

# TQFT - a new direction in algebraic topology\*

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December 7, 1999

## Abstract

We give an introduction for the non-expert to TQFT (Topological Quantum Field Theory), focussing especially on its role in algebraic topology. We compare the Atiyah axioms for TQFT with the Eilenberg Steenrod axioms for homology, give a few simple examples of TQFTs, and discuss some other approaches that have been taken to defining TQFT. We then propose a new formulation of TQFT, which is closer in spirit to the way conventional functors of algebraic topology, like homology, are presented. In this approach the fundamental operation of gluing is incorporated through the notion of a *gluing morphism*, which we define. It allows not only the gluing together of two separate objects, but also the self-gluing of a single object to be treated in the same fashion. As an example of our approach we reformulate and generalize a class of examples due to Quinn based on the Euler characteristic.

IST/DM/xx/99  
math.GT/yymmmnn

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\*Contribution to the proceedings of the workshop “New Developments in Algebraic Topology”, Faro, Portugal, July 13-14, 1998.

# 1. Introduction

Topological Quantum Field Theory, or TQFT for short, is a notion which originally arose from ideas of quantum physics. Since then the notion has developed considerably in a number of directions, and in particular has had a pervasive influence in mathematics. In this article we will be defending the point of view that TQFT is a new type of functor of algebraic topology, analogous to homology or homotopy.

We recall that a functor is a map between two categories, like the functor  $H_n$  from the category of topological spaces to the category of abelian groups, which assigns to each topological space  $X$  its  $n$ -th homology group  $H_n(X)$ , and *furthermore* assigns to each continuous map  $X \xrightarrow{f} Y$  a group morphism  $H_n(X) \xrightarrow{H_n(f)} H_n(Y)$ , with the assignments obeying certain natural conditions. This property of not only mapping mathematical objects, but also the morphisms between them, implies a very profound relationship between the two categories, a bridge between two often very different regions of the mathematical landscape.

For those without a physics background, a few words are also in order about the underlying notions of quantum physics. Fields are simply (a suitable class of) functions  $\phi : \Sigma \rightarrow X$  defined on the  $d$ -dimensional space manifold  $\Sigma$ , or  $\Phi : M \rightarrow X$  defined on the  $d + 1$ -dimensional space-time manifold  $M$ , into another space  $X$ , which may be, for instance,  $\mathbf{R}^n$ , or a Lie group  $G$ . A field  $\Phi$  describes an evolution of fields  $\phi$ , at least when  $M = \Sigma \times I$ , where  $I$  is an interval (of time), and is interpreted as some kind of multifingered evolution, when  $M$  is not of this form. The evolution is classical when  $\Phi$  is a critical point of a certain functional  $S(\Phi)$ , called the action. The corresponding quantum evolution is a superposition of evolutions known as the path integral,  $\int d\mu(\Phi) \exp(-iS(\Phi)/\hbar)$ , where  $d\mu(\Phi)$  is a rather elusive “measure” on the space of all evolutions, and  $\hbar$  is the quantum parameter. Roughly speaking, as this parameter tends to zero, the path integral reduces to the classical evolution in the stationary phase approximation. When  $S(\Phi)$  and  $d\mu(\Phi)$  happen to be invariant under diffeomorphisms, we have a so-called *topological* quantum field theory, giving rise to topological invariants of  $M$ .

The most famous example of a TQFT was Witten’s use of the Chern-Simons action to obtain a topological invariant for 3-manifolds (as well as, simultaneously, an invariant of embedded 1-dimensional submanifolds, i.e., knots and links) [1]. Numerous other constructions, both heuristic and rigorous, followed, including a class of “state-sum models”, involving piecewise-linear (PL) manifolds. [2, 3, 4]. For PL manifolds of dimension 4, a state-sum model due to Crane and Yetter [5] gave rise to a combinatorial formula for the signature. TQFT has had most impact in 3 and 4-dimensional topology, where the classical invariants are weak. For reviews and further background see [6, 7].

## 2. Atiyah’s axioms

The common features underlying a number of different TQFT constructions were formalized by Atiyah in a set of axioms for TQFT [8]. These are modeled on similar axioms for conformal field theory, due to Segal [9]. We will give a brief description of them. According to Atiyah, a  $(d+1)$ -dimensional TQFT is an assignment  $\Sigma \mapsto V_\Sigma$  and  $M \mapsto Z_M$ , which

assigns to every  $d$ -dimensional oriented manifold  $\Sigma$  a finite-dimensional vector space  $V_\Sigma$  (over some fixed field  $\mathbb{K}$ ), and to every  $(d+1)$ -dimensional oriented manifold with boundary  $M$ , an element of  $V_{\partial M}$ , such that the following axioms hold:

- A1) the assignment is functorial with respect to diffeomorphisms of  $\Sigma$  and  $M$ ,
- A2)  $V_{-\Sigma} = V_\Sigma^*$ , where  $-\Sigma$  denotes  $\Sigma$  with the opposite orientation, and  $V_\Sigma^*$  is the dual vector space of  $V_\Sigma$ ,
- A3) a)  $V_{\Sigma_1 \sqcup \Sigma_2} = V_{\Sigma_1} \otimes V_{\Sigma_2}$ , where  $\sqcup$  denotes the disjoint union,  
 b)  $Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2}$ ,  
 c) (gluing axiom)  $Z_{M_1 \sqcup_\Sigma M_2} = \langle Z_{M_1}, Z_{M_2} \rangle$ , where  $M_1 \sqcup_\Sigma M_2$  means  $M_1$  glued to  $M_2$  along  $\Sigma$  (see figure) and  $\langle \cdot, \cdot \rangle$  is given by evaluation of  $V_\Sigma^*$  on  $V_\Sigma$ ,
- A4) a number of non-triviality conditions.

The first axiom means, in particular, that when  $M$  and  $M'$  are diffeomorphic, there is a linear isomorphism  $V_{\partial M} \rightarrow V_{\partial M'}$  which takes  $Z_M$  to  $Z_{M'}$ . One of the non-triviality conditions is  $V_\emptyset = \mathbb{K}$ , which is related to axiom A3a):  $\emptyset \sqcup \emptyset = \emptyset \Rightarrow V_\emptyset \otimes V_\emptyset = V_\emptyset$ , hence  $V_\emptyset$  can only be  $\mathbb{K}$  or the trivial vector space  $\{0\}$ . Thus, in particular, a TQFT assigns to every closed  $(d+1)$  manifold  $M$  a numerical invariant  $Z_M \in \mathbb{K}$ .

The all-important gluing axiom may be reformulated by noting that, if  $\partial M = \Sigma_1 \sqcup (-\Sigma_2)$ , then  $Z_M \in V_{\Sigma_1} \otimes V_{\Sigma_2}^* \cong \text{Lin}(V_{\Sigma_2}, V_{\Sigma_1})$ . Thus  $Z_M$  may be regarded as a linear map, and with  $\partial M_1 = \Sigma_1 \sqcup (-\Sigma)$  and  $\partial M_2 = \Sigma \sqcup (-\Sigma_2)$ , the gluing axiom 3c) reads:

$$Z_{M_1 \sqcup_\Sigma M_2} = Z_{M_1} \circ Z_{M_2}.$$

We remark that  $\partial M$  can always be regarded as the disjoint union of two boundary components, since either or both components can be empty.

At this stage it is probably helpful to give a simple example. When  $d = 0$ , any non-empty oriented  $(d+1)$ -dimensional manifold with boundary  $M$  is isomorphic to either the oriented interval  $I$  or the oriented circle  $S^1$ , or to a disjoint union of copies of these. Suppose  $\mathbb{K}$  is the field of complex numbers,  $\mathbb{C}$ . To specify the TQFT, all we have to do is choose  $V_{\bullet+}$ , the vector space assigned to the positively oriented point  $\bullet+$ , and we choose it to be  $\mathbb{C}^2$ . All the remaining assignments follow from the axioms. From A2),  $V_{\bullet-} = (\mathbb{C}^2)^*$ , and the disjoint union of positively and negatively oriented points maps to the corresponding tensor product of copies of  $\mathbb{C}^2$  and its dual. Interpreting  $Z_I$  as a linear map from  $\mathbb{C}^2$  to  $\mathbb{C}^2$ , the topological diffeomorphism between two intervals glued together at one end and a single interval, gives rise to the equation  $Z_I \circ Z_I = Z_I$ , and thus  $Z_I$  is a projection. By another of the non-triviality axioms, the projection  $Z_I$  is actually taken to be surjective, i.e.,  $Z_I = \text{id}_{\mathbb{C}^2}$ .

All that remains is to identify  $Z_{S^1}$ , which is a linear map from  $V_\emptyset = \mathbb{K}$  to itself, and thus given by the number  $Z_{S^1}(1) \in \mathbb{K}$ . To this end, we look at the interval and  $Z_I$  in two new ways. First, if we regard  $\partial I$  as  $-(\bullet+ \sqcup \bullet-) \sqcup \emptyset$ , by folding the interval into an upturned U shape, oriented clockwise, and “reading” it from bottom to top (see figure),

we get  $Z_{\cap+} : V_{\bullet+} \otimes V_{\bullet-} \rightarrow \mathbb{C}$ , where we have denoted the folded interval with its clockwise orientation  $\cap+$ . In the same manner we may fold the interval the other way into a clockwise-oriented U shape (see figure above) and regard its boundary as  $\emptyset \sqcup (\bullet+ \sqcup \bullet-)$ . In this way we get  $Z_{\cup+} : \mathbb{C} \rightarrow V_{\bullet+} \otimes V_{\bullet-}$ , where we have denoted the folded interval with its clockwise orientation  $\cup+$ . To describe the various linear transformations, we introduce a basis  $\{e_1, e_2\}$  in  $V_{\bullet+} = \mathbb{C}^2$ , and let  $\{e_1^*, e_2^*\}$  denote its dual basis. The element of  $V_{\bullet+} \otimes V_{\bullet+}^* \cong \text{Lin}(V_{\bullet+}, V_{\bullet+})$  giving rise to  $Z_I = \text{id}_{\mathbb{C}^2}$  is  $e_1 \otimes e_1^* + e_2 \otimes e_2^*$ . This same element, regarded as belonging to  $(V_{\bullet+} \otimes V_{\bullet+}^*) \otimes \mathbb{C}$ , leads to  $Z_{\cup+}$  being given by  $1 \mapsto e_1 \otimes e_1^* + e_2 \otimes e_2^*$ . Regarding  $e_1^* \otimes e_1 + e_2^* \otimes e_2$  as belonging to  $\mathbb{C} \otimes (V_{\bullet+} \otimes V_{\bullet+}^*)^* \cong \text{Lin}(V_{\bullet+} \otimes V_{\bullet+}^*, \mathbb{C})$  leads to  $Z_{\cap+}$  being given by  $e_i \otimes e_j^* \mapsto \delta_{ij}$ , for  $i, j, = 1, 2$ . Now we may consider the clockwise oriented circle  $S^1$  as the result of gluing a clockwise oriented inverted U and a clockwise oriented U (see figure).

Thus we calculate  $Z_{S^1}(1) = Z_{\cap+} \circ Z_{\cup+}(1) = Z_{\cap+}(e_1 \otimes e_1^* + e_2 \otimes e_2^*) = 2$ . This result completes the description of the TQFT, since any disjoint union of intervals and circles maps to the appropriate tensor product of  $Z_I$  and  $Z_{S^1}$ .

The preceding example also hints at yet another frequently-employed reformulation of the Atiyah axioms for TQFT. This version may be regarded as a response to the following, seemingly bizarre, question: if  $M$  is mapped to  $Z_M$ , a (linear) function, why not make  $M$  itself into a function? In fact, this is achieved by introducing the cobordism category,  $(d+1)$ -**Cobord**, whose objects are oriented  $d$ -dimensional manifolds and whose morphisms are equivalence classes of  $(d+1)$ -dimensional manifolds with boundary. More precisely, if  $\Sigma_1$  and  $\Sigma_2$  are two objects, then the set of all morphisms from  $\Sigma_2$  to  $\Sigma_1$  is  $\text{Mor}(\Sigma_2, \Sigma_1) = \{M \mid \partial M = \Sigma_1 \sqcup (-\Sigma_2)\} / \text{Diff}$ , where *Diff* stands for diffeomorphisms which restrict to the identity on the boundary of  $M$ . Composition of  $[M_1] \in \text{Mor}(\Sigma, \Sigma_1)$  and  $[M_2] \in \text{Mor}(\Sigma_2, \Sigma)$  is given by  $[M_1] \circ [M_2] = [M_1 \sqcup_{\Sigma} M_2]$ , where the square brackets denote the equivalence class. This is associative, due to the identification of diffeomorphic manifolds, and furthermore, every object  $\Sigma$  has an identity morphism, namely  $[\Sigma \times I]$ .

In this framework, a TQFT is a functor from the cobordism category  $(d+1)$ -**Cobord** to the category of finite-dimensional vector spaces (over  $\mathbb{K}$ ) **Vect**, described by  $\Sigma \mapsto V_{\Sigma}$  on objects and  $(\Sigma_2 \xrightarrow{[M]} \Sigma_1) \mapsto (V_{\Sigma_2} \xrightarrow{Z_{[M]}} V_{\Sigma_1})$  on morphisms, which preserves the products or “monoidal structures” ( $\sqcup$  and  $\otimes$  respectively), the unit objects for these products ( $\emptyset$  and  $\mathbb{K}$  respectively), and the “involutions” ( $\Sigma \mapsto -\Sigma$  and  $V \mapsto V^*$  respectively).

This “cobordism definition” of TQFT is very elegant, but slightly unusual from the viewpoint of conventional algebraic topology in that the cobordism category is somewhat uncanonical as a category. We will return to discussing this point in section 5. It is very appropriate for describing TQFT-like constructions involving embedded manifolds, like braids and tangles, such as Turaev’s operator-valued invariant for tangles [2]. See [10] for a discussion by one of the authors of the cobordism approach for these embedded TQFT’s.

### 3. A comparison with homology

At this stage it is appropriate to return to the theme of the title, and inquire about the role of TQFT in the context of classical algebraic topology. It is illuminating to examine some of the similarities and differences between Atiyah's axioms for TQFT and the Eilenberg-Steenrod axioms for homology, as described in many textbooks on algebraic topology, e.g. [11].

	<b>Homology</b> <b>Eilenberg-Steenrod axioms</b>	<b>TQFT</b> <b>Atiyah axioms</b>
1	The topological objects are pairs of topological spaces $(X, Y)$ with $Y \subset X$ .	The topological objects are pairs $(M, \Sigma)$ with $\Sigma = \partial M \subset M$ .
2	A topological object $(X, Y)$ is sent to an abelian group $H(X, Y) = \bigoplus_n H_n(X, Y).$	A topological object $(M, \Sigma)$ is sent to a vector space (and a point belonging to it) $(V_\Sigma, Z_M)$ .
3	The theory is additive in the sense that $\sqcup$ maps to $\oplus$ .	The theory is multiplicative in the sense that $\sqcup$ maps to $\otimes$ .
4	One of the key axioms is an excision property, i.e. related to subtracting one space from another one (see figure below).	One of the key axioms is a gluing property, i.e. related to adding one space to another one (see figure below).
5	A single point ' $\bullet$ ' maps to $H(\bullet, \emptyset) = \mathbb{Z}$ , the simplest non-trivial free abelian group. When $\bullet$ does not map to $\mathbb{Z}$ , the theory is described as a generalized homology theory.	The empty set $\emptyset$ maps to $V_\emptyset = \mathbb{K}$ , the simplest non-trivial vector space over $\mathbb{K}$ . For $d = 0$ , the single point does not in general map to $\mathbb{K}$ , and for $d > 0$ it does not even make sense to speak of $V_\bullet$ .
6	The theory is not geared to any specific dimension. There is a connecting homomorphism relating $H_n(-, -)$ to $H_{n+1}(-, -)$ .	The theory is (usually) geared to a specific dimension. There are only hints of a dimensional ladder, linking TQFT's for different dimensions.
7	Homology is a functor and the topological morphisms are canonical maps.	TQFT is a functor only in the cobordism approach, and there the topological morphisms are equivalence classes of manifolds.
8	The theory can be applied to various topological categories.	The theory, as it stands, applies only to (differentiable) manifolds.

We leave these comparisons for the reader to contemplate, but cannot refrain from mentioning the observation, due to Louis Crane, that TQFT behaves in some ways like an exponentiated version of homology, bearing in mind especially properties 3 and 5.

The restriction to differentiable manifolds in the Atiyah axioms should not be taken too literally, since, as Atiyah himself stresses, the axioms are meant to be minimal and can be extended in a variety of ways. Indeed there are a number of important examples of TQFTs for other topological categories. We will proceed to describe two of these.

First there is a class of TQFTs known as “state sum models”, which are defined combinatorially using triangulated manifolds. One starts by assigning algebraic data to the simplices of the triangulation, subject to some admissibility conditions. These algebraic data may come from a variety of sources, e.g. groups, quantum groups, representations, categories, subfactors and so on. From the data one calculates a numerical weight by means of some rule, and then, given a triangulated manifold  $M$ , chosen to be without boundary for simplicity,  $Z_M$  is defined to be the sum over all admissible data of the corresponding weights. If the data and rules for assigning weights are suitably matched,  $Z_M$  is independent of the triangulation chosen.

A very simple example of a state sum model, which gives the flavour of the construction, is defined for triangulated 2-manifolds without boundary in the following fashion (this example is based on a construction of Dijkgraaf and Witten [12]). For any 2-simplex of the triangulation “colour” its oriented edges with elements of a fixed finite group  $G$ , subject to the admissibility condition that the group elements corresponding to the 1-cycle around the boundary of the 2-simplex multiply to 1 (the identity). Also, if the orientation of an edge is reversed the group element assigned to it is replaced by its inverse (see figure).

Now, given a 2-manifold  $M$ , we define:

$$Z_M = \sum_{\text{colourings}} \left( \frac{1}{\#G} \right)^{\#V}$$

where  $\#G$  is the number of elements of  $G$  and  $\#V$  is the number of vertices of the triangulation. Triangulation invariance corresponds to invariance under two local moves, the Pachner moves [13], shown in the figure below.

Invariance under these moves follows, for the first move, from the associativity of group multiplication, and for the second move, from the fact that the increase in the number of colourings by a factor of  $\#G$ , is compensated by the  $\frac{1}{\#G}$  factor coming from the extra vertex.

The second example of a TQFT not involving differentiable manifolds, is due to Quinn [14]. This TQFT is defined for a very general class of topological spaces, namely finite  $CW$ -complexes, which need not even be (topological) manifolds. Let  $M$  be a finite  $CW$ -complex, and let  $\Sigma_1^i, \Sigma_2^o$  be disjoint  $CW$ -subcomplexes of  $M$ , labelled  $i$  for “in” and  $o$  for

“out”. In the cobordism picture of TQFTs we can regard  $M$  (or rather an equivalence class of  $M$ s – for notational simplicity we will just write  $M$ ) as a morphism from  $\Sigma_1^i$  to  $\Sigma_2^o$ . If the “out” of  $M_2$  equals the “in” of  $M_1$ , both equal to  $\Sigma$  say, we can glue or compose:

$$(\Sigma^i \xrightarrow{M_1} \Sigma_1^o) \circ (\Sigma_2^i \xrightarrow{M_2} \Sigma^o) = (\Sigma_2^i \xrightarrow{M_1 \sqcup_{\Sigma} M_2} \Sigma_1^o),$$

where  $M_1 \sqcup_{\Sigma} M_2$  denotes  $M_1$  and  $M_2$  glued together along their common subcomplex  $\Sigma$ .

Given this setup, Quinn defines a TQFT as follows. Choosing  $\mathbb{K} = \mathbb{C}$ , the complex numbers, we set  $V_{\Sigma} = \mathbb{C}$  for every  $\Sigma$ , i.e. this TQFT does not distinguish anything at the level of objects of the cobordism category. (In fact, Quinn chooses a commutative ring  $R$  instead of  $\mathbb{C}$ , but this makes little difference to the example.) At the level of morphisms, however, the assignments are as follows:

$$(\Sigma^o \xrightarrow{M} \Sigma^i) \mapsto (\mathbb{C} \xrightarrow{Z_M} \mathbb{C}), \quad Z_M(c) = e^{i\alpha\chi(M, \Sigma^i)}(c)$$

where  $\chi(M, \Sigma^i) = \sum_{n=0}^{\dim M} (-1)^n \text{rank}(H_n(M, \Sigma^i))$  is the relative homology version of the Euler number, and  $e^{i\alpha}$  is a fixed element of  $\mathbb{C}$  of modulus 1. This assignment gives rise to a functor, and in particular it respects composition: ( $Z_{M_1 \sqcup_{\Sigma} M_2} = Z_{M_1} \circ Z_{M_2}$ ), because of the formula

$$\chi(M_1 \sqcup_{\Sigma} M_2, \Sigma_2^i) = \chi(M_1, \Sigma_1^i) + \chi(M_2, \Sigma_2^i),$$

which, incidentally, may be proved by using excision. Another feature of this example, relating to our comparison between TQFT and homology in the previous section, is that it illustrates well the observation about the  $\exp(\text{homology})$  structure of TQFT. We will be returning to a discussion of this interesting example from a different perspective in section 6.

## 4. Other definitions of TQFT

The definition Atiyah gave in [8] encapsulated the essential ingredients common to a large class of TQFT models, whilst at the same time restricting itself to a specific topological category, the category of differentiable manifolds. As Atiyah himself states, the axioms allow for numerous generalizations. Here we mention three other definitions which aim to provide a more general framework for TQFT.

The first one, due to Quinn [14], introduces the notion of “domain category”, being a category endowed with a collection of structures which are abstractions of topological notions, such as boundary, cylinder, or gluing. In particular the boundary of an object need not be the actual boundary, since the object in question need not be a manifold, as in Quinn’s example described above. The axioms for a domain category are so general that even purely algebraic examples, involving algebras over a commutative ring, fit the definition.

Next, Turaev’s definition, which appears in Chapter III of [2], achieves generality in the topological category in a rather different way, by introducing the abstract notion of “space structure”, which encompasses as special cases any kind of extra structure with which a topological space can be endowed, such as a choice of orientation, a differentiable structure,

the structure of a CW complex, etc. Both Quinn and Turaev adopt the “cobordism” approach as described above, i.e. equivalence classes of  $M$ ’s are the morphisms of the topological cobordism category, gluing corresponds to composition, and a TQFT is a functor from this cobordism category to a suitable algebraic category.

The cobordism theme was taken a step further by Baez and Dolan in [15], when they started a programme to understand the subtle relations between certain TQFT models for manifolds of different dimensions, frequently referred to as the dimensional ladder. This programme is based on higher-dimensional algebra, a generalization of the theory of categories and functors to  $n$ -categories and  $n$ -functors, where for instance a 2-category has not just objects and morphisms, but also 2-morphisms, being, roughly speaking, morphisms between morphisms. In this framework a TQFT becomes an  $n$ -functor from the  $n$ -category of  $n$ -cobordisms to the  $n$ -category of  $n$ -Hilbert spaces. Since the definition of  $n$ -category is itself rather elusive, this programme should still be described as being in a state of development, but nevertheless reveals a fascinating view of parallel developments in algebra and topology.

All three definitions are very much in the spirit of Atiyah’s original approach, which has thus proved to be rather influential. Also in practice most authors studying a specific TQFT model base themselves on the cobordism version of the Atiyah axioms.

## 5. An alternative approach to TQFT

As discussed in the previous section, most authors adopt the cobordism approach to TQFT, and thereby move away from the kind of framework which might facilitate a comparison with conventional functors of algebraic topology, such as homology and homotopy. Although the cobordism category is indeed a category, its morphisms are not “canonical maps”, whereas in the topological categories used to define homology and homotopy theories they are. More precisely, there is no forgetful functor from the cobordism category to the category of topological spaces. One practical consequence is that, in the cobordism framework, handling isomorphisms between  $M$ ’s or  $\Sigma$ ’s becomes somewhat delicate, since the role of morphisms in the category has already been occupied by (equivalence classes of)  $M$ ’s. A related observation is that composition of morphisms on the topological side and the gluing operation are inextricably related in the cobordism approach.

Another point we would like to make is that, in the cobordism approach, the gluing operation is inherently binary, i.e. necessarily involves gluing two distinct  $M$ ’s together. The possibility of gluing a single  $M$  to itself only enters at a later stage of development of the theory, for instance with the following general result about gluing the ends of a “generalized cylinder”  $\Sigma \times I$ , together, to make  $\Sigma \times S^1$  (see figure)

(here  $\Sigma$  is a  $d$ -dimensional manifold without boundary), namely:

$$Z_{\Sigma \times S^1} = \dim V_{\Sigma}.$$

We saw one instance of this TQFT theorem in section 2, in the example where  $\Sigma$  was a single point. However, a more fundamental view of gluing is as a *unary* operation, which

may of course be interpreted as binary when the single  $M$  actually consists of two separate parts.

These considerations led us to seek a different approach to defining TQFT theories, with the following aims:

- to incorporate a wide range of topological categories, as in the Quinn and Turaev definitions discussed in the previous section, and including cases of “embedded topology” like tangles;
- to formulate TQFT as a functor from a topological category, whose morphisms are genuine maps in the above sense;
- to separate the roles of the composition and gluing operations;
- to incorporate gluing as a fundamentally unary operation from the beginning;
- to provide a clean and efficient formulation for ease of calculation.

We will now proceed to describe the main features of this construction, without going into technicalities. A detailed version is currently under preparation. In a nutshell, the idea is as follows: starting with some topological category, restrict the morphisms to be just isomorphisms and so-called gluing morphisms, to be described shortly. A TQFT is then a functor from this category to an algebraic category with suitably matching structures.

The objects of the topological category  $\mathcal{C}$  are *pairs* of the form  $(M, \Sigma)$ , where  $M$  is an object of the aforementioned starting category, denoted  $\mathbf{M}$ , e.g. the category of oriented  $(d + 1)$ -dimensional manifolds with boundary, and  $\Sigma$  is an object in a related category, denoted  $\mathbf{\Sigma}$ , whose objects are subspaces of  $M$ 's with any additional structures that implies, e.g. the category of oriented  $d$ -dimensional manifolds without boundary. Thus  $\Sigma$  plays the role of the boundary of  $M$ , although it need not be the actual boundary in all cases (e.g. in Quinn's example, see sections 3 and 6). Regarded as topological spaces, i.e. ignoring any structures, the objects of  $\mathbf{\Sigma}$  are taken to be finite disjoint unions of connected and mutually separated components. Both  $\mathbf{M}$  and  $\mathbf{\Sigma}$  are monoidal categories with product the disjoint union  $\sqcup$ .

Furthermore there is a functor  $I$  from  $\mathbf{\Sigma}$  to itself, which can be thought of as “change of orientation”, and is such that  $\Sigma$  and  $I(\Sigma)$  are the same as topological spaces. Thus  $I$  only acts on the structures, not on the space itself.

Turning to the morphisms of the category  $\mathcal{C}$ , first we have isomorphisms which, for a pair of objects  $(M, \Sigma)$  and  $(M', \Sigma')$ , are given by isomorphisms  $f : M \rightarrow M'$ , such that  $f|_{\Sigma}$  is an isomorphism in the category  $\mathbf{\Sigma}$  from  $\Sigma$  to  $\Sigma'$ . The only other type of morphisms we will consider in  $\mathcal{C}$  are the gluing morphisms which we will now define.

**Definition** Let  $(M, \Sigma)$  and  $(M', \Sigma')$  be two objects of  $\mathcal{C}$  and suppose  $\Sigma_1$  and  $\Sigma_2$  are disjoint non-empty components of  $\Sigma$ , each being the disjoint union of one or more connected components of  $\Sigma$ . A *gluing morphism* from  $(M, \Sigma)$  to  $(M', \Sigma')$  is a pair  $(f, \varphi)$ , where  $f : M \rightarrow M'$  is a morphism of  $\mathbf{M}$ , and  $\varphi : \Sigma_1 \rightarrow I(\Sigma_2)$  is an isomorphism of  $\mathbf{\Sigma}$ , such that

- 1)  $f$  is surjective,

- 2)  $f|_{M \setminus (\Sigma_1 \sqcup \Sigma_2)}$  is injective,
- 3)  $f|_{\Sigma \setminus (\Sigma_1 \sqcup \Sigma_2)}$  is an isomorphism onto  $\Sigma'$ ,
- 4) for every  $y \in f(\Sigma_1)$  there is a unique pair  $(x, \varphi(x)) \in \Sigma_1 \times \Sigma_2$  such that  $f(x) = f(\varphi(x)) = y$ ,
- 5)  $f(\Sigma_1) \cap f(M \setminus (\Sigma_1 \sqcup \Sigma_2)) = \emptyset$ ,

where in conditions 1), 2), 4) and 5)  $f$  and  $\varphi$  refer to the set-theoretic mappings underlying the corresponding morphisms.

The intuitive content of the definition is that we are gluing two “boundary” components  $\Sigma_1$  and  $\Sigma_2$  of  $M$  together using the isomorphism  $\varphi$ , and the gluing morphism is from  $M$  “before gluing” to a copy of  $M$  “after gluing” (see figure).

>From 2),  $f$  is 1 : 1 except on  $\Sigma_1$  and  $\Sigma_2$ , where, from 4) and 5), it maps 2 : 1 onto their common image  $f(\Sigma_1)$ , which, from 3) and 5), is disjoint from  $\Sigma'$ . Also, from 3), the unglued remainder of  $\Sigma$  is isomorphic to  $\Sigma'$ . In condition 4), as  $\varphi$  is taken in the set-theoretic sense, the distinction between  $\Sigma_2$  and  $I(\Sigma_2)$  disappears, since they are the same as sets.

The key property of gluing morphisms is that we can combine two of them to get a new gluing morphism, defined to be their composition.

**Theorem/Definition** Let  $(f, \varphi)$  from  $(M, \Sigma)$  to  $(M', \Sigma')$ , and  $(g, \psi)$  from  $(M', \Sigma')$  to  $(M'', \Sigma'')$  be gluing morphisms. Then  $(g \circ f, \theta)$ , where  $\theta = \varphi \sqcup (I(f^{-1}) \circ \psi \circ f)$ , is a gluing morphism from  $(M, \Sigma)$  to  $(M'', \Sigma'')$ , and is defined to be the composition of  $(f, \varphi)$  and  $(g, \psi)$ .

The intuitive content of the theorem is that gluing in two stages is equivalent to gluing everything in one go (see figure)

*Sketch of proof:* Let  $\Sigma_3$  and  $\Sigma_4$  be the preimages under  $f$  of  $\Sigma'_1, \Sigma'_2$ , respectively, where  $\psi : \Sigma'_1 \rightarrow I(\Sigma'_2)$ . By 3) and the fact that  $\Sigma$  is monoidal,  $f|_{\Sigma_3}$  and  $f|_{\Sigma_4}$  are isomorphisms onto  $\Sigma'_1, \Sigma'_2$ , respectively. Then  $\varphi \sqcup (I(f^{-1}) \circ \psi \circ f)$  is an isomorphism of  $\Sigma$ , since  $\varphi, \psi, f|_{\Sigma_3}$  and  $(f|_{\Sigma_4})^{-1}$  are all isomorphisms of  $\Sigma$ , and  $I$  is a functor. The properties 1) to 5) are easily checked.  $\square$

It is straightforward to extend the above result to include the combination of a gluing morphism and an isomorphism, or an isomorphism and a gluing morphism, in both case giving rise to a new gluing morphism, defined as the composition. Thus the isomorphisms and gluing morphisms taken together close under composition. It is also easy to see that composition is associative and that there is an identity morphism for each object  $(M, \Sigma)$ . Furthermore the monoidal structures on  $\mathbf{M}$  and  $\Sigma$  coming from the disjoint union give rise to a monoidal structure on  $\mathcal{C}$ . Thus we have:

**Theorem** The above definitions yield a monoidal category  $\mathcal{C}$ , whose morphisms consist of isomorphisms and gluing morphisms.

To get a TQFT functor we first need to choose an algebraic target category  $\mathcal{D}$ . This category is endowed with structures to match those of  $\mathcal{C}$ , and we describe them very briefly. The objects of  $\mathcal{D}$  are pairs of the form  $(V, x)$ , where  $V$  is a finite dimensional vector space over a ground field  $\mathbb{K}$ , and  $x$  is an element of  $V$  (more general choices are possible but we do not go into this here). The morphisms from  $(V, x)$  to  $(W, y)$  in  $\mathcal{D}$  are linear maps  $f : V \rightarrow W$  such that  $f(x) = y$ .  $\mathcal{D}$  is a monoidal category with product the tensor product:  $(V, x) \otimes (W, y) = (V \otimes W, x \otimes y)$ . There is an endofunctor  $J$ , corresponding to the endofunctor  $I$  in  $\mathcal{C}$ , which acts only on the vector space part. For instance, if  $\mathbb{K}$  is  $\mathbb{C}$ , the complex numbers,  $J$  could be the functor that maps  $V$  to  $\overline{V}$ , being the space  $V$  with the same addition as  $V$  and the conjugate scalar multiplication ( $\overline{c} \cdot x := \overline{c} \cdot x$ ).

A TQFT functor  $Z$  is a functor from  $\mathcal{C}$  to  $\mathcal{D}$  of a certain special form. To describe it we need the following assignments:

- a) for each object  $(M, \Sigma)$  of  $\mathcal{C}$  an object  $(V_\Sigma, Z_M)$  of  $\mathcal{D}$ , such that  $V_\Sigma$  depends only on  $\Sigma$ .
- b) for each isomorphism (of  $\Sigma$ ),  $\Sigma_1 \xrightarrow{\varphi} \Sigma_2$ , a linear isomorphism  $V_{\Sigma_1} \xrightarrow{Z_\varphi} V_{\Sigma_2}$ , with this assignment being functorial (mapping identity maps to identity maps and compositions to compositions) and respecting the monoidal structures and endofunctors  $I, J$ .
- c) for each  $\Sigma$  of  $\Sigma$  a linear map (which we call evaluation)  $e_{V_\Sigma} : J(V_\Sigma) \otimes V_\Sigma \rightarrow \mathbb{K}$  satisfying various properties, including a multiplicative property connecting  $e_{V_{\Sigma \sqcup \Sigma'}}$  with  $e_{V_\Sigma}$  and  $e_{V_{\Sigma'}}$ .

A TQFT functor  $Z : \mathcal{C} \rightarrow \mathcal{D}$  is then given in terms of these assignments by:

- 1) on objects:  $(M, \Sigma) \mapsto (V_\Sigma, Z_M)$
- 2) on isomorphisms:  $((M, \Sigma) \xrightarrow{f} (M', \Sigma')) \mapsto ((V_\Sigma, Z_M) \xrightarrow{Z_{f|\Sigma}} (V_{\Sigma'}, Z_{M'}))$
- 3) on gluing morphisms:

$$((M, \Sigma) \xrightarrow{(f, \varphi)} (M', \emptyset)) \mapsto ((V_\Sigma, Z_M) \xrightarrow{Z_{(f, \varphi)}} (\mathbb{K}, Z_{M'})),$$

where, for  $\Sigma = \Sigma_1 \sqcup \Sigma_2$  and  $\varphi : \Sigma_1 \rightarrow I(\Sigma_2)$ ,  $Z_{(f, \varphi)}$  is given by

$$Z_{(f, \varphi)} = e_{V_{\Sigma_2}} \circ (Z_\varphi \otimes \text{id}_{V_{\Sigma_2}}),$$

with a similar but somewhat more complicated formula for the case when  $\Sigma' \neq \emptyset$ , involving  $e_{V_{\Sigma_2}}$ ,  $Z_\varphi$  and  $Z_{f|\Sigma \setminus (\Sigma_1 \sqcup \Sigma_2)}$ .

The main theorem is that the properties of the assignments a)-c) guarantee that  $Z$  is a functor from  $\mathcal{C}$  to  $\mathcal{D}$  respecting the monoidal structures. Essentially  $Z$  provides a representation of a severely restricted subclass of morphisms of the starting category  $\mathbf{M}$ , but a subset which includes the all-important class of gluing morphisms allowing for topology changes. In the next section we will re-examine one of our previous examples from this new perspective to illustrate how the definition works in practice.

## 6. Quinn's example revisited

In this section we return to Quinn's example of a TQFT for finite  $CW$ -complexes, rephrased in terms of our definition. The objects of the topological category  $\mathcal{C}$  are pairs  $(M, \Sigma)$ , where  $M$  is a finite  $CW$ -complex and  $\Sigma$  is a disjoint union of connected sub-complexes of  $M$ , with each connected component labelled  $i$  (in) or  $o$  (out). We denote the union of the in (out) components of  $\Sigma$  by  $\Sigma^{(i)}$  ( $\Sigma^{(o)}$ ). The functor  $I$  acts on  $\Sigma$  by changing the labels from  $i$  to  $o$  and vice-versa, i.e.  $I(\Sigma^i) = \Sigma^o$  (or  $I(\Sigma^o) = \Sigma^i$ ), but  $\Sigma^i$  and  $\Sigma^o$  as  $CW$ -subcomplexes are the same. The morphisms of  $\mathcal{C}$  are isomorphisms and gluing morphisms, where isomorphisms are given by isomorphisms in the category of finite  $CW$ -complexes  $f : M \rightarrow M'$ , such that  $f|_{\Sigma}$  is an isomorphism from the subcomplex  $\Sigma$  to  $\Sigma'$  preserving the labels of each component, and gluing morphisms are given by the general definition in the previous section.

The objects of the algebraic category  $\mathcal{D}$  are pairs of the form  $(V, x)$  where  $V$  is a vector space over the field of complex numbers  $\mathbb{C}$ , and  $x \in V$ . The endofunctor  $J$ , corresponding to the topological endofunctor  $I$ , is taken to be trivial, i.e.  $J$  is the identity functor. The morphisms of  $\mathcal{D}$  are linear maps preserving the respective elements.

A class of TQFT functors may now be specified as follows. For every  $\Sigma$  we take  $V_{\Sigma} = \mathbb{C}$ . The element of  $V_{\Sigma}$  corresponding to  $M$  is:

$$Z_M = u^{c_1\chi(M)+c_2\chi(\Sigma^{(i)})+c_3\chi(\Sigma^{(o)})},$$

where  $u$  is some fixed nonzero element of  $\mathbb{C}$ ,  $\chi$  denotes the Euler characteristic, and  $c_1, c_2$  and  $c_3$  are fixed unknowns. The assignment  $\varphi \mapsto Z_{\varphi}$  is taken to be trivial, i.e.  $Z_{\varphi}$  is the identity map on  $\mathbb{C}$  for every isomorphism  $\varphi$  of  $\Sigma$ . Finally the evaluation  $e_{V_{\Sigma}} : J(V_{\Sigma}) \otimes V_{\Sigma} = \mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C}$  is given by  $e_{V_{\Sigma}}(x \otimes y) = u^{c_4\chi(\Sigma)}xy$ . The multiplicative property mentioned in the general case, here corresponds to the statement  $\chi(\Sigma \sqcup \Sigma') = \chi(\Sigma) + \chi(\Sigma')$ .

>From these assignments we can determine how  $Z_M$  changes under topological isomorphisms and gluing morphisms. For an isomorphism  $f : (M, \Sigma) \rightarrow (M', \Sigma')$  we get  $Z_M = Z_{M'}$ , since  $Z_{f|_{\Sigma}}$  is the identity. To study the effect of gluing morphisms, let us start, in terms of the framework of section 3, with the case of two separate  $M$ 's,  $(M_1, \Sigma_1^i \sqcup \Sigma_2^o)$  and  $(M_2, \Sigma_2^i \sqcup \Sigma_3^o)$ , which are glued together along their common component  $\Sigma_2$  to give  $(M_1 \sqcup_{\Sigma_2} M_2, \Sigma_1^i \sqcup \Sigma_3^o)$ . In our terms this corresponds to a gluing morphism

$$(f, \varphi) : (M_1, \Sigma_1^i \sqcup \Sigma_2^o) \sqcup (M_2, \Sigma_2^i \sqcup \Sigma_3^o) \rightarrow (M_1 \sqcup_{\Sigma_2} M_2, \Sigma_1^i \sqcup \Sigma_3^o)$$

where  $\varphi$  is the identity on  $\Sigma_2$ . Now  $Z_{(f, \varphi)}$  acts here by multiplication by  $u^{c_4\chi(\Sigma_2)}$ , due to our choice of evaluation, so in the equation

$$Z_{(f, \varphi)}(Z_{M_1} \otimes Z_{M_2}) = Z_{M_1 \sqcup_{\Sigma_2} M_2},$$

we have the exponent of  $u$  on the left hand side given by:

$$c_4\chi(\Sigma_2^i) + c_1\chi(M_1) + c_2\chi(\Sigma_1^i) + c_3\chi(\Sigma_2^o) + c_1\chi(M_2) + c_2\chi(\Sigma_2^i) + c_3\chi(\Sigma_3^o)$$

and on the right hand side by:

$$c_1\chi(M_1 \sqcup_{\Sigma_2} M_2) + c_2\chi(\Sigma_1^i) + c_3\chi(\Sigma_3^o).$$

Due to the formula  $\chi(M_1 \sqcup_{\Sigma_2} M_2) = \chi(M_1) + \chi(M_2) - \chi(\Sigma_2)$ , which already appeared in a slightly different form in section 3, the equation is equivalent to  $(c_1 + c_2 + c_3 + c_4)\chi(\Sigma_2) = 0$ , and since  $\chi(\Sigma_2)$  is not necessarily zero, we get the following constraint on the constants:

$$c_1 + c_2 + c_3 + c_4 = 0.$$

Quinn in his original discussion [14] gave two examples of TQFTs, the one we described in section 3, which he called the Euler theory, and a modified example called the skew Euler theory. In terms of our approach these correspond to two special cases of the above constraint:

$$\begin{aligned} c_1 = -c_2 = 1 \quad \text{and} \quad c_3 = c_4 = 0 & \quad (\text{Euler theory}), \\ c_1 = -c_3 = 1 \quad \text{and} \quad c_2 = c_4 = 0 & \quad (\text{skew Euler theory}). \end{aligned}$$

Replacing  $u$  by  $u' = u^c$  means that one of the unknowns can be set to 1 and the most natural choice is to set  $c_1 = 1$ . So, from the equation above we get

$$c_2 + c_3 + c_4 = -1.$$

Apart from Quinn's two solutions there is a "balanced solution", with  $c_4 = 0$  and  $c_2 = c_3 = -1/2$ , which is halfway between them in the sense that the Euler characteristics of  $\Sigma^{(i)}$  and  $\Sigma^{(o)}$  appear on an equal footing in the formula for  $Z_M$ , but there are many other solutions, even with  $c_4 = 0$ . Any solution with  $c_4 \neq 0$  has the property that the corresponding TQFT does not map the topological morphisms trivially, i.e. does not map all morphisms to the identity map on  $\mathbb{C}$ .

In our framework we can go one step further and consider the effect of self-gluing. First we need a formula to replace the previous one for  $\chi(M_1 \sqcup_{\Sigma_2} M_2)$ . Let  $b_n(M) = \text{rank}(H_n(M))$ , where  $H_n(M)$  is finitely generated, since  $M$  is a finite CW-complex. We have

$$\chi(M) = \sum_{i=0}^{\infty} (-1)^i b_i(M).$$

Suppose we have disjoint subcomplexes  $\Sigma_1$  and  $\Sigma_2$  of  $M$ , and  $\varphi : \Sigma_1 \rightarrow \Sigma_2$  is an isomorphism. Let  $M \xrightarrow{\nu} M_\varphi$  be the canonical map, where  $M_\varphi$  is the identification space under the equivalence relation generated by  $x \sim \varphi(x)$  for any  $x \in \Sigma_1$ . The spaces  $M \setminus (\Sigma_1 \sqcup \Sigma_2)$  and  $M_\varphi \setminus \nu(\Sigma_1)$  are homeomorphic and thus  $H_n(M \setminus (\Sigma_1 \sqcup \Sigma_2))$  is isomorphic to  $H_n(M_\varphi \setminus \nu(\Sigma_1))$ . Now using excision we get  $H_n(M, \Sigma_1 \sqcup \Sigma_2) \cong H_n(M_\varphi, \nu(\Sigma_1))$  and hence  $b_n(M_\varphi) = b_n(M) - b_n(\Sigma_2)$ . Thus, the Euler characteristic formula for self-gluing is:

$$\chi(M_\varphi) = \chi(M) - \chi(\Sigma_2).$$

Now in our TQFT approach, we have a gluing morphism  $(f, \varphi) : (M, \Sigma_1^i \sqcup \Sigma_2^o) \rightarrow (M_\varphi, \emptyset)$ . The equation  $Z_{(f, \varphi)}(Z_M) = Z_{M_\varphi}$  gives rise to the equation for the exponents of  $u$  on either side:

$$c_4 \chi(\Sigma_2) + \chi(M) + c_2 \chi(\Sigma_1) + c_3 \chi(\Sigma_2) = \chi(M_\varphi)$$

and using  $\chi(\Sigma_1) = \chi(\Sigma_2)$  and the above formula for  $\chi(M_\varphi)$ , this corresponds to the same constraint as for mutual gluings

$$c_2 + c_3 + c_4 = -1.$$

It is straightforward to extend the previous discussion to cases where some out components of  $M_1$  and some in components of  $M_2$  remain after gluing, in the case of gluing two  $M$ s together, and some in and out components remain after gluing, in the case of self-gluing of a single  $M$ .

In conclusion, our approach to TQFT allows one to considerably increase the class of Quinn-type examples and extend their range of application. It is our hope that this approach will help clarify some issues in the general theory and specific TQFT models, and will inspire new types of TQFT construction. In future work, apart from giving full details of the definition, we intend to develop other examples, including ones involving embedded topology, like curves in manifolds, or some geometrical features. It is our belief that TQFT is a very profound structure offering a wide range of potential applications still to be explored.

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