulation (St-Pierre & Elrod, 2022). Recent contributions include multidimensional quality models (e.g., Barigozzi & Ma, 2018; Garella & Lambertini, 2014; Novo-Peteiro, 2020) and experimental VPD (Amaldoss & Shin, 2011; Alventosa et al., 2023).

As mentioned above, our work is mostly related to the few articles that derive the endogenous market configuration, assuming simultaneous quality choices followed by simultaneous price choices. Under nil costs and uniform distribution of consumers' quality valuation, Wauthy (1996) shows that the equilibrium market configuration depends on the heterogeneity in consumers' tastes. If consumers' preferences are very similar, a high-quality monopoly arises in equilibrium. For higher heterogeneity (defined by the ratio of the highest quality valuation to the lowest quality valuation), both firms are active. For intermediate levels of heterogeneity, the whole market is covered (with two subcases but both implying a corner solution in the pricing game), whereas for higher heterogeneity, a partial coverage duopoly holds. Under quadratic investment quality costs, Liao (2008) obtains similar results both regarding the inexistence of a strictly interior full coverage duopoly and the impact of heterogeneity, with corner full coverage duopoly happening for low heterogeneity and partial coverage duopoly holding for high heterogeneity. Chambers et al. (2006) assume variable costs that increase with quality, but a very different distribution of consumers' tastes that includes consumers who value infinitely quality and, hence, are willing to pay any price. Under this assumption, the authors show that, in equilibrium, the high-quality firm always chooses the highest feasible quality (a feature that is also present in Wauthy, 1996) and end up obtaining similar results as a strictly interior full coverage duopoly never occurs. Compared with Wauthy, there is however an interesting difference, as in Chambers et al. (2006); a monopoly never emerges as the equilibrium market configuration.Chambers et al. (2006) do not study the impact of changing tastes' heterogeneity, as their distribution is fixed, but explore the impact of changes in the curvature of the marginal costs, showing that the higher the curvature, the lower the difference between the high-quality and the low-quality.

There are two strong conclusions of this literature, which have not been emphasized enough in the VPD literature: (i) a strictly interior full coverage duopoly never arises in equilibrium and (ii) a full coverage duopoly with a corner solution in the price game happens for a large set of parameters. These conclusions question the relevance of assuming that a strictly interior full coverage duopoly holds and, on the contrary, stresses the importance of not ignoring full coverage duopolies with corner solutions in the price game.

We should acknowledge that there are some other works that have considered the possibility of different market structures in different setups which, however, are not directly related with our work. For instance, Mantovani et al. (2015) consider a model with two quality dimensions, hedonic quality and environmental quality, which are in conflict with each other. They derive the equilibrium configurations in the price game but do not endogenize quality choices. García-Gallego and Georgantzís (2009) study the equilibrium market configurations following an increase in the consumer's willingness to pay (WTP) for products sold by socially responsible manufacturers, assuming that investment costs increase with the level of social responsibility. They apply Liao's (2008) results to study the impact of changes in the WTP. In a sequential quality choices model, Noh and Moschini (2006) study entry-quality decisions and entry deterrence quality decisions, assuming the potential entrant faces fixed entry costs. The authors consider the possibility of a full coverage or a partial coverage monopoly but assume that, in case of entry, a duopoly with full coverage occurs.

In the current paper, we admit that marginal production costs depend quadratically on the quality and that consumers' quality valuations follow a uniform distribution. We derive the SPNE of the complete two-stage game where firms first simultaneously choose their quality and next simultaneously choose their prices. We identify when each possible market coverage configuration holds, considering also monopoly scenarios. Our analysis builds on the results of Pires et al. (2022) regarding the second-stage price competition game, using them as the starting point of our analysis and concentrating on the first stage of the game, where firms decide their quality levels. We show that as long as quality investment costs are infinitesimal, a monopoly will never be a subgame perfect equilibrium, a result also obtained by Chambers et al. (2006). In addition, we show that either partial or full market coverage duopolies may arise in equilibrium and that there are three distinct types of full coverage duopoly equilibria. The type of equilibrium is determined by how the lowest quality consumer valuation is related to the quality valuation heterogeneity in the population. When the level of the lowest quality valuation is low (with respect to the quality valuation heterogeneity), the subgame perfect equilibrium involves quality choices that are in the interior of the region where a partial coverage duopoly holds in the price game. For slightly higher lowest valuation levels, the firms choose qualities that are in the interior of the region where a full coverage duopoly with a corner solution holds. For even higher lowest quality valuation, there are multiple equilibria quality combinations that are located at the frontier between an interior and a corner full coverage duopoly. Finally, for high lowest quality valuations, the qualities chosen are in the region where a strict interior full coverage duopoly holds. The last two results are novel in the literature that endogenizes market configuration. Moreover, the last result is particularly relevant for the whole VPD literature as it shows that, under our assumptions, a strictly interior full coverage duopoly may arise as an equilibrium, validating analyses where this market configuration is assumed, as long as parameters are restricted to the interval where it holds.

As expected, in equilibrium, there is always quality differentiation. However, quality differentiation may decrease with the quality valuation heterogeneity, even though within a certain market configuration, quality differentiation increases with heterogeneity. This surprising result is due to the fact that changes in the quality valuation heterogeneity may lead to changes in the type of equilibria that occurs and there exists a discontinuity in the quality choices (and, consequently, a discontinuity in the other equilibrium variables) when market coverage changes from partial market coverage to corner full market coverage. This result highlights the importance of endogenizing the market structure and market coverage strategic choice, as comparative statics considering just a particular market configuration would lead to wrong results.

There are many examples of industries where our results may be relevant because consumers consider both quality and price in their decision and marginal production costs are increasing with quality. For instance, in the automobile industry, firms must make quality and price decisions, competing on factors like safety features or oil efficiency and also struggling to make the best price choices. Another good example may be the airline industry, where companies must find a balance between quality features such as in-flight services (entertainment, food options, and seat comfort) and reliability, on the one hand, and competition on ticket fares on the other. In the telecommunications industry, oligopolistic players compete on network quality, coverage, and speed, and adjust pricing plans to retain customers and attract new ones. In the oligopolistic mass media and entertainment industry, high-quality content tends to attract more subscribers, and firms adjust subscription fees and advertising rates to maximize profits.

Our paper has important implications in terms of firms' market positioning analysis and pricing decisions in such industries. Namely, our results tell us that monopoly is not a likely outcome in industries where quality matters for consumers' decisions and implies increasing marginal production costs. Oligopoly, either with partial or full coverage, is the equilibrium market configuration. Thus, firms that happen to be monopolists in such markets should count on being threatened and facing competition.

In our model, as the interval of feasible qualities is not restricted, maximal differentiation is never an equilibrium, contradicting the conventional wisdom according to which, by softening price competition, the strongest product differentiation should be chased by firms. Thus, firms should be careful in differentiating too much, and in particular, the high-quality firm should commit not to chose a too high-quality. There are some real-world examples where a nonequilibrium behavior seems to have happened and high-quality competitors had to leave the market or at least faced a market share decline due to having a too high-quality. Leica, a brand known for its high-quality cameras and lenses, is an example of this. With production costs significantly higher than the competitors, due to a specialized workforce, the brand experienced a significant market share decline. The bankruptcy of the retail chain C. Wonder is attributed to mounting operating costs associated with differentiating high quality. These real-world examples show the importance of firms thinking strategically and correctly evaluating the consequences of their quality-price choices.

Furthermore, our paper also provides important insights regarding the implications of changes in consumers' preferences, driven either by public policies or by public or private information campaigns.[3] In particular, the result that full coverage arises under low relative tastes' heterogeneity tells us that equity policies aimed at approaching income levels (that drive consumers' preferences) and information campaigns aimed at putting in evidence the advantages of higher safety patterns or greener options may lead to firms' decisions that allow the entire demand to acquire the product. Quality certificates, such as energy certificates in the housing market, can also be incentives for consumers to opt, in this example, for more energy efficient houses and thus shrink the preferences' interval. From a business perspective, market segmentation and targeting, based for example on demographics or recorded buying behavior, together with taste events or recommendation algorithms may change consumer's heterogeneity and thus influence the equilibrium market configuration.[4]

At a game theoretical level, our paper highlights the importance of considering all possible subgames and equilibrium candidates, when one derives the SPNE. Ignoring some of the subgames may lead us to conclude that the players will choose an action in the first stage of the game that in reality would not be chosen if we had considered all the possible subgames. Thus, the analysis of the whole game is essential to correctly derive the SPNE. A common approach has been to assume that a certain scenario holds, derive the SPNE restringing to that scenario, and, in the end, verify if the obtained solution satisfies the conditions for that scenario to hold, thus checking if the solution is «internally consistent ». However, our paper illustrates that an internally consistent solution may not be a SPNE, showing that it is wrong to analyze dynamic games assuming that a particular subset of all the possible subgames holds. In addition, it calls attention to the importance of not overlooking points where the profit functions are nondifferentiable, as those points are valid equilibrium candidates. These concerns have been ignored by the vast majority of VPD models, but we believe that this problem is more general and extends to other dynamic models.

The article is organized as follows. In the next section, we describe the model. In Section 3, we find internally consistent SPNE candidates, assuming an interior solution for each of the relevant market configurations. This section corresponds to the approach that is most commonly used, and hence, it is comparable with existing results for a given market configuration. In Section 4, we show that there are SPNE candidates in the frontier that separates two market regions. Section 4 identifies the subgame perfect Nash equilibria of our two-stage complete information game. Finally, Section 6 summarizes the conclusions. Appendix A contains all proofs.

2 | THE SETUP AND SPNE

2.1 | The model

We consider a standard VPD model (Gabszewicz & Thisse, 1979; Mussa & Rosen, 1978; Shaked & Sutton, 1982; Tirole, 1988) where there are two firms, indexed by $i=1,2$. Each consumer either buys one unit of the product or none, and if he buys a product from firm i, his net utility (or surplus) is given by the following:

$$U_i(\theta) = \theta k_i - p_i,$$

where k_i represents the quality of the product sold by firm i and p_i is the corresponding price. If the consumer does not buy, his utility is nil.

The parameter $\theta \geq 0$ is a taste parameter that reflects how much the consumer values quality-consumer quality valuation. This parameter is uniformly distributed across $[\underline{\theta}, \underline{\theta}+h]$, with density $f(\theta)=\frac{1}{h}$. The parameter $\underline{\theta}$ is the lowest quality valuation, and $h>0$ is the quality valuation heterogeneity. For fixed $\underline{\theta}$, a higher heterogeneity (higher h), implies a higher average quality valuation and an increase in consumers' valuation dispersion.

Total production costs for a given quality k_i are given by the following:

$$C(q_i)=ck_i^2 \cdot q_i$$

where q_i is the quantity produced and c is the marginal cost coefficient. In other words, firms face constant marginal production costs that increase with quality. This implies that the maximal surplus that a firm can offer to consumers, which holds when price is equal to marginal cost, $\bar{U}_i(\theta)=\theta k_i - ck_i^2$, is a concave function of quality (hence, firms are in a more symmetric position than when costs are nil, where the maximal surplus increases linearly with quality) and is nil for $k_i=0$ and for $k_i=\frac{\theta}{c}$ (hence, no firm will ever choose $k_i \geq \frac{\underline{\theta}+h}{c}$).

We analyze a two-stage complete information game where, in the first-stage, the duopolists simultaneously choose the quality of the product and, in the second-stage, they simultaneously choose prices. Without loss of generality, we assume that $k_2>k_1$, except when checking that a SPNE candidate survives large quality deviations and when we represent the SPNE to emphasize there are similar SPNE with $k_1>k_2$, where the roles of the firms are reversed. Hence, hereafter firm 2's product is high-quality, whereas firm 1's product is low-quality. We assume additionally that, in the first-stage of the game, there is an investment cost in quality given by the following:

$$I(k_i)=\begin{cases} 0 & \text{if } k_i=0 \\ F_i & \text{if } k_i>0 \end{cases},$$

where F_i is an infinitesimally small positive constant.

2.2 | Approach to find SPNE

To find the SPNE, we have to solve first the second-stage of the game (price competition) for all (k_1,k_2), determining the equilibrium prices and profits as a function of (k_1,k_2). Considering these, we then solve the first-stage of the game and determine the equilibrium vector of qualities. Conceptually, this is a standard procedure, but the analysis is not a trivial one, as we allow any quality to be chosen, leading to many different possible market configurations some of which with complex analytical solutions. In addition, profit functions are nondifferentiable in the frontier between different market regions, which further increases the complexity of the problem.[5]

The second-stage of the game has been solved by Pires et al. (2022) considering all possible market configurations and determining the set of quality vectors where each market configuration region holds. The paper shows that besides the classical cases of high-quality monopoly, partial coverage duopoly, and full coverage duopoly, there are regions where low-quality monopolies (constrained or unconstrained and with partial or full coverage) hold. Moreover, corner full coverage duopoly equilibria also exist. In these equilibria, the low-quality firm offers nil surplus to the lowest valuation consumer; thus, the constraint for full coverage is binding. The possible market configurations depend on how the lowest quality valuation $\underline{\theta}$ is related to h. For instance, for $\underline{\theta}$ low, partial coverage configurations are dominant and a high-quality monopoly can never hold, while for high $\underline{\theta}$, a partial coverage duopoly cannot hold whereas a high-quality full coverage monopoly is possible.

As the results of the second-stage game are known, the main emphasis in this paper is in solving the first-stage game. However, for completeness, in the proofs of our results, we present the Nash equilibrium in the second-stage game as a starting point to derive the equilibrium quality choices in the first-stage game. Moreover, the way we present our interior SPNE candidates follows the market configurations found in Pires et al. (2022).

We derive our SPNE quality results in three steps. Considering that the type of price equilibrium depends on (k_1,k_2), we first derive the equilibrium candidates in each price equilibrium region, by restricting the analysis to the set of qualities where that type of price equilibrium holds and by considering interior candidates. However, the SPNE may not occur in the interior of any of the market configurations regions but rather in the frontier that separates two market configuration regions. Thus, in the second step, we verify if that happens but still using local optimality arguments. Some of the equilibrium candidates found in the two first steps may not be Nash equilibria of the first-stage game, as there may exist profitable large deviations. Hence, in the third step of our analysis, we check if none of the firms gains by doing large deviations in its quality, including deviations where the low-quality firm leapfrogs the high-quality firm and becomes the high-quality firm or vice-versa and deviations to quality levels such that other type of price equilibrium holds. The candidates that survive these large deviations tests are SPNE.

3 | INTERIOR INTERNALLY CONSISTENT SPNE CANDIDATES

In this section, we compute the SPNE candidates in the interior of each market configuration region. That is, we derive candidates that can be found by solving the system of first-order conditions of the first-stage game (the quality choices game), which is the type of candidate that is normally assumed to hold. In doing so, we determine the parameter values such that the proposed equilibrium is internally consistent; that is, the SPNE candidate satisfies the conditions for that type of market configuration to hold.

We would like to emphasize that interior SPNE refers to the quality-choice game. This should not be confused with the interior and corner solutions in the duopoly full coverage case, which are related with the Nash equilibrium of the price game. For instance, we may have a SPNE in the interior of the (k_1, k_2) region where a corner full coverage duopoly holds in the second-stage of the game.

We do not consider the case of $k_1 = k_2$ as that would lead to homogenous product Bertrand competition, which implies nil operating profits and hence, considering the investment cost, implies negative profits. Thus, firms choosing the same positive quality can never be a SPNE. In addition, we do not explore the SPNE candidates when firms are restricted to choose qualities that imply a low-quality monopoly in the price game because this type of candidate can be immediately ruled out if we allow firms to choose a nil quality. For a low-quality monopoly to be a Nash equilibrium of the second-stage game, the high-quality firm would have to choose such a high quality that it would be very costly to produce the good. Hence, it would offer very low surplus to consumers, which would explain why the low-quality firm would be able to behave as a monopolist. But if the low-quality firm is a monopolist, then, to avoid the investment costs, the high-quality firm, in the quality choice stage, has an incentive to deviate from such high quality and choose $k_2 = 0$.

3.1 | High-quality firm monopoly

The SPNE candidate, when we assume that a high-quality monopoly holds in the second-stage of the game, is the following:[6]

Proposition 1. If firms are restricted to choose qualities such that, in the second-stage game, firm 2 can behave as a high-quality constrained monopolist covering the whole market; the SPNE candidate qualities are as follows:

$$k_1^{**} = 0 \text{ and } k_2^{**} = \frac{\underline{\theta}}{2c}$$

and the corresponding equilibrium prices and profits are as follows:

$$p_1^{**} = 0, p_2^{**} = \frac{\underline{\theta}^2}{2c}, \Pi_1^{**} = 0 \text{ and } \Pi_2^{**} = \frac{\underline{\theta}^2}{4c} - F.$$

This SPNE candidate holds for $\underline{\theta} > 2h$.

Proof. See Appendix A. ■

The high-quality monopoly SPNE candidate leads to a quality choice that only depends on the lowest quality valuation, $\underline{\theta}$, and the marginal cost coefficient, c. The reason is that the maximum profit of the monopolist is reached when it extracts all the surplus from the lowest valuation consumer, and thus, the full coverage constraint is binding. Considering this, the optimal quality is the one that is «ideal » for the lowest valuation consumer.[7]

It should be mentioned that the higher is the quality valuation heterogeneity (the higher is h), the higher has to be the lowest quality valuation for this candidate to be possible.

3.2 | Interior full coverage duopoly

The SPNE candidate under the assumption that an interior full coverage duopoly occurs in the second stage of the game is given by the following:

Proposition 2. If firms are restricted to choose qualities such that an interior full coverage duopoly equilibrium occurs in the second stage of the game, the SPNE candidate qualities are as follows:

$$k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c} \text{ and } k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$$

and the corresponding equilibrium prices and profits are as follows:

$$p_1^{**} = \frac{\underline{\theta}(2\underline{\theta} - h)}{8c} + \frac{25h^2}{64c}, p_2^{**} = \frac{\underline{\theta}(2\underline{\theta} + 5h)}{8c} + \frac{49h^2}{64c}$$
$$\text{and } \Pi_1^{**} = \Pi_2^{**} = \frac{3h^2}{16c} - F$$

This SPNE candidate holds for $\underline{\theta} \geq 1.25h$.

Proof. See Appendix A. ■

This result extends Proposition 1 in Lambertini (1996), as we consider any h (Lambertini assumes $h = 1$) and we provide the range of parameter where this SPNE candidate holds.

In the interior full coverage duopoly SPNE candidate, we have $k_2^{**} - k_1^{**} = \frac{3}{4}\frac{h}{c}$. Thus, the quality differentiation is increasing with the quality valuation heterogeneity, h, and decreasing with the marginal cost coefficient, c. Note that this quality differentiation does not depend on $\underline{\theta}$ and that both firms have precisely the same demand (each one covers half of the market) and get the same profit. That is, firms differentiate their products to soften price competition, and in the second-stage of the game, prices are such that the two firms have the same margin. The price of firm 2 is increasing with $\underline{\theta}$ and h and decreasing with c. On the other hand, the price of firm 1 may not be increasing with h as the quality chosen by this firm is decreasing with the quality valuation heterogeneity. However, as long as $\frac{\underline{\theta}}{h} < 400$, the low-quality price is also increasing with h.

3.3 | Partial coverage duopoly

Motta (1993) derived numerically the SPNE candidate under the assumption of partial market coverage for specific parameter values

(Proposition 1').[8] We extend his result to any $\underline{\theta}$ and h and derive an analytical solution and the interval where this candidate exists. The SPNE candidate under partial coverage is given by he following:

Proposition 3. If firms are restricted to choose qualities such that a partial coverage duopoly equilibrium occurs in the second-stage of the game, the SPNE candidate qualities are as follows:

$$k_1^{**} = 0.199361\frac{\underline{\theta}+h}{c} \text{ and } k_2^{**} = 0.40976\frac{\underline{\theta}+h}{c}$$

and the corresponding equilibrium prices and profits are as follows:

$$p_1^{**} = \frac{0.087848}{c}(\underline{\theta}+h)^2 \text{ and } p_2^{**} = \frac{0.22666}{c}(\underline{\theta}+h)^2$$
$$\Pi_1^{**} = \frac{0.0121}{ch}(\underline{\theta}+h)^3 - F \text{ and } \Pi_2^{**} = \frac{0.0164}{ch}(\underline{\theta}+h)^3 - F$$

This SPNE candidate holds for $\underline{\theta} < 0.60321h$.

Proof. See Appendix A. ■

In the partial coverage SPNE candidate, the quality differentiation, $k_2^{**} - k_1^{**}$, is equal to $\frac{0.21040}{c}(\underline{\theta}+h)$. Hence, quality differentiation is increasing with the lowest quality valuation, $\underline{\theta}$, and with the quality valuation heterogeneity, h, but decreasing with the marginal cost coefficient, c. Higher quality differentiation implies softer price competition, which explains why prices and profits are increasing in $\underline{\theta}$ and h.

Note also that increasing h implies a larger set of the lowest quality valuations for which the partial coverage duopoly SPNE candidate holds, as the region of validity of the equilibrium candidate is $\underline{\theta} \in [0, 0.60321h)$. This is a consequence of the increase in quality differentiation and of the associated price increase, which implies that more consumers will opt for not buying the good.

3.4 | Corner full coverage duopoly

In the price game, the partial coverage and the interior full coverage solutions are mutually exclusive, but for certain quality vectors, none of them holds. Instead, in the second-stage game, the equilibrium involves a corner solution where the low-quality firm full coverage constraint is binding (Pires et al., 2022). The interior SPNE candidate under corner full coverage is given by the following:

Proposition 4. If firms are restricted to choose qualities such that a corner full coverage duopoly equilibrium occurs in the second-stage of the game, the SPNE candidate qualities, (k_1^{**}, k_2^{**}), are a solution of the following system:[9]

$$\begin{cases} \left((\underline{\theta}-2ck_1)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)-h\left(\underline{\theta}k_1-ck_1^2\right)\right)(k_2-k_1)+ \\ \quad \left(\underline{\theta}k_1-ck_1^2\right)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)=0 \\ \frac{1}{6c}\left(\underline{\theta}+h+4ck_1+\sqrt{(\underline{\theta}+h+4ck_1)^2-12c(2\underline{\theta}+h)k_1}\right)=k_2 \end{cases}$$

and the corresponding prices and profits are obtained by replacing (k_1^{**}, k_2^{**}) in the second-stage equilibrium prices and profits. This candidate is valid for $0.5h \le \underline{\theta} \le 0.625h$.

Proof. See Appendix A. ■

To give an idea how this SPNE corner solution candidates behave, we exemplify this solution for $c = 0.5$ and $h = 1$ in Table 1. We consider different values of $\underline{\theta}$ (values that later on will be shown to be SPNE).

Note that k_1^{**} is decreasing with $\underline{\theta}$, which seems counterintuitive. However, this happens because we are in a price Nash equilibrium region where the lowest valuation surplus constraint, $p_1 \le \underline{\theta}k_1$, is binding. For a given quality k_1, when $\underline{\theta}$ increases, the lowest valuation consumer is willing to pay more, which allows firm 1 to increase its price. Firm 2 knows that, and therefore, it also increases p_2. Due to this strategic effect, firm 1 would like to increase its price further, but it cannot because it is constrained to offer a nil surplus to the lowest valuation consumer. This explains why, in choosing its quality, firm 1 ends up decreasing it as $\underline{\theta}$ increases (decreasing its quality is another way of extracting surplus from the lowest valuation consumer). On the other hand, the quality of firm 2 is nonmonotonic in $\underline{\theta}$. The direct effect of increasing $\underline{\theta}$ on the quality offered by the firm 2 is positive. However, as firm 1 decreases its quality when $\underline{\theta}$ increases, the strategic effect goes in the opposite direction. The total effect depends on which of these two effects dominates.

4 | FRONTIER SPNE CANDIDATES

From the previous sections, we know that for $0.625h < \underline{\theta} < 1.25h$, there is no interior SPNE candidate. To illustrate what happens in this

TABLE 1 SPNE candidates in the corner full coverage duopoly region for $c = 0.5$ and $h = 1$.

$\underline{\theta}$	k_1^{**}	k_2^{**}	p_1^{**}	p_2^{**}	Π_1^{**}	Π_2^{**}
0.55	0.5130	0.9969	0.2822	0.7645	0.0673	0.1480
0.56	0.5129	0.9962	0.2872	0.7687	0.0679	0.1536
0.57	0.5121	0.9958	0.2919	0.7736	0.0685	0.1595
0.58	0.5106	0.9957	0.2962	0.7791	0.0689	0.1656
0.59	0.5088	0.9959	0.3002	0.7853	0.0693	0.1719
0.60	0.5065	0.9966	0.3039	0.7923	0.0697	0.1784
0.61	0.5041	0.9977	0.3075	0.7999	0.0699	0.1851
0.62	0.5014	0.9991	0.3109	0.8082	0.0702	0.1919

Abbreviation: SPNE, subgame perfect Nash equilibrium.

parameters' region, Figure 1 shows the location of the SPNE candidates in the (k_1,k_2) space for $\underline{\theta}=1$ and $h=1$, considering the market regions found in Pires et al. (2022).[10] What we observe is that none of these candidates (points I and C in the figure) is internally consistent as the interior candidates are in the corner region and the corner candidates are in the interior full coverage region. What this means is that when considering qualities in the interior full coverage duopoly region, firms gain by differentiating their qualities the maximum they can (if they could, they would differentiate so much that they would no longer be in the interior full coverage region). On the other hand, when considering qualities in the corner region, firms gain by differentiating the minimum they can (if they could, they would differentiate so little that they would no longer be in the corner region). This suggests that the SPNE may occur at the frontier between the interior and the corner full coverage duopoly regions.

In fact, the profit functions are nondifferentiable in the frontier separating two different market regions, which implies that the optimal qualities may happen at those frontier points.

Proposition 5. For $0.625h<\underline{\theta}<1.25h$, there are multiple SPNE candidates which are in the frontier between the interior and the corner full coverage solutions. For given c,h, and $\underline{\theta}$, the set of SPNE candidates is the set of (k_1^{**},k_2^{**}) that satisfies the following system:

$$\begin{cases} (i) & \frac{1}{2c}(\underline{\theta}-h)+\frac{1}{2c}\sqrt{(\underline{\theta}-h)^2+4ck_1(2\underline{\theta}+h-2ck_1)}=k_2 \\ (ii) & \left((\underline{\theta}-2ck_1)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)-h\left(\underline{\theta}k_1-ck_1^2\right)\right)(k_2-k_1)+ \\ & \left(\underline{\theta}k_1-ck_1^2\right)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)>0 \\ (iii) & (h-\underline{\theta}+ck_1+ck_2))(\underline{\theta}-h-3ck_1+ck_2)<0 \\ (iv) & (\underline{\theta}+2h-ck_1-ck_2)(\underline{\theta}+2h-3ck_2+ck_1)>0 \\ (v) & \left(ck_2^2-h(k_2-k_1)-\underline{\theta}k_2\right)\left((4ck_2-2h-2\underline{\theta})(k_2-k_1)-\left(ck_2^2-h(k_2-k_1)-\underline{\theta}k_2\right)\right)<0 \end{cases}.$$

and the corresponding equilibrium prices and profits are obtained by replacing (k_1^{**},k_2^{**}) in the second-stage equilibrium prices and profits.

Proof. See Appendix A. ■

This result is based on the idea that if, for both firms, (k_1,k_2) is such that individual profit's left partial derivative is positive and the right partial derivative is negative, then the vector (k_1,k_2) is a SPNE candidate as, given the other firm's quality, each firm is (at least locally) maximizing its profit. So, finding the SPNE candidate is a matter of finding the (k_1,k_2) such that this property holds. What happens is that, for $0.625h<\underline{\theta}<1.25h$, there is not a unique SPNE candidate with $k_2>k_1$. There is a segment in the frontier that separates the interior and the corner full coverage regions such that the left derivatives are positive and the right derivatives are negative (the proof in Appendix A includes an example where these derivatives are calculated and it makes it evident that there are multiple equilibria). The multiple equilibria are shown in Figure 1 by the bold segment.

Table 2 characterizes the set of multiple SPNE candidates for $c=0.5, h=1$, and $F=0.001$. For each $\underline{\theta}$, the table indicates the range of equilibrium values for each variable. Note that in the case of Π_2^{**} we indicate the range from the highest to the lowest value because the profit of firm 2 decreases as one considers equilibria with higher qualities. All the remaining variables equilibrium values increase as one moves to equilibria with higher qualities.

Some interesting features should be highlighted. The first is that the segment in the frontier that corresponds to the multiple equilibria depends on $\underline{\theta}$. As $\underline{\theta}$ approaches $0.625h$ or $1.25h$, the segment in the frontier with multiple equilibria becomes small (in the limit, it converges to the interior corner duopoly SPNE candidate when $\underline{\theta}$ approaches $0.625h$ and to the interior full coverage duopoly candidate when $\underline{\theta}$ approaches $1.25h$). On the contrary, for $\underline{\theta}$ in the middle of interval $(0.625h, 1.25h)$, the set of multiple SPNE is quite large. For instance, for $\underline{\theta}=1$ and $h=1$, the range of possible SPNE qualities for firm 1 is [0.545,1].

It is also interesting to note that the multiple equilibria seem to be influenced by the characteristics of both the corner full coverage duopoly and the interior full coverage duopoly candidates. For instance, the lower limit of the range of possible qualities of firm 1

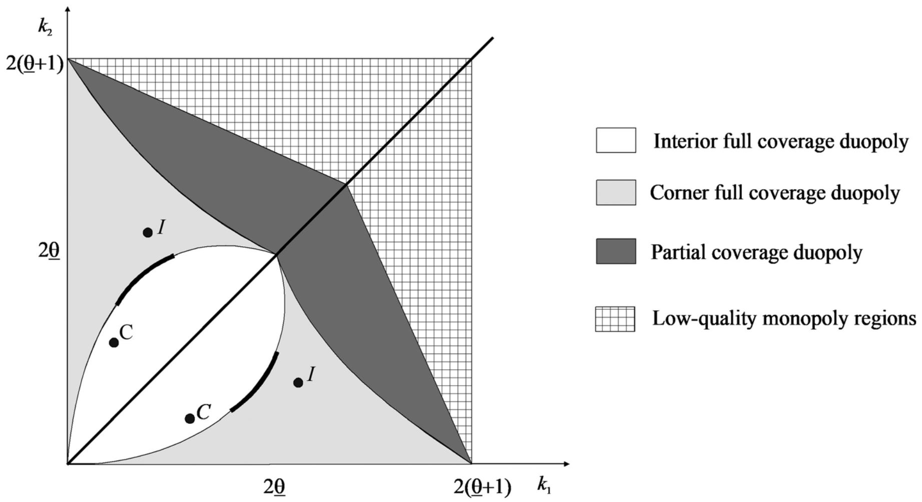


FIGURE 1 Subgame perfect Nash equilibrium (SPNE) in the qualities space, for $c=0.5$, $h=1$, and $\underline{\theta}=1$. The points I and C correspond to the interior and the corner full coverage duopoly SPNE candidates, but they are not internally consistent.

TABLE 2 Multiple frontier SPNE candidates for $c = 0.5, h = 1$, and $F = 0.001$.

$\underline{\theta}$	k_1^{**}	k_2^{**}	p_1^{**}	p_2^{**}	Π_1^{**}	Π_2^{**}
0.63	[0.498,0.510]	[1.005,1,018]	[0.313,0.321]	[0.823,0.833]	[0.070,0.071]	[0.198,0.195]
0.70	[0.473,0.603]	[1.083,1.203]	[0.331,0.422]	[0.977,1.082]	[0.078,0.095]	[0.250,0.214]
0.80	[0.458,0.733]	[1.215,1.468]	[0.366,0.586]	[1.234,1.493]	[0.089,0.137]	[0.323,0.235]
0.90	[0.443,0.865]	[1.348,1.733]	[0.398,0.779]	[1.513,1.964]	[0.099,0.187]	[0.403,0.246]
1.00	[0.545,1.000]	[1.635,2.000]	[0.545,1.000]	[2.031,2.500]	[0.143,0.249]	[0.441,0.249]
1.10	[0.735,1.118]	[2.005,2.260]	[0.808,1.229]	[2.742,3.091]	[0.227,0.319]	[0.421,0.252]
1.20	[0.913,1.245]	[2.340,2.525]	[1.095,1.494]	[3.487,3.749]	[0.322,0.403]	[0.392,0.245]
1.24	[0.983,1.048]	[2.468,2.510]	[1.218,1.299]	[3.794,3.862]	[0.363,0.384]	[0.378,0.346]

Abbreviation: SPNE, subgame perfect Nash equilibrium.

starts by being decreasing with $\underline{\theta}$ (which is a feature that holds in the corner SPNE candidate), but for $\underline{\theta}$ >0.945, it becomes increasing and tends to 1 as $\underline{\theta}$ converges to 1.25.

When $\underline{\theta} < 1$, in all the equilibria, firm 2 has higher prices, higher market shares, and higher profits than firm 1.

For $\underline{\theta} = 1$, the upper limit SPNE candidate is interesting because firms have precisely the same market share and get the same profit. Moreover, firm 1 offers the «ideal» quality of the lowest valuation consumer, whereas firm 2 offers the «ideal » quality of the highest valuation consumer (the «ideal» quality for consumer θ is $k = \frac{\theta}{2c}$ as explained in footnote 7). But, for all the other equilibria, firm 2 has higher prices, higher market shares, and higher profits than firm 1.

For $1 < \underline{\theta} < 1.25$, in all the equilibria with $k_1^{**} > 1$, firm 1 has a higher market share and a higher profit than firm 2. This is very interesting because it means that, for these equilibria, there exists a low-quality firm advantage.

Hence, we provide another instance where the result of a low-quality advantage may hold, besides the case of a low degree of nonlinearity (Schmidt, 2006) or the case of consumer's distribution skewed towards the low-end (Schubert, 2017; Wang, 2003).

5 | DETERMINING THE SPNE

5.1 | Checking the inexistence of profitable large deviations

In the previous sections, we determined the internally consistent SPNE candidates in each market region as well as the frontier candidates, assuming $k_2 > k_1$. However, these candidates may not be SPNE if we take into account that each firm can deviate to quality levels such that $k_1 > k_2$ or where other market configurations hold. Thus, for each equilibrium candidate (k_1^{**}, k_2^{**}) found in the previous sections, we need to check if considering k_j^{**} firm i gains by deviating to a quality level such that the quality ranking of the two firms changes and/or to a quality level where another equilibrium market configuration holds. Only the candidates that survive this analysis are SPNE.

Although there are internally consistent SPNE candidates where a monopoly holds, we now show that, under our assumptions regarding the investment cost F, the monopoly candidates are not SPNE:

Lemma 1. Assuming that F is positive but infinitesimally small, there does not exist a SPNE where the high-quality firm is a monopolist.

Proof. See Appendix A. ■

It should be highlighted that, with higher investment costs, the high-quality monopoly may become a SPNE. In fact, if $F > \frac{32}{243c}h^2$, there is no longer a profitable deviation by the low-quality firm, and consequently, a high-quality monopoly would be a SPNE (see proof in Appendix A). Moreover, the smaller is h and the higher is c, the easier is condition $F > \frac{32}{243c}h^2$ to be satisfied, and thus, the easier it is for a monopoly SPNE to exist.

From Section 3, we know that for $0.5h \le \underline{\theta} \le 0.60321h$, there are two candidates to be SPNE: one is the partial coverage candidate and the other one is the corner full coverage candidate. Which of these candidates is SPNE? Numerically, we can show the following:

Lemma 2. For $0.5h \le \underline{\theta} \le 0.545h$, the unique SPNE with $k_2 > k_1$ is the partial coverage candidate. For $0.545h < \underline{\theta} \le 0.60321h$, the unique SPNE with $k_2 > k_1$ is the corner solution candidate.

Proof. See Appendix A. ■

Considering the two previous lemmas and $k_2 > k_1$, for any $\underline{\theta} \le 0.625$ or $\underline{\theta} \ge 1.25$, there is now a unique SPNE candidate, whereas for $0.625 < \underline{\theta} < 1.25$, there are multiple SPNE candidates. To make sure that each candidate is indeed a SPNE, we still need to check that firms do not increase their profits by undertaking large quality deviations. In particular, we need to check that firm 1 does not gain by leapfrogging firm 2 and that firm 2 does not gain by choosing a lower quality than firm 1, assuming the same market configuration. In addition, we need to check that none of the firms wants to deviate to quality levels such that another price market configuration holds in the second-stage of the game. We exemplify this type of tests for the case of the interior full coverage duopoly candidate. We show first the inexistence of profitable large deviations within the interior full coverage duopoly region.

Lemma 3. If $\underline{\theta} \geq 1.25h$, none of the firms can profitably deviate from the SPNE candidate $k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c}$ and $k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$ to qualities where $k_1 > k_2$, and an interior full coverage duopoly holds.

Proof. See Appendix A. ■

The proof shows that, although it is possible for firm 1 to leapfrog firm 2 or to firm 2 to choose $k_2 < k_1^{**}$, these deviations are not profitable (see the details in Appendix A).

Next, we show that there are no profitable deviations from the interior full coverage duopoly to quality levels where a high-quality monopoly holds:

Lemma 4. If $\underline{\theta} \geq 1.25h$, there is no profitable deviation from the interior full coverage duopoly SPNE candidate to a high-quality monopoly region.

Proof. See Appendix A. ■

The proof shows that, considering the quality of firm 2, firm 1 cannot deviate and become a high-quality monopoly (the deviation is impossible), and firm 2 cannot gain by deviating to a quality such that a high-quality monopoly occurs in the second-stage game as that would imply that the other firm would be the one that becomes monopolist. Similar arguments also hold if we consider deviations to other types of monopoly: either a deviation is impossible or when it is possible, it is the other firm who becomes a monopolist. This is illustrated in Figure 2. Points *I* are the internally consistent full coverage duopoly SPNE candidates. For the case where firm 1 is the low-quality firm (above the diagonal), we represented the possible unilateral quality deviations. We see that firm 1 cannot deviate to a high-quality monopoly region, but it can deviate to a low-quality monopoly region (e.g., to point *A*). However, point *A* is in a region where firm 2 is the monopolist. Thus, a deviation to *A* is not profitable for firm 1. On the other hand, firm 2 can deviate to any of the monopoly regions (e.g., to points *B* and *C*). However, in both cases, firm 2 would be deviating to a region where firm 1 is the monopolist!

Similarly, we can show that, for $\underline{\theta} \geq 1.25h$, there are no profitable deviations from the SPNE candidate $k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c}$ and $k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$ to quality levels where either the partial coverage duopoly or the corner solution holds, which completes the proof that for $\underline{\theta} \geq 1.25h$, this candidate is a SPNE.

The remaining possible large deviations were checked numerically which, together with the previous results, allows us to conclude the following[11]:

Proposition 6. Assuming F_i are positive but infinitesimally small and $k_2 > k_1$, the SPNE of the quality-price game are as follows:

1. When $0 \leq \underline{\theta} \leq 0.545h$ (when $\frac{\underline{\theta}+h}{\underline{\theta}} \geq 2.835$), there is a unique SPNE in the interior of the partial coverage duopoly region.
2. When $0.545h < \underline{\theta} \leq 0.625h$ (when $2.6 \leq \frac{\underline{\theta}+h}{\underline{\theta}} < 2.835$), there is a unique SPNE in the interior of the full coverage corner duopoly region.
3. When $0.625h < \underline{\theta} < 1.25h$ (when $1.8 < \frac{\underline{\theta}+h}{\underline{\theta}} < 2.6$), there are multiple SPNE located in the frontier between the corner and the interior full coverage regions.
4. Lastly, when $\underline{\theta} \geq 1.25h$ (when $\frac{\underline{\theta}+h}{\underline{\theta}} \leq 1.8$), there is a unique SPNE in the interior full coverage duopoly region.

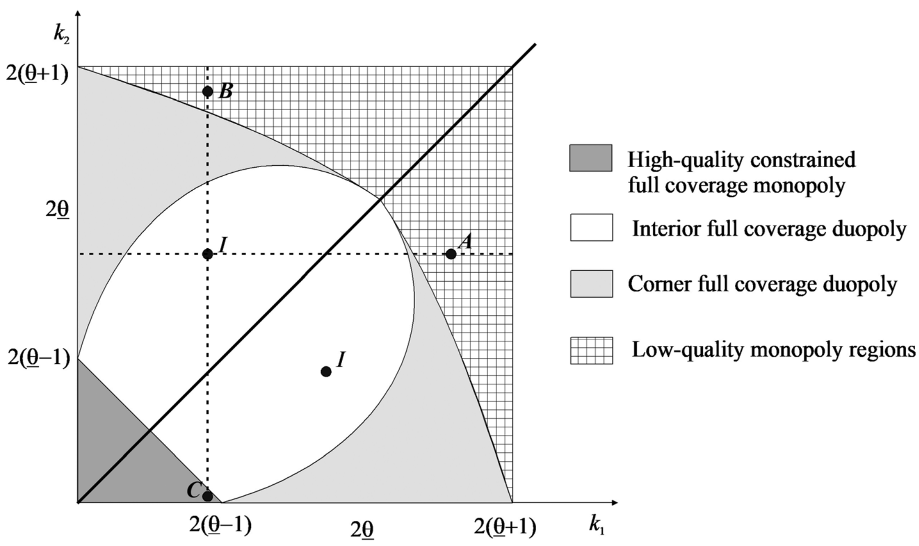
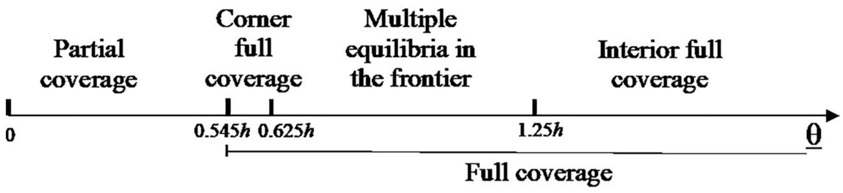


FIGURE 2 For $c = 0.5$, $h = 1$, and $\underline{\theta} = 2$, the points *I* are the internally consistent full coverage duopoly subgame perfect Nash equilibrium (SPNE) candidates. Keeping the quality of the other firm, we can check deviations to qualities on the dashed horizontal line, for firm 1 and on the dashed vertical line for firm 2.

FIGURE 3 Types of subgame perfect Nash equilibrium (SPNE), as a function of $\underline{\theta}$.



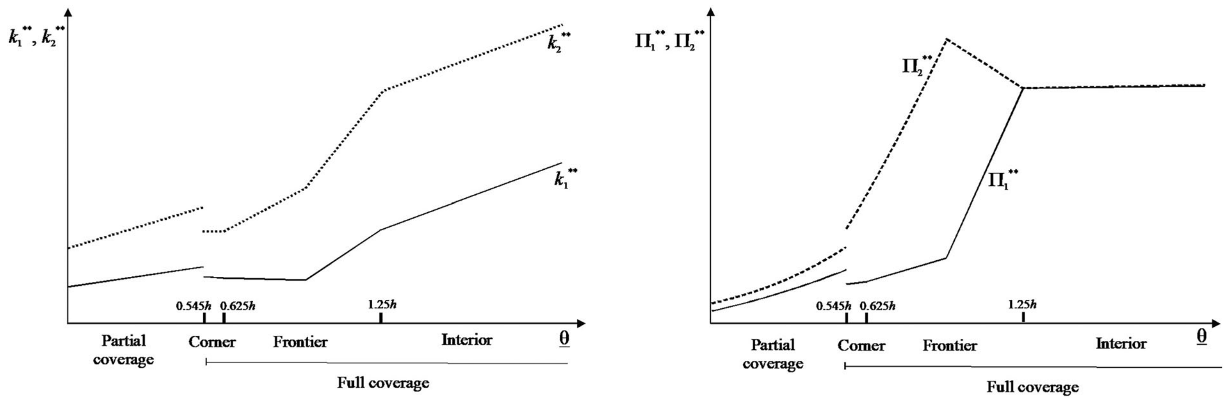


FIGURE 4 Subgame perfect Nash equilibrium (SPNE) qualities and profits as a function of θ, for given c and h.

In all four cases, a similar SPNE exists with $k_1 > k_2$, where the roles of the firms are reversed.

Proof. See Appendix A. ■

Figure 3 illustrates the previous proposition. It is interesting to note that the cutoff values under which each SPNE holds do not depend on the marginal cost coefficient, c. Moreover, the structure of the SPNE regions does not change with h, which has a multiplicative effect. Increasing the quality valuation heterogeneity increases the set of profitable quality choices and increases in a proportional way the size of each SPNE region.

The existence of several SPNE market configurations has some noteworthy implications. As previously mentioned by García-Gallego and Georgantzís (2009), in the case of investment costs increasing with quality, and by Wauthy (1996) under nil costs, in our setup, it also happens that the transition from duopoly partial coverage to corner full coverage implies discontinuous "jumps" in equilibrium qualities, prices, profits, and social welfare. We illustrate this for the case of equilibrium qualities and profits in Figure 4, which shows a discontinuity at $\underline{\theta} = 0.545h$ (the cutoff that separates the partial coverage and the corner duopoly equilibria).

Another implication is that one has to be very careful in doing comparative statics exercises, as a change in parameters may result in a change in the equilibrium market configuration. This is even more pertinent when we move from a partial coverage duopoly SPNE to a corner full coverage SPNE, as we may get counterintuitive comparative statics results due to the discontinuity. For instance, one would expect that, *ceteris paribus*, increasing consumers' heterogeneity, h, increases the equilibrium quality differentiation. However, changing h affects the cutoffs that separate the different market configurations and, because quality differentiation drops when we go from the partial coverage duopoly to corner full coverage, quality differentiation may go down when h increases, which is the opposite of what we would expect.

The last implication is well-illustrated in the equilibrium profits shown in Figure 4. We would like to call attention to the fact that, for $0.625h < \underline{\theta} < 1.25h$, we are in the frontier region where there are multiple equilibria. To ensure the legibility of the figure, we only represented the equilibrium qualities and profits corresponding to the lowest values for k_1^{**} and k_2^{**}. This figure shows that in the interior full coverage duopoly region, the high-quality and the low-quality firms have equal profits, a result that coincides with Schmidt (2006) (our setup is equivalent to $\delta = 2$ in Section 3 of Schmidt's paper). However, under partial coverage duopoly or in the corner region, one obtains the more usual result of a high-quality advantage. This shows that Schmidt (2006) results cannot be generalized to other market configurations and that a low-quality advantage may be harder to hold under partial coverage duopoly or in the corner full coverage duopoly. The frontier region is also interesting in this regard, as for $h < \underline{\theta} < 1.25h$, there are equilibria with high-quality advantage (as the ones shown in Figure 4), but as shown in Table 2, there are also equilibria where $\Pi_1^{**} > \Pi_2^{**}$ and thus a low-quality advantage holds.

5.2 | Discussion

In this section, we discuss and compare our results with previous results in the VPD literature that endogenizes market coverage.

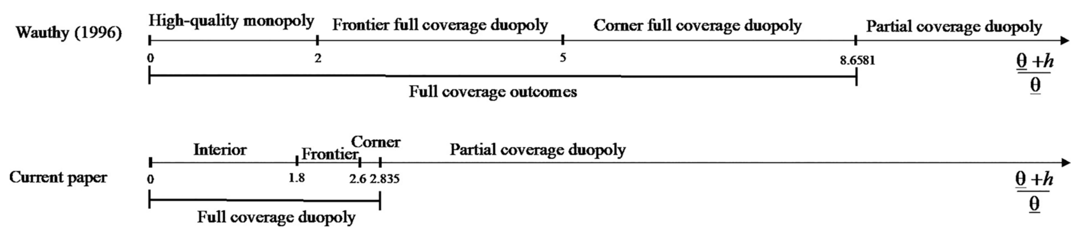


FIGURE 5 Comparing the equilibrium market configurations in the current paper with Wauthy (1996).

First, it should be highlighted that the existence of a strictly interior full coverage is a novel result in the endogenous market coverage literature. The inexistence of a strictly interior full coverage duopoly is clear in page 350 of Wauthy (1996) where it is explained that, under a covered market, the low-quality firm chooses the lowest quality possible, subject to the restriction that the market is covered. A similar result holds in Chambers et al. (2006) (see Theorem 3 and the discussion that follows it). Finally, the result is obvious in Proposition 1 of Liao (2008). Why is that, in our setup, there is a region where the SPNE is a strictly interior full coverage duopoly? Note that, in this region, the principle of maximal differentiation does not hold and all consumers get a strictly positive surplus. Firms are competing to attract consumers, which depends on the surplus they are able to offer them. But with marginal costs that depend quadratically on quality, the surplus that can be profitably offered is a concave function of quality, which means that, to offer a competitive surplus, a firm cannot offer too low or too high levels of quality. This fact limits the advantages of firms differentiating too much their qualities. Thus, we obtain lower differentiation in equilibrium, which also implies more competition and a positive surplus for all consumers. Another way of interpreting this result is by looking at the direct and strategic effects when choosing quality. For instance, if quality is costless, the high-quality always gains by increasing its quality, as it increases its demand and softens price competition. But under our assumptions, increasing quality also means higher marginal costs, which has a direct negative impact on the firms profit that may overwhelm the direct demand effect and the strategic effect, which are positive. So, it is not surprising that there are regions of parameters where maximal differentiation is not optimal.

Second, as the constraint that the market is covered is binding, the interval identified by Wauthy as corresponding to interior full coverage duopoly, in reality, corresponds to frontier SPNE! However, in Wauthy (1996), for given parameter values, there is a unique frontier SPNE, whereas in our case, there are multiple SPNE. In Wauthy (1996), the best response of the high-quality firm is always to choose the highest feasible quality. Considering this, the low-quality firm best response is to differentiate its quality the maximum possible, by decreasing it but subject to the constraint of being in the interior full coverage region, and there is a unique solution to this problem. In our setup, the surplus that a firm can profitably offer to consumers is a concave function of quality, and hence, choosing the highest feasible quality is no longer a best response for the high-quality firm. In the frontier SPNE, both firms gain by increasing differentiation subject to the constraint that they are in the interior full coverage region. But there are several points in the frontier where this occurs!

In Proposition 6, the intervals defined in parentheses allow a direct comparison of our results with Wauthy (1996) and Liao (2008).[12] Figure 5 illustrates the comparison with Wauthy (1996).

Another interesting result in our setup is that, for infinitesimal investment cost, a monopoly never arises in equilibrium, whereas in Wauthy (1996), a high-quality monopoly holds for low relative heterogeneity. How can we explain this difference in the results? With nil costs, when heterogeneity in consumers' preferences is low, the low-quality firm has to differentiate a lot to be able to attract the lower valuation consumers. As the level of heterogeneity decreases, the low-quality firm is able to attract less and less consumers, till we reach a point where all consumers prefer the high-quality good. This does not happen in our setup because if we were in such a case, the low-quality firm would gain by leapfrogging the high-quality firm, by choosing a quality above the high-quality one, a deviation which is not possible under nil costs as the high-quality firm always chooses the highest feasible quality.

However, our results and Wauthy (1996) also have some qualitative similarities. In particular, partial coverage duopoly occurs for higher values of relative heterogeneity while full coverage outcomes happen for lower relative heterogeneity. But partial coverage duopoly holds for a larger set of heterogeneity values in our model. This implies that concerns with the market being uncovered are more relevant when marginal costs vary quadratically with quality. One reason for having high heterogeneity of quality valuations is the existence of a very unequal income distribution.[13] Therefore, policy measures that reduce income inequality are expected to reduce consumers' heterogeneity. This general policy measure may actually be more effective at promoting inclusion than measures specifically directed to the market in question.

6 | CONCLUSION

Research using VPD models has widely ignored the fact that the market configuration that arises is endogenously determined through firms's choices. This paper analyzes this issue in a quality-then-price model with marginal production costs that vary quadratically with quality and uniformly distributed quality valuations.

To solve our model, we first analyzed the internally consistent interior SPNE candidates within each market structure, extending

Lambertini (1996) in the case of the full coverage duopoly, and the numerical result obtained by Motta (1993) in the case of the partial coverage duopoly and finding the SPNE candidate in the region where full coverage duopoly with a corner solution occurs. Thus, our results generalize previous results under quadratic marginal costs, in articles that assume a given market configuration.

The second step in our analysis was to check for the existence of SPNE candidates in the frontier between market regions where the profit functions are nondifferentiable. We show that there is a region of parameters where there are multiple SPNE candidates in the frontier between the corner and the interior full coverage region, which is a novel result in the VPD literature. A closer look at Wauthy (1996) reveals that what he denotes as interior full coverage duopoly are, in reality, frontier SPNE candidates. However, for given parameter values, Wauthy obtains a unique frontier SPNE candidate, while we obtain multiple SPNE candidates.

The final step in our analysis was to check for large deviations when choosing quality, including deviations to quality levels where other market structures arise. We found cases where the internally consistent SPNE candidate assuming a particular structure is not SPNE. For instance, in the high-quality monopoly SPNE candidate, the low-quality firm has incentive to deviate to a quality where an interior full coverage duopoly holds (the most profitable deviation is to leapfrog the high-quality firm). Thus, a high-quality monopoly is not a SPNE in our setup, while it is an equilibrium Wauthy (1996), because with nil costs leapfrogging, the high-quality firm is not possible as this firm chooses the highest feasible quality. This shows that verifying internal consistency is not enough to prove that we found an equilibrium solution.

Our results show that, in the SPNE, a monopoly never arises in equilibrium and that market coverage depends on how large is the lowest quality valuation with respect to the quality valuation heterogeneity. When the lowest quality valuation is low (with respect to the quality valuation heterogeneity), the SPNE quality choices lead to a partial coverage duopoly. For slightly higher values of the lowest quality valuation, a full coverage duopoly with a corner solution occurs, in which the low-quality firm is constrained to offer a nil surplus to the lowest valuation consumer. For even higher lowest quality valuation, there are multiple SPNE quality vectors, which are in the frontier that separates the corner and the interior full coverage regions. Finally, for high values of the lowest quality valuation, the equilibrium qualities are in the interior of the region where a strictly interior full coverage duopoly occurs. The last result is novel in the literature that studies endogenous market configurations. Considering that this is one of the market configurations more frequently assumed, this result is very relevant as it shows that this outcome, which has been frequently assumed *a priori*, is expected to be observed in equilibrium, for low levels of relative tastes' heterogeneity, if we assume marginal costs that vary quadratically with quality.

While our paper has some novel results, it also confirms, in qualitative terms, some of the previous results in the endogenous VPD literature. In particular, we obtain full coverage outcomes for lower levels of relative heterogeneity (measured by the ratio between the highest and the lowest quality valuation) and partial coverage duopoly for higher levels of heterogeneity. However, partial coverage results are more likely under our setup of convex marginal costs than under nil costs.

Therefore, this paper substantiates Wauthy (1996), since we confirm that in vertical product differentiated models one should not *a priori* impose any market coverage or any market configuration.

We show that Schmidt (2006) result of equal high-quality and low-quality profits under a quadratic marginal cost function only holds under interior full coverage equilibria. With partial coverage or in the corner equilibria, quadratic marginal costs lead to a high-quality advantage. Moreover, in the multiple frontier equilibria, there exist equilibria where the high-quality firm has higher profits, but there are also equilibria where the reverse happens and a low-quality advantage arises. This extends previous results of low-quality advantage obtained under full coverage, for low convexity of the marginal cost function (Schmidt, 2006) or under consumers' tastes distribution skewed towards lower quality valuations (Schubert, 2017). But, more important, our results show that whether there is high-quality or low-quality advantage is largely dependent on the equilibrium market configuration, a result that has been hinted by Wang (2003), further highlighting the relevance of considering endogenous market configurations.

Our paper also shows that comparative statics focusing on a single market configuration may lead to wrong conclusions, because parameter changes may imply changes in the type of equilibrium that will occur and that may change the comparative statics. For instance, one cannot conclude that, for a given lowest quality valuation, increasing the heterogeneity in the quality valuation leads to higher equilibrium quality differentiation. This result challenges possible previous presumptions and clearly emphasizes the importance of knowing when each type of equilibrium holds.

Finally, our results have interesting managerial implications. In particular, our model highlights that too much quality differentiation may harm firms' profitability and that the equilibrium product differentiation degree depends on the market configuration. In addition, our results suggest that the pursuit of monopolization strategies is likely to be unsuccessful as monopoly cannot be sustained in equilibrium.

Although our results were derived assuming that willingness to pay increases linearly with quality and that marginal costs vary quadratically with quality, we conjecture that similar qualitative results hold as long as the surplus under marginal cost pricing is a strictly concave function of quality, with nil surplus holding for nil quality as well as for a sufficiently high quality. Under these conditions, being the high-quality firm is not equivalent to being able to offer a higher net utility to the consumer and choosing a too high or too low quality is not profitable, limiting the level of quality differentiation that can profitably be sustained. A strictly concave surplus function can be obtained with linear willingness to pay and strictly convex marginal costs, as in our model. But a strictly concave consumer surplus, under marginal cost pricing, can arise under other assumptions, such as a willingness to pay that is a strictly concave function of quality and marginal costs depending linearly on quality.

Future avenues for research should include sequential quality-pricing decisions, exploring first mover advantage analysis and long-run entry strategies. Another interesting extension would be to consider n oligopolist firms.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing does not apply to this article as no new data were created or analyzed in this study.

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ENDNOTES

[1] Note that the market coverage assumption may be implicit in other assumptions. If one assumes that consumers' tastes are distributed on the interval $[0, \overline{\theta}]$, where $\overline{\theta}$ is the highest quality valuation, the market is implicitly uncovered because the lowest valuation consumer is not willing to pay anything for quality. This assumption has been used in many papers, including Aoki and Prusa (1996), Lehman-Grube (1997), Lambertini and Tedeschi (2007), and Niem (2019).

[2] Table on page 56 of Jorge et al. (2022) provides a good overview of the main assumptions used in previous theoretical VPD models.

[3] Changes in consumers' preferences, driven either by public or private decision-makers have been addressed, for instance, in García-Gallego and Georgantzís (2009) in a vertical differentiation model in the context of products sold by socially responsible manufacturers, taking into consideration information campaigns aimed at increasing ecological consciousness. As another example, the work of Tsagarakis and Georgantzis (2002) provides evidence of the possible effectiveness of information sessions to increase willingness to pay and reduce the heterogeneity of preferences for ecological products, in an application to the use of recycled water for irrigation.

[4] In modern societies social influencers also shape tastes and may moderate preferences heterogeneity.

[5] Wauthy (1996) and Chambers et al. (2006) analyses were largely simplified by the fact that the high-quality firm always chooses the highest feasible quality, which implies that only the low-quality firm best response needs to be determined. However, this is not true under our assumptions, where choosing the highest feasible quality is no longer the best response for the high-quality firm, substantially increasing the complexity of finding the SPNE.

[6] If the constraint that requires the monopolist to offer to consumers at least the same surplus than the one offered by the other firm is binding, the monopolist is a constrained monopolist. Otherwise, it is an unconstrained monopolist.

[7] For consumer with quality valuation θ, the «ideal » quality, considering the marginal costs, is the solution of $\max_k \theta k - ck^2$, hence it is $k = \frac{\theta}{2c}$.

[8] Proposition 4 of (Moorthy, 1988) also specifies the SPNE candidate in the partial coverage duopoly case but the result is not consistent with Motta (1993) and the current paper (it is not stated how the result was obtained).

[9] There exists a closed form solution (available upon request) to this system, however it is a very long expression and does not allow us to interpret it or do comparative statics. Thus, to get a feeling how this solution behaves, we have to resort to the numerical solution.

[10] The figure is adapted from Pires et al. (2022) as we do not distinguish the various types of low-quality monopolies. It is obtained by representing the limits of each market region presented in Pires et al. (2022) propositions. The regions that are relevant depend on $\underline{\theta}$ and h. In the figure we also represent the quality regions with $k_1 > k_2$ (below the diagonal) as they are relevant for checking deviations and, additionally, it emphasizes the existence of SPNE which are symmetric to each other.

[11] We present the intervals where each market configuration arises in two ways, to facilitate comparison with previous literature.

[12] Chambers et al. (2006) considers a fixed distribution of quality valuation, thus it does not provide the intervals where each market configuration holds.

[13] Our model can be shown to be equivalent to a model where all consumers derive the same utility from consuming one unit of a particular good with quality k_j, but have different incomes, which implies that they have different willingness to pay for good j. In such setup θ_i can be reinterpreted as the inverse of the marginal utility of income of consumer i with income y_i, that is, $\theta_i = \frac{1}{u'(y_i)}$. Assuming a decreasing marginal utility of income, richer consumers will have lower marginal utility, hence higher willingness to pay for the good of quality k_j. Thus, the distribution of θ is derived from the income distribution.

[14] This solution is available from the authors, upon request.

[15] When a firm deviates to a quality that is lower than the rival's quality, we denote the deviation by k_i^-. Similarly, if the deviation is to a quality above the rival's quality, we denote the deviation by k_i^+.

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APPENDIX A: PROOFS

Proof of Proposition 1. From Proposition 1 in Pires et al. (2022), we know that if $k_1 < k_2$ and $k_1 + k_2 < \frac{1}{c}(\underline{\theta} - h)$, firm 2 can behave as a high-quality constrained monopolist and covers the whole market. In addition, the equilibrium prices and profits are as follows:

$$p_1^* = ck_1^2 \text{ and } p_2^* = \underline{\theta}(k_2 - k_1) + ck_1^2$$
$$\Pi_1^* = -F \text{ and } \Pi_2^* = (k_2 - k_1)(\underline{\theta} - c(k_2 + k_1)) - F.$$

Thus, the first-order condition for firm 2 in the first-stage of the game is as follows:

$$\frac{\partial \Pi_2}{\partial k_2} = \underline{\theta} - c(k_2 + k_1) - c(k_2 - k_1) = 0 \Leftrightarrow k_2 = \frac{\underline{\theta}}{2c}$$

The second-stage game equilibrium profit of firm 1 is $\Pi_1^* = -F$ for all $k_1 > 0$, but then the best response of firm 1 in the first-stage is $k_1 = 0$ as this implies that $\Pi_1^* = 0$. Therefore, the SPNE equilibrium qualities, if we restrict to quality levels such that firm 2 is a monopolist, are $k_1^{**} = 0$ and $k_2^{**} = \frac{\underline{\theta}}{2c}$. Substituting in the equilibrium prices and profit, we get $p_2^{**} = \frac{\underline{\theta}^2}{2c}$ and $\Pi_2^{**} = \frac{\underline{\theta}^2}{4c} - F$.

This candidate holds for $k_1 + k_2 < \frac{1}{c}(\underline{\theta} - h)$, which implies that

$$\frac{\underline{\theta}}{2c} < \frac{1}{c}(\underline{\theta} - h) \Leftrightarrow \underline{\theta} < 2\underline{\theta} - 2h \Leftrightarrow \underline{\theta} > 2h$$

■

Proof of Proposition 2. From Proposition 4 in Pires et al. (2022), we know that if $k_1 < k_2 \le \frac{1}{c}(\underline{\theta} + h)$, $\frac{k_1}{3}(2\underline{\theta} + h - 2ck_1) + \frac{k_2}{3}(\underline{\theta} - h - ck_2) \ge 0$ and $\frac{1}{c}(\underline{\theta} - h) \le k_2 + k_1 \le \frac{1}{c}(\underline{\theta} + 2h)$, an interior full coverage duopoly holds in the second-stage of the game and the equilibrium prices and profits in the second-stage are as follows:

$$p_1^* = \frac{(h - \underline{\theta})(k_2 - k_1) + 2ck_1^2 + ck_2^2}{3} \text{ and } p_2^* = \frac{(\underline{\theta} + 2h)(k_2 - k_1) + 2ck_2^2 + ck_1^2}{3}$$
$$\Pi_1^* = \frac{1}{9h}(k_2 - k_1)(h - \underline{\theta} + c(k_1 + k_2))^2 - F \text{ and } \Pi_2^* = \frac{1}{9h}(k_2 - k_1)(\underline{\theta} + 2h - c(k_1 + k_2))^2 - F$$

This implies that, in the first-stage of the game, the system of first-order conditions is given by the following:

$$\frac{\partial \Pi_1}{\partial k_1} = \frac{(h - \underline{\theta} + ck_1 + ck_2)(\underline{\theta} - h - 3ck_1 + ck_2)}{9h} = 0$$
$$\frac{\partial \Pi_2}{\partial k_2} = \frac{(\underline{\theta} + 2h - ck_1 - ck_2)(\underline{\theta} + 2h - 3ck_2 + ck_1)}{9h} = 0$$

Note that each first-order condition is a product of the type $\frac{1}{9h} \cdot A_i \cdot B_i = 0$, where $A_1 = h - \underline{\theta} + ck_1 + ck_2$, $B_1 = \underline{\theta} - h - 3ck_1 + ck_2$, $A_2 = \underline{\theta} + 2h - ck_1 - ck_2$ and $B_2 = \underline{\theta} + 2h - 3ck_2 + ck_1$. Hence, there are four potential solutions. However, considering $k_2 > k_1$ and the second-order conditions of each firm, $\frac{\partial^2 \Pi_i}{\partial k_i^2} < 0$, we can show that each firm's best response is given by $B_i = 0$:

$$k_1 = \frac{\underline{\theta} - h}{3c} + \frac{k_2}{3}$$
$$k_2 = \frac{\underline{\theta} + 2h}{3c} + \frac{k_1}{3}$$

Hence, the SPNE candidate is

$$k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c} \text{ and } k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$$

and the corresponding equilibrium prices and profits are as follows:

$$p_1^{**} = \frac{\underline{\theta}(2\underline{\theta}-h)}{8c} + \frac{25h^2}{64c},\ p_2^{**} = \frac{\underline{\theta}(2\underline{\theta}+5h)}{8c} + \frac{49h^2}{64c} \text{ and } \Pi_1^{**} = \Pi_2^{**} = \frac{3h^2}{16c} - F.$$

In order for this solution to be valid, the lowest valuation consumer must have a nonnegative surplus. Hence,

$$\begin{aligned} U(\underline{\theta}) &\geq 0 \Leftrightarrow \underline{\theta}\left(\frac{\underline{\theta}}{2c} - \frac{h}{8c}\right) - \left(\frac{\underline{\theta}(2\underline{\theta}-h)}{8c} + \frac{25h^2}{64c}\right) \\ &\geq 0 \Leftrightarrow \frac{1}{64c}\left(16\underline{\theta}^2 - 25h^2\right) \geq 0 \Leftrightarrow \underline{\theta} \geq \frac{5}{4}h \end{aligned}$$

■

Proof of Proposition 3. From Proposition 5 in Pires et al. (2022), we know that, if $k_1 < k_2 \leq \frac{1}{c}(\underline{\theta}+h), k_2 \leq \frac{1}{c}(\underline{\theta}+h) - \frac{1}{2}k_1$ and $h(k_2-k_1) - 3\underline{\theta}k_2 + ck_2(2k_1+k_2) > 0$, a partial coverage duopoly price equilibrium occurs in the second-stage of the game and the equilibrium prices and profits are as follows:

$$\begin{aligned} p_1^* &= \frac{k_1(\underline{\theta}+h)(k_2-k_1) + 2ck_1^2k_2 + ck_2^2k_1}{4k_2-k_1} \text{ and} \\ p_2^* &= \frac{2k_2(\underline{\theta}+h)(k_2-k_1) + ck_1^2k_2 + 2ck_2^3}{4k_2-k_1} \\ \Pi_1^* &= \frac{k_1k_2(k_2-k_1)(\underline{\theta}+h+c(k_2-k_1))^2}{h(4k_2-k_1)^2} - F \text{ and} \\ \Pi_2^* &= \frac{k_2^2(k_2-k_1)(c(2k_2+k_1) - 2(\underline{\theta}+h))^2}{h(4k_2-k_1)^2} - F \end{aligned}$$

Computing the first-order conditions for the Nash equilibrium in the first-stage of the game and simplifying them, we obtain the following:

$$\frac{\partial \Pi_1}{\partial k_1} = \frac{k_2(\underline{\theta}+h+ck_2-ck_1)[(k_2(4k_2-7k_1))(\underline{\theta}+h+ck_2-ck_1) - 2ck_1(k_2-k_1)(4k_2-k_1)]}{h(4k_2-k_1)^3} = 0$$

$$\frac{\partial \Pi_2}{\partial k_1} = \frac{k_2(2ck_2+ck_1-2(\underline{\theta}+h)) \times [((3k_2-2k_1)(4k_2-k_1) - 8k_2(k_2-k_1))(2ck_2+ck_1-2(\underline{\theta}+h)) + 4ck_2(k_2-k_1)(4k_2-k_1)]}{h(4k_2-k_1)^3} = 0$$

Since we are looking for a solution with $k_2 > 0$, the solution of this system is equivalent to the solution of the following:

$$\begin{aligned} &(\underline{\theta}+h+ck_2-ck_1)[(k_2(4k_2-7k_1))(\underline{\theta}+h+ck_2-ck_1) - 2ck_1(k_2-k_1)(4k_2-k_1)] = 0 \\ &(2ck_2+ck_1-2(\underline{\theta}+h)) \times \\ &[((3k_2-2k_1)(4k_2-k_1) - 8k_2(k_2-k_1))(2ck_2+ck_1-2(\underline{\theta}+h)) + 4ck_2(k_2-k_1)(4k_2-k_1)] = 0 \end{aligned}$$

Note that both conditions can be written as the product of two factors ($A_i \cdot B_i = 0$). Therefore, one can analyze the potential Nash equilibrium by considering all combinations that guarantee $A_i \cdot B_i = 0$, for $i = 1,2$. However, only one of them simultaneously satisfies the condition that $k_1 < k_2$ and the second-order conditions for the two firms. Hence, only in that case are the two firms simultaneously in their best responses. This solution is the one where $B_1 = 0$ and $B_2 = 0$

$$\begin{aligned} k_2(4k_2-7k_1)(\underline{\theta}+h+ck_2-ck_1) - 2ck_1(k_2-k_1)(4k_2-k_1) &= 0 \\ ((3k_2-2k_1)(4k_2-k_1) - 8k_2(k_2-k_1))(2ck_2+ck_1-2(\underline{\theta}+h)) + 4ck_2(k_2-k_1)(4k_2-k_1) &= 0 \end{aligned}$$

This system has several solutions but the only valid one (real numbers with $k_2 > k_1$) is as follows:

$$k_1^{**} = 0.199361\frac{\underline{\theta}+h}{c} \text{ and } k_2^{**} = 0.40976\frac{\underline{\theta}+h}{c}.$$

The corresponding equilibrium prices and profits are as follows:

$$p_1^{**} = \frac{0.075}{c}(\underline{\theta}+h)^2, p_2^{**} = \frac{0.2267}{c}(\underline{\theta}+h)^2$$
$$\Pi_1^{**} = \frac{0.0121}{ch}(\underline{\theta}+h)^3 - F \text{ and } \Pi_2^{**} = \frac{0.0164}{ch}(\underline{\theta}+h)^3 - F.$$

In order for this solution to be valid, the lowest valuation consumer must have a negative surplus; otherwise, the market would be fully covered. Thus,

$$U(\underline{\theta}) < 0 \Leftrightarrow \underline{\theta}\left(0.199361\frac{\underline{\theta}+h}{c}\right) - \frac{0.075}{c}(\underline{\theta}+h)^2 < 0 \Leftrightarrow \underline{\theta} < 0.60321h.$$

■

Proof of Proposition 4. From Proposition 6 in Pires et al. (2022), we know that, if $k_1 < k_2 \le \frac{1}{c}(\underline{\theta}+h), \frac{k_1}{3}(2\underline{\theta}+h-2ck_1)+\frac{k_2}{3}(\underline{\theta}-h-ck_2) < 0, h(k_2-k_1)-3\underline{\theta}k_2+ck_2(2k_1+k_2) \le 0$ and $\underline{\theta}-\frac{h}{2} \le \frac{\theta(k_2-2k_1)+ck_2^2}{2(k_2-k_1)} \le \underline{\theta}+\frac{h}{2}$, a corner full coverage duopoly price equilibrium occurs in the second-stage of the game and the equilibrium prices and profits are as follows:

$$p_1^* = \underline{\theta}k_1 \text{and} p_2^* = \frac{(\underline{\theta}+h)(k_2-k_1)+\underline{\theta}k_1+ck_2^2}{2}$$
$$\Pi_1^* = \frac{\left(\underline{\theta}k_1-ck_1^2\right)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)}{2h(k_2-k_1)} - F \text{ and}$$
$$\Pi_2^* = \frac{\left(ck_2^2-h(k_2-k_1)-\underline{\theta}k_2\right)^2}{4h(k_2-k_1)} - F.$$

The first-order conditions in the first-stage game are equivalent to the following:

$$\frac{\partial \Pi_1}{\partial k_1} = \frac{\left((\underline{\theta}-2ck_1)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)-h\left(\underline{\theta}k_1-ck_1^2\right)\right)(k_2-k_1)+\left(\underline{\theta}k_1-ck_1^2\right)\left(h(k_2-k_1)+ck_2^2-\underline{\theta}k_2\right)}{2h(k_2-k_1)^2} = 0$$
$$\frac{\partial \Pi_2}{\partial k_2} = \frac{\left(ck_2^2-h(k_2-k_1)-\underline{\theta}k_2\right)\left((4ck_2-2h-2\underline{\theta})(k_2-k_1)-\left(ck_2^2-h(k_2-k_1)-\underline{\theta}k_2\right)\right)}{4h(k_2-k_1)^2} = 0$$

There are four solutions for the first-order condition of firm 2:

$$k_2 = \frac{1}{2c}\left((\underline{\theta}+h)-\sqrt{(\underline{\theta}+h)^2-4chk_1}\right)$$
$$k_2 = \frac{1}{2c}\left((\underline{\theta}+h)+\sqrt{(\underline{\theta}+h)^2-4chk_1}\right)$$
$$k_2 = \frac{1}{6c}\left(\underline{\theta}+h+4ck_1-\sqrt{(\underline{\theta}+h+4ck_1)^2-12c(2\underline{\theta}+h)k_1}\right)$$
$$k_2 = \frac{1}{6c}\left(\underline{\theta}+h+4ck_1+\sqrt{(\underline{\theta}+h+4ck_1)^2-12c(2\underline{\theta}+h)k_1}\right)$$

However, only the last one satisfies the second-order condition, $\frac{\partial^2 \Pi_2}{\partial k_2^2} < 0$, and hence, only

$$k_2 = \frac{1}{6c}\left(\underline{\theta}+h+4ck_1+\sqrt{(\underline{\theta}+h+4ck_1)^2-12c(2\underline{\theta}+h)k_1}\right)$$

can be a best response of firm 2. Considering this condition and firm 1 first-order condition, we get the equilibrium candidate under the full coverage corner duopoly. It should noted that, using Mathematica software, we were able to find a closed form solution.

However, it is a very long expression and does not allow us to interpret the solution or do comparative statics.[14] Thus, to get a feeling on how this solution behaves, we resort to the numerical solution.

Recall that, in order for this solution to be valid, we must have $\frac{k_1}{3}(2\underline{\theta}+h-2ck_1)+\frac{k_2}{3}(\underline{\theta}-h-ck_2)<0$ and $h(k_2-k_1)-3\underline{\theta}k_2+ck_2(2k_1+k_2)\leq 0$. It can be shown that the solution to the system that defines the corner SPNE candidate only satisfies the first condition for $\underline{\theta}\leq 0.625h$ and only satisfies the second condition for $\underline{\theta}>0.5h$. Thus, this SPNE candidate can only hold for $0.5h\leq\underline{\theta}\leq 0.625h$. ■

Proof of Proposition 5. From the proof of the interior full coverage case (Proposition 2), it is clear that, for $\underline{\theta}<1.25h$, if firms choose the interior full coverage SPNE candidate, (k_1^{**},k_2^{**}), they will no longer be in the interior full coverage region. In that solution, firms are differentiating too much their quality and the lowest valuation consumer would no longer buy the product. In addition, we also know that, for $\underline{\theta}>0.625h$, the corner SPNE candidate is not valid. In that candidate, firms would be differentiating too little and thus they would be in the interior full coverage region rather than in the corner region. This is illustrated in Figure 1, where we can see that neither the interior full coverage candidates (point *I*) nor the corner full coverage candidates (point *C*) are internally consistent. This suggests that SPNE qualities may be in the frontier between the corner and the interior full coverage regions. Considering that, at the frontier, the profit functions are nondiferentiable, for a vector of qualities to be a SPNE candidate for both firms, the profit's left partial derivative has to be positive and the right partial derivative has to be negative. When k_1 increases, one is going from the left to the right in the graph; thus, firm 1 is going from the corner region to the interior full coverage region. The reverse happens for firm 2: When k_2 increases, one is going from the bottom to the top in the graph; hence, firm 2 is going from the interior to the corner full coverage region.

The frontier between the interior and the corner regions is given by $\frac{k_1}{3}(2\underline{\theta}+h-2ck_1)+\frac{k_2}{3}(\underline{\theta}-h-ck_2)=0$. Solving this equation with respect to k_2 and considering that k_2 cannot be negative, we obtain condition (*i*), which guarantees that (k_1^{**},k_2^{**}) is in the frontier between the corner and the interior full coverage regions. The second condition, (*ii*), guarantees that the derivative of firm 1 profit in the corner region is positive when evaluated at the candidate point $\frac{\partial\Pi_1}{\partial k_1^-}(k_1^{**},k_2^{**})>0$ (note that the denominator of $\frac{\partial\Pi_1}{\partial k_1}$ is always positive, so the sign of the derivative is determined by the sign of the numerator). The third condition, (*iii*), requires $\frac{\partial\Pi_1}{\partial k_1^+}(k_1^{**},k_2^{**})<0$. Similarly, the fourth and fifth conditions, (*iv*) and (*v*), imply that $\frac{\partial\Pi_2}{\partial k_2^-}(k_1^{**},k_2^{**})>0$ and $\frac{\partial\Pi_2}{\partial k_2^+}(k_1^{**},k_2^{**})<0$. Therefore, as long as there exists a solution to the system, there will be SPNE candidates for $0.625h<\underline{\theta}<1.25h$. It turns out that, for given c and h and for $\underline{\theta}$ in this interval, there are multiple solutions.

To illustrate this result, we consider the case of $c=0.5, h=1$, and $\underline{\theta}=1$ (see Table A1). In this case, there is an infinite set of SPNE, which are located in the frontier that separates the interior full coverage duopoly region and the corner full coverage duopoly region. In fact, all (k_1,k_2) such that $k_2=\frac{1}{2c}(\underline{\theta}-h)+\frac{1}{2c}\sqrt{(\underline{\theta}-h)^2+4ck_1(2\underline{\theta}+h-2ck_1)}$ (points in the frontier) and with $0.55\leq k_1\leq 1$ are SPNE candidates:

For points in the frontier with $k_1<0.55$, the right derivative of firm 1' profit is positive, so firm 1 would have an incentive to deviate to higher qualities. For $k_1>1$, the left derivative of firm 1' profit becomes negative, so firm 1 would deviate to lower qualities. However, all quality vectors in the frontier between the interior and the corner regions such that $0.55\leq k_1\leq 1$ are SPNE candidates. ■

Proof of Lemma 1. From Proposition 1, we know that a high-quality monopoly SPNE can only hold for $\underline{\theta}>2h$. We showed that the unique candidate to such an equilibrium is $k_1^{**}=0$ and $k_2^{**}=\frac{\underline{\theta}}{2c}$, where $\Pi_1^{**}=0$. Assuming $k_2^{**}=\frac{\underline{\theta}}{2c}$, let us check if firm 1 can

TABLE A1 Example of multiple SPNE candidates in the frontier between the interior and the corner full coverage duopoly regions (for $c=0.5, d=1$ and $\underline{\theta}=1$).

k_1	k_2	$\frac{\partial\Pi_1}{\partial k_1^-}(k_1^{**},k_2^{**})$	$\frac{\partial\Pi_1}{\partial k_1^+}(k_1^{**},k_2^{**})$	$\frac{\partial\Pi_2}{\partial k_2^-}(k_1^{**},k_2^{**})$	$\frac{\partial\Pi_2}{\partial k_2^+}(k_1^{**},k_2^{**})$
0.55	1.6416	0.1152	−0.0007	0.1719	−0.1754
0.60	1.6971	0.1083	−0.0089	0.1552	−0.1939
0.70	1.7944	0.0897	−0.0262	0.1282	−0.2213
0.80	1.8762	0.0651	−0.0447	0.1082	−0.2383
0.90	1.9442	0.0351	−0.0639	0.0936	−0.2473
1.00	2.000	0.0000	−0.0833	0.0833	−0.2500

Abbreviation: SPNE, subgame perfect Nash equilibrium.

deviate to a quality level where the price equilibrium is an interior full coverage duopoly. Firm 1 can either deviate to quality levels below (we denote this deviations by k_1^-) or above k_2^{**} (we denote these deviations by k_1^+).[15] Let us assume that firm 1 deviates to the most profitable deviation in that region, which is given by the quality best response in the interior full coverage case. So firm 1 may deviate to the following:

$$k_1^- = \frac{\underline{\theta} - h}{3c} + \frac{k_2}{3} = \frac{\underline{\theta}}{2c} - \frac{h}{3c} \text{ or } k_1^+ = \frac{\underline{\theta} + 2h}{3c} + \frac{k_2}{3} = \frac{\underline{\theta}}{2c} + \frac{2h}{3c}$$

It is easy to check that both deviations satisfy the constraints $\frac{1}{c}(\underline{\theta} - h) \leq k_2 + k_1 \leq \frac{1}{c}(\underline{\theta} + 2h)$ which are required for an interior full coverage duopoly price to exist. This type of equilibrium also requires that

$$\frac{k_1}{3}(2\underline{\theta} + h - 2ck_1) + \frac{k_2}{3}(\underline{\theta} - h - ck_2) \geq 0$$

Substituting (k_1^-, k_2^{**}) and (k_1^+, k_2^{**}) in the previous condition, we get (note that with k_1^+ firm 1 becomes the high-quality firm):

$$\frac{\left(\frac{\underline{\theta}}{2c} - \frac{h}{3c}\right)}{3}\left(2\underline{\theta} + h - 2c\left(\frac{\underline{\theta}}{2c} - \frac{h}{3c}\right)\right) + \frac{\frac{\underline{\theta}}{2c}}{3}\left(\underline{\theta} - h - c\frac{\underline{\theta}}{2c}\right) \geq 0 \Leftrightarrow \underline{\theta}^2 \geq \frac{20}{27}h^2$$
$$\frac{\frac{\underline{\theta}}{2c}}{3}\left(2\underline{\theta} + h - 2c\frac{\underline{\theta}}{2c}\right) + \frac{\frac{\underline{\theta}}{2c} + \frac{2h}{3c}}{3}\left(\underline{\theta} - h - c\left(\frac{\underline{\theta}}{2c} + \frac{2h}{3c}\right)\right) \geq 0 \Leftrightarrow \underline{\theta}^2 \geq \frac{27}{40}h^2$$

So, with these deviations, firm 1 is within an interior full coverage price region. The profit of firm 1 would be given by the following:

$$\Pi_1(k_1^-, k_2^{**}) = \frac{1}{9h}\left(\frac{\underline{\theta}}{2c} - \left(\frac{\underline{\theta}}{2c} - \frac{h}{3c}\right)\right)\left(h - \underline{\theta} + c\left(\frac{\underline{\theta}}{2c} - \frac{h}{3c}\right) + c\frac{\underline{\theta}}{2c}\right)^2 - F = \frac{4}{243c}h^2 - F > 0$$
$$\Pi_1(k_1^+, k_2^{**}) = \frac{1}{9h}\left(\frac{\underline{\theta}}{2c} + \frac{2h}{3c} - \frac{\underline{\theta}}{2c}\right)\left(\underline{\theta} + 2h - c\left(\frac{\underline{\theta}}{2c} + \frac{2h}{3c}\right) - c\frac{\underline{\theta}}{2c}\right)^2 - F = \frac{32}{243c}h^2 - F > 0$$

Hence, both deviations are profitable, although firm 1 gains more by leapfrogging firm 2. Thus, in a high-quality monopoly, the low-quality firm would have an incentive to deviate to a quality level such that the interior full coverage equilibrium holds. Therefore, considering the assumptions regarding the investment costs, the high-monopoly cannot be sustained as a SPNE. ■

Proof of Lemma 2. These results were obtained numerically. We developed a simulation model where, for each set of parameters, the equilibrium second-stage profits were computed for all possible combinations of (k_1, k_2) considering very small steps (equal to 0.005) for the quality levels. Based on the equilibrium profit matrices, we determined the global best responses for each firm and determined the SPNE quality choices. The simulation model completely replicated our analytical results and Wauthy (1996) results, which validates the simulation model. For $0.5h < \underline{\theta} \leq 0.60321h$, there was always two symmetric SPNE (one with $k_1 > k_2$, the other one with $k_2 > k_1$). These SPNE were in the partial coverage region for $0.5h \leq \underline{\theta} \leq 0.545h$ and in the corner region for $0.545h < \underline{\theta} \leq 0.60321h$. ■

Proof of Lemma 3. We need to show that firm 1 cannot gain by deviating to $k_1^+ > k_2^{**}$ and firm 2 cannot gain by deviating to $k_2^- < k_1^{**}$. Considering $k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$ (the equilibrium quality in the interior full coverage SPNE candidate), if firm 1 chooses $k_1^+ > k_2^{**}$, its profit function would be (now firm 1 is the high-quality firm and hence we need to use the corresponding profit function):

$$\Pi_1 = \frac{1}{9h}\left(k_1 - \left(\frac{\underline{\theta}}{2c} + \frac{5h}{8c}\right)\right)\left(\underline{\theta} + 2h - c\left(\frac{\underline{\theta}}{2c} + \frac{5h}{8c}\right) - ck_1\right)^2 - F$$

Note that to remain in the interior of qualities region that leads to an interior full coverage duopoly in the price game, it must be $k_1 \le \frac{\underline{\theta}}{2c} + \frac{11h}{8c}$ as otherwise the indifferent consumer would be above $(\underline{\theta}+h)$. Considering that $k_1 \le \frac{\underline{\theta}}{2c} + \frac{11h}{8c}$, this function has a maximum at $k_1 = \frac{\underline{\theta}}{2c} + \frac{7h}{8c}$, but the corresponding profit is $\frac{1}{144c}h^2 - F$, which is lower than $\frac{3h^2}{16c} - F$, so firm 1 does not gain by leapfrogging firm 2.

A similar argument can be used to show that firm 2 does not want to deviate to $k_2^- < k_1^{**}$. Considering $k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c}$, the profit of firm 2 with such a deviation would be as follows:

$$\Pi_2 = \frac{1}{9h}\left(\frac{\underline{\theta}}{2c} - \frac{h}{8c} - k_2\right)\left(h - \underline{\theta} + c\left(\frac{\underline{\theta}}{2c} - \frac{h}{8c}\right) + ck_2\right)^2 - F$$

This function has a maximum at $k_2 = \frac{\underline{\theta}}{2c} - \frac{3h}{8c}$, but the corresponding profit is $\frac{1}{144c}h^2 - F$, which is lower than $\frac{3h^2}{16c} - F$, so firm 2 does not gain by deviating. ■

Proof of Lemma 4. Can firm 1 gain by deviating to become a high-quality monopolist? In that case, firm 1 has to leapfrog firm 2 (so the roles of the firms are reversed). From Proposition 1 in Pires et al. (2022), we know that this equilibrium only holds for $k_1 + k_2 \le \frac{1}{c}(\underline{\theta} - h)$. Considering $k_2^{**} = \frac{\underline{\theta}}{2c} + \frac{5h}{8c}$ and $k_1^+ > k_2^{**}$, we get $k_1^+ + k_2^{**} > 2\left(\frac{\underline{\theta}}{2c} + \frac{5h}{8c}\right) = \frac{1}{4c}(5h + 4\underline{\theta})$. However, this violates the condition for a high-quality monopoly to exist. So, it is not possible for firm 1 to deviate and become a high-quality monopolist.

Let us now check if firm 2 can deviate to become a high-quality monopoly. Considering $k_1^{**} = \frac{\underline{\theta}}{2c} - \frac{h}{8c}$, to have a high-quality monopoly k_2, must be such that:

$$k_2 \le \frac{1}{c}(\underline{\theta} - h) - \left(\frac{\underline{\theta}}{2c} - \frac{h}{8c}\right) \Leftrightarrow k_2 < \frac{\underline{\theta}}{2c} - \frac{7h}{8c}$$

But then $k_2 < k_1$, which means that firm 1 would be the monopolist firm. But then firm 2 does not gain by deviating. ■

Proof of Proposition 6. This result is a consequence of all the previous analytical results, together with a numerical verification that there are no profitable large deviations. We developed a simulation model where, for each set of parameters, the equilibrium second-stage profits were computed for all possible combinations of (k_1, k_2) considering very small steps for the quality levels. Based on the equilibrium profit matrices, we determined the global best responses for each firm and determined the SPNE quality choices. The simulation model replicated all our analytical results as well as Wauthy (1996) results, which validates it. ■