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Nélia Amado and Susana Carreira (Editors)

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PREFACE

Innovation, *inclusion*, *sharing* and *diversity* are some of the words that briefly and suitably characterize the ICTMT series of biennial international conferences – the International Conference on Technology in Mathematics Teaching. Being the twelfth of a series which began in Birmingham, UK, in 1993, under the influential enterprise of Professor Bert Waits from Ohio State University, this conference was held in Portugal for the first time. The 12th International Conference on Technology in Mathematics Teaching was hosted by the Faculty of Sciences and Technology of the University of Algarve, in the city of Faro, from 24 to 27 June 2015, and was guided by the original spirit of its foundation.

The integration of digital technologies in mathematics education across school levels and countries, from primary to tertiary education, together with the understanding of the phenomena involved in the teaching and learning of mathematics in technological environments have always been driving forces in the transformation of pedagogical practices. The possibility of joining at an international conference a wide diversity of participants, including school mathematics teachers, lecturers, mathematicians, mathematics educators and researchers, software designers, and curriculum developers, is one facet that makes this conference rather unique. At the same time, it seeks to foster the sharing of ideas, experiences, projects and studies while providing opportunities to try-out and assess tools or didactical proposals during times of hands-on work. The ICTMT 12 had this same ambition, when embracing and welcoming just over 120 delegates who actively and enthusiastically contributed to a very packed program of scientific proposals and sessions on various topics.

The overall theme of ICTMT 12 – *Re-visioning teaching and learning with technology in mathematics* – echoes the importance and the opportunity for reflection on the significant advances already achieved in this area and for their reaffirmation as solid grounds on which to foresee future directions. Subordinate to the more general theme, the contribution of proposals were divided into five specific themes, most of which representing the continuity of previous conferences. A new theme was this time included in relation to the reporting of on-going or concluded technology-based projects.

The four days of sessions at ICTMT 12 offered a rich scientific programme composed of 4 invited plenary talks, 81 paper presentations, 9 workshops and 20 posters, covering the five specific themes – curriculum, teachers, students, resources, and projects.

An inspiring part of the results and insights shared during the conference was provided by the contribution of the plenary speakers – Alison Clark-Wilson (UK), Manuel Santos-Trigo (Mexico), Nathalie Sinclair (Canada), and Susana Carreira (Portugal) who brought forward-thinking on topics as varied as:

- devising strategies to scale students access to dynamic mathematical technology in lower secondary mathematics;
- characterising ways of reasoning that emerge during the construction and exploration of dynamic models of mathematical tasks;
- addressing the affective dimension of students' and teachers' experiences with expressive technologies;

- realizing the extent to which young students are able to take advantage of commonly available technological tools to engage in mathematical problem solving.

For the first time in the history of ICTMT a Portuguese-English Exchange Strand was set up in this conference. The high number of participants from Portuguese-speaking countries and the very importance granted to the full participation and collaboration of Portuguese teachers and researchers were strong reasons for this novelty at a conference which seeks to discuss and improve the quality of mathematics education with technologies, beyond the geographical or language barriers. Thus, submissions were called in Portuguese and the accepted proposals were also presented in Portuguese at the conference, which allowed for parallel sessions in the two languages to be offered in the programme.

All contributions published in these proceedings – whether articles or extended abstracts of workshops and posters – were subject to a process of peer review, led by the members of the International Scientific Committee and carried out by a sizeable pool of reviewers, to whom a sincere recognition is due. The conference proceedings are organized in two parts, the first of which contains all the articles written in English language – Part A – and the second which includes the articles in Portuguese language – Part B. For the sake of inter-understanding and connection between both parts, the reader may find an English translation of the title and abstract of each article that was written and presented in Portuguese.

Altogether, we firmly believe that these proceedings embody multiple perspectives, advances, results, theoretical approaches and innovative proposals on how the learning of mathematics, its teaching, assessment and task design can be transformed and improved both inside and outside school or higher education, through an intelligent, inventive, and stimulating use of the increasingly powerful and widespread technologies of the present.

On behalf of the Organizing Committee, we wish to express our gratitude to all the sponsors that in a committed way accepted to add value to ICTMT 12. We are also truly thankful to the University of Algarve and to its Faculty of Sciences and Technology for the constant support given to the organization of this conference.

Finally, we want to compliment all the authors, including those whom we like to call friends of ICTMT who kindly presented special sessions in the conference. We thank all who came to share their relevant work and their lively and friendly participation, on a few sunny days of the Algarve summer, around the common goal of advancing technology in mathematics education.

The ICTMT 12 Co-Chairs,

Nélia Amado and Susana Carreira

PART A

ENGLISH LANGUAGE CONTRIBUTIONS

PLENARY TALKS

FROM TECHNOLOGICAL INNOVATION IN INDIVIDUAL CLASSROOMS TO LARGE-SCALE TRANSFORMATION OF TEACHING PRACTICES – MIND THE GAP

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The ICTMT conference proceedings chart the development of technology use in mathematics education from the 1990s to the current day. Throughout this period, the prevailing topics for plenaries, papers and workshops have been focused on the development of innovative classroom practices involving 'new' technologies. Alongside this, there has been a slow but emergent theme that has brought aspects of teachers' professional development to the fore - as attempts to scale the widespread use of technology by students have proved both challenging and expensive. In this plenary, I will draw on some personal contributions to ICTMT conferences from the past in order to highlight how my own work now focuses on the design and evaluation of technology-focused professional development for teachers of mathematics. This work is set against the backdrop of the Cornerstone Maths project in England, which is aiming to scale student access to dynamic mathematical technology in lower secondary mathematics in hundreds of schools.

Keywords: Mathematics education, Dynamic mathematical technology, Scaling professional development, Cornerstone Maths

INTRODUCTION

I begin by thanking the conference Chairs, Nélia Amado and Susana Carreira and the International Scientific Committee for the invitation to give a plenary at ICTMT, a series of conferences that I first attended in 2001 as a classroom teacher - and I have not missed an ICTMT conference since that date. I feel a real sense of community at these conferences, which I think is enhanced by the presence of practising teachers and lecturers, technology designers and researchers in both mathematics and mathematics education – a combination that is not often present at other academic conferences. At the heart of this keynote is the notion of innovation, meaning 'to introduce change and new ideas', which has been a common starting point for many of ICTMT's participants over the years since the very first conference in 1993. In this plenary, I chart my personal experiences of innovative technology in mathematics education through my role as a secondary school mathematics teacher, a Head of department and to my current role as an educational researcher whose work focuses on the iterative design of dynamic mathematical technologies, teaching resources and teacher professional development programmes. Of course I do not work alone and although my colleagues over the years are too numerous to mention, my own thinking and practice have been greatly influenced by the enthusiasm, knowledge and insights that Adrian Oldknow, Celia Hoyles and Richard Noss have stimulated.

For me, one innovation in technology that impacted on my mathematics classroom practices came in the form of dynamic geometry software (The Geometer's Sketchpad), that Warwick Evans and Adrian Oldknow had introduced me to in 1997 during my Master's degree course in mathematics education. In my role as a Head of Mathematics Department in an inner-city secondary school, I worked with my colleagues to devise tasks that encouraged students to work collaboratively in pairs at the computer to foster an inquiry-based approach to explore concepts in 2-D geometry (See Figure 1). At that time, I genuinely believed that I was at the beginning of an exciting rethinking of the school mathematics curriculum and, had I been asked to predict what pupils' mathematical experiences in classrooms and examinations would look like in 2015, I would have envisaged some form of dynamic geometry software in widespread use.

Figure 1. Exploring 'z-angles' – A task from my classroom in 1999.

DEFINING SOME IMPORTANT TERMINOLOGY

The research on technology use in mathematics education has tended to evaluate its impact based on one of three aspects: the affective - concerning ideas such as time on task, enjoyment, and motivation; the socio-cultural – including levels of collaboration, group work and the enabling of communities; and cognitive, which concerns the human interactions with mathematical objects such as sliders, syntax, geometric shapes etc. Of course these three aspects are not mutually exclusive and any attempt to evaluate the impact of an innovation is likely to touch on all three. However, in my own work I have always been drawn to the cognitive domain whereby the digital tools shape the mathematics that is learned whilst also opening a new landscape of 'learnable mathematics' (See Kaput, Hoyles, & Noss, 2002).

The ICTMT12 abstracts reveal a wide range of mathematical technological environments in use: Mathematica, GeoGebra, Geometer's Sketchpad, MathPen, CATO, Cubes & Cubes, Dessiner les formes, Gummii, Moodle, Graphing Calculator, TouchCounts, ALEX, Cabri, WinPlot, Wiimotes, Cabri Elem, Tinkerplots, Simply Geometry, Cornerstone Maths, Rhinoceros, R code, spreadsheet… and also some more general environments: Mobile devices, e-learning environment, intelligent support system, blended learning, YouTube clips, applet, widget, Web 2.0, computer games, c-book, CD-ROMs, Khan Academy… This diversity of interpretations of 'technology' can be problematic when we fail to define clearly the characteristics of our chosen technological tools, tasks, and pedagogic approaches – so I begin with some characteristics that underpin my own work.

My colleagues and I feel it necessary to define the technology that we use as 'dynamic mathematical technology' thus:

> transformative computational tools through which students and teachers can (re-)express their mathematical understandings, understandings which are simultaneously externalised and shaped by the interactions with the tools. (Clark-Wilson, Hoyles, Noss, Vahey, & Roschelle, 2015, See also Hoyles and Noss (2003))

So by default, if such technology is transformative, it has to disrupt knowledge and practice!

Alongside this, sits 'innovation'. Although it is easy to make a personal claim that something is innovative, this judgement is actually made by the receiver of the innovation. This explains in part why some many 'innovations' seem to appear and reappear albeit in slightly different guises at subsequent ICTMT conferences – sometimes to the annoyance of more knowledgeable or experienced conference participants. The need for all participants to substantiate any claims of innovation by paying attention to the content of past ICTMT conference proceedings, referencing the wider literature base and building on existing knowledge and practices seems an important one! However, this demand should not be at the expense of encouraging innovation at grass roots level by classroom teachers and lecturers who may not be yet engaged with the more-established research community!

A LONGITUDINAL PROGRAMME OF DESIGN BASED RESEARCH - CORNERSTONE MATHS

The Cornerstone Maths Project is a collaborative design-based-research project between colleagues at SRI, US and London Knowledge Lab, UCL Institute of Education that begun in 2011 with generous funding from the Li Ka Shing Foundation (LKSF). It aimed to capitalise on the outcomes of a number of programmes of research to exploit the dynamic and visual nature of digital technology (DT) in hundreds of mathematics classrooms to stimulate engagement with mathematical ways of thinking by:

- focusing on the 'big mathematical ideas' (linear function, algebraic variable, geometric similarity) that are often considered hard to teach;
- making links between key representational forms;
- providing an environment for students to explore and solve problems within guided structured activities;
- embedding activities within realistic contexts.

The resulting technology-enhanced curriculum units combine specially designed software, pupil workbooks, teacher guides and accompanying synchronous and asynchronous professional development.

The iterative design and evaluation processes have been undertaken in several phases[1]. The research reported in this plenary refers to outcomes of the final phase of the LKSF-funded work and some early findings from the current Nuffield Foundation funded research, which take place in England.

In both cases the theoretical foundations for the Cornerstone Maths curriculum and its accompanying models for teachers' professional development are reported in publications by the respective project teams (Clark-Wilson, Hoyles, Noss, et al., 2015; Geraniou, Mavrikis, Hoyles, & Noss, 2011; Hoyles, Noss, Vahey, & Roschelle, 2013; Mavrikis, Noss, Hoyles, & Geraniou, 2012; Roschelle & Shechtman, 2013; Phil Vahey, Roy, & Fueyo, 2013).

EVIDENCE FROM 'AT-SCALE' USES OF DYNAMIC MATHEMATICAL TECHNOLOGY

The products and processes of scaling a technological innovation

Since 2010, the LKL project team has worked with 18 PD 'multipliers' organised into 13 project networks. These networks have involved at least 417 teachers from 183 secondary schools and over

9500 pupils. This context has enabled us to develop our understand of the process of scaling student access to dynamic mathematical technology.

Scaling (or widening/increasing use) involves both ‘products’ (or measurable outcomes) and ‘processes’ (the means through which these are achieved) (Hung, Lim, & Huang, 2010) and it is highly influenced by the context and culture in which it takes place. In England in 2015, this means:

- There are no recommendations for technology use in mathematics (5-16) – this is a ‘pedagogical’ decision for individual schools and teachers.
- Schools determine their own pathways through the curriculum (using a localised ‘scheme of work’).
- School inspection processes do not focus on the use of technology at a subject level.
- There is very little localised support for teachers of mathematics (i.e. mathematics advisers/consultants).

Our research has concluded a set of quantifiable outcomes that are key to understanding whether our innovation has indeed scaled that are shown in Figure 2.

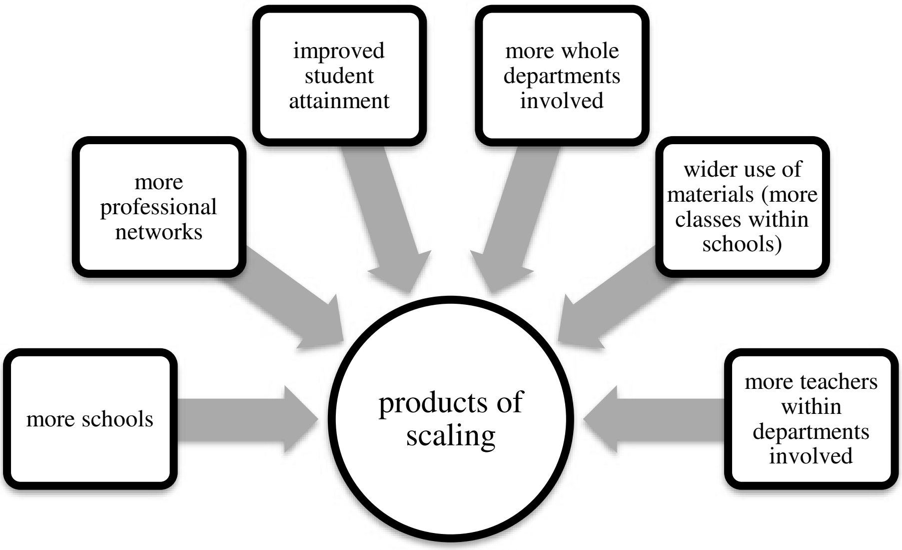

Figure 2 Quantifiable outcomes of scaling the Cornerstone Maths innovation

Alongside this, the all-important processes of scaling are shown in Figure 3. Some of these processes may be common across different countries and cultures; for example, the web-based curricular activity system has proved effective in US studies (P. Vahey, Knudsen, Rafanan, & Lara-Meloy, 2013) and the need for ‘PD multipliers’ (Rösken-Winter, Schüler, Stahnke, & Blömeke, 2015). Each of these processes is articulated in more detail in Clark-Wilson et al (2015).

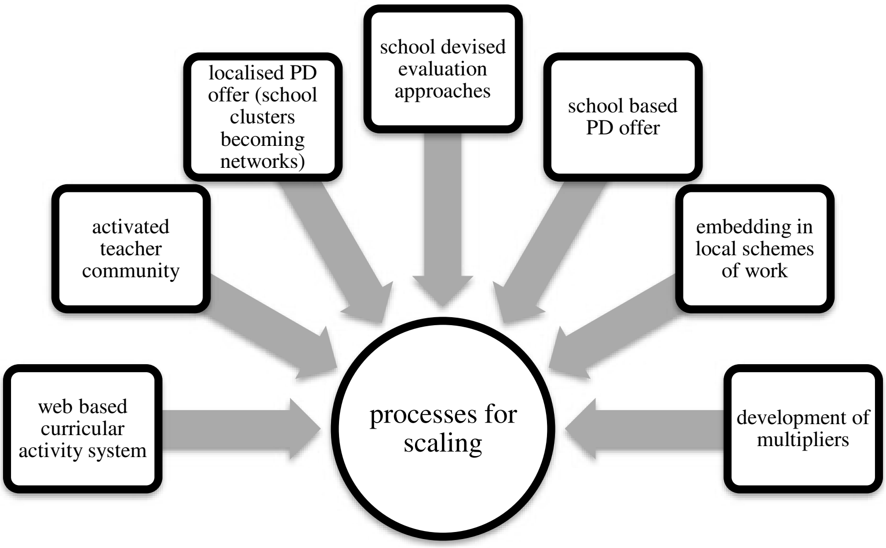

Figure 3 Processes of scaling the Cornerstone Maths innovation

Researching teacher development through a technological innovation

The current research project (funded by Nuffield Foundation[2]) aims to analyse the development of teachers' mathematical knowledge for teaching and associated mathematics pedagogical practice related to the three Cornerstone Maths topics; algebraic generalisation, geometric similarity and linear functions as they participate in professional development and classroom experimentation.

There are many theories that conceptualise aspects of teacher knowledge: pedagogical content knowledge (PCK, Shulman, 1986); it's elaboration as technological pedagogical content knowledge (TPaCK, Mishra & Koehler, 2006); the Knowledge Quartet (Turner & Rowland, 2011); Mathematical Knowledge for Teaching (MKT, Ball, Hill, & Bass, 2005); and Horizon Content Knowledge (Hill, Ball, & Schilling, 2008). The methodological challenge is to devise ways to access such knowledge using survey, interview, classroom observation, video analysis, using critical incidents etc.

In our methodology we begin with Hill et als' MKT: content knowledge relating to topic; teachers' understanding of students' topic-specific knowledge and we add, key representations within the technology and how these relate to each other (mathematically). We focus our research on teachers' (re)design of selected Cornerstone Maths' tasks, which stimulate 'landmark activities' (using a Lesson Study approach) to provide a window on teachers' knowledge and practice as they reflect on ***disruptions*** when embedding digital technology. (The construct of a 'landmark activity' is extended in Clark-Wilson, Hoyles and Noss (2015)).

EMERGING FINDINGS ON TEACHERS' MKT RELATING TO ALGEBRAIC VARIABLE

We report some early findings on teachers' MKT relating to algebraic variable for a cohort of 72 teachers who represent a group that is skewed towards younger (< 10 years) and less experienced teachers (mathematics teaching experience) teachers. (See Figures 4 and 5).

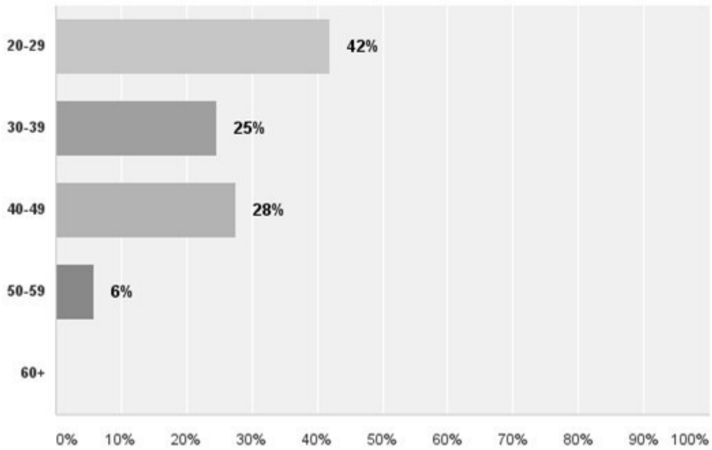

Figure 4 Age demographic of teachers (n=72)

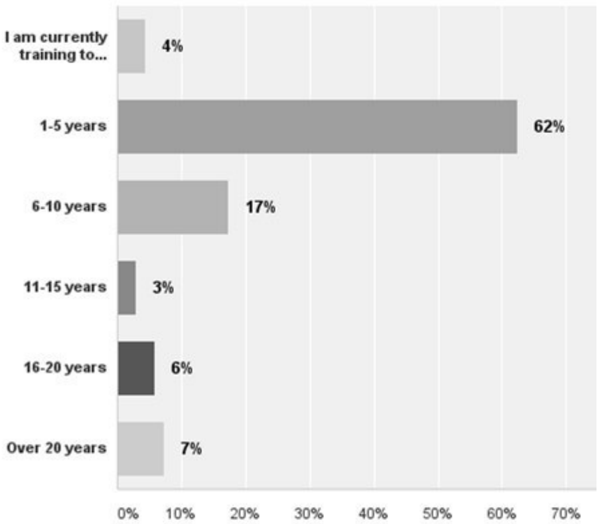

Figure 5 Years of mathematics' teaching experience (n=72)

Perhaps surprisingly for a younger group of teachers, they report little use of dynamic mathematic technology in their teaching, with 88.4% of the teachers reporting that they had *never* or *only occasionally* given their students such opportunities in lessons.

A baseline item to assess teachers' MKT related to algebraic variable asked for their personal definitions, which were categorised using Küchemann's interpretations of the use of letters in school algebra (Küchemann, 1981).

Of the 69 teacher respondents:

- 19% gave definitions that suggested that letters represented a specific unknown - 'An algebraic variable is when a letter is used to express a number we don't yet know. The variable is usually separated by an operation symbol such as plus, minus multiply or divide sign. Example x + 3 = 7 where x is the variable;
- 67% gave definitions that suggested that the value of a letter can vary – 'A real life value that can change (eg. Temperature, time, cost etc), represented by a letter';

- 6% gave richer definitions that considered the mathematical domain – 'An unknown that can take on a range of values'.
- 8% of the teachers gave incorrect or ambiguous responses – 'Algebraic variables consist of letters and numbers. Take for example 2a + 3b. 2a is an algebraic variable 3b is another algebraic variable'.
- Furthermore, when asked to rate their level of confidence in the response that they had provided for this item, 48% of the teachers stated that they were 'not at all confident' or only 'quite confident' with their definitions. One teacher clarified this by saying 'I feel much more confident in my understanding of how algebra works than my ability to explain it. There are so many different and interconnected ideas that I find it hard to choose an explanation, which is both accessible and correct'.

A key element of the initial face-to-face PD event for these teachers was discussion time during which a range of definitions were presented and discussed. This discussion was extended during a PD task where teachers worked within the Cornerstone Maths 'Patterns and expressions' software in an activity that required them to create a simple repeating pattern and 'unlock' the value of the 'No of (repeating) blocks' such that this number can be varied by using a dynamic slider, which again revealed uncertainties in the action of 'naming' of this algebraic variable and the teachers' interpretations of its mathematical meaning. Further more, we research how teachers lessons plan for the same task are conceived as a window on their developing pedagogic practices when using dynamic mathematical technology with pupils. Although these are tentative early findings, our early work is promising with respect to our project's aims and further outcomes from this project will feature in future publications.

TOWARDS ICTMT 22… NEW VISIONS FOR MATHEMATICAL TECHNOLOGY

In concluding my plenary paper, I return to my own starting point (some 14 years ago) when I first presented my own 'innovative' classroom practice to an ICTMT conference audience. In the intervening time my professional pathway has led me away from my own classroom practice to focus more on how large-scale teacher development projects and processes might provide the best conditions possible for more students of mathematics to experience a dynamic technology-enhanced mathematics curriculum.

I challenge the ICTMT audience to consider the following questions[3]:

What impact do you want your work (research, innovation, products) to have on learners'/teachers' mathematical experiences?

What will be <u>your</u> legacy at ICTMT22 in 2035? - and for the younger members of the ICTMT community, what will you be presenting?

Notes

1. **Li Ka Shing Foundation funded work**: Planning Phase 1 (Jun-Jul 2011); Pilot Phase 1 (Jul – Dec 2011); Pilot phase 2 (Jan – Jul 2012); Phase 3 (Dec 2012 – Nov 2014)

Nuffield Foundation funded work: Dec 2014 – Nov 2016 *Researching impact on teachers' mathematical knowledge for teaching (MKT) and practice.*

2. See http://www.nuffieldfoundation.org/developing-teachers-mathematical-knowledge-using-digital-technology.

3. You are welcome to contribute your own reactions by posting a comment to the web-page http://bit.ly/ictmt12Vision

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TRADITIONAL MATHEMATICAL PROBLEMS UNRAVELLED WITH UNTRADITIONAL TOOLS: HINTS FROM THE PROBLEM@WEB PROJECT

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Abstract: The fact that using digital technologies for tackling mathematical problems has a transformative effect on the nature of problem solving has been emphasised by many researchers in mathematics education. But how such transformation takes place in the solving of mathematical problems is still vague. At the same time we are witnessing the growing impact of digital technologies in the lives, activities and daily forms of communicating and accessing information of the young generation. They are seen as fluent in the use of digital tools, and their familiarity with such tools is increasingly common. What we still need to realise is the extent to which they take advantage of commonly available technological tools to engage in mathematical problem solving. The Problem@Web project gave us an opportunity to address this question by looking at children involved in two online mathematical competitions – the SUB12 and the SUB14. In this talk I will bring the idea of solving-and-expressing as a fundamental unity for examining the role of technologies in the ways students *see* the solution to a problem and *express* it with digital media.

THE COORDINATED AND SYSTEMATIC USE OF DIGITAL TECHNOLOGIES TO FOSTER, REFINE AND EXTEND STUDENTS' PROBLEM SOLVING EXPERIENCES

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Abstract: Mathematical problem solving is, has been, and will continue to be a prominent and evolving research and practice domain in mathematics education worldwide. What does it mean for students to engage in problem solving activities? What are the common problem solving principles that support and guide students' development of mathematical competences? What tasks are important to promote students' problems solving experiences? To what extent the students' coordinated use of different digital technologies offers them affordances and opportunities to represent, explore, and solve mathematical tasks? I will address these types of questions presenting and discussing exemplars to illustrate how both multiple purpose and ad hoc technologies can be used to frame, foster, and promote students problem solving approaches. In particular, I will characterise ways of reasoning that emerge during the construction and exploration of dynamic models of mathematical tasks.

TECHNOLOGY TO THINK AND FEEL WITH

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Abstract: Beginning with Papert, mathematics educators have acknowledged that the power of digital technologies in mathematics learning involved both cognitive and affective dimensions of experience. While many researchers who study the use of expressive technologies care strongly about the affective dimension, their papers and articles tend to focus on the cognitive - often with only a footnote or concluding remark on the students' pleasure or excitement. In this talk, I will discuss some of the challenging of doing research that adequately addresses the affective dimension of students' and teachers' experiences with expressive technologies, and then present some new theories that could support this work. Using examples of research involving young children engaging with TouchCounts, I will illustrate how these theories can provide insight into the way students make mathematical sense.

PAPERS

Theme: Curriculum

COMPUTER-AIDED EXPLORING THE MATHEMATICS BEHIND TECHNICAL PROBLEMS – EXAMPLES OF CLASSROOM PRACTICES

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The mathematics in technical problems can be discovered by computer-aided experiments. The framework is the Bachelor of Science (BSc) scheme in Computational Mathematics for "Virtual Product Development". The required mathematical competencies as well as the required competencies in the application field of engineers have to be taken into account. The didactical concepts and the learning environment play an important role. Examples are presented from four different courses in the areas of statics, elasticity, finite elements and partial differential equations. It will be reported on the implementation within the curriculum at Beuth University of Applied Sciences Berlin, the classroom experiments and the teacher's role.

Keywords: Mathematics, computer experiments, simulations, Mathematica software system®

BSC COMPUTATIONAL MATHEMATICS FOR "VIRTUAL PRODUCT DEVELOPMENT" (VDP)

Beuth University of Applied Sciences Berlin offers an undergraduate course in Computational Mathematics. The programme is offered in 7 semesters and leads to the award of a Bachelor of Science. The study aim is to qualify for a job opportunity in industry as a mathematician. The first three semesters consist of the basic modules Calculus, Linear Algebra, Computer Science/Programming, Numerical Methods, Probability Methods, and Discrete Methods. In the following three semesters students can specialize in computational engineering that offers job opportunities in the area of virtual product development as a simulation expert. Industrial branches are e.g. biomechanics/medical physics and automotive industry. The required mathematical competences are modelling, programming, simulation/calculation, visualization of big data, image processing, and computer aided geometric design (CAGD) as well as finite element methods (FEM). The required competences of the application field of engineers are modelling, computer aided design (CAD), FEM and the language/notion and way of thinking of engineers.

Within this specialization there is a didactic unit of four modules comprising 20 European credit points (ECTS): The contents of the module Engineering Mechanics (10cr) are Statics and Mechanics of Materials, as well as the language/notion of engineers, and also mathematical models in this area. The contents of the module Finite Element Methods (10cr) are one- and two-dimensional FE-structures as rod, beams, membranes, plates as well as numerical methods for general boundary value problems.

The learning environment is computer based: 50% lectures and 50% exercises in a computer laboratory of maximal 20 students. The author has 10 years industrial experience at Airbus Company in Hamburg, Germany in the department of structural dynamics including aeroelasticity.

Within these modules Engineering Mechanics and Finite Element Methods mechanical problems will be analyzed with different means: analytical/algebraic solutions by hand and by computer algebra systems (CAS), programming and visualizing using CAS (algebraic, graphical) as well as professional Finite Element software (numerical, graphical) as a substitute for experiments in the area of engineering mechanics. With computer simulations one can illustrate mathematics by examples from mechanics, show in mechanical problems mathematical structures and discover mathematical questions behind real life situations.

This will be demonstrated by four examples of classroom experiments that students have to perform with the computer algebra system MATHEMATICA® of Wolfram (2015). The university department of Mathematics has Mathematica® network licenses in the computer laboratory. Other examples of experiments are given by Kalus (1998) and Kalus, Karsai, Rács & Schwenk (2006).

ORDINARY DIFFERENTIAL EQUATIONS OF BEAM STRUCTURES – BOUNDARY CONDITIONS

In the module Engineering Mechanics beam structures are analyzed. If the deflection w is four times differentiable in the domain of definition then the equilibrium differential equation of a Bernoulli beam is $EI\, w^{(4)}(x) = q(x)$, with x as the coordinate along the beam axis, $EI \in \mathbb{R}^+$ as bending stiffness and q as the line force function acting vertically to the beam. A unique solution requires boundary conditions. These can be interpreted mechanically. A learning target is to set up or to model these boundary conditions mathematically from an engineering sketch.

In figure 1 an example of a beam structure with a hinge, a single force F and a constant line force $q_0 \in \mathbb{R}$ acting only in a part of the structure is given. Furthermore at both ends the beam is fixed or clamped suppressing the deflection and bending at the ends.

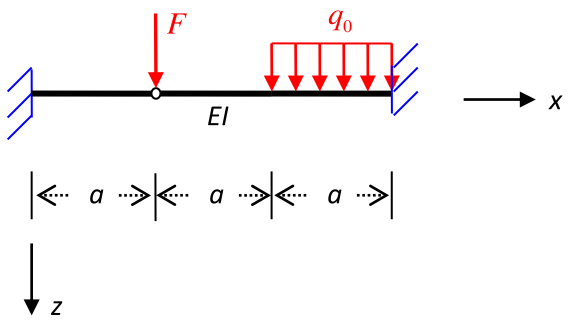


Figure 1. Beam structure

The two types of boundary conditions can be well distinguished in mechanical language as well as in mathematical language: geometric and static resp. Dirichlet and Neumann boundary conditions. Students can benefit from these two languages in solving the modelling task. The mechanical interpretable shear force Q is proportional to the third derivative of the deflection w : $Q(x) = -EIw'''(x)$. The mechanical interpretable bending moment M is proportional to the second derivative of the deflection $w : M(x) = -EIw''(x)$. At the position of a hinge the first derivative of w is continuous but not differentiable and no bending moment will be transmitted. At positions of a single force the shear force Q is not continuous and the value of the force is the jump height.

The mathematical model consists of three differential equations in the three open intervals:

$$EI\, w^{(4)}(x) = 0, \forall x \in \left]0, a\right[;\; EI\, w^{(4)}(x) = 0,\; \forall x \in \left]a, 2a\right[;\; EI\, w^{(4)}(x) = q_0, \forall x \in \left]2a, 3a\right[$$

The connection is given by 8 transition conditions. Furthermore there are four boundary conditions. These 12 conditions match with the number of integration constants:

$$w(0)=0,\ w'(0)=0,\ w(3a)=0,\ w'(3a)=0,\ \ w^{(i)}(2a^-)=w^{(i)}(2a^+),\ \ i=1,2,3,4$$
$$w(a^-)=w(a^+),\ -EIw''(a^-)=0, -EIw''(a^+)=0,\ \ -EIw'''(a^-)=-EIw'''(a^+)+F$$

where $w\left(a^-\right)=\lim\limits_{\substack{x\to a\\ x<a}} w(x)$, $w\left(a^+\right)=\lim\limits_{\substack{x\to a\\ x>a}} w(x)$ denote the left- respectively the right side limit of the function w at the point a.

The students have to set up these conditions and solve the system algebraically with MATHEMATICA®. Furthermore they are asked to plot the deflection function w, the bending moment function M and the shear force function Q for different values of the load variables. The result can be seen in figure 2. The deflection, bending moment and shear force are shown from left to right. Students are asked to match these graphics with their knowledge from engineering mechanics.

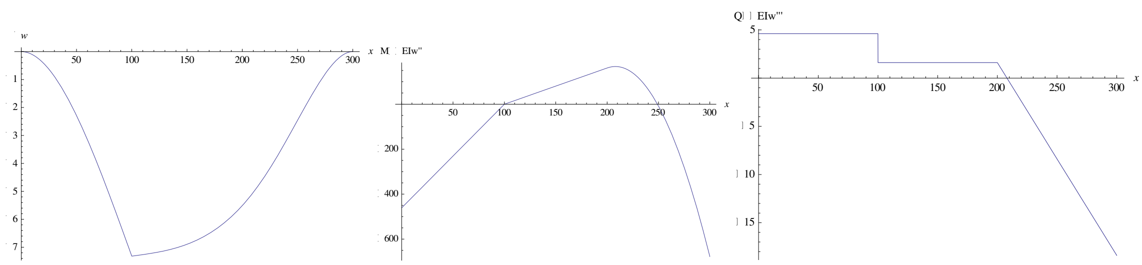


Figure 2. Deflection, bending moment, shear force

The MATHEMATICA® code of this example can easily be modified for problems of similar type.

MULTI-BODY STRUCTURES - DETERMINACY

A fundamental concept in engineering mechanics is the classification of multi-body structures by statical and kinematical determinacy resp. indeterminacy. The criterion is the question of the solution of the unknown support and joint reactions from the equilibrium equations of all parts of the structure. Students should be able to give a reliable classification by looking directly at the engineering sketch. This classification in engineering language corresponds to a similar classification in mathematical language namely the rank of a matrix that is built up from the equilibrium equations. On the other hand the abstract notion of the rank of a matrix can be exemplified by multi-body-structures.

A multi-part structure is called kinematically determinate if it doesn't allow any rigid body motions as a whole and of its parts. Otherwise it is called kinematically indeterminate. Structures that are kinematically determinate can further be subdivided as statically determinate und statically indeterminate. A special case of a kinematically indeterminate structure is called wobbly. Figure 3 shows a single-body structure consisting of a rigid rectangular membrane subjected to a typical load case. The left picture shows a hinged and a simple support at the lower corners that make the structure statically determinate while the membrane in the right picture has two hinged supports making it statically indeterminate.

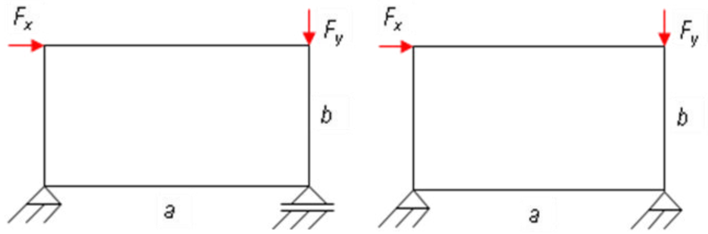


Figure 3. Kinematically determinate: $\text{Rank} = m = 3$**, statically determinate:** $m = n = 3$**, statically indeterminate:** $m < n = 4$ **i.e.** $\det \neq 0$

Figure 4 shows same structure with three different supports making it kinematically indeterminate. These three cases represent typical situations. The left has insufficient supports allowing a rotation of the structure. The middle has too many but redundant supports. An exceptional case depicts a wobbly structure allowing an infinitesimal rotation about the lower left corner.

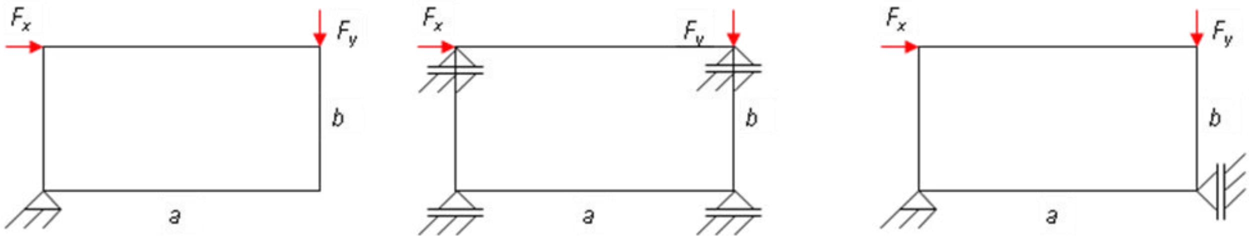


Figure 4. Kinematically indeterminate: $2 = \text{rank} < m = 3$**, rotation:** $n < m = 3$**, displacement:** $m < n = 4$**, wobbly:** $m = n = 3$ **i.e.** $\det = 0$

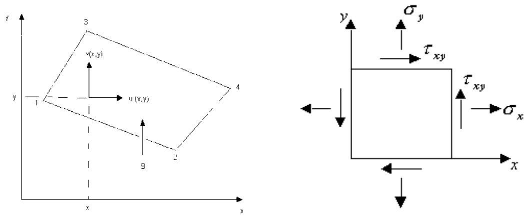
The corresponding mathematical concept is the rank of a (n, m) - matrix where n is the number of equilibrium equations and m the number of unknown supports and joint reactions. In two (three) dimensions the number of equilibrium equations is three (six) times the number of parts in the multi-body structure. There are translations in the direction of the axes and rotations about the axes.

If the rank is equal to m, implying $m \leq n$ then the structure is kinematically determinate. If $m = n$, i.e $\det = 0$ it is statically determinate otherwise if $m < n$ it is statically indeterminate. If the rank is less than m there is a mechanism in the structure meaning kinematical indeterminacy. The three cases $m > n,\ m < n,\ m = n$ can be distinguished.

The students have to set up the equilibrium equations, determine the rank and connect the mathematical results to the physical situation and vice-versa.

ELASTIC MEMBRANE – SOME ANALYTICAL ASPECTS

The left picture of figure 5 left shows an elastic membrane consisting here of a quadrilateral in the x-y-plane. The vector valued displacement function is $\begin{pmatrix} u \\ v \end{pmatrix}$ defined on the set of points of the membrane. Thus $u(x, y),\ v(x, y)$ are the displacements in x- respectively in y-direction of a point (x, y) of the membrane. Only classical linear theory is considered.



$$\frac{\partial u}{\partial x} = \varepsilon_x, \ \frac{\partial v}{\partial y} = \varepsilon_y$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}$$

Figure 5. Elastic membrane: Deformation, stress, strain

The equations that determine the displacement vector consist of three sets of coupled equations:

At each point (x, y) of the membrane there are the equilibrium conditions of the local stress functions $\sigma_x, \sigma_y, \tau_{xy}$, the normal stresses in the *x*- and *y*-direction and the shear stress, with an external volume force. They form two linear partial differential equations of first order for the three stress functions. The meaning of these stress quantities is indicated in middle picture of figure 5 which shows an infinitesimal element of plane stress.

The kinematic equations relate the strain functions $\varepsilon_x, \varepsilon_y, \gamma_{xy}$, the normal strains in the *x*- and *y*-direction and the shear strain, with the displacement functions u, v. The right picture of figure 5 shows the linear relationship, i.e. linear combinations of normal derivatives, consisting of three equations between 5 unknown functions.

Finally the experimentally verified material law relates the strain functions to the stress functions. This results in three linear equations between 6 unknown function variables.

Altogether there are 8 equations for 8 functions $u, v, \varepsilon_x, \varepsilon_y, \gamma_{xy}, \sigma_x, \sigma_y, \tau_{xy}$ of two variables *x* and *y*. These 8 equations can be put together resulting in two coupled linear partial differential equations of second order for the two scalar deformation functions *u* and *v*. In order to obtain a unique result boundary conditions are needed. For a special geometry, e.g. a rectangular domain and special boundary conditions an analytical solution of this system of partial differential equations can be found. The discovery of the mathematical solution can be directed by a good understanding of the physical problem. On the other hand the mathematical solution techniques for these special cases are well known and can be learnt. The point is that these mathematical techniques are quite natural from the physical point of view.

The four cases of boundary conditions are of the Neumann type in mathematical language. In engineering language these are load conditions due to the fact that the stresses are linear combinations of derivatives of the scalar displacements which can be seen from the kinematic equations and the material law.

Figure 6 shows the rectangular under uniform tension in the left picture while the right picture shows the second load case of uniform shear loading.

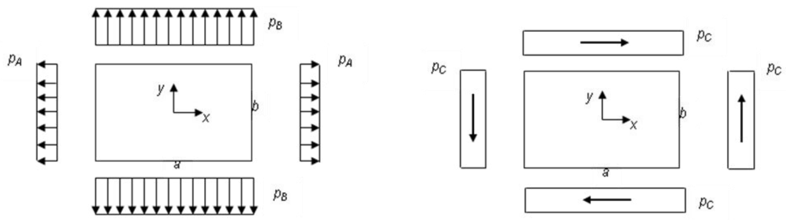


Figure 6. Elastic membrane: Tension, shear loading

The third and fourth set of boundary conditions is given in figure 7: Bending loading on the left and a special shear bending relating p_E and p_F on the right.

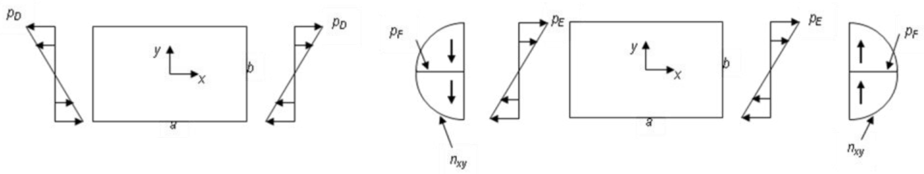


Figure 7. Elastic membrane: Bending, special shear bending loading

Technical considerations immediately show that the solution in any of these cases is not unique because rigid body motions are possible, i.e. translation in x - and y - direction as well as a rotation. This means that there will be three integration constants in the solution. This is due to the fact that there are no geometric boundary conditions in engineering language or Dirichlet boundary conditions in mathematical language. Mathematically speaking the boundary problem is not well posed. Students have to get this link between the two languages and shall see the advantage of understanding the technical problem for finding a mathematical solution.

Secondly, technical considerations lead in each case to appropriate mathematical assumptions on the stress functions in the interior of the rectangle which is possible because of the special case of boundary conditions that give the stress functions on the boundary. An easy interpolation can be made. However, the mathematical correct formulation of the boundary conditions from the picture has first to be set up in the *x*-*y*-coordinate system. Students have to remember the local sign convention of the stress taking into account the positive and negative face of a cutting line characterized by the normal vector that points outward from the interior of the rectangle. This is a modeling aspect. The assumption for the stresses has to be matched with the 8 primary governing equations to solve for die displacements u and v by direct integration.

For example the analytical solution of the displacement function for the forth load case of special shear bending is shown in figure 8. The rigid body motion parameters can clearly be identified: α for the rotation and a_x, a_y for the translations. E, G and ν are material parameters.

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \frac{4\,p_F}{E\,b^2} \begin{pmatrix} x^2 y - (2+\nu)\,y^3/3 \\ -\nu\,x\,y^2 - x^3/3 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ p_F/G & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Figure 8. Elastic membrane: Analytical solution

Figures 9 and 10 give the graphical illustrations of the solutions in the four loading cases however, with appropriate geometrical, i.e. Dirichlet boundary conditions. The vector valued displacement function is represented as a vector field. These graphical representations fit to the technical concept of the boundary loading cases. The students use MATHEMATICA®.

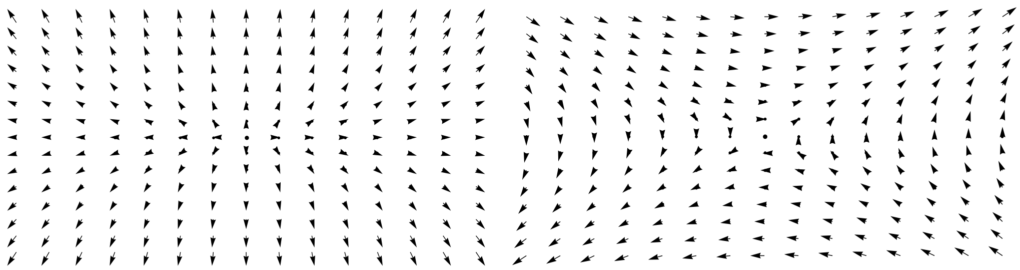


Figure 9. Elastic membrane: Deflection under tension and shear loading

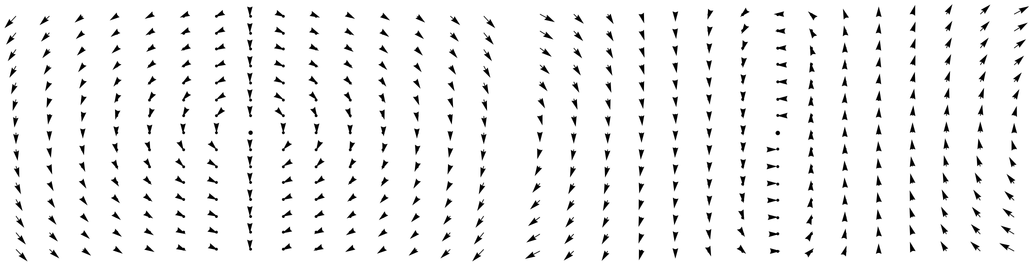


Figure 10. Elastic membrane: deflection under bending and special shear bending loading

ELASTIC MEMBRANE – SOME ASPECTS OF FINITE ELEMENT METHODS

The rectangular membrane of the last two sections is again used but in the context of finite elements. Figure 11 shows a four-noded membrane element in two dimensions. Each node has two degrees of freedom, i.e. displacements in *x*- and *y*-direction denoted by $u_i, v_i,\ i = 1,2,3,4$. This can be written as a displacement vector $\{U\} \in \mathbb{R}^8$. The interaction to the surrounding is given via the nodes. A force can act on each node $Fx_i, Fy_i,\ i = 1,2,3,4$. The total force vector on all nodes is noted by $\{F\} \in \mathbb{R}^8$. Hooke's law establishes a linear relationship between the displacement and the force vector: $[K]\{U\} = \{F\}$ where $[K] \in \mathbb{R}^{8\times 8}$ is denoted as stiffness matrix.

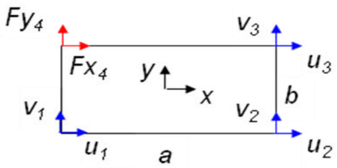


Figure 11. Elastic membrane: Finite Element Method

The students connect their knowledge in engineering science with their knowledge in Linear Algebra. The stiffness matrix $[K]$ is symmetric, positive semidefinit and the dimension of its kernel is three. From mechanical engineering they realize without calculating that a basis for the kernel of the stiffness matrix $[K]$ consists of the three rigid body motions, i.e. translations in *x*- and *y*-direction and a rotation. Hooke's law determines the possible displacement vector holding the structure in an equilibrium state with a given force vector. In the kernel of $[K]$ are those displacement vectors $\{U\}$ that hold the structure in an equilibrium state for the external force vector being the zero vector. This means that no internal deformation of the structure is possible but only rigid body motions.

Students also have to calculate all eigenvalues and eigenvectors and to visualize the eigenvectors with MATHEMATICA®. The result is shown in figure 12. The eight pictures correspond to the eight eigenvectors. The undeformed square (blue) is shown together with the deformed shape (red).

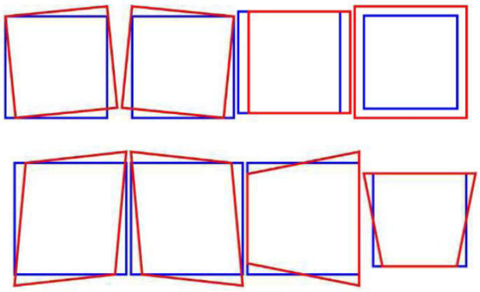


Figure 12. Elastic membrane: Exact eigenvectors, eigenvalues 0,0,0,2,2/3,2/3,5/9,5/9

The corresponding eigenvalues from left to right starting with the upper row are

$0,0,0,2,\frac{2}{3},\frac{2}{3},\frac{5}{9},\frac{5}{9}$ where academic material properties and geometric data have been chosen in order to get nice numbers. The third picture clearly shows the horizontal rigid body motion. Students should remember that the basis for the zero eigenspace is not unique and that one can chose another basis. The vertical displacement and the rotation shown in figure 13 can be constructed by an appropriate linear combination from those MATHEMATICA® has calculated. The sum of the first two gives the vertical translation and the subtraction of the second and third from the first gives the rotation.

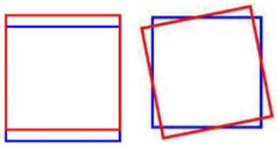


Figure 13. Elastic membrane: Linear combination for rigid body modes

In finite element theory the stiffness matrix $[K]$ for a rectangular membrane is calculated by a two-dimensional integration of polynomials of maximal degree of two in any coordinate direction. For a general quadrilateral the integrand is a rational function of polynomials and a numerical integration is required. Appropriate is the Gauß integration formula. For huge systems with several hundred thousands of elements a lot of computer time is needed. A transient crash calculation that uses an additional loop over the time steps will need some days of execution time. To reduce the calculation time an underintgration for the calculation of the stiffness matrices is used. One uses only one integration point in the middle of the element area. This speeds up the calculation time however, one has to control the error. This is state of the art in automotive industry.

In our teaching environment students integrate the above mentioned stiffness matrix numerically with a one point Gauß rule and get a different stiffness matrix $\{K1\}$. The calculated eigenvectors are shown in figure 14. The observation is that the dimension of the kernel of $\{K1\}$ is 5. Again one can calculate the rigid body motions shown in figure 15 from those MATHEMATICA® has calculated.

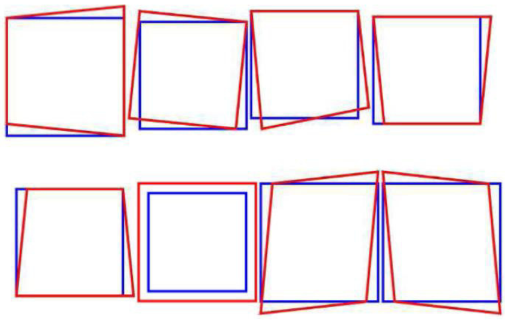


Figure 14. Elastic membrane: Eigenvectors due to underintegration, eigenvalues 0,0,0,0,0,2,2/3,2/3

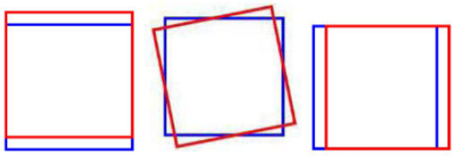


Figure 15. Elastic membrane: Linear combination for rigid body modes

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INCORPORATING GAME APP A.L.E.X. INTO EXISTING MATHEMATICS CURRICULA: AN EXAMPLE FROM PRIMARY SCHOOL GEOMETRY

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This paper accounts on the main experiences gained from a study which incorporated A.L.E.X., an educational puzzle game available on iPad or Android tablet devices, within the primary school geometry. The study took place in a public primary school in Cyprus. A group of fifteen (n=15) Grade 6 pupils (8 boys and 7 girls; aged 11-12), was randomly selected to consist the sample. The A.L.E.X. application, accompanied with a worksheet, constituted the official medium of teaching. The design of the worksheet was such as to integrate technology with the measurement of the perimeter and area of rectangles. While working with A.L.E.X., children identified and processed geometrical principles that emerged spontaneously. These results concur with those of previously conducted studies which suggest that game apps can be used in the mathematics classroom as the machinery for children to become reflective and self-directed learners.

Keywords: Mobile technologies, apps, primary mathematics

INTRODUCTION

Mobile mathematics learning has lately attracted considerable attention by the mathematics education community. The significant potential of tablets and other mobile devices as ubiquitous tools that can radically transform and enrich both formal and informal mathematics learning is underlined in the existing literature (Clark & Luckin, 2013; Melhuish & Falloon, 2010). However, the amount of available primary research studies on the integration of mobile devices such as tablets and android tablets in the classroom and beyond is still relatively small due to the novelty of these technologies.

The current paper summarizes the experiences gained from a study which incorporated A.L.E.X., an educational puzzle game available on tablet devices, within the primary school geometry. A.L.E.X., which uses programming logic in a game setting, belongs to the constantly growing list of educational apps that aim at developing young children's rudimentary programming concepts through games. In this study, we explored ways of using this coding game app as a tool for engaging students in authentic geometry problem solving activities that can help raise their intrinsic interest in mathematics, and promote the attainment of important competencies essential in modern society.

LITERATURE REVIEW

The ease of use of tablet devices stimulates more exploratory behaviour and engagement compared to desktop computers (Chau, 2014), resulting in a very large percentage of even very young children being frequent users of smart mobile devices. As research indicates, the main activity in which children engage when using mobile devices is to play games (Common Sense Media, 2013). Responding to this trend, there has been an explosive growth in the number of educational games apps targeting children available on the market (Chau, 2014).

The increased popularity and proliferation of digital games, has led to a widespread interest in their integration into the mathematics curriculum. Several mathematics educators (e.g. Ke, 2008; Meletiou-Mavrotheris, 2013) have been experimenting with digital games, investigating the ways in which this massively popular worldwide youth activity could be brought into the mathematics classroom in order to capture students' interest and facilitate their learning of mathematical concepts. The research literature suggests that digital educational games have many potential benefits for mathematics teaching and learning. One of their foremost qualities is the capacity to motivate, to engage and to immerse players (Felicia, 2009). Educational use of games is an effective means of improving students' attitudes towards mathematics. It has been shown that educational games captivate students' attention, contributing to their increased motivation and engagement with mathematics (e.g. Ke, 2008). When playing games, children have to diagnose problems, make conjectures, plan and carry out investigations to test their conjectures, distinguish alternatives, construct models, debate with peers, and form coherent arguments (Felicia, 2009). This supports the development of valuable mathematical problem-solving skills such as strategic thinking, planning, multi-tasking, self-monitoring, communication, negotiation, group decision-making, pattern recognition, accuracy, speed of calculation, and data-handling (Miller & Robertson, 2010).

While digital educational games do have lots of potential benefits for mathematics teaching and learning, not all the available game apps are designed to promote optimal development among children. Nonetheless, although less commonly developed than hoped, some exceptional exemplars of developmentally meaningful mobile education game apps that can help create constructive and valuable learning experiences for children do exist (Chau, 2014). One promising type of game apps, are coding apps which teach children the concepts behind programming in a playful context. With an increasing focus on programming and coding finding its way onto the curriculum in many different countries across the world, some innovative, educationally sound game-based apps that support the development of computer programming skills from a young age have begun to appear. Several educational apps are currently available for helping children with no coding background or expertise, grasp the basics of programming through the exploration and/or creation of interactive games and other applications (e.g ScratchJr, ALEX, Hopscotch, Move the Turtle, Light-Bot, Bee-Bot, Daisy the Dinosaur, Kodable, etc.). Often, coding game apps enable children to share their games with others, and to play or edit games programmed by others.

Having taken their inspiration from Logo (Papert, 1980), educational coding apps promote a constructionist approach to tablet use, with the emphasis being on students using tablets to become creators instead of consumers of computer games. In addition to the provision of a highly motivational and practical approach for introducing children to computer programming and developing their computational thinking (Wilson, Hainey & Connolly, 2012), coding game apps provide rich opportunities for the reinforcement of problem-solving, critical thinking, and logical thinking skills (e.g. sequencing, estimation, prediction, metacognition) that apply across domains. At the same time, they can also be helpful in developing subject-specific knowledge in different domains including mathematics. As findings of several studies conducted in the past using mainly the programming language Logo have indicated, there are strong connections between thinking processes of learners during writing their own computer programs and many aspects of mathematical thought (Aydin, 2005). Programming provides an ideal environment for expressing and experimenting with mathematical ideas, for making abstract mathematical ideas more concrete (Aydin, 2005). The design, coding, revision, and debugging of computer commands, helps students develop higher order mathematical problem solving skills such as deductive reasoning (Subhi, 1999) and metacognition (Clements & Nastasi, 1988), while at the same time improving their conceptual understanding of key mathematical ideas. Researchers have found that programming using constructionist environments like Logo increases, among others, students' understanding of arithmetic and measurement processes (Clements, Battista, & Sarama, 2001), their algebraic reasoning (Sutherland, 1994), and their general geometry abilities (Clements, Battista, & Sarama, 2001). Thus, it becomes crucial to incorporate computer programming into existing mathematics curricula. Coding game apps provide an ideal opportunity for doing so in an engaging, non-threating, and child friendly manner.

METHODOLOGY

The exploratory study described in the current paper took place in a public primary school, located in a village of Cyprus. The majority of its students come from families with low socioeconomic status and income, who have negligible or no experience with mobile technology. Dropouts before high school graduation constitute a usual phenomenon among the area population and this stance is mirrored in parents' limited interest in their children's educational attainment. The researchers knowingly selected such a context to orchestrate a teaching intervention. Their goal was to explore the potential of tablet technologies for providing students with fundamental geometry knowledge, skills and confidence in doing mathematics. Among the school community, a group of fifteen pupils (8 boys and 7 girls) in Grade 6 (aged 11-12), were randomly selected to consist the sample. One of the authors assumed the role of the teacher and organized at random the class intro three groups of five. Each group was given an iPad through which participants could have access to the A.L.E.X application.

A.L.E.X. Game App

A.L.E.X. is a fun programming puzzle game that lets players control a robot along a path. It is free educational app suitable for downloading on iPad or Android tablet. The game is all-ages friendly. The lower levels of the games are suitable for children as young as six, while the higher levels might be challenging even for high school students or adults. It has the potential to tacitly promote a number of concepts and procedures embedded in the school mathematics curriculum. This becomes feasible by offering the user the opportunity to think and plan logically as he or she programs robot A.L.E.X. (see Figure 1) with a sequence of commands, in order to get through each level from start to finish.

Figure 1: Robot A.L.E.X.

The game has two modes, Play and Create. In the Play mode, players complete standard puzzles using the pieces provided to them. The Create Mode includes feature for players to create their own puzzle. In this mode, players can devise their own levels by structuring the pathways they would like A.L.E.X. to follow (see Figure 2), and play through their own levels.

Figure 2: Users' potentiality to "create" their own levels

The directions A.L.E.X. could follow are simple and symbolically expressed. For instance, the commands "turn left, right" or "go forward" could be given when one touches the game's screen on the particular arrow pointing to the respective direction (see Figure 3).

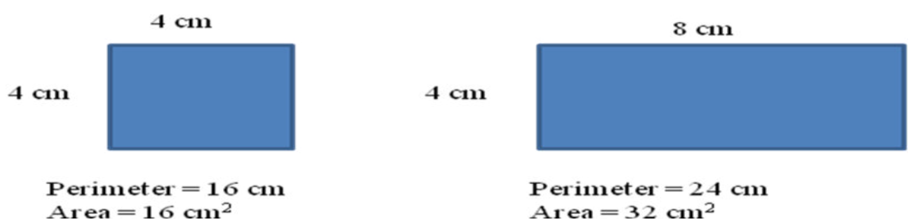
Figure 3: Screen's display of commands

The application A.L.E.X. accompanied with a worksheet (see Figure 4) constituted the official medium of teaching. The design of the worksheet was such as to integrate technology with the measurement of the area and perimeter of a rectangle. Geometry, closely related to the constructionist approach of game app A.L.E.X., was supposed to serve as an attractive basis for revealing children's connected, but temporarily hidden, mathematical concepts and thinking skills.

Task 1

Read the following scenario and answer the questions that follow.

Imagine that during a mathematics lesson one of your classmates raises her hand very excited. She tells your teacher that she has figured out a theory that nobody ever told the class. She explains that she has discovered that as the perimeter of a rectangle increases, the area also increases. The girl goes up to the board and draws the following pictures to prove what she is saying:



Your teacher asks the rest of the class to respond to this girl. What would you say to her? Do you agree or disagree with her "theory" and why?

TASK 2

Using the symbols ↑ ↶ ↷ write down the steps A.L.E.X. should follow, so that rectangular pathways of perimeter 12 units each could be constructed.

TASK 3

Construct the rectangular pathways you wrote in Task 2. Calculate the area of each rectangle in square units. What do you observe?

TASK 4

How do you think TASK 2 and TASK 3 could help you respond to the scenario of TASK 1? Solve TASK 1 again by using A.L.E.X.

Figure 4: The worksheet given to students

The role of the teacher/researcher was restricted solely to coordinating the function of the groups' discussions. For strengthening the reliability of collected data, a tape recorder was placed at each group's desk. Adding children's authentic voice to the data collected through the completed worksheets was deemed critical by the researchers, in terms of helping them decode the arisen mathematical ideas and skills. Data collection spanned four consecutive 40-minute teaching periods.

For the purpose of analysis, what we have done was, after reading the transcripts and comparing the various interactions for similarities and differences, to identify themes or patterns in the data, which were repeated a number of times. Our list of recurrent themes appears in the next section.

FINDINGS

Children were first asked to respond individually to Task 1 of the provided worksheet (see Figure 4). Task 1 is a slightly modified and translated version of a scenario used in Ma's work (1999, p. 84). All fifteen sixth graders participating in the study replied that they agreed with the girl's proposed theory (see Task 1 of Figure 4). Then, the teacher/researcher invited them to work in groups to solve Tasks 2, Task 3 and Task 4. The most worthy of note themes highlighted in children's dialogues are listed below. Each theme is accompanied by selected quotes.

Understanding that a square is a special form of a rectangle / Overcoming the barrier of the most prevalent rectangle image

The children constructed a square with 12 cm perimeter. Each side was 3 cm. Soon after, they started wondering about the correctness of their solution.

T: I asked you to construct a rectangle with a perimeter 12 cm.

S1: But this is a square.

T: Do you mean that you've made it wrong?

All: Yes.

S3: The rectangle is longer.

S4: The length is bigger and the width is smaller.

T: Who would like to tell us what the characteristics of a rectangle are?

S1: A parallelogram…

S5: It has four right angles.

S1: The top and the bottom are different from the left and the right.

S2: Yes, the left and the right are the same and the top and the bottom are the same.

T: So, what is your conclusion? Is there any relationship between the rectangle and the square?

S3: There is a relationship. In both shapes, the right and the left are the same as well as the top and the bottom are the same. What's going on is that the left and the right of the rectangle are the same with those of the square. But the top and the bottom are longer.

S4: Yes, but both shapes have four right angles.

For students to extend their example space of a rectangle, the researcher problematized them about the rectangular image that tends to prevail in their minds. Watson and Mason (2005) explain that "examples are usually not isolated; rather, they are perceived as instances of a class of potential examples. As such they constitute what we call an example space" (p. 51)

T: When I ask you to construct a rectangle what image comes first to your mind?

S1: The long one…either horizontally or vertically because if you reverse it, it's the same narrow, long shape.

T: Haven't you ever heard that the characteristics of a rectangle are a quadrilateral that has the opposite sides…

S2: Parallel…

S3: Equal ones.

T: Right. Let's have a look at the square you have constructed (The teacher points at the square 3x3, children had drawn earlier).

S4: It has four right angles…

S2: And its opposite sides are parallel and equal…

S1: It's a rectangle, as well! We simply haven't thought about that…because there is a difference in the length of the top and the bottom sides. Whereas, in a square all sides are the same.

S2: Yes, this is the only difference but both shapes meet the same characteristics.

T: Coming back to our initial question now…is the square you have constructed here is correct?

All: Yes.

Distinguishing among the rectangles with the same perimeter that the square has the biggest area and that the rectangle whose one dimension is 1cm long has the smallest area.

After the children have constructed three rectangles (3cm x 3cm, 5cm x 1cm, 4cm x 2cm – See Figure 5), the teacher asked them to say what they notice.

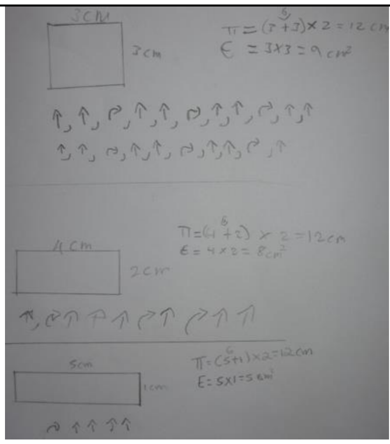


Figure 5: Rectangles of perimeter 12 cm

S2: The biggest area is the one of the square. Because all sides are equal and it is a bit bigger than the rest of the shapes.

S4: The width and the length are the same.

T: Yes, and what does that mean?

S4: That the area will be greater than the area of the rectangle five times one.

S12: When the width is not 1 cm then the area of that rectangle is more. And this is because every number when multiplied by one, remains the same.

T: And what happens when one side is 1cm?

S12: We have the smallest area.

DISCUSSION

One pervasive challenge in mathematics education at the school level is the identification and use of authentic contexts to motivate student inquiry and learning. Findings suggest that it is possible for pedagogically sound game apps like A.L.E.X. to support learning of the mathematics curriculum in educationally powerful and interactive ways. The children participating in the current study engaged themselves in authentic geometry problem solving activities and built higher order critical thinking skills. The recognition, for example, of the rectangle with the greatest and least area, among the rectangles with the same perimeter might be an indication of the activity's potential to help children become reflective and self-directed learners.

Despite the potential of appropriate game apps for transforming mathematics teaching and learning, their success as an instructional tool in formal situations will ultimately depend upon the abilities of teachers to take full advantage of their affordances. Teachers are the ones who decide what technological tools to use, and how to employ them in their classrooms (Becker, 2007). The classroom discussion listed in the previous section about whether a square could also be named as rectangle is indicative of the key role a teacher could play in planning classroom game-based activities, and in supporting and scaffolding pupils by providing appropriate feedback.

Considering the positive experiences of the current study's participants with the coding game app A.L.E.X., further research exploring the ways in which the integration of coding game apps within the mathematics curriculum can impact students' motivation and learning of mathematics should be conducted. Future effect studies taking place in regular classroom settings can help to determine the actual potential of coding game apps as learning tools, by shedding light into both facilitating and inhibiting factors to their successful implementation in formal learning settings. Research focusing on the integration of coding game apps in the mathematics classroom can provide useful insights to mathematics teachers on how to best utilize the affordances provided by digital games to motivate their students, and to scaffold and extend their mathematical reasoning. It can also inspire future content developers and influencers to cultivate the mobile apps space to the benefit of young learners (Chau, 2014).

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FORMATIVE FEEDBACK IN THE NUMBER STORIES PROJECT

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The University of Chicago Number Stories project aims to enhance student engagement in solving real-world problems in a Cabri environment through the provision of effective feedback. This paper illustrates some of the current possibilities for feedback that are being developed and discusses the issue of providing feedback in open-ended situations.

Keywords: formative feedback, Cabri, real-world, on-line

INTRODUCTION

The *Number Stories* (NS) project (CEMSE, 2014) involves the development of an online database of number stories, which are real-world questions based on real-world contexts supported by factual sources, targeted to individual users (or solvers) including school students at any level, teachers, teacher-educators, home-schoolers, district supervisors, and the general public. Unlike a traditional curriculum project where real-world problems may be used as a means to achieve specific mathematics learning goals, the main aim of the NS project is to promote understanding about how mathematics is used in daily life and to enable solvers to gain in their confidence and ability to apply mathematics in real situations. Field testing will commence in spring 2015, and it is hoped that ultimately the resource will be freely available.

Each number story consists of a collection of Cabri files[1] in which the context is established, a question or problem is posed, and a solution is given. There is a wide range in the mathematics required. For example, questions from the "Chain Letters" context shown in Figure 1 below, in which one person sends a letter to n people who then send letters to a further n people, may be solved by techniques ranging from dragging representations of letters into mailboxes to using geometric series. Questions may be specific or more open-ended, such as "Who got the better end of the deal when Manhattan was purchased by the Dutch from the Indians in 1626?", which allows a number of approaches and does not have a well-defined solution.

Screenshots from an early version of "Chain Letters" are given in Figure 1 below. This illustrates that real-world contexts also include culturally relevant contexts. A number of questions can come from a particular context; another question in this context might involve doing a survey on people's beliefs regarding chain letters. A number of solutions can also be given to a particular problem: two other author solutions involve using the sum of a geometric series and repeatedly multiplying by 6 using a calculator. Solvers may also post their own solutions to the database and explore the solutions posted by others. However, in the question posed below, the only feedback given is whether the answer is correct or incorrect: the solver must consult the solutions in order to find out more about how to solve the problem.

Example of a CONTEXT Peanuts Chain Letter

Suppose Charlie Brown sent the chain letter to six of his friends, and each of them sent copies to six friends, and the chain continued, with each person who received the letter in turn sending a copy to six people.

A Peanuts cartoon is shown in which Charlie Brown shows a chain letter to Linus, who challenges Charlie Brown's belief that breaking the chain could bring bad luck.

We have not yet negotiated the rights to use this cartoon in any publication.

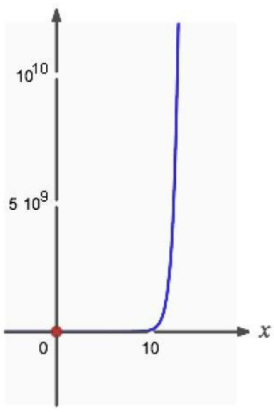
Example of a QUESTION Chain, Chain, Chain

When Charlie Brown sends the letter to 6 friends,that's one step in the chain. When his friends all send the letter to 6 more friends, that's another step. How many chain letters would be sent after 12 steps, assuming no one broke the chain?

Example of a SOLUTION Author Solution 2: Chain, chain, chain

Each step in the chain creates an additional 6 letters for each letter that was mailed in the previous step. I set up a spreadsheet to multiply the number of letters by 6 for each step and to add up the total number of letters after each round. I got the following outcomes.

Step	No of letters	Total number of letters mailed
1	6	6
2	36	42
3	216	258
4	1296	1554
5	7776	9330
6	46656	55986
7	279936	335922
8	1679616	2015538
9	10077696	12093234
10	60466176	72559410
11	362797056	435356466
12	2176782336	2612138802



I conclude that 2,612,138,802 letters would have been sent after twelve steps.

Figure 1. A Number Story context, question, and solution

We have highlighted that this is a difficulty; simply giving correct/incorrect feedback in the question file may not motivate the solver to explore the problem further. We are hence considering ways in which more engaging feedback may be given, and this paper will illustrate some of our approaches.

THEORETICAL FRAMEWORK

Based on Shute (2008), we define formative feedback as feedback given to the learner with the purpose of enabling the learner to modify their thinking or behavior in order to meet the goals of the activity in which they are engaged. We note that such feedback may be deliberately designed to be evaluative or may emerge from the affordances of the environment, when an object behaves in a particular way when manipulated. Laborde (2014) stresses the importance of such feedback in convincing students that they are wrong and also giving information that can be used to discover a more appropriate strategy.

Hattie and Timperley (2007) categorize feedback according to its purpose and have developed the following framework for feedback:

- Feedback at the *task* level (FT) provides information about how well a task is being accomplished in relation to a goal.
- Feedback at the *process* level (FP) provides information about the processes being used to accomplish a task.
- Feedback at the *self-regulation* level (FR) provides information to help learners monitor and regulate their own actions towards a goal.
- Feedback at the *self* level (FS) provides an evaluation of the student as a person. This is generally problematic and we are hence avoiding it.

This framework aligns well with the aims of the NS project, highlighting task completion, but in the context of developing solvers' problem-solving abilities in an environment where solvers are self-directed in their choice of problems to solve. In particular, for more open-ended tasks, FP and FR are crucial to sustain engagement, to experience successful task completion and to enhance problem-solving ability.

The first of the two major issues that need to be considered in designing feedback in a digital environment is that of communication:

The computer is limited in the amount of information it can obtain from the student, and is also limited in the amount and types of feedback it can provide (Mavrikis and Gutierrez-Santos, 2010, p. 642).

In contrast, in face-to-face communication in a technology environment, a human facilitator can, for example, speak, point to screen objects, take control of actions, and draw inferences based on facial expressions and gaze direction. However, the computer can provide immediate feedback and may be perceived as less judgmental.

The second issue is that of designing feedback to support various ways solvers might approach the problem. Even for an apparently straightforward task such as the Chain Letter problem, if all solutions are to be supported then all possible solution processes and possible misconceptions would need to be identified, and this is simply not possible for a more open-ended task. This points to the necessity of FR.

A more detailed literature review concerning feedback and formative assessment in technology environments and the capacity of Cabri to provide such feedback may be found in Mackrell (2015).

DEVELOPING FORMATIVE FEEDBACK IN NUMBER STORIES

Any one Number Story is likely to contain FT, FP and FR feedback, and it is not possible to precisely delimit the type; good feedback at any level will identify errors, but also enhance the solver's ability to identify the processes in which they are engaged and to self-assess.

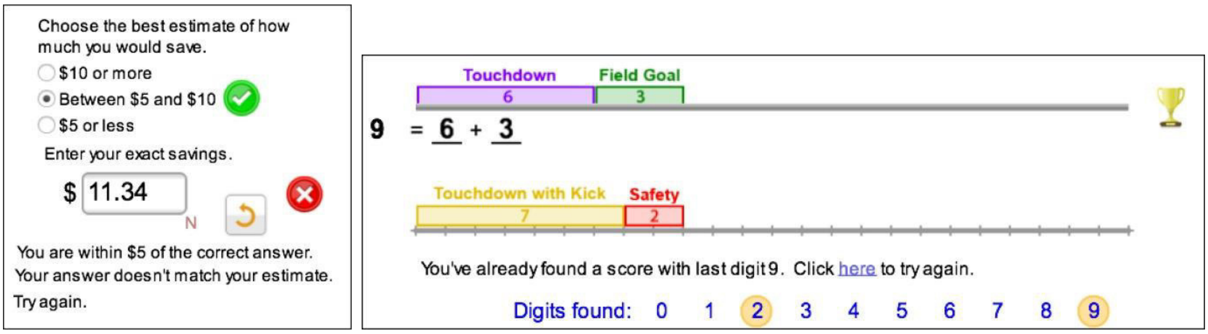


Figure 2. Examples of feedback on the task

The first example in figure 2 above shows FT that gives further information about a solver response. FP is also involved through the comparison between the estimate and the exact savings. The second example checks whether a particular response has already been given, and the information on digits found enables FR.

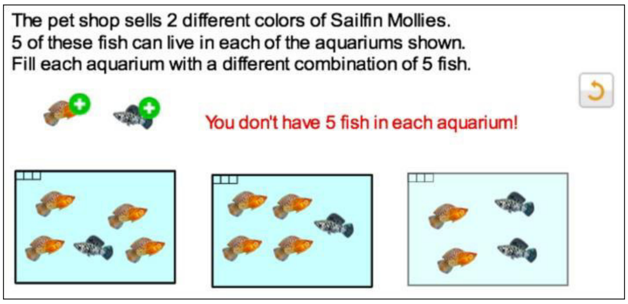


Figure 3. Delayed feedback on the task

Figure 3 above gives an example in which precise FT is not given immediately; the first time the mistake is made, only the text appears. The second time, the incorrect aquarium flashes. This gives an opportunity for self-assessment: the solver is made aware that there is a problem, but needs to identify where this occurs.

Figure 4 below gives an example of a question where the solver is first asked just to calculate the answer. On making a mistake, the question is scaffolded to show the steps involved in finding the answer. If further mistakes are made, visual aids are shown. For example, to find the number of years between 2003 and 2012, a number line is shown. A yet further mistake will add the 1 yr marks to this number line.

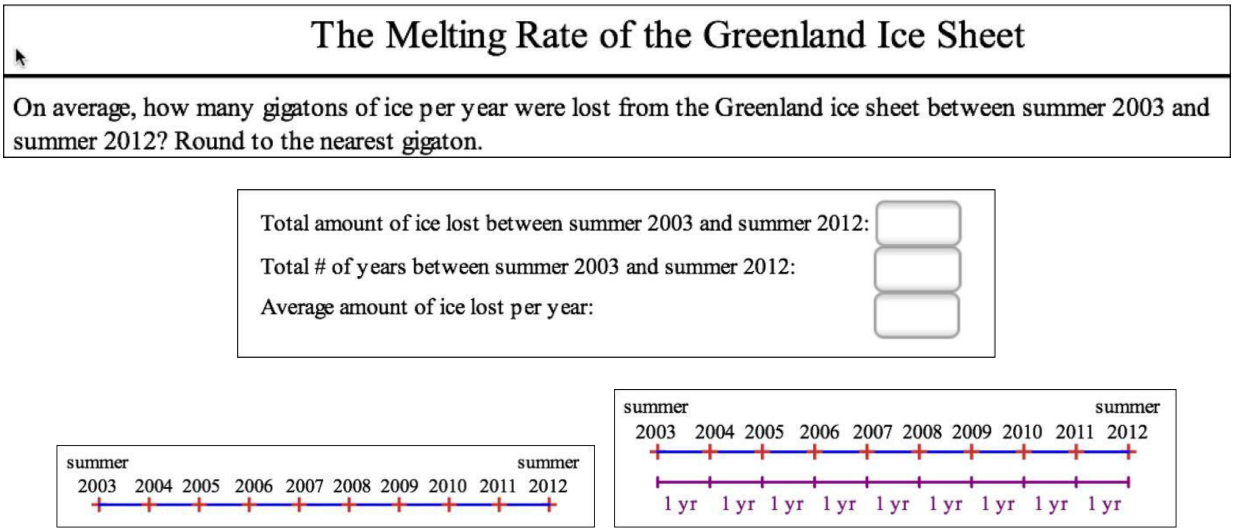


Figure 4. A question and the scaffolding that progressively appears after mistakes

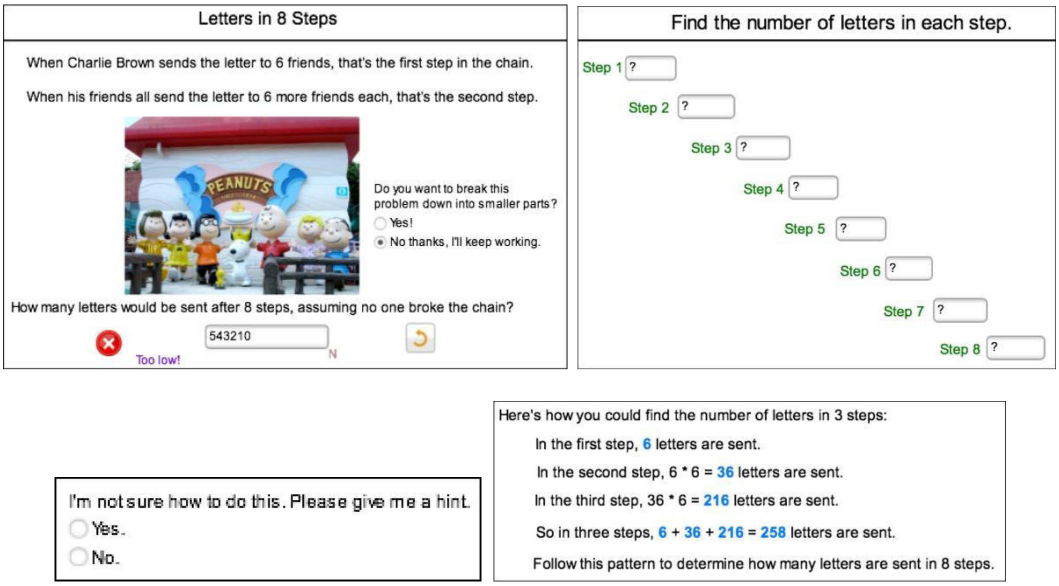


Figure 5. Scaffolding on demand

There are two tensions in this type of scaffolding: first, to not overly scaffold the problem, and secondly to not confuse the solver with too many choices.

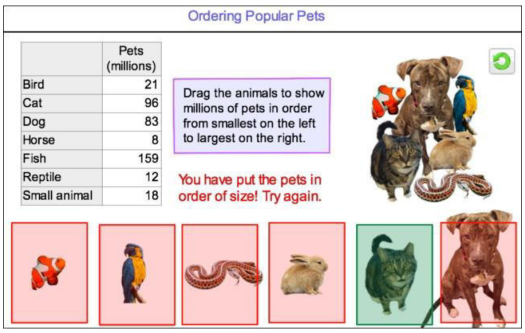


Figure 6. Identification of incorrect strategy

In figure 6 above, FP takes the form of directly identifying an incorrect strategy.

An important strategy for enhancing self-assessment is to use feedback that shows the solver the consequences of their response. For example, solvers are shown the end of a swimming race, in slow motion, to evaluate whether they have chosen the pair of swimming times that are closest together. Solvers may also set parameters, such as those governing the motion of a satellite, and then view a model of the satellite in motion.

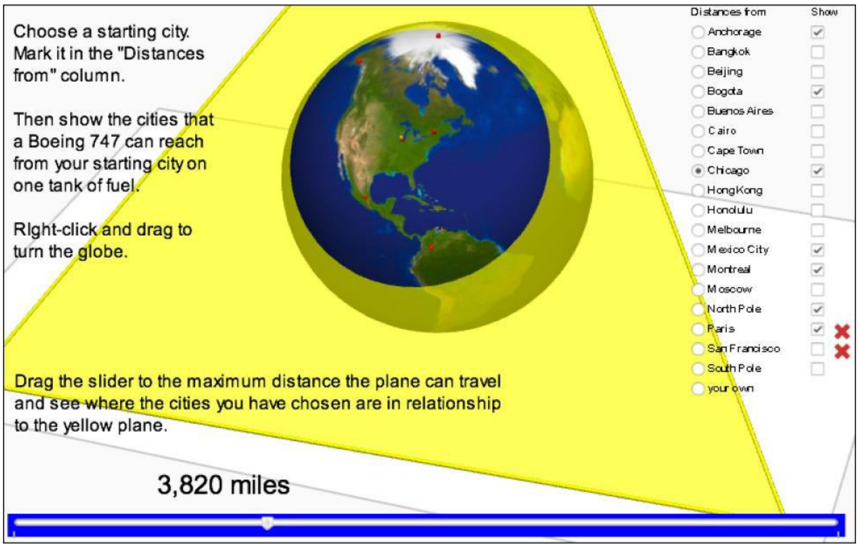


Figure 7. Self-assessment through showing solvers the consequence of their choices

Figure 7 above shows a plane cutting the Earth at a chosen distance from a starting city. Direct feedback is given as to whether cities within this range have been chosen correctly, but this is made more meaningful by the ability to drag the plane. A second phase of the problem checks whether the chosen distance is correct.

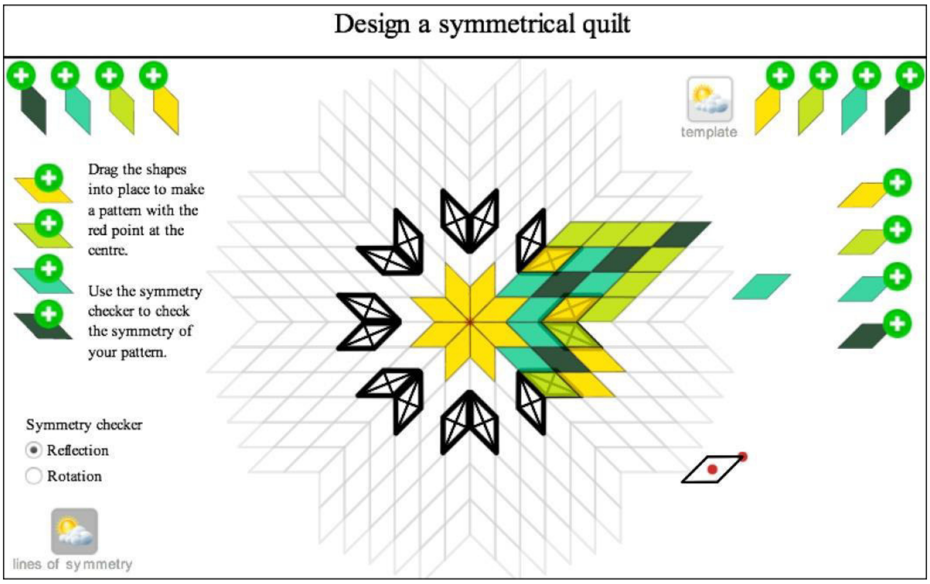


Figure 8. A tool to enable self-assessment

We are also including Number Stories that involve designing objects, such as star quilts as shown in Figure 8 above. In this situation we have provided a symmetry checker tool to enable the solver to self-assess by checking the properties of their design. Not all the squares covered by the black rhombi are the same colour; hence the design as shown does not have eight lines of symmetry.

Our main challenge is that many of the most interesting and engaging real-world questions involving mathematics are both open-ended in approach and do not have well-defined answers. An example is the NS mentioned above about the purchase of Manhattan. One possibility in such a situation is to structure the question to give specific choices of direction, or specific sub-questions to solve.

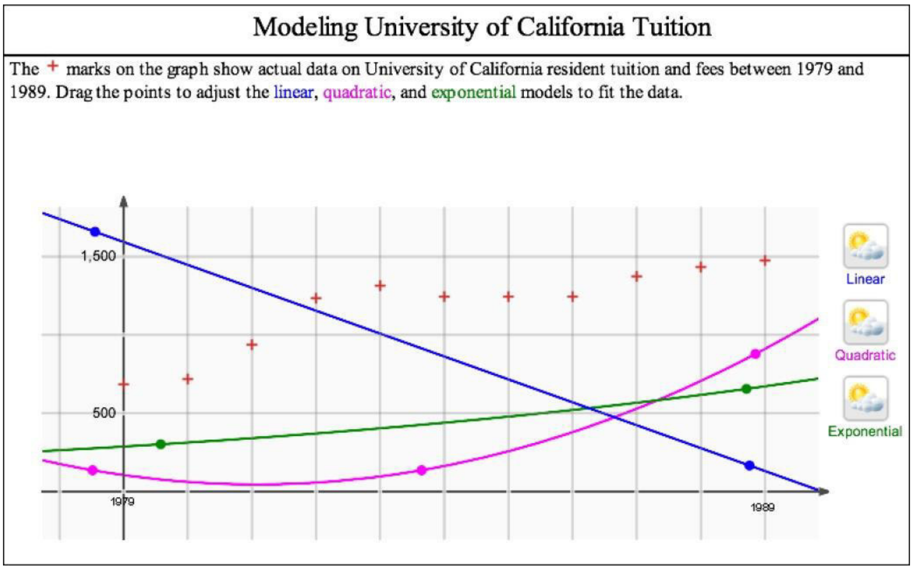


Figure 9. A question involving modelling

An example is shown in figure 9 above. Solvers drag points to fit curves to a set of data and are then given feedback if their curve does not lie relatively close to the data. It appears that the quadratic curve is the best fit. However, the next stage in the problem involves dragging the axes to explore the models beyond the original date range, which makes it clear that the quadratic model is problematic:

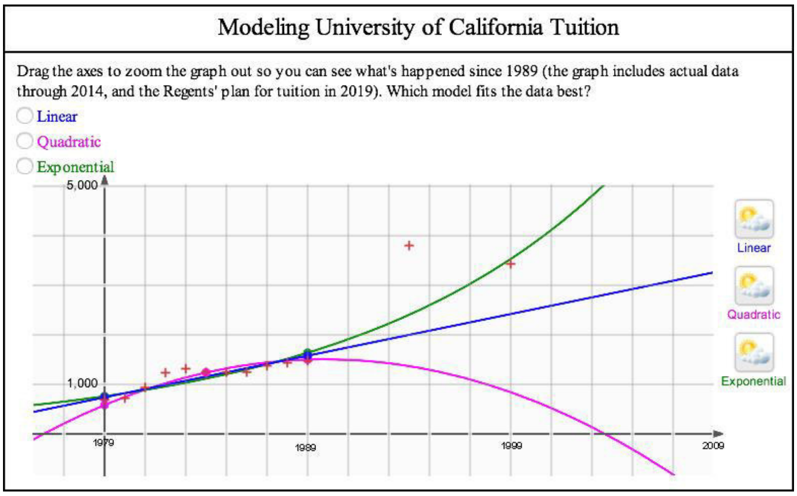


Figure 10. Problems with a quadratic model

The solver is then asked to write an equation for the exponential model, and is shown the graph of the function they have created to compare with the original curve.

Another possibility is to give no feedback, but instead to invite solvers to explore solutions posted to the database by other solvers in order to self-evaluate and evaluate others.

CONCLUSION

We have developed a number of feedback possibilities and are recognizing the importance of designing feedback that encourages the solver to self-assess. In a technology environment, strategies that give feedback at the self-regulation level (FR) are not just desirable, but necessary, as the most informative feedback possible must come from the solvers themselves.

An advantage of our main Cabri development environment is that it is possible to further elaborate the feedback provided as our ideas develop. We hence look forward to developing our feedback provision further, in particular in response to the results of our imminent field testing.

NOTES

1. We are likely to include Number Stories written in other software as well as Cabri.

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FOR THE LOVE OF STATISTICS: APPRECIATING AND LEARNING TO APPLY EXPERIMENTAL ANALYSIS AND STATISTICS THROUGH COMPUTER PROGRAMMING ACTIVITIES

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For the past 4 years, we have been involved in a project that aims to enhance the teaching and learning of experimental analysis and statistics, of environmental and biological sciences students, through computational programming activities (using R code). In this project, through an iterative-design, we have developed sequences of R code-based activities, that have been implemented in three institutions in Mexico and Portugal, in 8 postgraduate and 4 undergraduate courses; these are hands-on sets of tasks in R script that include computer programming work and are meant to be carried out collaboratively (a sample of an ANOVA activity is given). General results indicate that students tend to enjoy the courses; lose their fear of statistics; develop competencies for applying statistical methods and using computational tools, such as R, on their own data that deepens their understanding of the biological phenomena they have to analyse.

Keywords: Statistics education, experimental analysis, technology-enhanced learning, computer programming, R code

INTRODUCTION: DIFFICULTIES IN THE TEACHING OF STATISTICS

This paper deals with the teaching of experimental data analysis, probability and statistics, to students in the environmental sciences. Probability and statistics are commonly recognised as difficult both to teach and learn. Their inclusion in graduate programs in biology and related sciences have often had unsuccessful results (Bishop & Talbot, 2001), with learners having difficulty in understanding and/or applying the concepts. Batanero (2001) found that even researchers have a poor understanding and use incorrectly many statistics concepts, including basic ones, and consequently their students also fall into similar misunderstandings.

However, competency in carrying out experimental analysis, and applying probability concepts and statistical methods, is crucial in scientific disciplines; environmental sciences are not the exception. Researchers (and science students) must be able to use these mathematical tools (probability and statistics) to interpret their data, make decisions, and communicate and defend their findings using statistical arguments, with good understanding of the methods and language used (Holmes, 2002).

For the past decade, the main author of this paper has been teaching experimental analysis and statistics courses, to undergraduate and post-graduate students, and researchers in environmental sciences (specifically in the areas of Marine and/or Biological sciences) at several institutions in Mexico, Portugal and Chile (see details in Mascaró, Sacristán & Rufino, 2014). Before 2011, her teaching approach was a traditional one, akin to that of her colleagues, and students had difficulties understanding the abstract and relatively complex statistical concepts. As detailed in Mascaró et al. (2014), we identified that many observed difficulties stemmed from 3 aspects:

1. Students' general aversion to mathematics and statistics: some common statements are "we study biology because it doesn't have much mathematics" with many students failing to grasp why the

need of statistics in their field of study. Furthermore, students that encountered difficulties in previous courses tend to become frustrated and convinced of their inability to learn and apply those concepts, which in turn (as a vicious circle) leads to further rejection of these topics.

2. The type of computational tools and software used in statistics courses and for analysing data in Biology and Environmental Sciences. Some issues related to this are that many of these software are black boxes (e.g. Statistica) or unfriendly and unreliable (e.g. Excel), so that it is difficult to assess what is happening when a function is applied and detect errors; there is little control in the construction of graphs, which creates a disassociation between numerical data and the graphical output; specialised software is expensive and thus not easily accessible by students. Thus, the technological tools usually used, often hamper (instead of help) the learning of the statistics concepts, and their applications.

3. The syllabus conception and approach. Among other things, the usual approach usually presents sets of algorithmic skills for using predetermined software tools (which leads students to confuse software-use skills, with analysis strategies); tends to include few or very superficial discussions (e.g. on the meaning of the error in a statistical inference) and opportunities for exercising strategic decision-making; and tends to be separate from the real, natural world that it is meant to analyse. These lead to a further alienation of students from the problem solving context in which statistical knowledge is applied, making statistical modelling and data analysis a "field of specialists".

AN INNOVATIVE APPROACH FOR TEACHING STATISTICS: THEORETICAL FRAMEWORK AND METHODOLOGY

In order to address the above issues, four years ago we began a project [1] to develop a new teaching strategy for our experimental analysis and statistics courses. For this, we take into account research in the field of statistical education from which we draw the following recommendations:

- Contextualising concepts by using concrete examples with data from real research situations and creating statistical models by translating problem statements to (abstract) mathematical formulations (Batanero, Díaz, Contreras & Arteaga, 2011).

- Emphasis on the use of diagrams, since graphic representations are essential in data organisation, statistical reasoning and analysis (Wild & Pfannkuch, 1999): The importance of graphic comprehension (e.g., as defined by Friel, Curcio and Bright, 2001), and of being able to change from one representational register to another, are well known in mathematics education.

- Teaching methodologies that shift the role of students from passive to active, with emphasis on developing statistical reasoning, rather than a blind application of statistical tests.

In this regard, computer technologies can be used for simulating and visualising stochastic phenomena, and in general for statistics education (e.g. see Ben-Zvi & Friedlander, 1997). But in order to integrate technology in a meaningful way (instead of how statistics software was previously used) and using it to give students a more active role, we were inspired by the Logo programming and constructionist philosophy (Papert & Harel, 1991), which suggests that learning – and access to relatively advanced abstract ideas – can be facilitated if students explore ideas and concepts through construction, such as that involved in computer programming activities. Constructionism also promotes the sharing of ideas and products; thus, another principle for our seeked approach was for the computer-based tasks to be carried out through collaborative work leading to reflective interactions (e.g. that involve explanation, justification and evaluation – Baker & Lund, 1997).

Thus, our approach was to design probability, statistics and experimental analysis courses that centred on sequences of meaningful constructionist, and collaborative, computer-based and computer-programming activities. As main digital computer programming tool, and statistics software, we chose R. The R software [2] (from the *R Project for Statistical Computing*) has been used as a pedagogical tool for the analysis of various kinds of quantitative data. It allows for more control in the handling of objects and their representations, as well as transparency and reliability, than the commonly used Statistica or Excel. Using the R console, one can directly run commands, or scripts (programs) created in an editor, and computational instructions can have direct correspondence with the behaviour of a particular phenomena (e.g. the R code can mirror a mathematical equation) which has the further potential of facilitating the understanding of the statistical concepts involved. Thus we found it very suitable for the approach we seeked. (For more details of R, see Paradis, 2005; and for its features that we found useful, see Mascaró et al., 2014). The use and programming of R, like that of any other programming language, requires initiative in order to look for different strategies to reach the same result; it encourages experimentation which lead students to lose their fear of failure; different strategies can be compared and analysed; it involves memory and understanding of previously learned commands in order to apply them in new problems. All of these factors invite students to reflect on how they think about a determined problem and on their approach, giving them an active and responsible role in their own learning.

Our courses need to cover various elements of experimental design (which may, or not, be carried out in strict sequence). The aim is for students to understand how to carry out and apply the theoretical and experimental design, and develop criteria for selecting tools and types of tests in a research approach; learn to use the basic concepts of experimental design that help to build a statistical model to be used in a research study; and apply statistical computing software (in this case, the programming language R) to carry out the calculations related to the design of experiments; and learn how to interpret the results given by the software. In our approach, all the activities for teaching statistical reasoning are designed to be carried out using R (see Mascaró et al., 2014, for more specific criteria underlying the design of the activities); and are even presented through R-code "worksheets" with instructions, guidelines, examples, programming tasks, questions for reflection, comments, and commented solutions to the activities which are made available to students at the end of each session. The understanding of statistical models is facilitated by creating objects in R, to represent them. Thus, students need to develop familiarity with the programming language R and its libraries.

The activities are designed to be carried out in teams of 2-3 students and moderated by an instructor. They are of two types: Those for introducing the basics of R functioning and use. And those on different topics of statistics courses (e.g. activities that deal with frequency distributions; activities on binomial, Poisson, and normal distributions; activities on ANOVA and comparisons between means; activities on linear regression; etc). These include theoretical and practical components, so that students can apply the concepts to problems related to their subject of study. Students need to draw and interpret graphs (i.e. visualise the models) relating numerical data to graphical representations, as well as to mathematical formulae. They need to predict what a change in the code would produce. Thus, students go back and forth in the analysis of the data, and even suggest changes in the datasets for obtaining different results. The final aim is for students to be able to produce their own statistical models or modify those given (depending on the complexity of the activity) and fully explore their results and interpretations.

The design of our courses and tasks, and their implementation and research, follows an iterative methodology (Plomp, 2013), in which each is revised and redesigned according to the experiences and results of its implementation, and thus informs the design of the next implementations. So far, we have designed and implemented 34 computer-based activities, continuously revised, that have shaped *8 postgraduate and 4 undergraduate courses* in Mexico (at the Sisal Academic Unit of the National Autonomous University of Mexico) and in Portugal (at the Portuguese Institute for the Sea and Atmosphere – IPMA – in Lisbon; and at the Interdisciplinary Centre for Research in Marine and Environmental Sciences – CIIMAR – in Porto). Undergraduate courses are semester-long courses; postgraduate ones, are intensive 1-2 week courses. (See details of the first 34 tasks, and 7 courses, in Mascaró et al., 2014.) Our first (pilot) course was in 2012 at IPMA, Portugal, on Univariate Statistics (covering the topics of probability distribution of random variables; statistical inference and hypothesis testing; and linear regression); it was an intensive 30-hour, week-long course for postgraduate students and researchers in Marine Biology and Environmental Sciences.

SAMPLE TASK: ACTIVITY 11 ON ANOVA

We present here the structure of the 11th activity (as in its last iteration) related to the concept of ANOVA, included in a 2015 post-graduate course on Experimental Design and Data Analysis:

1. This activity starts with a research problem statement, an extract of which reads:

> It has been observed that an increment in water temperature produces an increment in the metabolic rate of *Maxquil* crabs *Libinia dubia*, an important prey of several top predators considered key species in the commercial fisheries in the Yucatán Peninsula. The experiment consisted of exposing 12 crabs in each of 3 temperatures during 30 days and recording their oxygen consumption (mg O2/g dry weight/minute) individually by means of an open flux respirometer. Water temperature in treatments were: 18, (low); 25(medium) and 30ºC (high). Results are in the sheet 'respir' of the Excel file 'datosDEAD.xlsx'.

The statement is collectively read out loud (drawing students attention to the problem), with adequate accentuation to words and phrases, and correct pronunciation of the named species. This helps make the context and objective of the investigation clear, and assists in the identification of the essential information elements (e.g. variables, where the data set can found, etc) for an adequate understanding of the problem. From this point on, students work in pairs.

2. In the R worksheet, instructions are then given to identify the cause-effect relationship implicit in the problem, the response and explanatory variables within the data set, and the statistical nature of these variables. This constitutes a particularly important step in the activity because it leads students to develop skills for formulating a research question; translating the plausible answer into the underlying hypothesis of the model and it's logical opposite (the null hypothesis); and identifying a statistical null hypothesis to be used as a reference in the test procedure. For example, some of the included instructions are:

> 1. Import the data to R and save it as 'dat'. Verify its characteristics and structure. How many dimensions does 'dat' have? How does 'dat' differ from previous problems where the means of only two samples were compared?

2. What is the cause-effect relation to be tested with this experiment?

- Identify the variable that would explain the response if the cause-effect relation was to be confirmed. Is it a continuous or a discrete variable?
- Write a sentence stating the null hypothesis, using mathematical or logical terms.

In order to carry out these tasks, among other things, students would have to type several commands in R, such as those to import the data set, rename it and display its structure, summary and names.

3. The next step has the objective of guiding students through a short yet effective exploration of data, such as is exemplified in the following item:

3. Explore the data both numerically and graphically in order to answer the following questions (TIP: remember functions such as 'boxplot', 'hist', 'mean', 'sd', and how to do subsetting).

- Are the mean values of the three levels different? Are their dispersions different?
- How is the distribution of the response variable? Is a single histogram of the response variable useful to answer this question?

Once they have identified what is to be compared, they are instructed to obtain graphical and numerical outputs that will enable them to make such comparisons both visually and numerically. Using R code, students can generate graphs such as the one in Figure 1:

```
boxplot(dat$oxi~dat$tempf); hist(dat$oxi)
mean(dat[dat$tempf=="Low",3])
mean(dat[dat$tempf=="Med",3])
mean(dat[dat$tempf=="High",3])
```

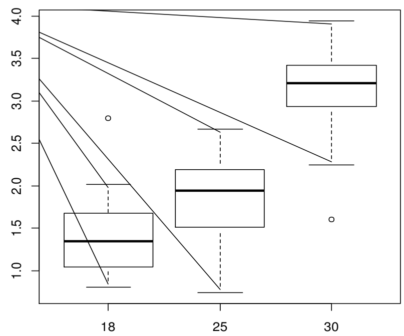

Figure 1: Sample boxplots generated by the students by typing the R code displayed on the left-side.

This step trains students to find differences, trends and patterns in the data and relate them to predictions stated in the hypotheses. This is an important activity for developing a sense of the magnitude of differences, both in a relative and an absolute manner. In terms of the pedagogical approach, it constitutes an enhanced means to construct and manipulate objects in the programming environment (by adding arguments in the code and verifying its effects).

4. Next, students are required to adjust a statistical model to the data and explore the numerical outputs. Students will practice with the specific R code needed to complete instructions, and will learn to identify the information of interest, within different lists returned. Sample questions are:

4. Apply an ANOVA to the data, by first adjusting a linear model using the 'lm' function and analyse what R returns.

- Do you recognise any of the values under the title 'Coefficients'? What do you think they are?

5. Use the function 'anova' on the object you just created and answer the following questions:

- Are the d.f. values consistent with those obtained in class?
- What are the values in the Sum Sq column?
- Calculate a measure of the total variation of the response variable in the experiment.
- What does the F value represent? Is it a large or a small value?
- What does the p value mean?
- What is the conclusion of the test? How do you interpret it?

This step is an important learning task in which operations previously reviewed in class, and their results, are made evident in such a way that students can follow the algorithms of the statistical procedure and retrieve the information regarding the different sources of variation in the data. Having to write, and modify, the R code, in order to obtain different pieces of information allows students to learn through 'trial and error' providing robust, self-constructed knowledge with an added value of personal fulfilment.

5. The final steps in the activity aim to obtain the expected values of the parameters estimated by the model and its visualization. This is presented using items such as the following, which produces a graph like the one shown in Figure 2.

10. Copy the following code to obtain a visualization of the model you adjusted.

```
nmes<-data.frame(tempf=unique(dat$tempf))
nva<-data.frame(names, unique(round(as.data.frame(predict(mod1, se.fit=T)),2)))
ggplot(dat, aes(y=oxi, x=tempf))+ stat_summary(fun.y = mean, geom="point",
size=3)+ geom_point(position = position_jitter(width = .1), aes(shape=tempf,
col=tempf), size=3)+ geom_errorbar(data=nva,aes(x=tempf,y=fit,ymin=fit-
se.fit,ymax=fit+se.fit), colour="black", width=.2)+ theme_bw() +
ylab(expression(paste("Oxygen consumption   ", "(mg ", O[2], "/g/min)")))+
xlab(expression(atop(paste("Temperature (", degree , "C)"))))
```

- What do the black and coloured points represent? What do the whiskers represent?
- Can you identify the code which produces the different elements in the graph?

Once a conclusion has been drawn, it is important to complete the objective of the research question by predicting expected results if the experiment were to be repeated. This step implies the need to interpret results in the terms specified in the context of the problem and asses the probability of error in such statements, hence the limitations of such conclusions. Most importantly, it leads to an explicit recognition of the need to return to the research question to improve knowledge on the topic and drive the investigation further either to a more universal or a better specified hypothesis. An approach for generating scientific knowledge based on successive approximations, where the answer in each round is never certain and is thus rarely complete, is very closely emulated by the actual process in which students build their statistics knowledge through the activities in the course.

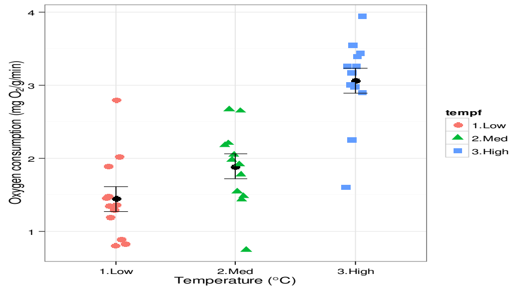

Figure 2: Graph produced by the code given in item 10.

6. After the previous steps, which are carried out as collaborative work in pairs, are finished, the different teams share their solutions, and one team is asked to describe the process they went through to complete all the items in the activity, including sharing and running the R code used. Answers are given and discussed in a collective manner. The instructors help moderate the discussion by intervening when necessary. During this collective exercise, instructors review what has been learned by making explicit comparisons of how students might have answered before and after to some of the questions. They also provide alternative R coding to reach similar results, further expanding specific features of R. Solutions are thereby constructed collectively.

SOME GENERAL RESULTS OF THE COURSES IMPLEMENTATIONS

For the assessment of the courses, we have used a combination of qualitative and quantitative techniques, including in-class field observations, student questionnaires evaluating the courses; informal interviews; and examining students' possible learning, thru students' commented and modified R-worksheets; and through their grades and tendencies in mean achievement (as compared with equivalent statistics courses at the same institution, using traditional approaches). The latter show that students' score higher in R programming based courses.

More recently, for the last postgraduate course on Multivariate Statistics, we used a multiple-choice pre- and post-test (which included theoretical statistics questions such as: what kind of calculations can be applied to matrices; when can a data set be defined as multivariate; why is it better to apply a multivariate procedure than many univariate; for what are measures of association in multivariate sets of data used; what are exploratory methods, what are the concepts, respectively, of Principal Component Analysis (PCA), non-metric dimensional scaling (NMDS) and MANOVA, and their respective characteristics). Twenty two students answered the pre-test and 19 the post-test. The highest scores were 78.3 on the pre-; and 83.3 on the post-test. The lowest scores were 15 on the pre-; and 30 on the post-test. The average scores rose from 43.6 to 60.2 on the post-test. These quantitative results show a mild improvement, but we consider that they do not reflect enough our qualitative appreciations. We are looking for better methodologies and instruments to assess the impact of the courses. Qualitative and anecdotal data indicate strong positive effects and possible long-term comprehension of statistical concepts with an appropriation of the R tool.

Related to the latter, most students feel they are now capable of using R on their own, and for their data, either creating new scripts or modifying pre-programmed scripts to suit their needs. In fact, in Mascaró et al. (2014) we related the anecdote of a Honduran student who claimed to have been "infected" with R, and had been self-compelled to use multivariate techniques on his data and write his own R scripts for analysing it; he is now a PhD student in Marine Sciences and Limnology, and uses R to analyse most of his data. Something similar happened to a former Masters student who attended one of our 2014 courses. He began a sampling scheme after the course and is now analysing his data using R to adjust a Generalised Linear Model: a procedure to analyse data that is not normally distributed, rarely implemented in point-and-click statistical software.

Comments and reaction of students in different courses (as also presented in Mascaró et al., 2014) tend to show their appreciation and enjoyment (with several students expressing how much fun they had), for the courses' approach, with many rating it as excellent and requesting continuations. Most students appreciate the classroom dynamics that include teamwork and group discussions: "Sessions were light and didactic." "When we correct the activities altogether using the beamer, I think about my mistakes and won't make the again." Another student said he liked becoming aware of "the different ways in which you can arrive to a solution to the problems, and the use of different tools to achieve that." Students tend to particularly appreciate the practical activities combined with theory. Many students comment on their surprise by the potentiality and versatility of R as a tool to analyse data (also appreciating the feedback of the environment, which contrasts with software such as Statistica or Excel where students either don't know what to do, or don't even realise a mistake was made). A couple of more recent comments in that regard are: "I enjoyed using R to make calculations instead of wasting time doing calculations by hand….that made time to analyse results and think on their interpretation." And: "I would have liked to see a work published with some of our exercises, so we could relate our knowledge with its application."

A thing we have now observed, is that our courses have become popular and enrolment has increased. In the Experimental Design and Data Analysis (EDDA) postgraduate courses, enrolment was 14 in 2013, increasing to 18 in 2014, with a record of 25 students in 2015. Students now apply for scholarships to be able to stay in order to take the Multivariate Analysis course after EDDA.

We would like to end with the overall motivation and appreciation for statistics achieved though our courses. Most students seem able to overcome their initial fears and difficulties, not only of computer programming, but also of statistics. There seems to be a deeper understanding of the rationale behind most statistical procedures, and a recognition of the advantages and limitations case by case. Statistics thus looses its supposed unreachability (i.e. being reserved for only specialists on the topic), becoming a universal tool that can be used and applied in efficient and useful ways. Moreover, throughout our courses, students often become better and more dedicated, readily desiring to learn a wider variety of quantitative tools to analyse and describe biological findings. For example, an undergraduate student –who in our 2015 Probability and Statistics course, tended to remain mostly silent and passive during the activities– now attends another statistics course and not only has a newfound assertiveness, but gives accurate answers and corrects mistakes made by her classmates; she commented: "I liked that you believe in us… you believe we can learn and that really motivated me, because I know I am slow in learning and even though it´s difficult I try hard. I discovered that I can learn maths, or at least stats, and that makes me really happy…" We have also received emails from former students with comments like these: "I wanted to tell you that the course helped to correct conceptual mistakes and fill in the blanks in my stats formation (multivariate), and helped me get 'friendly' with R; everything it left me, was essential." Another former student who initially had said she understood very little of univariate statistics, initially showing difficulties in our course, a month later wrote: "I just wanted to thank you for your course. I am here [at a U.S. university] not feeling completely lost in stats. I have been working on my data using R and, well, I'm delivering results, and I am feeling very happy, and very grateful".

NOTES

1. This project is financed in part by the PAPIME PE204614 grant from DGAPA-UNAM.

2. R (http://www.R-project.org) is a cross-platform free software environment (distributed under the GNU Licence) for statistical computing and graphics (Ihaka & Gentleman, 1996). Its development and distribution are carried out by several statisticians known as the "R Development Core Team" (Paradis, 2005).

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SUPPORTING THE DEVELOPMENT OF COLLEGE-LEVEL STUDENTS' STATISTICAL REASONING: THE ROLE OF MODELS AND MODELING

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The transition from descriptive to inferential statistics is a known area of difficulties for students. This article shares the experiences from a teaching experiment in a graduate-level quantitative research methods course, which adopted a non-conventional approach to teaching statistics that put models and modelling at the core of the curriculum. Findings indicate that the informal approach to statistical inference adopted in the course, which focused on modelling and simulation using the dynamic statistics software Tinkerplots2 as an investigation tool, promoted powerful ways of thinking statistically, while at the same time also developing students' appreciation for the practical value of statistics. The affordances offered by the technological tool for building data models and for experimenting with these models to make sense of the situation at hand, were instrumental in supporting student understanding of both informal and formal inferential statistics.

Keywords: Modelling, Dynamic Statistics Software, Model Eliciting Activities, Inferential Statistics, Informal Statistical Inference

INTRODUCTION

Statistics, the science of learning from data, is divided into two main areas: descriptive statistics and inferential statistics. Descriptive statistics is devoted to the organization, summarization, and presentation of data. It involves using tabular, graphical, and numerical techniques to analyse and describe a dataset. Inferential statistics, on the other hand, is intended to reach conclusions that extend beyond the immediate data. It improves decision-making in a variety of real-world situations by providing tools that enable the drawing of causal inferences, or inferences to populations using sample-based evidence.

While the basic data analysis techniques of descriptive statistics seem to be understandable by most students taking an introductory statistics course, the transition from descriptive to inferential statistics is a known area of difficulties (e.g. Rubin, Hammerman, & Konold, 2006). Research in statistics education has long suggested that students have difficulty using inferential statistics methods appropriately in applied problems. For example, research on introductory college-level statistics courses suggests that even students who can successfully implement procedures for hypothesis testing and parameter estimation are often unable to use these procedures appropriately in applications (e.g. Gardner & Hudson, 1999).

Advances of technology have, in recent years, provided instructors with new tools for adopting informal, data-driven approaches to statistical inference that can help lay the conceptual groundwork for formal inferential reasoning. The appearance of dynamic statistics learning environments (e.g., TinkerPlots2 (Konold & Miller, 2011), and Fathom© (Finzer, 2002)), which are designed explicitly to support integration of exploratory data analysis approaches and probabilistic models, and to allow for generation of data (e.g., drawing samples from a model) and experimentation (e.g., improving models, conducting simulations), have provided an enormous potential for making inferential reasoning accessible to students. Several researchers have been exploiting the affordances provided by these dynamic learning environments for promoting learners' ability to reason and argue about data-based inferences, with very encouraging results (e.g. Gil & Ben-Zvi, 2011; Konold & Lehrer, 2008; Meletiou-Mavrotheris, 2003; Paparistodemou & Meletiou-Mavrotheris, 2008). Some of the conducted studies have demonstrated that even young children can develop powerful notions about inference when using appropriate data visualization tools (e.g. Meletiou-Mavrotheris & Paparistodemou, 2014).

The current article shares the experiences from a teaching experiment that took place in a graduate level, quantitative research methods course, which put models and modelling at the core of the curriculum. The study, which adopted an informal, data-driven approach to statistical inference using the dynamic statistics software Tinkerplots2 as an investigation tool, sought to answer the following question: *How can the model building affordances provided by a technological learning environment like Tinkerplots2, be utilized in instruction to scaffold and extend students' reasoning about key ideas related to statistical inference?*

METHODOLOGY

Context and Participants

The teaching experiment took place in a Quantitative Research Methods course targeting graduate students enrolled in an M.A. in Educational Studies program offered at European University Cyprus. The course began on the first week of October 2014 and was completed at the end of January 2015. The first author was the course instructor.

There were nineteen (*n=19*) students enrolled in the course. Participants were either pre-service or in-service teachers, who were characterized by diversity in a number of parameters including age, and professional and academic background. Their age ranged from 23 to 42. Some had an academic background in primary education, while the rest were secondary school teachers in different domains (languages, humanities, natural sciences, physical sciences etc.). Students also had a varied background in statistics. Most of the older participants had very limited exposure to statistics prior to the course and had never formally studied the subject, while the younger ones had typically completed a statistics course while at college. However, even those students who had formally studied statistics in the past, had attended traditional lecture-based statistics courses that made minimal use of technology. Thus, upon entering the course, almost all of the students had very weak statistical reasoning and a tendency to focus on the procedural aspects of statistics.

Nature of the Teaching Experiment

The teaching experiment adopted a non-conventional approach to the teaching of the Quantitative Research Methods course, which put models and modelling at the core of the curriculum. In designing the teaching experiment, we ensured that our intervention covered the set curriculum included in the course syllabus. However, we expanded the curriculum by including, throughout the semester, activities that aimed at raising students' awareness of models and modelling, and of their usefulness in research settings involving statistical investigations.

The focus on modelling and simulation – along with inference – was being facilitated by having students use the dynamical statistical software package TinkerPlots2 for all modelling and analysis.

This software has been selected because it is designed explicitly to support integration of exploratory data analysis approaches and probabilistic models, and to allow for generation of data (e.g., drawing samples from a model) and experimentation (e.g., improving models, conducting simulations). Throughout the semester, students were using TinkerPlots2 to work on a set of carefully designed open-ended Model-Eliciting Activities (MEAs) (Lesh et al., 2000) in which they had to create and test statistical models in order to solve a complex, real world problem of statistics and provide answers to their research questions (Garfield, delMas & Zieffler, 2010). The activities were carefully designed to support but, at the same time, also explore students' evolving understandings of fundamental ideas related to statistical inference in the context of engaging in models and modelling for simulating data and evaluating their research claims and hypotheses. Some of the MEAs were completed individually, and others collaboratively in groups of 3-4 students. For some of the collaboratively-completed MEAs, groups presented and critiqued each other's solutions. The MEA *"How many tickets to sell?"* (adapted from http://new.censusatschool.org.nz/resource/using-tinkerplots-for-probability-modelling/), shown in Table 1, is a typical example of the activities in which students engaged during the course.

A progressive formalization approach was employed. The first part of the course focused on building a teaching pathway towards formal inference by helping students experience and develop the 'big ideas' of informal inference. Through their engagement with the open-ended MEA activities, students learned where these ideas apply and how. Later in the course, students were introduced to confirmatory or formal inference methods, and began comparing empirical probabilities with the theoretical ones. They learned the formal procedures for building sampling distributions, constructing confidence intervals, and conducting hypothesis testing.

The course content and structure was such that it encouraged "statistical enculturation". Statistical thinking was presented as a synthesis of statistical knowledge, context knowledge, and the information in the data in order to produce implications and insights, and to test and refine conjectures. The teaching of the different statistical tools was being achieved through putting students in a variety of authentic contexts where they needed the tools to model and make sense of the situation at hand. Probability was not presented as a body of clear and unambiguous generalizations free of any concrete interpretations, but as a modelling tool. Probability distributions were presented as models based on some assumptions which, when changed, might lead to changes in the distribution. The emphasis was not on teaching their formal properties, but on helping students understand why and where one could use these probability distributions to model a certain phenomenon, and in what ways this is useful. The similarities and differences between these ideal, mathematical models of reality, and statistical models based on experimental data were emphasized throughout the course. From informal uses of models early in the course to formal uses as part of significance tests at the later part the course, we were encouraging explicit discussion of how every model is essentially an oversimplification of reality which involves loss of information, and of how the success of probability models depends on their practically, and on their potential to give useful answers to our research questions.

Methods of Data Collection

Using classroom observation, videotaping, interviews of selected students, and student work samples, the study investigated students' interactions with Tinkerplots2, and documented the different ways in which students' engagement with data modeling activities influenced their understanding of key ideas related to inferential statistics. Currently, we are at the stage of analysing the collected data, in order to provide an in-depth investigation of students' interactions with TinkerPlots2 and with each other, and to document the different ways in which ideas related to models and modelling were understood and utilized by students in the context of making informal statistical inferences (ISIs) from data. We examine the ways in which students' engagement with Tinkerplots2, but also with MEAs scaffolded and extended their understanding of models and modelling. At the same time, we also investigate the ways in which the development of students' reasoning about models and modelling impacted their understanding of the big ideas related to statistical inference.

Air Zland has found that on average 2.9% of the passengers that have booked tickets on its main domestic routes fail to show up for departure. It responds by overbooking flights. The Airbus A320, used on these routes, has 171 seats. How many extra tickets can Air Zland sell without upsetting passengers who do show up at the terminal too often? 1. Why might people not arrive for a flight? 2. How many people, do you think, will not arrive for a particular Airbus A320 flight on which they have booked if, on average, 2.9% of passengers do not show up? 3. Draw a picture of what you think the distribution of the number of people that do not arrive for a flight would look like. 4. How many extra tickets do you think it would be sensible for the airline to sell? 5. What are the consequences of selling too many tickets? 6. What are the consequences of selling too few tickets? 7. You are going to set up a Tinkerplots simulation to model the number of people who do not arrive, (or the number of people who do arrive), for a flight on an Airbus A320 when they have booked a ticket. 8. What assumptions do you need to make? Are these assumptions likely to be true? 9. Describe how you would set up the simulation. 10. What will you use in the Sampler? 11. What are the possible outcomes and how will they look like in Tinkerplots2? 12. Should be sampling with or without replacement? 13. What is a single trial (DRAW = Run =) 14. What results will be collected? 15. How many trials? 16. Set up the Tinkerplots2 simulation and run it. 17. Do you need to adjust your model and run it again with different values to answer your question? 18. Sketch the distribution of the number/proportion of people who do not [or do] arrive for their flight. 19. Describe the distribution. 20. How does this distribution compare with the one you predicted? 21. Is there anything about the distribution from the simulation that surprised you? 22. What recommendation would you give to Air Zland for the number of extra tickets they should sell? 23. What are the consequences of this decision? 24. Explain why and how Tinkerplots2 has been useful for describing your recommendation that you would give to Air Zland for the number of extra tickets they should sell? 25. Compare your answer with those of other students in your class. 26. Reason why you have different answers. 27. Is there a theoretical distribution, which could be used to model the number of people that do not arrive for a flight? Justify your answer. 28. What assumptions do you need to make? Are these assumptions the same or different to the assumptions you made for the simulation? How likely are they to be true? 29. What are the parameter(s) of the distribution? 30. Using this distribution, write down the probabilities that you would use to deduce an answer for the airline. 31. Compare these probabilities with your simulation. 32. Would you change your recommendation to Air New Zealand regarding the number of extra tickets they should sell? 33. What different thinking did you need for the simulation approach and for the theoretical approach? 34. Is there an alternative distribution that could be used? Explain. 35. To what extent has Tinkerplots2 helped you (or not) to construct the theoretical probabilistic distribution?

Table 1: "How Many Tickets to Sell" Model-Eliciting Activity

RESULTS

Preliminary analysis of the data obtained during the teaching experiment, indicates that the instructional approach adopted by the study did foster students' ability to reason and make appropriate data-based inferences. Making modelling, generalization and justification an explicit focus of statistics instruction, helped learners develop more coherent mental model of key statistical ideas related to statistical inferences. The informal, data-driven inferences on which instruction focused on the first part of the course, helped students develop understandings of fundamental aspects of informal inference and argumentative reasoning (Ben-Zvi, 2006) that served as foundations for the formal study of inferential statistics in the latter part of the course.

The affordances offered by the dynamic software Tinkerplots2 for building data models and for experimenting with these models to make sense of the situation at hand, were instrumental in supporting student understanding of both informal and formal inferential statistical ideas. Use of the software promoted active knowledge construction and extended students' ability to produce reasonable inferences. Through gathering and analysing real data and/or performing computer-based simulations, students actively experimented with statistical ideas, examined the interaction between the data at hand and the theoretical model, and constructed more powerful meanings about key ideas related to inferential statistics.

Of course, similarly to other researchers, we also witnessed a number of challenges associated with the adoption of a modelling approach (Konold, Harradine, & Kazak, 2007; Pratt, 2011), and different levels of student reasoning and understanding of the role of models and modelling, and of the key assumptions underlying the models simulated by the computer (Zieffler, delMas, & Garfield, 2014).

Next, we provide a short description of the ways in which participants approached the "*How many tickets to sell?"* MEA (see Table 1). The preliminary analysis of the data collected on this activity, has revealed important differences in teachers' approaches to the problem – these differences will be the focus of a more detailed analysis of the data to be presented at the conference.

Modelling the 'How many ticket to sell?' scenario

Students used first the provided information that "on average 2.9% of the passengers that have booked tickets on its main domestic routes fail to show up for departure" to estimate the mean number of students who do not show up for departure. They knew that the aircraft Airbus A320 has 171 seats and thus estimated that, on average, around 5 passengers do not show up on each flight.

Students' next step was to model the Air Zland flight. Two students, Jason and Eleni, who worked as a team explained the procedure they followed: *"Using Tinkerplots, we made a model with 176 passengers, where 97.1% (0.971) of the passengers showed up, and 2.9% (0.029) did not show up, and we run some cases"* (see Figure 1). The important outcome of students' experimentation was that, after observing how different repetitions of this experiment worked for the particular model, they realised that "*the data changed each time, as there is the concept of chance*".

Acknowledging the need to repeat the experiment a very large number of trials to draw safer conclusions, students next chose the "Collect Statistic" feature of Tinkerplots2 to keep track of the number of students picking the "Helper" toy each time. They asked the software to repeat the experiment a large number of times and to draw the resulting distribution of sample statistics. This was done for the scenario in which AirZland sold five extra tickets (i.e. booked 176 tickets). In Figure 2, we see the distribution of sample statistics drawn by Jason and Eleni.

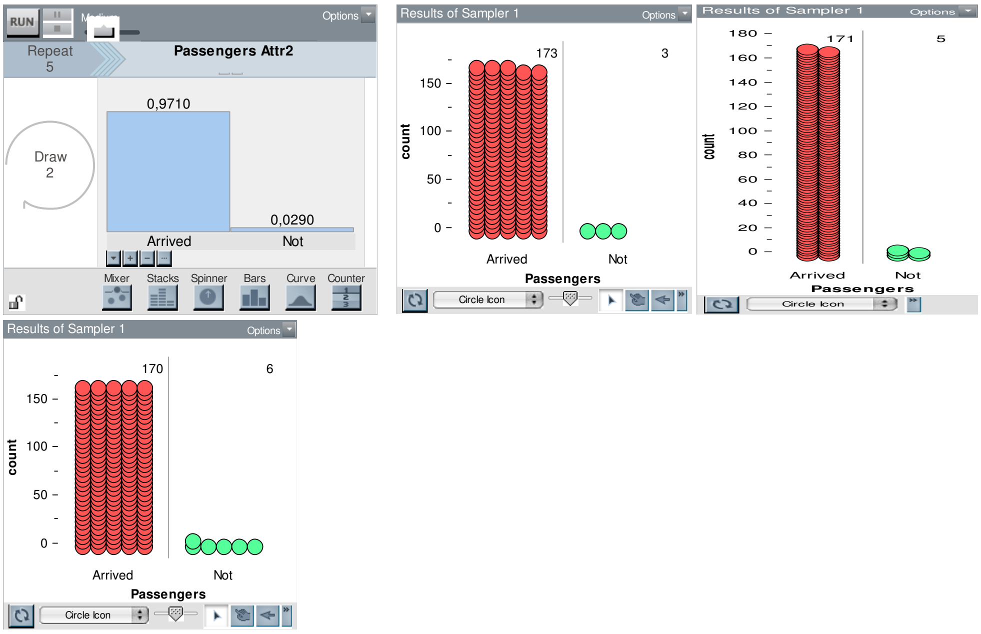

Figure 1: Outcomes of single sample simulations, under the null model, of the "How many tickets to sell?" MEA

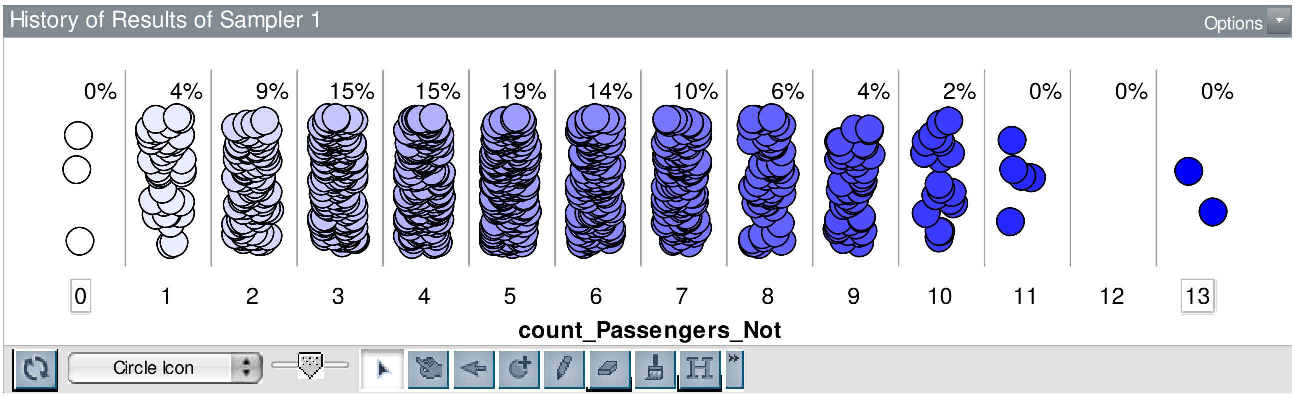

Figure 2: Outcomes of 1000 simulations for 176 tickets

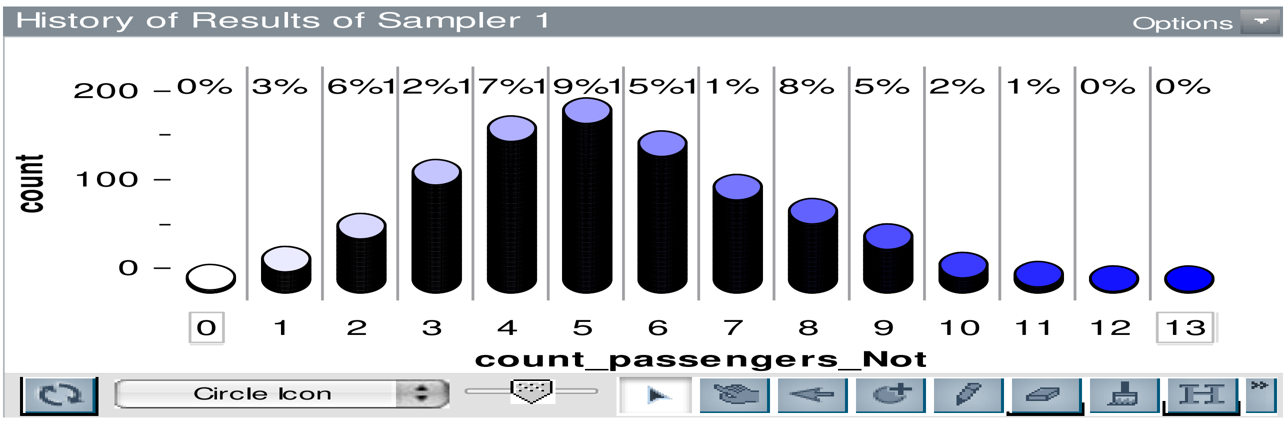

Figure 3: Outcomes of 1000 simulations for 175 tickets

Students were then asked to decide whether their model should be adjusted or not (see question 17 in Table 1). Jason and Eleni explained that they decided to run their model for 175 tickets instead of 176, as they found from Figure 2 that "*the number of cases of 4 and 5 passengers not showing up each time are very close*". In Figure 3, we see the distribution of sample statistics drawn by this pair of students after repeating the simulation of a flight with 175 booked tickets 1000 times.

$$P(0)=\frac{n!}{x!*(n-x)!}*p^0*q^{176}=\frac{176!}{(176-0)!}*(0.029)^0*(0.971)^{176}=0.0056=0.56\%$$

$$P(1)=\frac{n!}{x!*(n-x)!}*p^1*q^{175}=\frac{176!}{1!*(176-1)!}*(0.029)^1*(0.971)^{175}=0.0296=2.96\%$$

$$P(2)=\frac{n!}{x!*(n-x)!}*p^2*q^{174}=\frac{176!}{2!*(176-2)!}*(0.029)^2*(0.971)^{174}=0.0773=7.73\%$$

$$P(3)=\frac{n!}{x!*(n-x)!}*p^3*q^{173}=\frac{176!}{3!*(176-3)!}*(0.029)^3*(0.971)^{173}=0.1339=13.39\%$$

$$P(4)=\frac{n!}{x!*(n-x)!}*p^4*q^{172}=\frac{176!}{4!*(176-4)!}*(0.029)^4*(0.971)^{172}=0.1730=17.30\%$$

$$P(5)=\frac{\mathrm{n!}}{\mathrm{x!*(n-x)!}}*\mathrm{p}^5*\mathrm{q}^{171}=\frac{176!}{5!*(176-5)!}*(0.029)^5*(0.971)^{171}=0.1778=17.78\%$$

Table 2: Theoretical probabilities when booking 176 seats in the "How Many Tickets to Sell" MEA

Comparing the histograms of the resulting sampling distributions in Figures 2 and 3, Jason and Eleni made observations such as, for example, that: *"Although the possibility of 5 passengers not showing up has the highest chance, when booking 176 seats there is a 43% of chance for someone not being able to find a seat due to overbooking. On the other hand, when booking 175 tickets rather than 176, the chance of someone not being able to get a seat due to overbooking is reduced from 43% to 21%. So, we decided to recommend to AirZland to book 175 tickets, i.e. 4 extra tickets, in order to have a smaller percentage of complaining passengers. It is important for the company to have solutions for these passengers, as having a refund or a confirmed seat to the next fly, a free meal and/or a hotel to stay."*

Finally, students used the properties of the binomial distribution to determine theoretical probabilities when booking 176 seats (see Table 2). They concluded: "*The results from the binomial distribution are very similar to what we got when doing the Tinkerplots simulation. The theoretical probabilities show that our decision to recommend to the company to overbook 4 tickets is correct".* They went ahead to add: *"It was important how we experimented with TinkerPlots in order to reach a decision. The simulations helped us a lot to understand these theoretical probabilities".*

The study participants engaged in several MEAs similar in nature and format to the "How many tickets to sell" task. Using Tinkerplots2, students built and modified their own models of real world phenomena, and used them to informally test hypotheses and draw inferences. Their engagement with data-driven inferences helped them to develop sound informal understanding of the logic of hypothesis testing and its related statistical ideas (significance level, p-value, null and alternative hypothesis etc.), and served as a foundation for the formal study of inferential statistics.

Despite the fact that the simulation approach did give students rich insights that helped them better understand and appreciate the meaning and power of fundamental inferential concepts such as the central limit theorem, sampling distribution, and hypothesis testing, the introduction to the formal procedures for conducting hypothesis testing still caused difficulties to several of the study participants. Whereas almost all students in the course seemed very comfortable with the process of informally building sampling distributions of sample statistics (i.e. sampling distributions) and making inferences, several of them got very puzzled and intimidated when abstract notation and procedures were first introduced. In the beginning, students had difficulties in using traditional tools such as z, and t. Eventually, the modelling, simulation-based approach gave them insights that helped them better understand and appreciate the meaning and power of these formal inferential tools. Comparing empirical probabilities with the theoretical ones helped them make direct connections between the formal and the informal. There were some students in the class who, due a poor mathematical background, even at the end of the semester still confused basic statistical notation. However, the conceptual understanding of the logic of statistical inference that even these students had was much better compared to what is typically observed in students completing similar courses that employ more conventional, lecture-based approaches.

CONCLUSIONS

Statistics education research has long suggested that most adults, even ones with substantive formal training, tend to have poor reasoning about the stochastic and difficulty in using statistical methods appropriately in applied problems. Several studies examining learning outcomes of college-level statistics courses have indicated an alarming lack of statistical reasoning and thinking (e.g. delMas, Garfield, Ooms & Chance, 2007) among students who have completed such courses. Even students who can successfully implement procedures for hypothesis testing and parameter estimation might not be able to use them appropriately in applications (e.g. Gardner & Hudson, 1999). Thus, there is much room for improvement in statistics teaching and learning at the tertiary level (Garfield et al., 2012). The current study provides some useful insights on the ways in which making modelling, generalization and justification an explicit focus of statistics instruction, can help learners develop appreciation of the important role of models and modelling in statistical work and a more coherent mental model of key ideas related to statistical inference. Insights on the role of technology in scaffolding and extending students' reasoning about models and modelling are also provided.

As Garfield and Ben-Zvi (2008) point out, models are one of the most important and yet least understood ideas in introductory statistics. Despite the importance of data modelling and fitting models to data in statistical practice, the majority of students taking statistics courses are not given the opportunity to gain understanding of statistical model is or how it is used. Moreover, research illuminating how students come to learn and use statistical models is still very limited. Thus, findings from the current study are of both practical and theoretical importance to the statistics education community.

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TASKS PROMOTING PRODUCTIVE MATHEMATICAL DISCOURSE IN COLLABORATIVE DIGITAL ENVIRONMENTS*

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Rich tasks can be vehicles for productive mathematical discussions. How to support such discourse in collaborative digital environments is the focus of our theorization and empirical examination of task design that emerges from a larger research project. We present the theoretical foundations of our task design principles that developed through an iterative research design for a project that involves secondary teachers in online courses to learn discursively dynamic geometry by collaborating on construction and problem-solving tasks in a cyberlearning environment. In this study, we discuss a task and the collaborative work of a team of teachers to illustrate relationships between the task design, productive mathematical discourse, and the development of new mathematics knowledge for the teachers. Implications of this work suggest further investigations into interactions between characteristics of task design and learners mathematical activity.

Keywords: Collaboration, Dynamic Geometry, Mathematical Discourse, Task Design, Technology

INTRODUCTION

Mathematical tasks shape significantly what learners learn and structure their classroom discourse (Hiebert & Wearne, 1993). Such discussions when productive involve essential mathematical actions and ideas such as representations, procedures, relations, patterns, invariants, conjectures, counterexamples, and justifications and proofs about objects and relations among them. Nowadays, these mathematical objects and relations can be conveniently and powerfully represented in digital environments such as computers, tablets, and smartphones. Most of these environments contain functionality for collaboration. However, in such collaborative, digital environments, the design of tasks that promote productive mathematical discussions still requires continued theorization and empirical examination (Margolinas, 2013). To theorize and investigate features of tasks that promote mathematical discussions, we are guided by this question: What task-design features support productive discourse in collaborative, digital environments? Knowing these features will inform the design of rich tasks that promote mathematical discussions so that engaged and attentive learners build mathematical ideas and convincing forms of argumentation and justification in digital and virtual environments.

In virtual collaborative environments, the resources available to teachers to orchestrate collaboration and discourse among learners are different from those in traditional presential classroom environments. The salient difference is that in presential classroom environments the teacher is physically present, whereas in a virtual learning environment the teacher is artificially present; that is, the teacher exists largely as an artifact of digital tools. Consequently, the design of the tasks that are to be objects of learners' activities in virtual environments need to be constructed in ways that support particular learning goals such as productive mathematical discourse.

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We share Sierpinska's (2004) consideration that "the design, analysis, and empirical testing of mathematical tasks, whether for purposes of research or teaching, is one of the most important responsibilities of mathematics education" (p. 10). In this paper, we focus on the design of tasks that embody particular intentionalities of an educational designer who aims to promote and support productive discourse in collaborative, digital environments. Our work employs a specific virtual environment that supports synchronous collaborative discourse and provides tools for mathematics discussions and for creating graphical and semiotic objects for doing mathematics. The environment, Virtual Math Teams (VMT), has been the focus of years of development by a team led by Gerry Stahl, Drexel University, and Stephen Weimar, The Math Forum @ Drexel University, and the target of much research (see, for example, Stahl, 2008; Stahl, 2009). Recently, research has been conducted on an updated VMT with a multiuser version of a dynamic geometry environment, GeoGebra, (Grisi-Dicker, Powell, Silverman, & Fetter, 2012; Powell, Grisi-Dicker, & Alqahtani, 2013; Stahl, 2013). Our tasks are designed for this new environment—VMTwG. Though the environment and its functionalities are not the specific focus of this paper, we will later describe some of its important features to provide context for understanding our design of tasks. Our focus here is to describe how we address challenges involved in designing tasks to orchestrate productive mathematical discourse in an online synchronous and collaborative environment. We first describe the theoretical foundation that guides our design of tasks to promote potentially productive mathematical discourse among small groups of learners working in VMTwG. Afterward, we describe our task-design methodology and follow with an example of a task along with the mathematical insights a small team of teachers developed discursively as they engaged with the task. We conclude with implications and suggestions of areas for further research.

THEORETICAL PERSPECTIVE

The theoretical foundation of our perspective on task design rests on a dialogic notion of mathematics (Gattegno, 1987), what we call epistemic tools (Ray, 2013), and a sociocultural theory both of task and activity (Christiansen & Walther, 1986) and of instrument-mediated activity (Rabardel & Beguin, 2005).

Our notion of productive mathematical discourse rests on a particular view of what constitutes mathematics. From a psychological perspective, Gattegno (1987) posits that doing mathematics is based on dialog and perception:

> No one doubts that mathematics stands by itself, is the clearest of the dialogues of the mind with itself. Mathematics is created by mathematicians conversing first with themselves and with one another. Still, because these dialogues could blend with other dialogues which refer to perceptions of reality taken to exist outside Man…Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered *per se*. (pp. 13-14)

Mathematics results when a mathematician or any interlocutor talks to herself and to others about specific perceived objects, relations among objects, and dynamics involved with those relations (or relations of relations). For dialogue about these relations and dynamics to become something that can be reflected upon, it is important that they not be ephemeral and have residence in a material (physical or semiotic) record or inscription. On the one hand, through moment-to-moment discursive interactions, interlocutors can create inscriptions and, during communicative actions, achieve shared meanings of them. On the other hand, inscriptions can represent encoded meanings that—based on previous discursive interactions—interlocutors can grasp as they decode the inscriptions. Thus, inscriptive meanings and the specific perceived content of experience are dialectically related and mutually constitutive through discourse.

To increase the probability that the discourse of interlocutors is mathematically productive, it is useful that they employ individual and collaborative discursive means to make sense of mathematical situations. For this purpose, we invite interlocutors to employ particular epistemic tools. That is, to ask questions of themselves and of their interlocutors that query what they perceive, how it connects to what they already know, and what they want to know more about it. Specifically, these tools include three questions that interlocutors explicitly or implicitly engage: (1) What do you notice? (2) What does it mean to you? (3) What do you wonder about? The first and third questions come directly from work of The Math Form @ Drexel University (see, Ray, 2013). The second question is one that we have added. The purpose of these questions is to foster generative discussions within small groups of interlocutors that are grounded in their attention on perceivable, not necessarily visible, contents of experience that can be described as objects, relations among objects, and dynamics linking different relations. Using the epistemic tools, interlocutors' responses become public, relevant, and accountable. The idea is for interlocutors' to practice consciously these epistemic tools and over time become incorporated into their mathematical habits of mind.

The epistemic tools, among other things, are useful for enabling reflection on perceived infrastructural reactions of a dynamic geometry environment to interlocutors' actions in the environment. As they drag (click, hold, and slide) a base point of an object in a constructed figure, the environment redraws and updates information on the screen, preserving constructed geometrical relations among the figure's objects. This reaction to learners' dragging establishes a dialectical co-active relationship as the learner and the environment react to each other (Hegedus & Moreno-Armella, 2010). As learners attend to the environment's reaction, they experience and, since it responds in ways that are valid in Euclidean geometry, may become aware of underlying mathematical relations among objects such as dependencies.

Another role of the epistemic tools is to scaffold interlocutors' activity directed to understand and solve a mathematical task. We view tasks and activity from a sociocultural perspective. Within this perspective, Christiansen and Walther (1986) distinguish between task and activity in that "the *task* (the assignment set by the teacher) becomes the object for the student's activity" (p. 260). A task is the challenge or set of instructions that a teacher sets. An activity is the set of actions learners perform directed toward accomplishing the task. The activity is what learners do and what they build and act upon such as material, mental, or semiotic objects and relations among the objects. The task initiates activity and is the object of learners' activity.

Given the new digital, collaborative environments in which teaching and learning can occur, we find it theoretically useful to extend Christiansen and Walther's (1986) distinction of task and activity beyond analog environments: The purpose of a mathematical task in collaborative digital environments is to initiate and foster productive mathematical, discursive activity. The discursive activity is what learners communicate and do, what they build and act upon such as material, mental, or semiotic objects and relations. The digital, mathematical task is the object of learners' collective and coordinated activity.

Learners' activity directed toward a task is mediated by instruments. Before an instrument achieves its instrumental status, it is an artifact or tool. According to Rabardel and Beguin (2005) "the instrument is a composite entity made up of a tool component and a scheme component" (p. 442). The scheme component concerns how learners use the tool. Therefore, an instrument is a two-fold entity, part artifactual and part psychological. The transformation of an artifact into an instrument occurs through a dialectical process. One part accounts for potential changes in the instrument and the other accounts for changes in learners, respectively, instrumentalization and instrumentation. In instrumentalization, learners' interactions with a tool change how it is used, and consequently, learners enrich the artifact's properties. In instrumentation, the structure and functionality of a tool influence how learners use it, shaping, therefore, learners' cognition (Rabardel & Beguin, 2005). The processes of instrumentalization, instrumentation, and activity as well as the interaction of learners with themselves and the task reside within a particular, evolving context that is cultural, historical, institutional, political, social, and so on (see Figure 1).

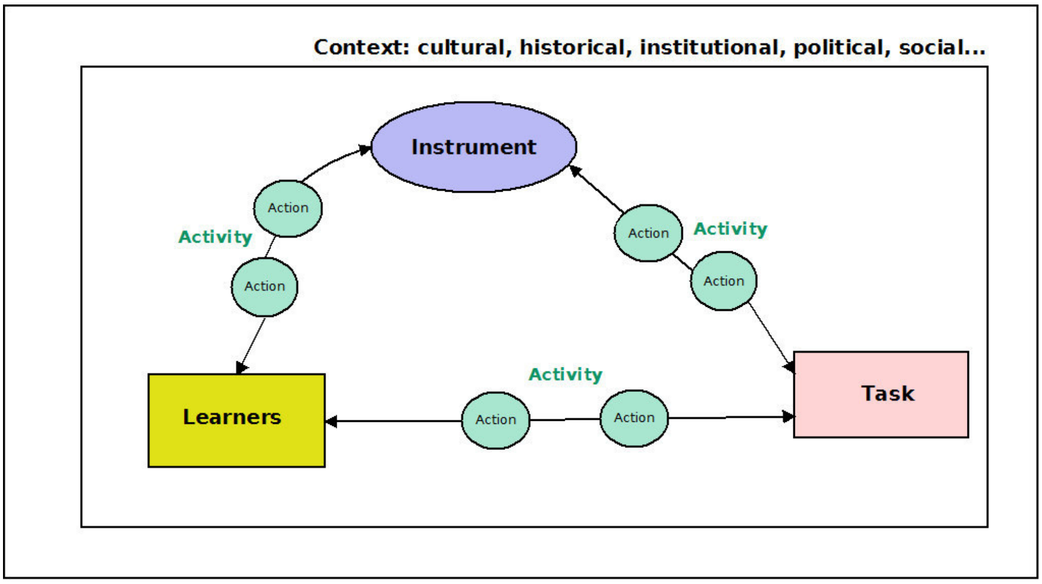

Figure 1. Relational model of learners engaged in instrument-mediated activity initiated by a task.

TASK-DESIGN METHODOLOGY

Our methodology of task design embodies particular intentionalities for a virtual synchronous, collaborative environment, such as VMTwG, that has representation infrastructures (GeoGebra, a dynamic mathematics environment) and communication infrastructures (social network and chat features). The intentions are for mathematical tasks to be vehicles "to stimulate creativity, to encourage collaboration and to study learners' untutored, emergent ideas" (Powell et al., 2009, p. 167) and to be sequenced so as to influence the co-emergence of learners instrumentation and building of mathematical ideas. To these ends and to respond to our research question, we attended to our theoretical perspective and infrastructural features of VMTwG to develop and test the following seven design principles for digital tasks that are intended to promote productive mathematical discourse by encouraging collaboration in virtual environments:

1. Provide a pre-constructed figure or instructions for constructing a figure.

2. Invite participants to interact with a figure by looking at and dragging objects (their base points) to notice how the objects behave, relations among objects, and relations among relations.
3. Invite participants to reflect on the mathematical meaning or consequence of what they notice.
4. Invite participants to wonder or raise questions about what they notice or the mathematical meaning or consequence of it.
5. Pose suggestions as hints or new challenges that prompt participants to notice particular objects, attributes, or relationships without explicitly stating what observation they are to make. Each hint has one or more of these three characteristics:
   a. Suggest issues to discuss.
   b. Suggest objects or behaviors to observe.
   c. Suggest GeoGebra tools to use to explore relations, particularly dependencies.
6. Provide formal mathematical language that corresponds to awarenesses that they are likely to have explored and discussed or otherwise realized.
7. Respond with feedback based on participants' work in the spirit of the following:
   a. Pose new situations as challenges that extend what participants have likely noticed, wondered, or constructed or that follow from an earlier task and that involve the same awarenesses or logical extensions of awarenesses they have already acquired.
   b. Invite participants to revisit a challenge or a task on which they already worked to gain awareness of other relationships.
   c. Invite participants to generalize noted relationships and to construct justifications and proofs of conjectures.
   d. Invite participants to consider the attributes of a situation (theorem, figure, actions such as drag) in order to generate a "what if?" question and explore the new question.

The purpose of hints is to maintain learners' engagement with a task and to encourage them to extend what they know. The hints support participants' discourse by eliciting from them statements that reveal what they observe and what they understand about the mathematical meanings or consequences of their observations. The challenges are available to provide opportunities for learners to explore further by investigating new, related situations. Hidden initially, learners can reveal the hints and challenges by clicking a check box.

These design principles guided how we developed tasks in our research project, a collaboration among investigators at Rutgers University and Drexel University. We employed VMTwG, which contains chat rooms for small teams to collaborate with tools for mathematical explorations, including a multi-user, dynamic version of GeoGebra. Team members construct geometrical objects and can explore them for relationships by dragging base points (see Figure 2). VMTwG records users' chat postings and GeoGebra actions. The project participants are middle and high school teachers in New Jersey who have little to no experience with dynamic geometry environments and no experience collaborating in a virtual environment to discuss and resolve mathematics problems.

The teachers took part in a semester-long professional development course. They met for 28 two-hour synchronous sessions in VMTwG and worked collaboratively on 55 tasks, Tasks 1 to 55.

Using our design principles, we developed dynamic-geometry tasks that encourage research participants, who were mathematics teachers, to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications.

TASK EXAMPLE

We present the work of a team of two teachers who worked as learners on a task. The task, Task 10, is one that the research team posed. While the teachers worked on it, they posed a wondering that led us to provide feedback of type 7a, inviting them to explore that wondering. Our analysis reveals how using the epistemic tools the teachers noticed and discussed geometric relations and completed a construction task, wondered about the necessity of a foundational object of the construction, and in the following session resolved their wondering, all through the use of the epistemic tools.

In the fourth week of the professional development course, the team worked on Task 10. Employing procedures of Euclid's second proposition (Euclid, 300 BCE/2002), the task engaged the team in constructing the copy of a line segment, without using the built-in compass tool, only using line segments, rays, and circles. The task also requested that they discuss dependencies and other relations among the objects (see Figure 2).

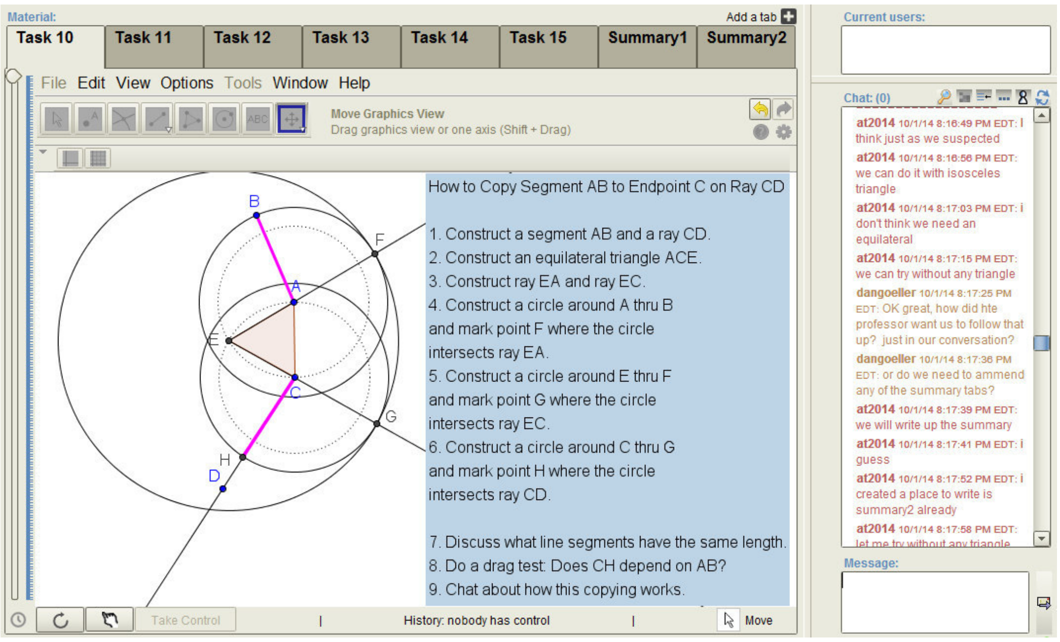

Figure 2: Task 10: Copying a line segment.

In the first synchronous session, the teachers successfully followed the construction instructions to copy segment AB onto ray CD. They used the epistemic tools to respond to this task and were attentive to co-active responses of VMTwG to their actions. In their noticings, they chatted about constructed dependencies and other relations among the geometric objects that they constructed. Below, an excerpt of the teachers' discussion illustrates their use of the epistemic tools and how they trigged productive mathematical discourse about a foundational aspect of the construction:

155	at2014:	o what we wonder about
156	at2014:	let's talk about it before we move on
157	at2014:	i am still trying to understand so i am not quite sure whether the equilateral triangle is necessary
158	at2014:	o maybe it does
159	dangoeller:	i agree lets get the others done before sketching this one again
160	at2014:	to get that big circle
161	at2014:	ok
162	dangoeller:	that’s a good question
163	at2014:	i am not sure why the equilateral triangle is necessary if it is at all
164	dangoeller:	it appears that it is, but the "why" behind it is unclear to me
165	at2014:	that would be the question for us to put in what we wondered about

In this excerpt, they employed the epistemic tools by wondering about whether an equilateral triangle is necessary in the construction procedure to copy a line segment (see lines 157, 163, and 164). In their session summary, they explicitly stated “We wonder whether the equilateral triangle is necessary or not and if it is necessary, why is it so.” In our written feedback, their wondering encouraged us to invite them to explore it in their next synchronous session. In that session, they explored copying a length with an equilateral triangle, an isosceles triangle, and without using any specific type of triangle, which was essentially using a scalene triangle (see Figure 3).

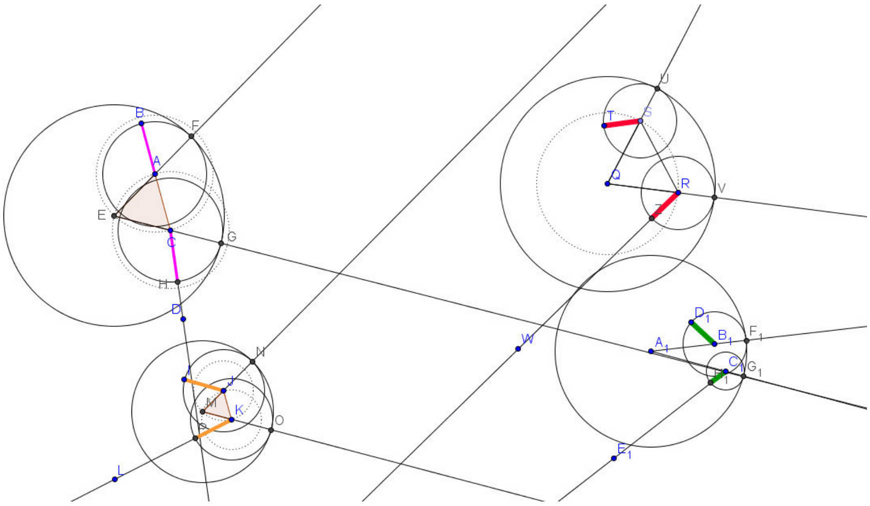

Figure 3: Teachers’ investigation of minimal condition for copying a segment length.

The teachers wrote in their session summary that after conducting drag tests on their constructions, “we found out that if we want the length of one segment to be dependent on another, we need at least the isosceles triangle”. Their constructions in Figure 3 include copying a length with an equilateral triangle (lower left corner), using an isosceles triangle (top right corner), and “with no triangle” (lower right corner). They justified their findings by discussing the dependencies each construction has. They make the point that having an equilateral triangle “is only keeping points A and C apart a certain distance, and we can do without it.” That is, they demonstrated that to copy the length of the segment AB the distance between A and C is immaterial and that only two congruent sides of a triangle matter.

DISCUSSION

Our focus was to describe task-design features that promote productive mathematical discourse among interlocutors working in an online synchronous environment. In the virtual environment, a teacher is present largely as an artifact of the environment's digital tools and most specifically in the structure and content of tasks. An important feature of our task design is the questions of our epistemic tools since when collaborating interlocutors respond to them they generate propositional statements that can become the focus of their discussions. Their discussions are mathematically productive as their noticings, statements of meaning, and wonderings involve interpretations, procedures, patterns, invariants, conjectures, counterexamples, and justifications about objects, relations among objects, and dynamics linking relations.

Our guiding task-design principles aim to engage learners in productive mathematical activity through inviting them to explore figures, notice properties, reflect on relations, and wonder about related mathematical ideas. The design provides support through hints and feedback to help learners with certain parts of the tasks. The tasks also include challenges that ask the participants to investigate certain ideas and extend their knowledge. The example provided above shows that the teachers moved from conjecture to justification through the use of our epistemic tools. Their wondering about fundamental construction concepts prompted them to build ideas that were new to them. Further investigation is needed to understand how the task-design elements, the affordances of collaborative digital environments, and learners' mathematical discourse interact to shape the development of learners' mathematical activity and understanding.

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USING TINKERPLOTS SOFTWARE TO LEARN ABOUT SAMPLING VARIABILITY AND DISTRIBUTIONS AS A BASIS FOR MAKING INFORMAL STATISTICAL INFERENCES

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We report on a study involving an instructional sequence that engaged a class of high school students in using TinkerPlots™ software to make informal statistical inferences on the basis of distributions of a sample statistic. The sequence involved a scenario and tasks entailing the comparison of multiple samples of two groups of organisms on a common attribute. Students engaged in 1) making sense of the scenario and a TinkerPlots simulation that produced distributions of a sample statistic, 2) interpreting a sequence of such distributions in relation to increasing sample size, and 3) inferring a value of the sampled population attribute. We highlight aspects of students' understandings of what an empirical sampling distribution represented in terms of the scenario's context, and aspects of their abilities to track the multi-tiered re-sampling process that began with a population and culminated with distributions of the sample statistic on which they based their inferences.

Keywords: Statistical inference, variability, sampling distributions, TinkerPlots

BACKGROUND

Our information and data-driven age makes statistical decision-making and inference arguably one of the most important schemes of ideas to target in school mathematics instruction (Garfield & Ben-Zvi, 2008). The apparent logic of inference centers on the idea that the value of an attribute of a population of interest—the whole of which is usually not directly accessible — can be inferred only indirectly, by examining the same attribute for a sample that is randomly chosen from that population. In practice, such inferences are typically made on the basis of a single sample drawn from a population. Yet, randomly selected samples have outcomes (values of a particular statistic) that typically vary from sample to sample. As Rubin, Bruce and Tenney (1991) have argued, the ability to balance and coordinate these two seemingly antithetical ideas — that of individual sample representativeness and the idea of sample-to-sample variability — is key to a coherent understanding of inference. The challenge for instruction, as Rubin et al. (1991) frame it, is to have students integrate these contrasting ideas of representativeness and variability into the unified notion of a sampling distribution. Saldanha and Thompson echoed Rubin et al.'s argument in that "…we do not see how the normative practice of drawing an inference from an individual sample to a population can be understood deeply without reconciling the ideas of sample-to-sample variability and relative frequency patterns that emerge in collections of values of a sample statistic…" (Saldanha & Thompson, 2007, p. 275).

Over the last fifteen years, the development of software designed specifically to support the learning of statistics with a focus on interactivity and dynamic visualization (e.g., Finzer, 2012; Konold & Miller, 2011) has made the use of sampling simulations and the generation of associated sampling distributions increasingly common in statistics instruction at the high school level and beyond. Statistics educators and researchers have recommended and explored the use of simulation-based statistics instruction and curricula to support the development of students' understanding of inference (e.g. Garfield & Ben-Zvi, 2008; Zieffler & Garfield, 2007). This paper reports on an effort in that vein involving a group of high school students engagement with simulation-based instructional activities using *TinkerPlots* — an interactive and dynamic data exploration and simulation software — to support their ability to make distribution-based informal statistical inferences.

THE STUDY AND INSTRUCTIONAL SEQUENCE

Participants, instructional context, and data corpus

We report on the second phase of an instructional intervention that engaged an intact class of twenty- three 9th-grade students (14 and 15 year-olds) in three 65 to 80-minute lessons in a school located in a suburb of a large city in the southwestern United States. Instructional activities involved the use of *TinkerPlots*™ software (Konold & Miller, 2011) to explore and analyze data sets in a first phase, and subsequently to simulate re-sampling and use the resulting sampling distributions as a basis for inferring a population parameter's value in a second phase. Students came into the study with some prior statistical knowledge and skills acquired both in their previous coursework and outside of school. For instance, most of the students understood a *sample* to be a "small part" of a larger collection of items that could be used to indicate information about a characteristic of the latter. Students knew how to compute the arithmetic mean of a set of values, and they knew a procedure for finding the median ("the middle value") of a data set. Students also demonstrated an ability to construct and use dot plots and histograms to compare and draw conclusions about two data sets. Students had not previously been exposed to *TinkerPlots*, nor had they previously engaged with re-sampling activities or been exposed to distributions of a sample statistic as products of repeated sampling.

The author oversaw and orchestrated the unfolding of the instructional sequence and class discussions that emerged therein, while an assistant observed and took field notes. Instruction was generally organized around whole-class discussions in which the author first introduced issues and ideas targeted in instruction, and demonstrated intended uses of *TinkerPlots* relevant to an investigation of those ideas. *TinkerPlots* files were projected onto a screen at the front of the class for all students to view and refer to in the discussions. Such discussions were subsequently interspersed with students' individual and paired work on structured activity sheets addressing those ideas; this involved using *TinkerPlots* to operate on data files provided to them (laptop computers were provided for each student) and answering reflection questions about the results of their actions on these data files. This implementation encouraged students to express their thinking about those ideas and aspects made salient to them through their use of *TinkerPlots* in a whole-class context. The lessons and class discussions were recorded with digital video cameras, and all students' written work on activity sheets and a final exam were recorded. Each student took part in a video-recorded exit interview conducted within one week of the end of the instructional sequence. The overarching research goal was to gain insight into students' thinking and understandings of ideas promoted in instruction in relation to their engagement with that instruction.

The instructional sequence: Lessons 1-3

The phase of the instructional sequence reported here engaged students in an investigation designed to foster their ability to make inferences to a population by considering the variability amongst outcomes of samples of a common size chosen from that population, and by comparing the variability across distributions of a sample statistic generated from collections of such samples of different sizes. The investigation centered on two big statistical ideas: 1) random sampling can be used to draw conclusions about the sampled population, and 2) larger samples lead to sampling distributions that tend to be less variable and hence lead to more confident conclusions about the population. The context for the investigation revolved around the following opening scenario that involved testing whether a species of genetically engineered fish tends to grow longer than normal fish (Key Curriculum Press, 2011):

> *A fish farmer stocked a pond with a new type of genetically engineered fish. The company that supplied the new type claims that these fish will grow to be longer than normal fish. The farmer decided to test the company's claim by stocking the pond with 625 fish, some normal and some genetically engineered. When the fish were fully-grown the farmer caught a sample of 130 fish from the pond and measured the length of each fish in his sample.*

TinkerPlots' sampling simulator tool was introduced and used throughout this part of the sequence to efficiently generate data and collections of random samples of various sizes from the population of fish, and to graphically display the distributions of a resulting sample statistic.

Lesson 1 The first lesson introduced students to the *Fish Farmer* scenario shown above (adapted from Key Curriculum Press, 2011, and Konold, 2005), framing the issue as one of a skeptical fish farmer wanting to test the claim that a genetically engineered type of fish tends to grow longer than a normal type. In the opening part of the investigation, prior to divulging the fish farmer's approach, students were first asked to consider how the farmer might go about testing the claim. The ensuing class discussions focused on issues of data collection and selecting a representative sample, providing occasion for students to consider possible ways in which the farmer might proceed. Students then examined a *TinkerPlots* data file showing the lengths of the two types of fish in the farmer's sample of 130 fish. They explored the data using *TinkerPlots* graphical tools and techniques that they had learned in the preceding phase of the sequence (see Figure 1 for an example of such); students considered the group differences and what they suggested about the lengths of the two types of fish in the population on the basis of what they observed in this single random sample. The big statistical idea promoted in this lesson was that if a randomly selected sample is assumed to be representative of its parent population, then it can be used as a basis for making claims about that population.

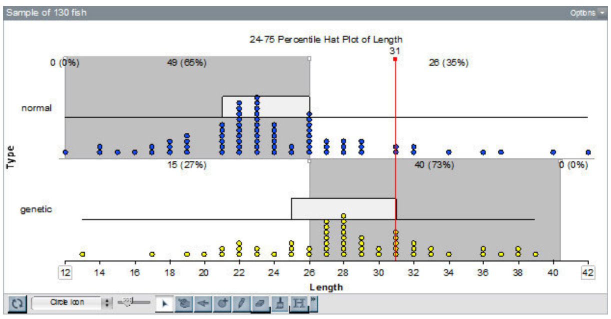
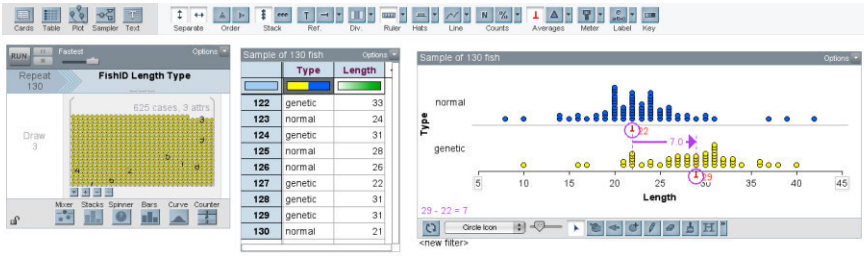


Figure 1. Two students' use of *TinkerPlots*' divider, percentile hat, and reference line tools to analyze a sample of 130 fish-lengths separated into lengths of normal fish and genetically engineered fish.

Lesson 2 The second lesson introduced students to the idea that sampling outcomes are expected to vary from sample to sample, were the sampling process repeated under essentially the same conditions, and that such variability therefore poses a problem for making inferences to an underlying population on the basis of any individual sample. At the same time, the lesson aimed to help students begin to understand that the variability amongst samples exhibits patterns that are predictable over the long run, and that such patterns can be discerned by analyzing collections of sampling outcomes. The lesson began by having students reflect on whether they would expect to obtain similar or different results if another sample of 130 fish were randomly drawn from the fish farmer's pond. Students were then prompted to reflect on whether a sample of 130 fish is big enough to make a confident claim about the sampled population, and to share and justify their intuitions about this question. The lesson then moved to a more systematic exploration of these questions by having students use *TinkerPlots*' sampling simulator to generate several samples of size 130, and then of size 15, from the simulated fish population (see Figure 2). Students used the *median* and *ruler* tools within *TinkerPlots* to record and visually highlight the median length of each type of fish in a sample, and the difference between these medians as a measure of the group differences (i.e., difference between the median length of two types of fish in a sample). Figure 2 displays the *TinkerPlots* set-up that the instructor and students used in this investigation; the simulator (left hand tool) used a mixer to represent the population of mixed fish. Each selected sample of 130 fish was represented in a case table displaying each fish's type and length. The case table was linked to an ordered and stacked dot plot showing the distribution of lengths of fish in the sample, separated by type and displaying the median of each type as well as the difference between medians using *TinkerPlots*' ruler tool. Each row of the table at the bottom of Figure 2 recorded the value of the three measures for a sample (median length of each type of fish and the difference between those medians) generated by running the simulation once. The four representations displayed in Figure 2 were dynamically linked and automatically updated with the results of each new repetition of the simulated sampling experiment (enacted by pressing the "run" button in the mixer tool). As such, students were able to observe the emergence of the value of the sample statistic (difference between median lengths of the two types of fish) and to track the variability in its values as the repeated sampling process unfolded.

The design of this simulation leveraged *TinkerPlots'* capacity to visually display the repeated sampling process as the sequential composition of linked sub-processes unfolding dynamically in "real time", with the aim of fostering students' imagery and understanding of the ending distribution as the product of such a composition. Indeed, it was an explicit goal of instruction that students conceptualize the sequential sampling process that began with the population and culminated with the results displayed in the bottom table (and later with a dot plot displaying the resulting distribution of the sample statistic, as shown in Figure 3). This goal was addressed more explicitly in Lesson 3, as described in the next section of the paper.



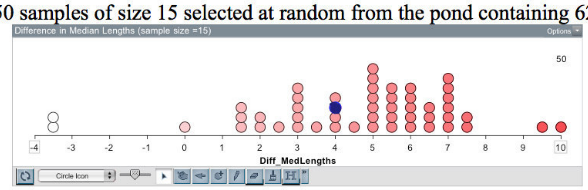
	MedLength_Genetic	MedLength_Normal	Difference_MedLengths
1	28	23	5
2	27	23	4
3	28	23	5
4	28	23	5
5	28	23	5
6	28	22	6
7	29	22	7

Figure 2. A *TinkerPlots* simulation of sampling 130 fish from a population of 625 mixed fish. The table at the bottom shows the three measures recorded for seven trials of the sampling experiment.

Students explored the patterns in these measures generated by simulating the sampling experiment seven times; they identified similarities and differences among the resulting medians and difference in medians for the collection of seven samples, and they used their observations as a basis for proposing how this might help test the claim that genetically engineered fish tend to grow longer than normal fish in the larger population. Class discussions around this exploration showcased students' perceived patterns, culminating with a general consensus that the genetically engineered fish in the population were inferred to be "between 4 and 7 centimeters longer" than the normal fish.

Lesson 3 The final lesson built on the activities and issues raised in Lesson 2 by having students examine the effects of sample size on the variability of the difference between median lengths of fish that they had previously explored only for seven samples. The lesson began with a demonstration and discussion of the *TinkerPlots* simulation of selecting 50 samples of size 15 from the simulated fish population (following the set up shown in Figure 2), culminating with the presentation of a distribution of the difference in median lengths for each of the 50 simulated samples as shown in Figure 3.

I. Here is a distribution of the difference between the median lengths of genetic and normal fish for 50 samples of size 15 selected at random from the pond containing 625 fish.



1. Describe what the darkened dot represents:

2. What information is shown by this dot?

3. How was this information obtained? Describe the sequence of steps in the sampling process that produced the information shown by the darkened dot.

Figure 3. A sampling distribution and accompanying prompts of the opening activity of Lesson 3.

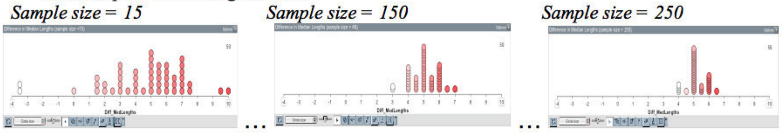
Discussions around this demonstration centered on having students track and explain the process of how the dot plot of the sampling distribution resulted from the sampler in terms of the various intermediate objects produced by the simulation and shown in the *TinkerPlots* window, as displayed in Figure 2. This activity aimed to help students build and solidify their imagery of the repeated sampling process and their meaning for the resulting sampling distribution displayed in Figure 3.

The accompanying activity prompts also assessed the strength and robustness of students' imagery by having them work backwards from a particular point on the dot plot and explain what it represented and the process that produced it. The activity prepared students for the subsequent part of Lesson 3, which required that they be able to decode and interpret a sequence of such dot plots coherently.

In the second part of Lesson 3 students examined and interpreted a sequence of five distributions of the difference in median lengths, each for 50 simulated samples of a different size drawn from the fish population. Students examined and interpreted these sampling distributions in relation to the increases in sample size. A subset of these distributions and the accompanying activity prompts are displayed Figure 4.

II. Here is a sequence of distributions. Each one is of the difference between the median length of genetic and median length of normal fish for 50 samples of a given size selected at random from the pond containing 625 fish.

Sample size = 15 ... *Sample size = 150* ... *Sample size = 250*



4. Compare these distributions for the various sample sizes. What do you notice about these distributions as sample size increases?

5. What is a big enough sample of fish for the farmer to pick from the pond in order to confidently test the company's claim that genetic fish tend to grow longer than normal fish? Please explain why you think this.

6. Estimate how much longer the genetic fish in the pond tend to be than the normal fish. How confident are you about this estimate? Please explain.

Figure 4. Task prompts and a subset of the distributions of the final activity of Lesson 3.

The first prompt (Question 4) aimed to orient students' attention to the fact that the clustering of the sample statistic becomes increasingly compact (its variability decreases) with increasing sample size. A group discussion of this observation ensued which involved eliciting students' ideas about how to describe and measure the pattern of observed variability. This discussion was followed by prompting students to use this pattern as a basis for choosing a sufficiently large sample in order to confidently infer whether genetically engineered fish tend to grow longer than normal fish in the population (Question 5). The final prompt (Question 6) asked students to estimate *how much longer* genetic fish tend to grow than normal fish. These questions culminated in a group discussion about the trade-off between the competing interests of maximizing sampling accuracy and minimizing sample size.

Three days after Lesson 3, students took an exam designed to assess their thinking and conceptions of the ideas addressed in the instructional sequence. Students responded to a set of questions nearly identical to those from Lesson 3, but couched in a different story context (i.e., testing whether a genetically modified variety of cucumbers tended to grow longer than a normal variety). Follow-up interviews conducted within a week of the exam queried students' responses to this analogous task.

SUMMARY OF SELECTED FINDINGS AND CONCLUSION

Two salient findings regarding students' thinking were revealed from an initial analysis of their responses to the activity prompts of Lessons 1 and 2 and the classroom discussions that emerged around them:

- Class discussions proved to be productive vehicles for helping students notice and reflect on key ideas targeted in the sequence of activities. The whole-group discussions around the orienting prompts and reflection questions of the activity sequence turned out to consistently provide rich opportunities for students to notice patterns of dispersion in the distributions that resulted from re-sampling simulations. As a particular case in point, the discussions around the activity of Lesson 2 (see Figure 2) highlighted several students' observations (derived from the patterns they observed in the bottom table of Figure 2) that "the samples [median lengths] don't vary by much" and that the median length of genetically engineered fish in a sample was larger than that of the normal fish in every sample chosen. This last was taken as evidence, and led to a consensus among students, that the genetically engineered fish in the population tended to grow longer (by between 4 and 7 centimeters) than normal fish. These discussions thus evidenced students' abilities to coordinate two things: 1) their expectation that the sample statistic's value will vary among samples, and 2) their emerging understanding that such variability is not haphazard but is instead constrained to a (possibly) predictable range of values.
- Students generally exhibited an appreciation of two seemingly competing ideas: although an individual random sample can be used to make an inference about the sampled population, repeating a sampling experiment shows that sampling outcomes (i.e., a statistic's values) vary from sample to sample. Significantly, a majority of the students seemed to readily appropriate the idea (promoted in instruction) of using a distribution of multiple values of the sample statistic generated for samples of a particular size as a basis for informally assessing their level of confidence in their inference about the population. For instance, most students asserted that they were "completely" confident in their inference that the genetically modified species of fish were generally between 4 and 7 centimeters longer than normal fish because the value of the statistic of interest invariably fell between those values in all trials of the sampling experiment.

Students' written responses to the first activity prompts of Lesson 3 (Figure 3), and their explanations of the responses to the analogous task that they provided in the exit interviews, indicated that many students tended to experience the following challenges:

- Difficulty understanding what the distributions of the sample statistic (difference between medians lengths) represented in terms of the scenario(s) in which they were embedded. Although most (12) students were able to correctly explain that an individual point in the distribution shown in Figure 3 (Question I.1) represented a particular sample of 15 fish, only 8 of those students had a clear sense of the full information convey by that point. A common difficulty amongst those having an unclear sense involved not seeing that the point indicated the *difference in median lengths of the two types of fish* in a sample, and instead thinking that it showed either an actual length or median length of fish. Such confusion and other difficulties were seen even among some of those students who evidently understood a point to represent a sample of 15 fish. The following responses to Questions I.2 and I.3 given by one such student illustrate the tendency to interpret the information indicated by the darkened point as an average of the 50 data values, rather than thinking of the relation between the point and the sampling process that produced it:

S8: "The information shown by this dot is that all of the data added together and divided by 50 is 4" and "This information was obtained by the position of the dot and the information around the dot and from the graph."

The above response is in contrast to one of the most coherent responses to the same questions, as illustrated below:

S9: "For this sample, the median of the genetic fish was 4 cm higher than the median of the normal fish." and "A sample of 15 fish was retrieved by the mixer. The program found the medians of the normal and genetic fish and then found the difference between them, which it represented with a dot."

S9's response clearly relates the darkened point to the sampling process that produced it, suggesting her having a vivid imagery of that process and an understanding of how the process generated the information indicated by the point. Moreover, her identification of the sample statistic as a difference between the median lengths of two subsets of the sample suggests her ability to parse and hold the three quantities in mind as distinct entities without confounding them. All of this is indicative of an ability to keep mental track of, and describe, the multi-tiered simulated re-sampling process (i.e., conceiving of the coordinated sequence of actions and resulting statistical objects) that begins with an identifiable simulated population, involves a random sampling process, and culminates in the generation of a distribution of the appropriate sample statistic.

Regarding *TinkerPlots'* potential impact on students' thinking and learning, student S9's direct references to objects entailed in the *TinkerPlots* simulation (as seen in Figure 2) suggest that the software's visual and dynamic display of the repeated sampling process and its linkage to the graph of the resulting sampling distribution may have supported the development of S9's coherent imagery as indicated above. This may also be true for other students who exhibited similar responses and abilities. However, I must point out that relatively few of the students evidenced having developed this ability to the same extent as S9.

These findings underscore challenges in designing instruction to foster the development of (informal) distribution-based inferential reasoning, and they orient me to consider possible refinements to the instructional activities. One possible refinement currently under consideration is inspired by Harel's repeated reasoning instructional principle (Harel & Koichu, 2010). The refinement would consist of an elaboration of the activity shown in Figure 3 designed to have students practice imagining and explicitly identifying and distinguishing the processes and intermediate objects generated by the *TinkerPlots* re-sampling simulation that produces the distributions of the sample statistic shown in Figures 3 and 4. The aim of such an elaboration would be to help students build stable and robust mental images of the re-sampling scheme and its products, so that they would be better positioned to hold the ensemble of coordinated ideas in mind as they attempt to make distribution-based inferences and quantify their confidence in such inferences.

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MULTI-REPRESENTATION BASED OBJECTIFICATION OF THE FUNDAMENTAL THEOREM OF CALCULUS

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This study was designed to elaborate learning trajectory for the fundamental theorem of calculus. The learning trajectory is based on the structural decomposition of the fundamental theorem of calculus, and on tasks to be learned with multiple representational technology-based artifacts. The study was guided by the objectification theory, which considers learning as a matter of actively endowing the conceptual objects made available by the artifact with meaning. The analysis of the data identifies the focuses involved in the elaborated learning trajectory.

Keywords: Learning trajectory, multiple representational technology, objectification, calculus, integral

INTRODUCTION

Constructing a learning trajectory for students is one of the most daunting and urgent issues facing mathematics education today. Simon (1995) offered the Learning trajectory as a way to explicate an important aspect of pedagogical thinking involved in teaching mathematical understanding. According to Simon, learning trajectory consists of three components: (a) the students' learning goal, (b) a hypothesis about the process of the students' learning of mathematical concept, and (c) tasks to be used to promote the students' learning (Simon, 1995). The issues of how to promote students' development of new mathematical concepts, especially those concepts where development not yet studied, is one of the significant problems facing mathematics educators. One of these concepts is the fundamental theorem of calculus (FTC) known as one of the important discoveries in the development of calculus. Many students still lack the opportunity to learn it conceptually (Thompson & Silverman, 2008; Bressoud, 2011).

Utilizing multi-representational artifacts in learning mathematical concepts has been extensively practiced, and special attention has been paid to their use in learning mathematical concepts based on the function concept as the cognitive root in algebra. Teaching and learning calculus having multi-representations of functions as its cognitive root and operationalized in interactive technological environment is an important example of this approach. Recently, researchers have shown interest in understanding the learning processes of calculus concepts when taught with multi-representational artifacts (Thompson & Silverman, 2008). For example, Yerushalmy and Swidan (2012) examined the objectification process involved in learning the accumulation function with interactive and multi-representational artifacts. These studies, like other studies in the field, are interested in understanding the learning process of isolated concepts. Because the FTC combines a variety of concepts, I believe that, to understand the progression processes of learning the FTC, we need to consider a learning trajectory that consists of a sequence of concepts rather than learning concepts in isolation. However, the major question is how to construct a learning trajectory that affords meaningful learning of the FTC, which allows students to participate in meaning making of the theorem components, and to foster their awareness of the relationships between the components.

My aim in this paper is to identify learning focuses involved in the process of multi-representation based objectification of the FTC through a hypothetical learning trajectory. Specifically, I am interested in answer the following research question

(1) What are the learning focuses involved in learning the FTC using multiple linked representational tools?

THEORETICAL FRAMEWORK

I consider learning to be a process of objectification (Radford, 2003). This means that learning is a matter of actively endowing with meaning the conceptual objects made available by the artifact used in the study. From a pragmatic point of view, Radford suggests a semiotic tool to analyse educational mathematical activities in cultural artifacts. The basic components of the semiotic tool are the students' attention and awareness of the mathematical object. Varieties of semiotic means of objectification that have a representational function attract the students' attention to mathematical objects. Furthermore, the properties of the artifact can help students to attend to the mathematical objects related to the activity under consideration. Paying attention to the necessary aspects of the mathematical phenomenon and using various semiotic means of objectification (e.g., words, action with artifact), students become aware of the attributes of mathematical objects within that phenomenon. By being aware, students attain objectification of the mathematical objects, which then become apparent to them through various devices and signs.

Graphs in a Cartesian system and table of values are considered to convey a cultural meaning. In this context, mathematical signs in general and graphs in particular play two roles. Radford, Bardini, Sabena, Diallo, and Simbagoye (2005) defined these roles as "social objects in that they are bearers of culturally objective facts in the world that transcends the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (p. 117). However, objectification as a theoretical tool enables a thorough analysis of the interaction between conceptual objects deployed in the artifact and the subjective meanings of the student. This kind of analysis is essential for my purpose, which aims to characterize the process of objectification throughout the suggested learning trajectory.

My theoretical assumption asserts that through social interaction with the artifact, students will pay attention on and become aware of the mathematical phenomena deposited in the artifact. Hence, I use the terms *attention* and *awareness* as a tool for characterizing the process of objectification. My aim is to identify focuses in the process of objectification as a strategy for elaborating the suggested learning trajectory of FTC using multiple representational artifacts.

THE ARTIFACT USED IN THE STUDY

The interactive accumulation artifact (IAA), which is at the heart of our study, is part of the Calculus UnLimited software (Yerushalmy and Schwartz, 1996). The IAA is shaped by pedagogical considerations and culturally accepted mathematical meaning. The objects in IAA (Cartesian graphs, table of values, etc.) are considered signs that carry meaning accepted in mathematical culture.

The IAA, as a multi-representational artifact, contains different types of signs and tools that I have grouped into four categories.

Graphing signs: Two Cartesian systems coordinated and aligned vertically. The curve in the upper Cartesian system signifies a function. The function is defined symbolically by the free input of a single variable expression. When the user chooses rectangles representation by clicking the icon, the lower system displays a graph that signifies Riemann accumulation function values $\sum_{i=0}^{n} f(a \pm i\Delta x) \cdot \Delta x$ and rectangles should appear in the upper Cartesian system (Fig. 1). When the user chooses continuous area representation by clicking the icon, the lower Cartesian system displays a graph that signifies the accumulation function $\int_a^x f(u)du$ and the upper Cartesian system displays a filled cover area. Clicking the two icons, the lower Cartesian system displays Riemann accumulation function and accumulation function graphs (Fig. 1). Displaying the function and the accumulation function graphs in the same interface represents the mutual relationship between the graphs. Up–down observation signifies the relationship between a function and its accumulation function. Down–up observation signifies the relationship between a function and its derivative function.

Numeric signs: The associated table of values contains at least four columns (Fig. 1). The column furthest to the left signifies (a) the upper limit values, (b) the delta x values, and (c) the Riemann sums values and the column furthest to the right (d) signifies the accumulated values.

Accumulation signs: Five methods of accumulation are available by clicking an icon on the upper side of Figure 1. Because my study focuses on the computation of rectangles, and on continuous area, the rectangles that appear on request in the upper Cartesian system signify the product $f(x_i + \Delta x) \cdot \Delta x$; the continuously covered area represents the limit of the sum of products $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i + \Delta x) \cdot \Delta x$.

Boundaries tool: Three parameters determine the value of the accumulation at a given point: the lower limit, upper limit, and the width of each interval (value control box Fig. 1). Students control the bounded area by using arrows to move a marker in intervals of Δx to the left or to the right of the lower limit. Reducing Δx value in the controlling values is reflected in the rectangles display by reducing its width. It is also reflected in the accumulation functions graph by bringing the discrete points closer to each other, and reflected in the table of value by approximating the numbers in the sigma to the numbers in the integral columns.

THE PROPOSED VERSUS TRADITIONAL FTC LEARNING TRAJECTORIES

Table 1 demonstrates the proposed learning trajectory compared with the typical approach for teaching the FTC.

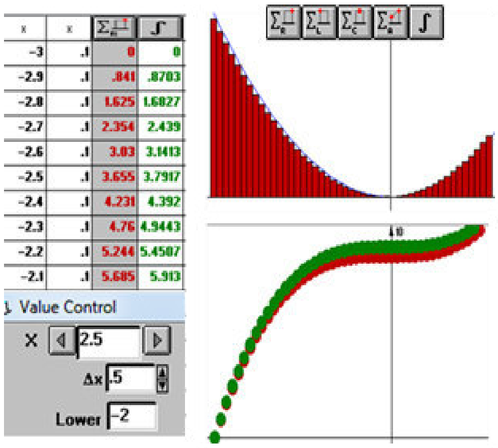


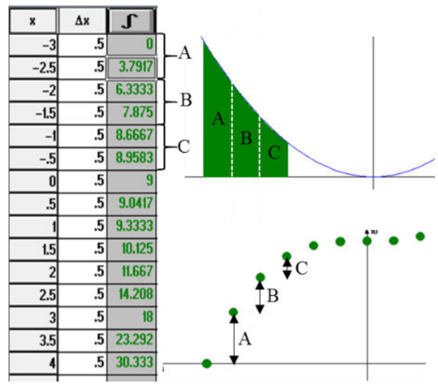
Figure 1. IAA interface

Typical teaching sequence of the FTC	Proposed learning trajectory
The anti-derivative concept $\int f(x)dx$	The Riemann accumulation function of a function f(x) represented as $$F_{\Delta x,a}(x) = \sum_{i=0}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} f(i\Delta x + a)\Delta x, a \le x \le b.$$
The definite integral concept $\int_a^b f(x)dx$	The accumulation function as convergence of the Riemann accumulation functions $$g(x) = \lim_{\Delta x \to 0} F_{\Delta x,a}(x)$$
Evaluating the definite integral concept by the Newton Leibnitz formula. $$\int_a^b f(x)dx = g(b) - g(a), In\ which\ g'(x) = f(x)$$	The connection between the derivative of the accumulation function with the accumulation function g'(x) = f(x)
	The evaluation part of the fundamental theorem of the calculus $\int_a^x f(u)du = g(x) - g(a)$

METHOD

The learning tasks: To design a hypothetical learning trajectory, the researcher must suggest activities that facilitate the intended learning, and address exceptions to the learning outcomes (Simon and Tsur, 2004). Therefore, four open-ended tasks were prepared. Each task deals with a different component of the FTC as shown in Table 1. The first task was designed to allow students to engage in the learning of the Riemann accumulation function. The second task deals with objectifying the definite integral concept as a convergence of the Riemann accumulation functions. This task was prepared to teach convergence idea to students informally, to allow them to engage in the limit concept in a way they should be able to perceive. The third task was designed to allow students to make a connection between the definite integral and the indefinite integral. The fourth task was designed to allow students to objectify the accumulation function symbolically and to apply the FTC to evaluate the definite integral. The left column of Table 2 exemplifies the third task and instruction given to the students. The Figure in the right column illustrates the IAA interface.

Explore and explain the relationship regarding the manner in which the graphs in the upper and lower Cartesian are changing. You should be able to input function expressions (from the functions listed below) and obtain function graphs such as $f(x) = x^2$, x^2-9, x^3. To create graphs in the lower graph window, select the rectangle option by clicking the icon or the filled cover area option by clicking the icon ∫. While working on the task you may use the software to change the value of the parameters of the value box (tool for value-control), as needed.



Participants and producers: The present study explored approximately 55 hours of learning by 11 pairs of 17-year-old students from two public schools in Israel; a rural one and an inner-city one. All participants were high achievers in mathematics and studied mathematics at the highest level in their schools. At the time the meetings took place, the students had already learned the concepts of function, derivative, and indefinite integral. They were familiar with using the derivative symbolically and were able to analyze functions by finding extreme points. The students were familiar with the indefinite integral concept graphically, as the inverse of derivative.

The participants volunteered to participate in four after-school meetings. The learning took place in the computer lab at the schools. Each pair of students shared one computer. The author introduced them briefly to the interface and illustrated how to use it. In particular, the students were told about the technical functionality of the artifact. The first author was present as an observer and provided technical and miscellaneous clarifications. To examine the focuses involved in objectifying the fundamental theorem of calculus, I asked the students to complete one task at each meeting.

Data collection and analysis

I video-recorded all the pairs of students in each session as they engaged in solving the given task. The corresponding computer screens were also captured. In total, 44 clips documenting the entire learning process were video-recorded.

In the spirit of grounded theory (Strauss & Corbin, 1998), data analysis began with repeated watching of the videos and transcripts to detect focuses in the process of objectifying the FTC. Focuses were defined as segments of discourse in which the students sought to discover possible connections between the graphs and to come up with conjectures about the mathematical relationship inherent in FTC. Descriptions were then written for each student. The transcripts and descriptions of all the students were used to create codes for students' explanations of the mathematical relationships involved in the FTC. The structural decomposition of FTC was used to detect the mathematical component used by the students in the process of objectification. This coding process led to a list of foci. For example, statements such as "It is the multiplication of the rectangle height with half" was coded as "objectifying the product." Statements that did not fit any component of FTC but were used by the students were coded by names that describe their practices, for example, "we add negative areas*;* here we start to add positive areas. It still negative, but the y-value is getting bigger. After the sum of the positive areas is bigger than the negative the sum of the areas becomes positive." The students' practices are guided toward exploring the positivity and negativity of the Riemann accumulation function. Therefore, statements such as this were coded as "objectifying the positional of Riemann accumulation function graph." At the end of this phase, the entire collection of video-clips was split into episodes. Each episode addresses the students' attempts to come up with conjectures about the mathematical relationship involved in the FTC. The overall episodes were organized in categories based on the focus under consideration.

FINDINGS

In total, the data analysis reveals 12 focuses that were involved in objectifying the fundamental theorem of calculus. In Table 3, the detected focuses and description for each focus is presented. The shaded grey rows represent focuses that are not outlined in the mathematical structure of the fundamental theorem of calculus; the non-shaded rows represent focuses that are outlined explicitly in the mathematical structure of the fundamental theorem of calculus.

Table 3: The detected focuses throughout the learning trajectory and description for each focus

Session	Objectified Category	Awareness
A	1. delta x.	Being aware of delta x while splitting the x-axis into equal length segments.
	2. The lower limit as relative zero.	Being aware of the lower limit as relative zero in the accumulation function, which splits the x-axis into two sections. The values of Δx are positive in the domain in which x is bigger than the lower limit value, otherwise Δx is negative.
	3. The product.	Being aware of the product meanings as represented by the initial point in the Riemann accumulation function.
	4. The sum of products.	Being aware of the sum of products meanings.
	5. The accumulation function properties.	Being aware of the properties of the Riemann accumulation function graph, such as the positivity–negativity, and increasing– decreasing of the Riemann accumulation function graph.
B	1. The differences between the Riemann accumulation function and the accumulation.	Being aware of the reasons which cause the differences between the two graphs.
	2. The convergence process of the Riemann accumulation function.	Being aware that the continuous area is consisted of a very small width rectangle and become aware of the sign area (positive or negative).
C	1. The relationship between the change of products and the accumulation function rate of change.	Being aware that the change of the areas under a curve when varying the upper limit is like the rate of change of the accumulation function.
	2. The relationship between the change of products, the function change, and the accumulation function rate of change.	Being aware that the function change is like the rate of change of the accumulation function.
D	1. The accumulated formula.	Being aware of the symbolic expression of the accumulation function in which the lower limit is zero.
	2. The accumulation function when the lower limit is positive.	Being aware of the symbolic expression of the accumulation function in which the lower limit is positive.
	3. The accumulation function when the lower limit is negative.	Being aware of the symbolic expression of the accumulation function in which the lower limit is zero.

The following excerpts aim to illustrate the objectification of the relationship between the function and the rate of change of the accumulation function (The 3rd task – table 2). At this stage of learning, the students were assumed to have objectified the Riemann accumulation function and the accumulation function as convergence.

Maram splits the continuous area into chunks of areas using the upper limit parameter. By each click on the upper limit parameter, she creates one chunk.

Maram: [Figure 2a appears on the screen, she clicks the upper limit parameter, Figure 2b appears.] (1) The y-value is about four [read from the table of values]. [Clicks the upper limit again] (2) It is about seven. It adds three. (3) The value, when added, is less because the new chunks are smaller than the first one. (4) The third chunk is less and the fourth is less. (5) It is increasing but at a decreasing rate.

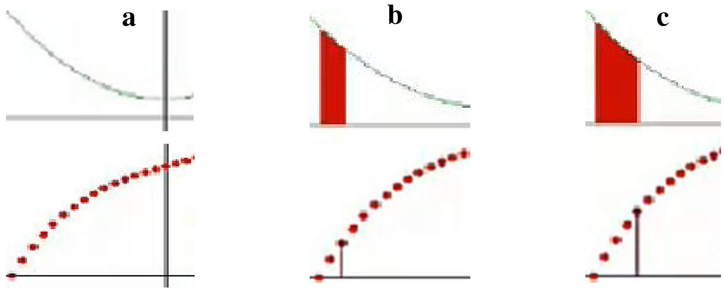


Figure 2. The accumulation function of the function x^2+1. (a), (b), and (c) Students use the upper limit value to relate the continuously covered area to the rate of change of the accumulation function

Varying the upper limit click by click drew Maram's attention to the manner in which the chunks of areas are changing. She evaluates the accumulation value as x = 2.5 and at x = 2 using the table of values. She compares the accumulated value in the interval (2.5, 2) with the accumulated value in the interval (3, 2.5). Maram paid attention to the fact that the accumulated value in the second interval is smaller than the accumulated value in the first interval. She ascribes this finding to the second chunk of area, which is smaller than the first chunk area. Maram's third utterance suggests her awareness that the accumulation function display in the lower Cartesian system signifies the continuously covered area [B2]. Moreover, Maram's fifth utterance, "it is increasing," suggests her awareness of the tendency of the accumulation function graph [A5]. It seems that she uses this focus to verify the idea of accumulation as summation. However, her utterance "but at a decreasing rate" suggests the connection between the manner in which the continuous area changes and the rate of change of the accumulation function [C1].

In the course of the learning process, Maram and her partner input the function $f(x)=x^2$. They paid attention to the correlation between the extreme point in the function graph and the inflection point in the accumulation function graph.

Maram: (**1**) The extreme point corresponds to the inflection point in the lower function. (**2**) Because there is no area at the origin, we add zero, (**3**) hence the lower graph is still stable there.

Narmin: (**4**) The slope at the inflection point is zero because the upper function is zero in the origin. (**5**) As the function decreases the slope of the lower function decreases also; when the function is zero, the slope of the lower is zero also. (**6**) It is the same principle as the area.

The correlation between the extreme point in the function graph and the inflection point in the accumulation function graph drew Maram's attention. She ascribes this correlation to the areas should be added in a neighbor of the extreme point. Maram's second utterance suggests her awareness of the relationship between the changes of the areas and the rate of change of the accumulation function [C1]. It seems that, objectifying the latter allows her to explain the correspondence of the extreme point with the inflection point. In addition, her third utterance suggests her awareness of the stability of the accumulation function as the upper limit is progressing forward [B2]. Narmin takes a different point of departure to objectify the tangent slope and ascribes it with meanings. She relates the zero slope of the inflection point in the accumulation function to the zero value of the function. It seems that the distinction Narmin makes allows her to make a connection between the change of function and the rate of change of the accumulation function [C2]. Furthermore, Narmin's sixth utterance suggests that she connects the change of the function, the change of the areas, and the rate of change of the accumulation function.

CONCLUDING REMARKS

Based on the mathematical structural decomposition of the FTC and utilizing multiple linked representational artifacts, a learning trajectory for learning the FTC was proposed.

Careful analysis detected 12 focuses marking the process of objectification of the FTC. Eight of the detected focuses were expected a priori since they implicitly appear in the mathematical structure of the FTC. This finding indicates that the tasks and the affordances of multi-representational artifacts allow the FTC and its components to be conceptualized. In addition, four focuses that were not mentioned in the mathematical structure decomposition of FTC were detected in the process of objectification. These four focuses were involved in objectifying other focuses (see for example the involvement of A5 and C1 in objectifying C2). Hence, it is reasonable to conclude that detected focuses should be considered as essential elements of the learning trajectory for teaching the FTC which emphasize the conceptual learning of it.

Becoming aware of the detected focuses identified in this study can assist teachers to plan learning tasks that support smooth transitions throughout the learning trajectory. It should help teachers to plan their instruction not as a means of moving through a curriculum, but as means of helping students move through the detection of focuses. I believe that, constructing a learning trajectory and identifying the learning focuses throughout the learning trajectory should have implications for improving practices of teaching concepts of calculus. The findings assert that introducing technological artifacts into the educational setting may change the ways students learn advanced mathematical concepts.

Using a learning trajectory as a learning tool and applying it in a real educational setting, however, needs further clarification, and more research is needed to shed light on this issue. The present paper identifies the learning focuses involved in learning the fundamental theorem found by observing 11 pairs of students. We do not claim that the learning trajectory proposed in this paper is unique or the best, therefore further research is needed to construct additional learning trajectories for calculus and to examine their influence on improving teaching practices. Extensive research is needed also to examine the application of the proposed learning trajectory among older and younger students, as well as lower achievers students.

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FROM AN INTUITIVE-ORIENTED TO A CONTENT-ORIENTED UNDERSTANDING OF THE BASICS OF CALCULUS

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In recent decades, the approach to calculus in mathematics classrooms has changed: a quite formal approach—closely linked to the teaching of calculus at university and based on the sequence concept—has been transformed to an intuitive access to the concepts of limit and derivative. Nowadays empirical investigations with freshmen at the university show that knowledge and beliefs even of mathematics students stay on an intuitive level concerning the basic concepts of calculus. This thesis of this article is that a technology-supported discrete access to the basics of calculus might close the gap between an intuitive- and a content-oriented understanding. This article is on the one hand a theoretical consideration concerning the basic concepts of calculus, the concept of limit and the concept of derivative. On the other hand it gives a strategy for a discrete access to calculus which can be seen as a content-oriented access to the basics of calculus.

Keywords: Calculus, Sequences, Derivative, Digital Technologies

Nowadays, calculus is an important subject in school mathematics for students intending to enter college or university. But calculus as a subject at school has–compared to algebra or geometry–a short history. Demanding ways of mathematical thinking, in particular regarding the understanding of the limit concept, were the main reason that calculus had not been part of the mathematics curriculum at secondary high schools until the beginning of the 20th century. Mathematicians were concerned that schools might not be able to teach calculus with a sufficient degree of strictness, which would lead to the formation of misleading views and opinions by the students. Following the "Merano reform" (1905), proposed by Felix Klein, a question relating to the demand for emphasizing the functional thinking in mathematics teaching came up for discussion: Is it not necessary to include calculus, the "culmination of functional thinking," into the mathematics curriculum? This discussion was extremely controversial. It took until the 1950s for calculus to be established as a part of mathematics education.

In the following the difficulties of students with the basic concepts in the last decades are the starting point for an alternative access to calculus. The ideas are further developments of concepts presented in Weigand (2014) and Weigand (2015).

THE UNDERSTANDING OF THE LIMIT CONCEPT

The understanding of the *concept of limit* has quite often been the subject of theoretical reflections and empirical investigations. It is well known that many students have problems with the formal definitions of the concepts of limit and derivative. They are either not able to use the definition properly in a given context, or they are able to solve problems on a formal level, but lack an advanced understanding of the concepts (e. g. Tall and Vinner 1981; Çetin 2009, Swinyard 2011). The main results of the investigations concerning the learning, teaching, and understanding of the limit concept in the last decades are:

a) A *conceptual understanding* of the formal limit concept is challenging for high school students as well as for some college and university students and requires explanations and visualizations using different representations (beyond the symbolic representation).

b) The understanding of the *process of the construction or calculation of limits* in the sense of *step-by-step processes on numerical and graphical levels* is essential for the understanding of the limit concept beyond a formal definition of limits and this can be supported by computer visualizations.

The following considerations are based on the (hypo)thesis that technology can support the relationship between the – necessary and important – representations and the understanding of the limit concept and the follow-up concept of derivative.

CONCERNING THE UNDERSTANDING OF THE CONCEPT OF DERIVATIVE

To understand the *concept of derivative*, it is necessary—besides understanding limit processes—to have adequate conceptions of the *rate of change* and to understand—in relation to limit processes—the transformation from the *average* rate of change to the *local* rate of change.

Nowadays it is well accepted that understanding the concept of derivative requires a wide and broad intuitive base of examples and related perceptions, especially concerning the concept of the rate of change in real-life problems, for example the average velocity while driving a car, the average inflow to a water basin, the average slope while climbing up a hill, or the average air temperature over a day (see, e.g., NCTM 2000). It is, however, also well-accepted that e. g. the transition from the *average velocity* (of a car) in an interval of time to the *local* or *instantaneous velocity* is a theoretical concept, which needs an extension or abstraction from real-life situations.

There are numerous propositions concerning the use of digital technologies and their dynamic possibilities of visualization of this approximation processes on a numerical and graphical level (Kidron & Zehavi 2002;; Martinovic & Karadag 2012; Caballero-Gonzales & Bernal-Rodriguez 2011; Henning and Hoffkamp 2013). All those suggestions have in common that they work with real-valued continuous functions and visualize—with programs such as Geogebra 1 or Cinderella2—the limit processes dynamically with a sequence of secants converging to the tangent in a point of the graph of the function and/or the numerical process of convergence in the frame of a table (in a spreadsheet). The necessary transition from the continuous perspective to the discrete stepwise process—the discretization process—concerning the limit process, which includes selecting either a sequence of points on the graph or a sequence of numerical values converging to a selected value of the function or to a point on the graph, has to be made by the learners on their own.

The detailed (re-)construction of the limit process and the possibility of step-by-step thinking in the frame of this process has always been the strongest argument for working with sequences and their limits before starting to work with the limit of real-valued functions and their limit processes, for example the first derivative.

1 www.geogebra.org.

2 www.cinderella.de.

THE INTUITIVE LIMIT CONCEPT

During recent decades, teaching the concepts of limit and derivative in mathematics classrooms has changed. In the seventies and eighties of the last century, especially in continental European countries such as France, Italy, Germany, and the Eastern European countries, the limit concept was based—in close relationship to college or university mathematics—on extensive work with sequences. A formal definition of the limit of a sequence and the proving of certain theorems concerning the convergence of sequences were the basis for the definition of the derivative of real-valued functions.

In the middle of the last century the mathematicians Emil Artin (1957) and Serge Lang (1964/1973) proposed a new concept in their college courses, the "intuitive limit concept," which was adopted as a concept for high schools (Blum 1975) and has since then been widely accepted in schools and is the dominant concept today. In this concept the slope of a curve $y = f(x)$ at a point P is discussed without any preceding introduction of the concepts of limit or sequences. E. g. concerning the function with $y = f(x) = x^2$ the slope of the line between two points $(x, x + h)$ and $(f(x), f(x + h))$ with an arbitrary point (x, y) on the curve and a "small positive number $h \neq 0$ is calculated:

$$\frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{2xh + h^2}{h} = 2x + h.$$

As h approaches 0, the slope of the curve $y = x^2$ at an arbitrary point (x, y) is $2x$, which gives the "derivative" of this function (at this point).

This method of calculating the derivative of a function by cancelling h out of the numerator and the denominator is possible for all polynomial functions. However, this method reaches its limit when it comes to the derivatives of exponential and trigonometric functions.

Nowadays, this concept is widely used and recommended for upper secondary high school and it is used in nearly all calculus schoolbooks, especially in Western Europe. As a consequence in many new curricula (such as in Germany), sequences are no longer part of calculus in mathematics lessons.

It becomes clear that this has been a turning point regarding the concept of calculus in mathematics lessons. The changes in the access to the derivative concept changed the contents and the structure of the entire calculus curriculum. A concept-oriented approach to calculus was substituted by an application-oriented approach. There is a danger that learners stay on an intuitive and technical level and that basic ideas or conceptions for a content-oriented or integrated understanding of the mathematical concepts are not given.

The following ideas criticize on the one hand the intuitive limit concept of the access to the derivative by proposing an alternative strategy based on the concept of sequences and using digital technologies, one the other hand these ideas should be seen as an extension of this intuitive limit concept by adding an introduction unit to this concept.

DIGITAL TECHNOLOGIES AND THE REVIVAL OF THE DISCRETE

As a consequence of the increasing role of digital technologies in mathematics and mathematics education, discrete mathematics, and hence sequences, have gained importance. This was emphasized by the NCTM *Standards for School Mathematics* (1989), which included discrete mathematics as a separate standard for grades 9 to 12: "Sequences and series ... should receive more attention, with a greater emphasis on their descriptions in terms of recurrence relations." 3 Sequences are prototypes of discrete objects in mathematics. In the *Principles and Standards for School Mathematics* (NCTM 2000), however, discrete mathematics is no longer a separate standard but is now distributed across the standards and spans the years from kindergarten through twelfth grade. *Iteration and Recursion* are explicitly emphasized as one of the three important areas of discrete mathematics.

Even though sequences are not explicitly defined or introduced as a separate concept in the mathematics curriculum, they are used quite often implicitly or in an intuitive way: Many real-life problems allow mathematical representations with sequences, for example growth processes or problems involving goods and their cost, or approximation algorithms such as the Heron-method for calculating irrational numbers or the Newton-method for calculating zeroes of functions are based on iteration sequences.

Nowadays, computers or digital technologies make it possible to generate sequences, to create symbolic, numerical, and graphical representations, and to switch between different representations—by just pressing of a button. In the following digital technologies are tools allowing a discrete access to the concept of limit and derivative as a preliminary stage working with these concepts on a continuous level.

A STEP-BY-STEP CONCEPT FOR A DISCRETE APPROACH TO CALCULUS

We will now present a concept of a discrete access to calculus, which develops the concept of the average rate of change based on a discussion of various sequences by looking at discrete functions. By gradually changing the step size of the discrete actions at hand, limit processes are prepared by comprehensible step-by-step actions and are thus easier to understand. Here, the computer is used both as a tool for the representation and visualization of sequences and functions, and as well as a tool for creating recursively defined sequences in particular, which allows the user to switch between symbolic, numerical, and graphical representations (see Weigand 2014).

Level 1: Getting to know (recursively defined) sequences in the frame of growth processes

Sequences can be explained or defined on a formal level via an explicit mapping a_n: $\mathbb{N} \to \mathbb{R}$, or they can be defined recursively. This is widely used for the representation of growth processes, for example *linear* growth by $a_{n+1} = a_n + d$, *exponential* growth by $a_{n+1} = A \cdot a_n$, and *limited* growth by $a_{n+1} = a_n + P \cdot (B - a_n)$, $n \in \mathbb{N}$, while all other variables are being real numbers. These sequences can easily be visualized using a spreadsheet or a computer algebra system such as Geogebra. The main goal of this first level is to become acquainted with the recursive kind of definition of sequences, to see the relationship between local aspects, between successive elements, and global aspects of the whole sequence, and to see the dependence of elements of the sequence on the initial value and the parameters (see Weigand 2004).

3 http://standards.nctm.org/Previous/CurrEvStds/9-12s12.htm

Level 2: Introducing the concept of difference sequences

The aim of this second level is the introduction of the concept of difference sequences $(\Delta a_n)_{\mathbb{N}}$, $\Delta a_n := a_{n+1} - a_n$, of a given sequence $(a_n)_{\mathbb{N}}$. Δa_n can be seen as the rate of change–concerning $\Delta n = 1$–of the sequence. The concept may be introduced in connection with real-life problems, for example the average air temperature per year, which may be presented in a table and a graph.

Level 3: The concept Z-functions and their difference functions

Level 3.1: Quadratic Z-functions

Starting with sequences or functions defined on the domain $\mathbb{N}$, we gradually extend the concept of sequence to functions defined on $\mathbb{Z}$, $f\colon \mathbb{Z} \to \mathbb{R}$, and advance to more subdivided discrete domains. We call functions $f\colon \mathbb{Z} \to \mathbb{R}$ "Z-functions." These functions f with $y = f(z)$ are "extended sequences," defined on integers $z \in \mathbb{Z}$, for example $f(z) = z^2 - 2z + 3$. We will now look at these Z-functions in relation to their difference-Z-functions D_f: $D_f(z) = f(z + 1) - f(z)$.

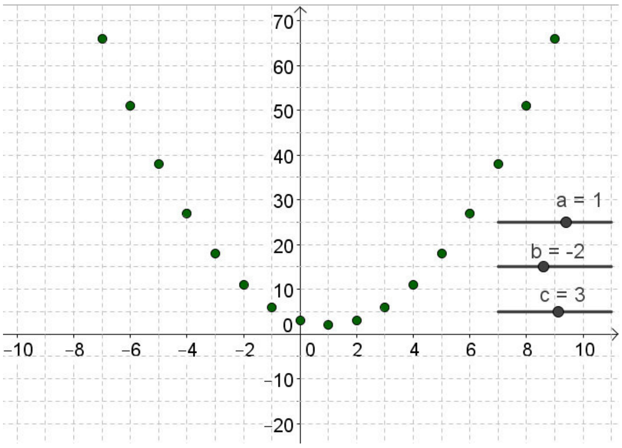

Fig. 1 $f(z) = z^2 - 2z + 3$

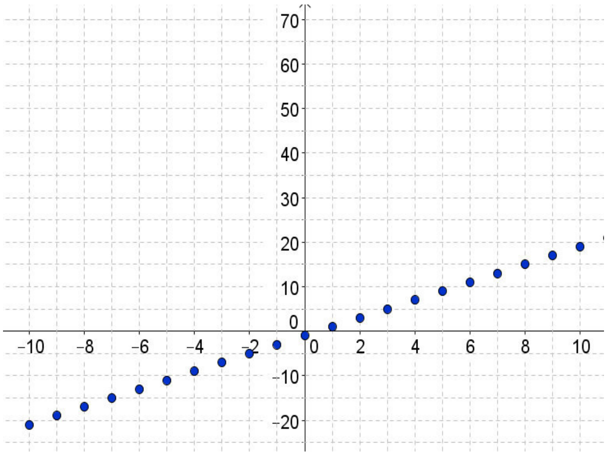

Fig. 2 $D_f(z) = f(z + 1) - f(z)$

$D_f(z)$ is the *rate of change* of the graph between the points $(z, f(z))$ and $(z + 1, f(z + 1))$. Digital technologies are used to visualize the dependence of D_f on the used parameters of f: $f(x) = a\cdot z^2 + b\cdot z + c$ graphically and to give reasons for the behavior of D_f.

Level 3.2: Polynomial Z-functions

The concept of Z-functions can be extended to polynomial functions of a higher degree, as the respective difference functions can be obtained algebraically in an equally simple manner. For the Z-function

$$f(z) = a\cdot z^3 + b\cdot z^2 + c\cdot z + d,$$

For example, we get the difference-Z-function D_f with

$$D_f(z) = 3az^2 + (3a + 2b)z + a + b + c.$$

We can see in particular that D_f is a quadratic function, which is also apparent in the graph.

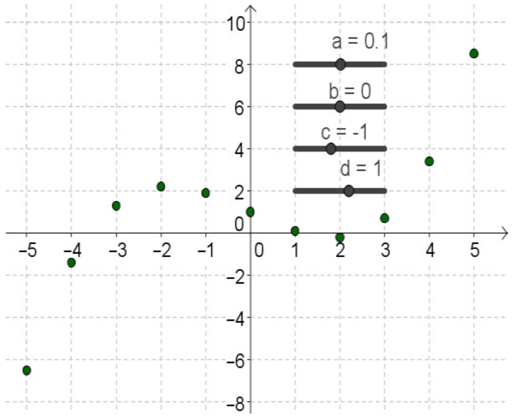

Fig. 3 Z-function $f(z) = 0.1 \cdot z^3 - z + 1$

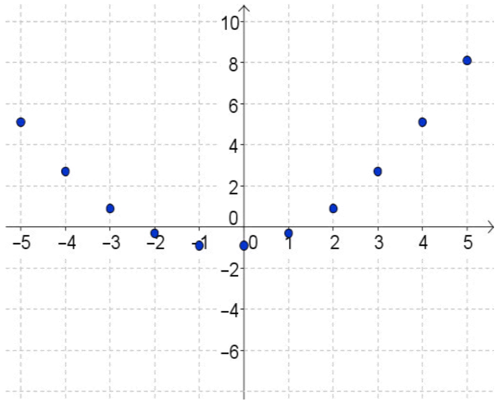

Fig. 4 $D_f(z) = f(z + 1) - f(z)$

The calculations can easily be extended to difference functions of higher order Z-functions, especially by using a CAS. It can be especially useful for calculating the difference functions of higher order Z-functions.

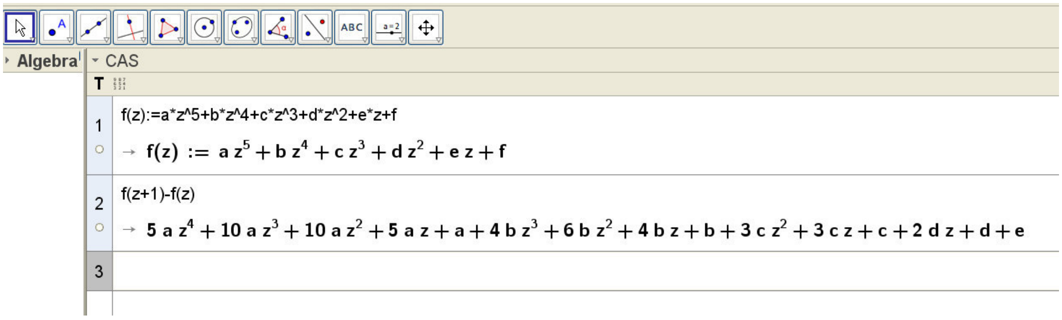

Fig. 12 CAS-calculation of D_f of a polynomial f of grade 5

The advantage of working with these discrete functions is the possibility of obtaining the *rate of change of discrete polynomial functions* only through algebraic transformations and the possibility of *step-by-step argumentations* concerning the properties of the function, especially concerning the rate of change and the difference function.

Level 4: Seeing difference quotients as (discrete) functions

The next step of an expansion of the Z-function is considering a domain with non-integer values, but we will still remain within discrete domains. The idea of the difference function as well as the calculation of the slope can be used as long as the domain consists of discrete values.

In a first step the domain $\mathbb{Z}$ of the Z-function f is expanded by considering the values $z_{10} = \frac{z}{10}$, $z \in \mathbb{Z}$. This means $z_{10} \in \mathbb{Z}_{10} = \{\ldots, -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, \ldots\}$ and we obtain the Z_{10}-function $f_{10}: \mathbb{Z}_{10} \to \mathbb{R}$. To get the rate of change of successive values, we restrict the calculation to an interval of the length $\frac{1}{10}$, and get the *difference-quotient-Z_{10}-function*

$$D_{f10}(z_{10}) = f\left(z_{10} + \frac{1}{10}\right) - f(z_{10}),\ z_{10} \in \mathbb{Z}_{10} = \{\ldots, -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, \ldots\}.$$

This can be generalized to an interval of the length $\frac{1}{n}$, $n \in \mathbb{N}$, and the difference-quotient-Z_n-function (see Weigand 2014).

Level 5: The relation between the difference (quotient) function and the local rate of change

The preceding steps to the access to the derivative emphasized the *global view* of the function and the difference-quotient-Z_n-function. The next step will be the concentration on the local view of a function while seeing the relation to the local rate of change of a function.

We continue with any real function f, choose a fixed value $z_0 \in \mathbb{Z}_n$, or even a generalized value $x_0 \in D \subseteq \mathbb{R}$, and consider the sequence of the difference quotient for a real-valued function f with respect to the value of x_0 for $n = \{1, 2, 3, ...\}$:

$$n \to D_n(x_0) = \frac{f\left(x_o+\frac{1}{n}\right)-f(x_o)}{\frac{1}{n}}.$$

Now, the sequence $D_n(x_0)$ can also be interpreted—considering the graph of f—as the sequence of the slopes of the secants through the point $(x_0, f(x_0))$. Seeing the construction of the derivative of a function f in a special point of the graph of f as a sequence of slopes of secants, the discrete-Z_n-functions with growing n provide a basis for the calculation of the *local rate of change*.

CONCLUSION AND PERSPECTIVES

It is expected that the proposed strategy prepares the concept of *local derivative* of a function (at one point) and gives a chance of a better understanding of this approximation process because of the possibility of the stepwise construction of this process. But it is also expected that the parallel presentation of sequences (or discrete functions) and their difference sequences (or functions) allow also a well-founded understanding of the concept of a *(global) derivative* function. The aim of the proposed concept is the better understanding of the concept of derivative. It is part of the project *"ABC – A discrete Approach to the Basics of Calculus"* (see Weigand 2014). At the moment, teaching units are developed and the next goal is an empirical evaluation of the step-by-step concept in real classroom settings. It is expected that first results will be given at the ICTMT 12.

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Theme: Projects

THE NET GENERATION AND THE AFFORDANCES OF DYNAMIC AND INTERACTIVE MATHEMATICS LEARNING ENVIRONMENTS: WORKING WITH FRACTIONS

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In this study, we investigated how the learners benefit from affordances of a dynamic and interactive mathematics environment—GeoGebra. Grade 6 and 7 students worked in the university computer laboratory in Turkey on a number of mathematical tasks, and their work was recorded with screen capturing software. Moreover, two graduate students videotaped and observed the participants working on their tasks. Analysis of data reveals that students' way of benefiting from this environment falls behind our expectations, although some advance uses were recorded.

Keywords: Visual Learning, Dynamic Learning, Explorative Learning

INTRODUCTION

The affordances of digital tools used in mathematics education have been explored and discussed by many scholars (Karadag & Aktumen, 2013; Karadag, Martinovic, & Freiman, 2011; Leikin & Grossman, 2013; Leung, Baccaglini-Frank, & Mariotti, 2013; Prusak, Hershkowitz, & Schwartz, 2011). Visual learning, dynamic learning, and explorative learning have been the affordances granted in various settings such as DIMLE discussion group in the PME 2011, in Reno, US; ICTMT Conference 2011, in Portsmouth, UK; and IDEAL Conference 2012, in Bayburt, Turkey. It has been claimed that the new generations of learners were born in a digital era and thus raised in the technologically-rich environments. Also, it has been proposed that their learning habits have evolved through technology and that these technological contexts better meet their expectations.

In order to better understand the aforementioned affordances of dynamic mathematics learning environments (DIMLE) and the extent to which these environments meet the learning expectations of the new generations of learners, a series of projects have been designed to investigate "how do Net Generation students benefit from the affordances of DIMLE while working on their mathematical tasks?" First, data were collected in Bayburt, Turkey, and their analysis is in progress. For the second step, data will be collected in Windsor, Ontario, Canada and the analysis will be performed accordingly. The subsequent steps are at the planning stage.

This paper describes the project expectations and goals, as well as the data and findings from the first step, conducted in Bayburt. The main focus was to observe and document the problem solving processes of students, rather than their ways of learning of any specific mathematical content. This project is unique not only because of its focus, but because of the type of data collected—the main data are the screen-recordings of students' work in DIMLE, which were then analysed using the Frame Analysis Method (Karadag, 2013; details are given in Methodology section).

THEORETICAL CONSIDERATIONS

We prefer using the term DIMLE, rather than Dynamic Geometry Systems (DGS) or Dynamic Geometry Environments (DGE). The reason is that recent inventions made these environments move further and cover other mathematics disciplines as well. Therefore, we developed a broader term in a personal discussion (Martinovic & Karadag, 2010). A detailed discussion on this topic was presented elsewhere (Karadag & Aktumen, 2013; Karadag, Martinovic, & Freiman, 2011; Martinovic & Karadag, 2012; Martinovic & Karadag, 2011). In this paper we present the research and our perspectives on visual learning, dynamic learning, and explorative learning.

Visual Learning

Visualization is widely accepted as a certain type of representation that may also be named geometrical or graphical representation (Rivera, 2011; Zimmermann & Cunningham, 1991). As a type of representation (i.e., 'visual'), it may be created by using concrete manipulatives or by using paper and pencil, as well as digital tools. Each of these cognitive tools has its own advantages and limitations. For example, creating 3D figures in a paper-and-pencil environment may be comparatively challenging compared to creating them in the specialized digital environments. From the cognitive psychology stance, each tool brings its own affordances and may provide opportunities for visual learning. Furthermore, a tool supporting visual learning is more than simply a device, it is "a tool for mathematical thinking" (Rivera, 2011, p. 36) and "plays a fundamental role in any account of concept or process development, including problem solving" (p. 43).

Despite the skepticism around the possible dangers of over-reliance on visualization in mathematics education, especially because "thinking visually makes higher cognitive demands than thinking algorithmically, and thus it is quite natural for students to gravitate away from visual thinking" (Eisenberg & Dreyfus, 1991), some scholars look at it differently. Goldenberg (1991), for example, eagerly writes: "Beyond affording exciting opportunities for students to engage in new mathematical ways of thinking, [graphical] environments...open up previously inaccessible mathematical domains, allowing...students to investigate topics...often dismissed as too advanced" (p. 41).

Dynamic Learning

Dynamism is usually associated with the dragging feature of DIMLE, regardless of the use of a slider. Leikin and Grossman (2013) argue that dynamic change on the screen help learners develop *what if* questions and encourage them to think through changes. Therefore, we define dynamic learning as the learning associated with dynamic change on the screen, the change obtained by dragging a slider or another object which controls/initiates the change.

Thinking process alongside with dynamic change of objects on the screen has been addressed in a number of studies (Leung, Baccaglini-Frank, & Mariotti, 2013; Leung & Lee, 2013; Pelczer, Singer, & Voica, 2014). They all agree that dynamic change engages learners in the various types of mathematical thinking procedures such as conjecturing, testing conjectures, experimenting, and inducing. Moreover, Pelczer, Singer, and Voica (2014) claim that, "[d]ynamic thinking allows a student to 'put a magnifying glass' in order to 'see' some details otherwise negligible" (p. 218).

Explorative Learning

Explorative learning is seen in the literature as a process alongside with aforementioned affordances. For example, Giaquinto (2005) uses the term "discovery" and states that, "the visual way of reaching the theorem illustrates the possibility of discovery" (p. 77). In our research, we prefer to distinguish discovery from exploration because we are not really interested in whether the result of the process is positive or not. We believe that learners will develop some understanding even if they do not succeed in discovering something. Thus, we define explorative learning as a way of learning through exploration, and identify the context of exploration as not dealing with obvious.

METHODOLOGY

Prior to introducing the study, we wish to emphasize one important distinction. Although the ultimate goal for this research series is to identify to what extent DIMLE meet expectations of the new generations of learners and to what extent new generations benefit from these learning environments, this study is not about testing how students learn (e.g., visually, dynamically, and exploratively). Rather, the goal of this research, at least presently, is to investigate if there is a ground for claims that DIMLE provide students with affordances for learning.

Research Question

The main research question for the study is to investigate, “how do Net Generation students benefit from the affordances of DIMLE while working on their mathematical tasks?” To investigate the students’ interactions with DIMLE, we recorded the ways in which they made use of affordances of DIMLE. We also sought evidence of visual learning, dynamic learning, and explorative learning in various curriculum contents, such as fractions, transformational geometry, patterns, 2D/3D geometry, and ratio/proportion. For this paper we analysed students’ work on tasks with fractions.

Tasks with Fractions

The tasks featured in this paper relate to equivalent fractions and ordering of fractions. The Figure 1 has screenshots of two different tasks with equivalent fractions. A Fraction 11 task requires that students drag the fractions equivalent to ½ into a circle that contains this fraction. In contrast to the numerical nature of this task, a visual version was also developed. For the Fraction 21 task, students were expected to identify the figure which best represents ½ (the fraction given above the triangles) and type their answer (i.e., A, B, or C) in a Word document, where the task was introduced.

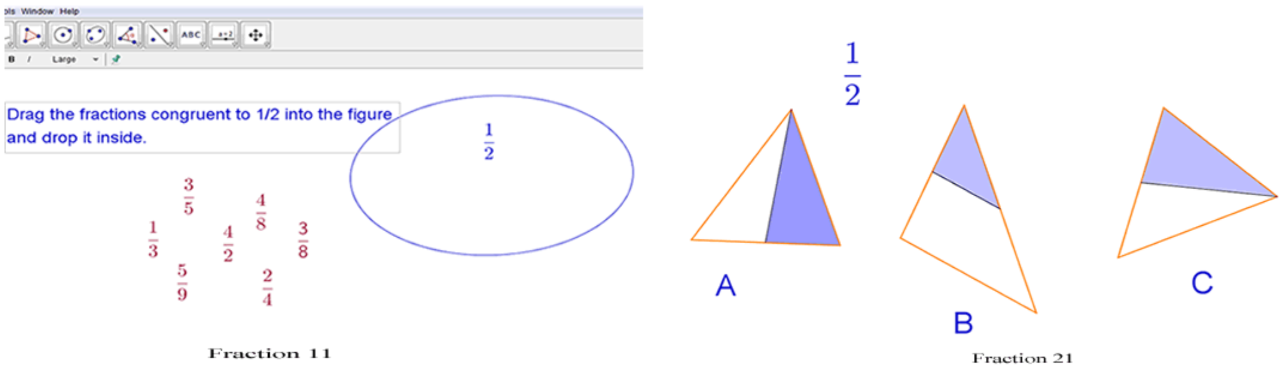

Figure 1: Screenshots of two different tasks on equivalent fractions (translated to English)

The Figure 2 has screenshots of tasks that involve ordering of fractions. The goal in the Fraction 41 task was to order the given fractions by placing them in increasing order (from left to right). Some of the fractions had equal numerators, while the denominators were equal for some others. For the visual version of this task, students were expected to compare the shaded areas to the whole areas of the triangles (see Fraction 42 with visual representations of various fractions) and to type their names in the order from smallest to largest in a Word document, which had instructions for the task.

Ten tasks in total, six tasks on equivalent fractions and four tasks on ordering fractions, were created for the study. The principal investigator, Karadag who has 20 years of teaching experience, and one of the graduate students, Birni, who has 1.5 years of teaching experience collaborated in developing ideas for the tasks and worked together to convert the ideas to GeoGebra worksheets.

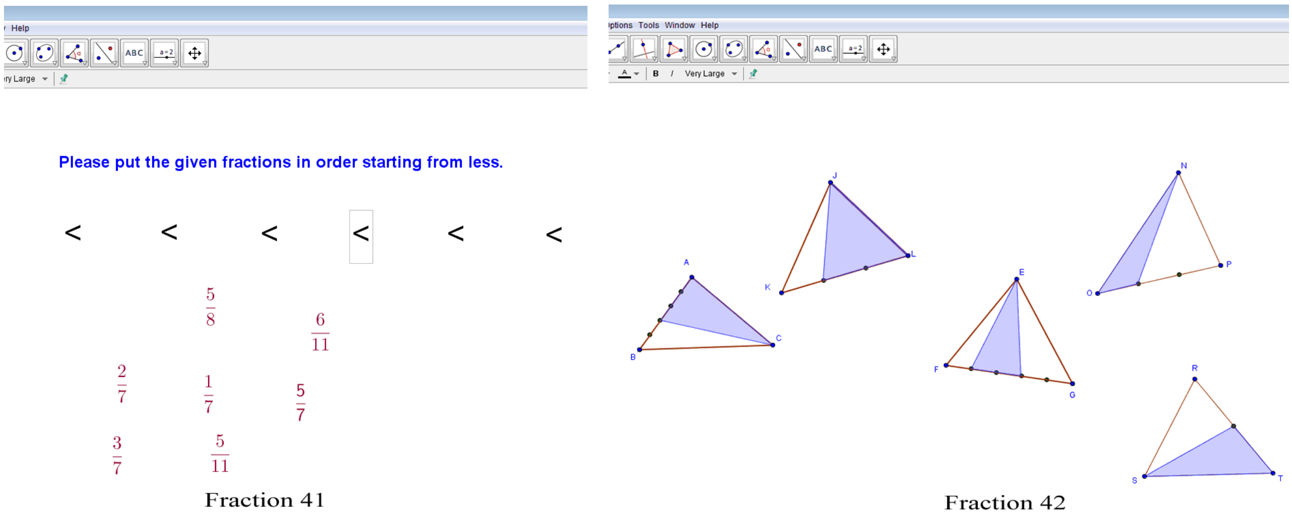

Figure 2: Screenshots of two different tasks on ordering fractions (translated to English)

Participants

Participants were 14 Grade 6 and Grade 7 students (7 in each) from two different schools (6 from one school and 8 from another), in Bayburt, a small city located in Turkey. There were five boys in the sample, and all children were selected by their school teachers with a criterion to be *technologically savvy students*. They spent four complete Saturdays, from 10 AM to 6 PM, at the university. A free launch and free transportation were provided. Two Graduate and six undergraduate students helped with organizing the study and collecting data. They were all students of the Faculty of Education at the Bayburt University.

Data Collection

The study began with instructions on using Wink and GeoGebra software on the first day. Wink is free software for screen-capture, while GeoGebra is a free DIMLE. The instructions covered only basic features of both software employed in the study. The main data consisted of screen-captured student work on mathematical tasks. The work was recorded by each student and saved onto the computer, which they used for the study. The study took place in the university computer lab. One of undergraduate students was videotaping the students by walking around the lab and focusing on a different student each time. Two graduate students were observing the students and taking notes. The principal investigator was present to provide instructions, introduce software and the research study, introduce each task, and to monitor the data collection. Screen-captures, videos, and observation notes serve as our data. Moreover, a focus group meeting with seven participant students was organized on the third day and the discussion was also videotaped.

Data Analysis

Given a wide range of data, that were to be analysed differently, we narrowed the data analysis to what we define as *high-quality demonstrations* of student work—the work that was recorded properly and with enough evidence to support or hinder the claims included in this paper.

To begin with, we reviewed screen-capturing data without going into detail, identified the degree of the quality of demonstrations, and coded them using three criteria: the number of frames-recorded, the degree of completeness of each work, and the degree of correctness of each work (see Figure 3). While doing so, we also developed an understanding of the quality of data in demonstrating the visual, dynamic, and explorative approaches that students employed while working on the tasks.

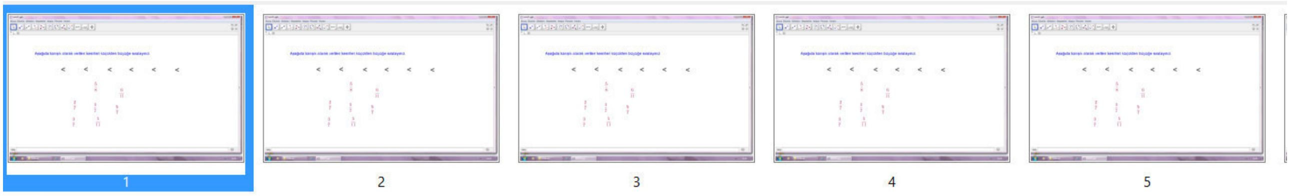

Figure 3: Screenshot of screen-capturing data (for illustration purposes only)

Then, we applied the Frame Analysis method to develop a rich understanding of data (Karadag, 2013). The Frame Analysis method is a multi-step method to allow the researcher to look at the data through various perspectives. First, the Wink raw files were reviewed (initial screening) in entirety. Following, we delved deeper into data by describing each process recorded on the screen and then by interpreting its meaning frame by frame (see Figure 4), in a close and layered way.

Frame Analysis Method

Wink provides a collection of screenshots—frames—captured for each half second of students' problem solving process. The Figure 3 contains first five frames—2.5 seconds—of a recording while the Figure 4 demonstrates a spreadsheet developed during the analysis of this data.

Eylem no	Başlangıç	Sonlanma	Kare sayısı	Süre	Görülenin		Yorumun	
					Tasviri	Anahtar kelimesi	Tasviri	Anahtar kelimesi
1	1	37	37	18,5	Fare sayıların üzerinde dolaşıyor	Fare .... dolaşıyor	soruyu okuyor ve sayıları gözden geçiriyor gibi	okuyor ve .. gözden geçiriyor gibi
2	38	46	9	4,5	Payı 5 olan kesirler yanyana getiriliyor	Payı 5 olan kesirler	Payı 5 olan kesirler gruplanıyor	gruplanıyor
3	47	50	4	2	6/11 kesri 5/11 kesrinin yanına getiriliyor sonra yerine geri götürülüyor	6/11 kesri 5/11 kesrinin	6/11 kesri 5/11 kesrinin yanına getiriliyor ama diğer kesirlerle aynı grupta olmadığı düşünülerek yerine geri götürülüyor	6/11 kesri 5/11 kesrinin yanına
4	51	54	4	2	5/11 kesri 6/11 kesrinin yanına taşınıyor	5/11 kesri 6/11 kesrinin	5/11 kesri 6/11 kesrinin yanına taşınarak paydası 11 olan kesirler olarak gruplanıyor	gruplanıyor
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	geriye kalan paydası 7 olan kesirler yanyana	paydası 7 olan	geriye kalan paydası 7 olan kesirler rasgele olarak	[illegible]

Figure 4: Screenshot of analysis of Wink data

The frame analysis method starts with chunking the data into pieces, and we define each of these pieces as activities. The numbers, seen in the first three columns of the Figure 4, state the order of activity, the number of the frame with which the activity starts or finishes, respectively. For example, the third activity, which is seen on the fourth line, started at the 47th frame and ended at the frame number 50. Therefore, 4 frames were captured during this activity (number on the fourth column of the same line), and it took 2 seconds to get this activity completed (number on the fifth column). Upon documenting chunks, the frame analysis method suggests describing (sixth and seventh columns) and interpreting (eighth and ninth columns) what is seen on the screen. For the third activity, it was described that the fraction 6/11 was dragged next to the fraction 5/11 and then back. As for interpretation, it was stated in the eighth column that 6/11 was brought next to 5/11 but returned, because the student realized that this fraction was not in the same group of those fractions. We put complete descriptions and interpretations in the sixth and eighth columns whereas keywords drawn from these notes were written in the seventh and ninth columns respectively.

After completing this deep analysis process, we directed our attention to the keywords and synthesised them to better understand what the data suggested. This synthesis usually took place in in searching to answer research questions while looking for new themes and patterns. Alongside with the screenshots, observation notes, and videotapes were used for triangulation purposes, as suggested by qualitative research experts (e.g., Creswell, 2007). The analysis of a focus group discussion is not included here because the focus of discussion is beyond the scope of this paper.

FINDINGS

Figure 5 illustrates two different moments of the task-related work performed by one of the participants. Given that Wink was set to capture 2 frames per second, frames numbered 113 and 141 illustrate 56.5th and 70.5th seconds of the student's work. The frame 113 illustrates the moment when the student BEK just started arranging the fractions while the following frame 141, is a record of what happened 14 seconds later, when BEK brought the fractions having 5 in their numerators together and the fractions having 11 in their denominators together. We concluded that BEK knows that one should look at the numerators and denominators separately, to put fractions in order. Please note that we have 28 more frames between these frames, therefore, we interpret the actions by reviewing all the frames, not only the frames presented here.

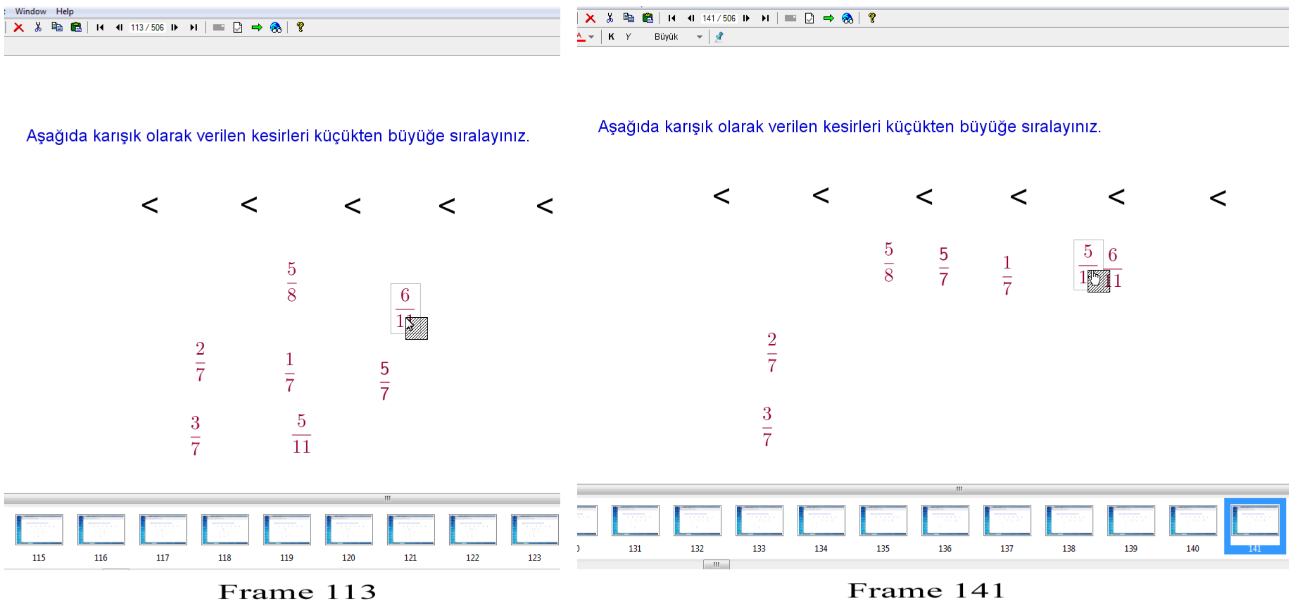

Figure 5: Screenshots of frames 113 and 141 of BEK, Fraction 41

The Figure 6 presents two frames from the solution of the same task by the same student. The frame 392 documents that BEK put the fractions having 7 in their denominators in one row while the fractions with 8 and 11 in their denominators in different rows. It is interesting where the fractions 5/11 and 5/8 were placed. It seems that BEK knew that those two fractions are less than 5/7 and therefore aligned them vertically but slightly left of 5/7. This alignment led us to conclude that she used the computer screen visually and moreover in two dimensions (horizontally and vertically), rather than horizontally, as represented by the less-than signs (<). However, we also noted that BEK arranged the fractions in a wrong order when she placed them between the less-than signs. Did she get confused when using the signs? We do not know—we may ask her in the follow-up interview.

While interpreting the data, illustrating BEK's work, we referred back to our video records and observation notes to confirm our interpretation. The video recording confirms that BEK got easily bored during the research and tried to finalize the process as soon as possible. The observation notes state that she had some degree of basic knowledge of fractions, but lacked making connections.

DISCUSSION

In this study, we sought to understand how much students benefit from the affordances of DIMLE. We documented that students appreciate the affordances of DIMLE and that they do not demonstrate any struggle with the use of visual, dynamic, and explorative affordances of the GeoGebra environment. This result seems consistent with our prior theoretical discussions (Karadag, Martinovic, & Freiman, 2011). However, we realize that they lack experience for employing these affordances in working with mathematics tasks in DIMLE. We view this study as contributing to the limited, but growing, body of research that explores how students benefit from the affordances of DIMLE. Consistent with literature (Eisenberg & Dreyfus, 1991; Leikin & Grossman, 2013; Leung, Baccaglini-Frank, & Mariotti, 2013; Leung & Lee, 2013; Pelczer, Singer, & Voica, 2014; Rivera, 2011), we documented students' benefits from visual and dynamic features of the DIMLE and that they employed these affordances to explore the tasks. However, their way of exploration falls short from our expectations, at least with respect to the tasks about fractions. We conclude that students' lack of exploration may stem from three major reasons: (1) lack of skill in using digital tools in mathematics education, (2) students' habits of working with test questions rather than explorative tasks, and (3) our lack of experience in designing proper tasks.

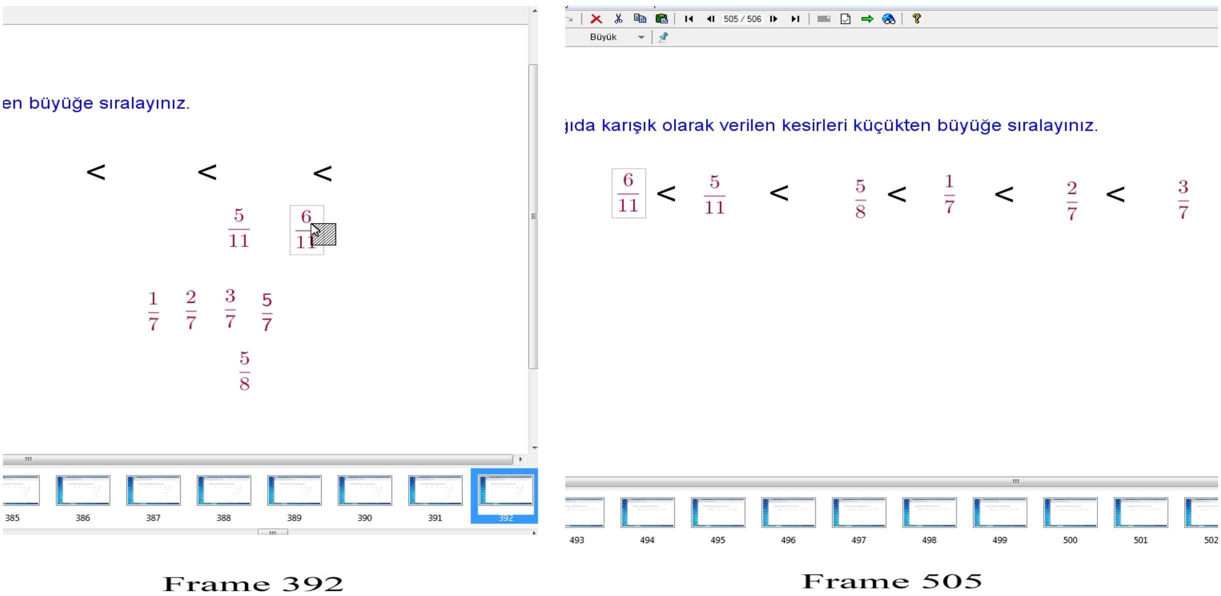

Figure 6: Screenshots of frames 392 and 505 of BEK, Fraction 41

Still, we have enough evidence that students intuitively benefit from visual and dynamic features of the environment. For example, when they were asked to put some fractions in order they grouped the fractions by dragging and grouping them, and then put them in order in each group. Interestingly, many did not use the less-than (<) signs provided for them, rather they put them in order as if there were the less-than signs between them. Moreover, some aligned the fractions not only horizontally but vertically, by imagining the existence of a second dimension or a second layer (see the Figures 5 and 6).

Referring back to research question, *how Net Generation students benefit from the affordances of DIMLE while working on their mathematical tasks*, we may conclude that they well-benefitted from visual and dynamic features of DIMLE whereas the degree of their use of exploration was low. Although there might be many reasons explaining this finding, our experience with these specific participants and the environment they live in suggest that their lack of doing mathematics in DIMLE may have played an important role. Besides the lack of enough experience in doing mathematics in DIMLE, their familiarity with standardized tests and expectation to get only one correct answer for each problem may have inhibited them to move further and explore more.

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ENGAGING STUDENTS IN ONLINE COLLABORATIVE PROBLEM SOLVING: TWO CASE STUDIES

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This paper presents two case studies of engaging secondary school students in online collaborative problem solving activities. The activities were carried out in two online learning environments. The first one was based on a threaded-discussion asynchronous forum and used inquiry-based tasks in geometry. The second one was based on a social network and used highly-challenging proof tasks in geometry. Both environments were implemented during the school year by the teachers. The findings showed that students were engaged in autonomous and meaningful problem solving activities and for long-term period. The concept of student sense of achievements was used to explain sustained engagement in the activities.

Keywords: collaborative problem solving, engagement, geometry, online learning environments.

INTRODUCTION

Information and communication technologies have opened a venue for potential online learning in mathematics. Evidence gathered from students' activities in social networks such as Facebook (Biton, Hershkovitz, & Hoch, 2014), web-based platforms (Stahl, 2009a), specifically designed environments such as Knowledge Forum (Moss & Beatty, 2006; Hurme & Järvelä, 2005) and threaded-discussion asynchronous forums (Wentworth, 2009) has demonstrated the potential of engaging students in meaningful collaborative problem solving activities in online learning environments.

In this paper I present two online learning environments (OLEs) designed for secondary school students. In designing these environments my colleagues and I were motivated (1) to support autonomous and meaningful problem solving activities of secondary school students in geometry, (2) to engage students in long term problem solving activities mostly after school time and with minimal efforts put forth by the teacher and (3) to use available, common and free technological tools as well as to implement simple designing principles. These goals were set so teachers will be able to replicate these environments for their curricular needs and to implement them as supplementary and helpful tools for their classes without allocating much time. Together these goals created different environments than those mentioned in previous studies. The goals of this study were to characterize student engagement in collaborative problem solving activities in these environments and to characterize situations of sustained engagement.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Azevedo, diSessa and Sherin (2012) referred to student engagement in classroom activities as the intensity and quality of participation. Past studies on students' mathematical activities in online environments pointed to several indicators regarding the intensity and quality of participation. I classified these indicators into three characteristics of engagement and will review them next.

(1) *Autonomy*. One indicator for autonomous activity is the sharing of students' posts. In several studies students contributed more than 90% of the posts and teacher intervention was limited. A more profound way to characterize students' autonomy is to analyze their type of interaction. Zhu

(2006) distinguished between a star type and interconnected web type of networked interaction. The star type is characterized by one participant (e.g., the teacher) who takes a central position connecting members in the network and potentially maintaining and dominating the discussion. In the interconnected web type the discussion is decentralized and the activity is maintained in a more equal manner, and thus in a more autonomous manner. An interconnected web type was observed in Hurme, Palonen and Järvelä's (2006) study.

(2) *Meaningful activity*. One indicator for meaningful activity is the accomplishment of the goals of the tasks. It was shown that in online environments student's encountered highly-challenging tasks with high success (Stahl, 2009b; Moss & Beatty, 2006). An additional indicator is the number of posts contributed by students and the portion of posts that bear mathematical content. In the studies that used asynchronous forums the number of posts for one class working on a single task was about a few dozen posts. In chats it may reach a few hundred short chat lines (Stahl, 2009a). In most of the studies about 70% of the posts were mathematical. The distribution of participation may indicate who among the participants probably gains learning outcomes from the environments (Weinberger & Fischer, 2006). In small groups of 3-4 members it may be expected that all participants will participate more or less equally (e.g., Stahl, 2009a). However, in large groups the phenomenon of lurkers or passive participants may occur (e.g., Wise, Speer, Marbouti & Hsiao, 2013). For example in Wentworth's (2009) study, among 25 participants only 13 were active. Beyond the statistical description, in analyzing the content itself Stahl (2009b) and Zemel, Xhafa and Çakir (2009) characterized student participation as expository or exploratory participation. In expository participation "participants reported on work they had already completed, whether it was work done prior to the chat or work done offline and without the participation of others in the production of that work during a chat" (Zemel et al., 2009, p. 429). In exploratory participation, "participants engaged each other (as a group) both in the investigation of the problem and in the production of possible solutions" (ibid., p. 429). In one case of successful group activity, Stahl (2009b) found that participants merged between expository and exploratory forms of participation.

(3) *Time duration*. Students' activities reported in past studies mostly lasted about a couple of lessons (90 min.). One exception is Moss and Beatty's (2006) study in which the environment was accessible to students during eight weeks and they worked between 30-45 min. per week.

This study addresses the following research questions in reference to students' collaborative problem solving activities in our two OLEs:

1. What characterized student engagement in terms of autonomy, meaningful activity and time duration?
2. Which forms of interaction were found as supportive to sustain student engagement?

METHODOLOGY

Participants

In the first case study nine 9th grade students from the same class participated in the first OLE and worked on the Mid-segment activity. It was students' first time experiencing the environment. This was one group among 11 groups that participated and was chosen for the study since it seemed to represent an average level of activity; students in this group were not the most successful but also did not underperform. In the second case study a single 10th grade class consisting of 16 students participated in the second OLE. They worked on the Two Circle problem. This class worked on several highly-challenging proof tasks in geometry within this OLE throughout the school year. The Two Circle problem was posed in the middle of that process after students gained some experience in the second OLE.

Online learning environments[1]

Following Lachmy et al.'s (2012) suggestion I will describe the two OLEs by referring to three aspects: technological (platform), content (tasks) and pedagogical (implementation in schools).

The first OLE was designed within the popular Moodle (http://moodle.org) learning management system. The environment consisted of two main technological components: (a) the threaded-discussion asynchronous forum used as the virtual place for maintaining the online discussions, and (b) GeoGebra dynamic geometry free software (http://www.geogebra.org) used as a tool supporting students' geometry explorations. Inquiry-based geometrical tasks were posed to students encouraging them to explore geometrical configurations, generate conjectures and justify them. Students' work began after they logged-in to the Moodle from their personal computers.

The second OLE was designed within the Google Plus social network as a closed community. The tasks consisted of highly-challenging proof tasks in geometry in a closed formulation (i.e., "prove that"). Students' work in Google Plus was mostly done from their personal smartphones and students were alerted when new posts came in. Several students had already used Google Plus in their personal life.

Both OLEs were implemented during the school year in full collaboration with the teachers as part of their regular geometry classes. The tasks were designed with the teachers to meet their curricular needs. Students worked in the environments mostly after school time, though in some cases openings and closures of activities were held in classrooms. Students were encouraged to collaborate and to share their ideas. Teachers were encouraged to refrain from intervening and to support students' autonomous learning. In the first OLE the participation was part of class duties and there were no time constraints. In the second OLE: the participation was not compulsory and the incentive was to promote geometry skills; the work was limited to one week after which the teacher would intervene in case no solution is found, and; students were instructed not to upload full solutions in order to give other students an opportunity to solve the problem. In both cases the teachers and the moderator (the author) followed the activity and contributed posts.

The tasks

The Mid-segment activity (Fig.1) was based on the geometrical configuration of Varignon's theorem. This geometrical configuration is well-recognized in the research on students' geometrical dynamic explorations (see review in Lachmy & Koichu, 2014). The task was split into several items to enhance the likelihood of meaningful inquiry by focusing students on specific geometrical relationships. Item 1 was intended to encourage students to discover and prove Varignon's theorem (that the internal quadrilateral is a parallelogram) and to apply its conclusion in Item 2. Items 3 and 4 were intended to encourage students to discover and prove both logical directions of the following "if and only if" statement: the internal quadrilateral is a rectangle if and only if the external one is orthodiagonal (that is, a quadrilateral whose diagonals are perpendicular). Item 5 was intended to encourage students to explore the geometrical situation further, generate additional conjectures and prove them. In that sense, Items 3 and 4 demonstrated the type of relationships we intended students to produce. For example, students could explore the conditions in which the internal quadrilateral is a rhombus or a square.

The Two Circle problem (Fig.2) was taken from Sharugin and Gordin (2001, p. 235). This task was chosen by the teacher because it looked accessible and created a deceptive feeling of immediate solution. Specifically, it was tempting to prove that EFHG was a rectangle but eventually it appeared as an impasse. A more productive way was to use similarity properties.

Consider the following situation: in an arbitrary quadrilateral, the midpoints of the adjacent sides are connected so that the four segments form an internal quadrilateral.

Item 1: Which quadrilaterals can be obtained as an internal quadrilateral? Convince us that the quadrilaterals that you indicated are indeed appropriate. What statement can you make regarding the internal quadrilateral? Explain.

Item 2: Yuval claimed: "I succeeded somehow to obtain a kite as an internal quadrilateral, but I cannot do it again". Do you think that Yuval had indeed obtained a kite? If you do, convince us that the internal quadrilateral can be a kite. If you don't, convince us that it is not possible.

Item 3: Which external quadrilaterals have a rectangle as their internal quadrilateral? For each of the external quadrilaterals, convince us that its internal quadrilateral is a rectangle indeed.

Item 4: We are given that an internal quadrilateral of some external quadrilateral is a rectangle. What can be claimed with certainty about the outer quadrilateral?

Item 5: Find additional relationships between external and internal quadrilaterals.

Fig. 1: the Mid-segment activity

Given are two extrinsic circles (that is, they do not have intersection points). From the center of each circle two tangent segments to the other circle are constructed. Prove that the chords obtained by the intersection points of the circles with the tangent lines are equal (prove that GH=EF).

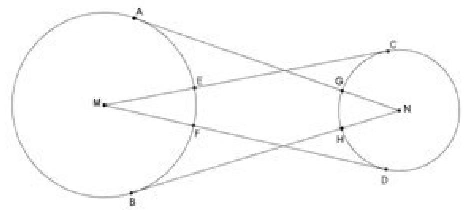

Fig. 2: the Two Circle problem

Data collection and analysis

The data consisted of students' posts and the files they attached. Secondary data consisted of communications and interviews with the teachers, reflective questionnaires filled out by the students and interviews with several students. The secondary data was used to enrich the analysis context.

The units of analysis were comprised of three types: (1) a single post served as the unit for the statistical analyses. (2) For the content analysis the unit was "a chunk" of activity through which participants organized their activities (Zemel et al., 2009). Chunks were delineated by first referring to the structures of a thread with all of its descendant posts (in the threaded-discussion asynchronous forum) or with all of its comments (in the social network) and then identifying openings and closings. (3) Claims and propositions were another unit for content analysis.

Autonomy. Student engagement in autonomous activity was characterized through: (1) the ratio between the number of students' posts and teacher's posts. (2) the centrality property, that is, the number of posts that were addressed to a specific participant. The centrality properties of all the participants indicated the type of networked interaction whether it was more a star type or an interconnected web type. (3) teachers' intervention - whether they supported or inhibited students' autonomy.

Meaningful. Student engagement in meaningful activity was characterized through: (1a) the number of posts students contributed and the share of mathematical posts, (1b) the number of words and files attached to the posts, (1c) the time students allocated for the task in and out of the OLE as they reported in the questionnaires (applied to the second case study only), (2) the distribution of posts among students, (3) the accomplishment of the tasks' goals and the progress they made in terms of their claims and propositions, (4) the number of chunks that consisted of exploratory or expository participation. Characteristics of expository participation included: reports that constituted the problem as solvable and positioned the expositor as an authority who supplied explanations; individualistic work which was not addressed to a certain participant, the teacher or even the group; responses that consisted of questions, solicitation for cues or appreciation. Characteristics of exploratory participation included: a conjoint effort to understand the problem and reached a solution; sharing suggestions of problem formulations, possible solution strategies, candidate solutions, resources, propositions and drawings; addressing posts to specific participants or the group, asking for opinion and expressing the degree of certainty in the proposals.

Time duration. It was calculated as the time that passed since the first post to the last one.

After examining the results for the first question, I then looked for forms of interaction that sustained student engagement.

FINDINGS

First case study: Inquiry-based tasks in a threaded-discussion asynchronous forum

Students were quite autonomous. Students contributed 75% of the posts containing 90% of the typed words. In addition, the centrality properties were almost equal between three students and the moderator indicating an interconnected web type of networked interaction. Also, the teacher's intervention was limited and aimed to promote exploratory activity. However, half of the posts were not addressed to a specific participant or the group and contained detailed answers for the tasks. This finding implied that those posts were implicitly addressed for the teacher's eye and thereby positioned her as an authority.

During the activity five out of nine students were active and contributed 21 posts with average of 60 words per post. In addition, they attached 7 files including GeoGebra files and pictures of GeoGebra-made drawings. Students' work in the environment lasted 18 days and consisted of only mathematical content. It can be described in two phases (Table 1). The first phase was shorter and contained the majority of the activity. Supported by the following findings this phase was completely characterized as expository participation: the chunks consisted of about 1.5 posts in average giving the impression of individualistic activity; chunks were consisted of reports which detailed conclusions and solutions and included drawings from GeoGeobra; as claimed previously, it seemed that students actually directed their answers to the teacher to show their accountability in accomplishing the task; in the few responses where students referred to their peers' work they expressed their appreciation or agreement regarding this work. In the few cases the teacher responded she intended to elicit more exploratory work but with no success.

During the first phase students exhibited their results with exploring different cases of quadrilaterals and raised five claims. Students raised correct claims in reference to the first two items and some of them provided correct proofs. In regards to Item 3 students did not discover the general case and considered the kite as the general case whose internal quadrilateral is a rectangle. They also did not discern between the direct and converse statements in Items 3 and 4. In addition, in regards to Items 3 and 4, students constituted their arguments on empirical grounds using GeoGebra drawings as sufficient for warranting their claims. In Item 5 students restated their previous conclusions and did not discover any new relationships. Therefore, the moderator intervened in the forum after three days of inactivity and presented a drawing of a (orthodiagonal) quadrilateral which was not a kite but whose internal quadrilateral was a rectangle. This intervention was meant to confront students with a surprising result which hopefully would encourage students to employ deductive reasoning and formulate the intended statements (more details on this kind of intervention see Lachmy & Koichu, 2014). The second phase started with this intervention and produced one chunk. This chunk consisted of more posts than the previous chunks and was spread over almost two weeks. It consisted of the work of two students who shared their results in exploring the new geometrical situation. They contributed four new claims including the intended claim for Item 3. Thus, this phase was characterized as exploratory participation.

Second case study: highly-challenging proof tasks in a social network

Students were quite autonomous. Students contributed 87% of the posts containing 74% of the words. The type of networked interaction was of an interconnected web type; four students held central positions such that together 51% of the posts referred to them. In addition, 21% of the posts were generally addressed to the group, setting it in the most central position. The teacher did not dominate the discussion and got less than 7% of the posts.

During the activity students contributed 217 posts and attached 18 files. The files were mostly pictures of their paper-and-pencil work and contained drawings of the geometrical situation, propositions and steps of a proof. Four students were active, contributing 47.5 posts per student with average of 11 words per post; eight students were much less active, contributing 3.8 posts per student with average of 7 words per post; three students only voted (i.e., "liked" others' posts) and one student was inactive. In addition, the four active students and another less-active one reported between 135-300 min. of involvement in the task. Additional eight students reported on average of 40 min. of work. Students' work in the environment lasted 13 days and produced mainly mathematical content. It can be described in two phases (Table 2). The first phase was shorter and contained the majority of the mathematical work in the environment. Supported by the following findings this phase was characterized as exploratory participation: students shared drawings including several auxiliary constructions, propositions, resources and candidate strategies; they referred to their work as suggestions which contributed to the collective effort in solving the problem; they addressed their ideas almost entirely to their peers or the group and in many cases the ideas were elaborated on during the chunk. In addition, in the few cases the teacher intervened she was mostly responsive and supportive for students' initiatives ("It is worth adding it to your previous ideas", "I think the drawing helps others"). At this stage students raised more than 70 new claims and propositions and exposed key ideas for solving the problem. However, they did not reach any solutions.

The second phase started when Maya (a pseudonym) announced that she reached a solution. This phase was rather sparse; about half of the mathematical posts compared to the first phase were spread over three times as many days as the first phase. Supported by the following findings this phase was characterized as expository participation: two students and a group of five students reported two different ways of solution; two additional students reported the work they had done, one of them suggested a third way of solution which was later supplemented by the teacher; students did not leave tracks on the processes they went through and did not explicitly address their reports to other participants or the group; responses from peers were mainly requests for explanations; although there were eight exploratory chunks, they were very short and most of them were initiated by the teacher or the moderator but elicited no further work from the students.

1st case study	EXS	EXR	Total
Phase 1 (4 days)	13 (20)	0 (0)	13 (20)
Phase 2 (13 days)	0 (0)	1 (7)	1 (7)
Total (20 days)	13 (20)	1 (7)	14 (27)

2nd case study	Non	EXS	EXR	Total
Phase 1 (3 days)	1 (7)	0 (0)	18 (147)	19 (154)
Phase 2 (10 days)	4 (18)	6 (50)	8 (27)	18 (95)
Total (13 days)	5 (25)	6 (50)	26 (174)	37 (249)

Tables 1 and 2: Number of chunks of activities according to phases and content. The content is characterized as non-mathematical (Non) or mathematical which further denoted as expository (EXS) or exploratory (EXR) participation. The number of posts is denoted in brackets.

Supportive forms of interaction

The findings above show that exploratory participation was supportive for sustaining student engagement in the online collaborative problem solving activities. In the Two Circle problem task the exploratory phase was more intensive and productive for solving the problem than the expository one. In the Mid-segment activity although the expository phase was more intensive than the exploratory one it did not progress much further than its initial point. In comparison, the short exploratory episode yielded new claims including the intended claim for Item 3. In addition, in both case studies, the chunks of the exploratory phases were longer. To explain what motivated exploratory participation, I will capitalize on the points where expository and exploratory phases were exchanged.

In both cases, the common denominator of these points was the change of *student sense of achievements*. Until the moderator's intervention in the Mid-segment activity, students were sure they had completely solved the tasks. The encounter with a surprising result that did not fit their conclusions changed student sense of achievements and prompted a pair of students to explore the new geometrical situation. The case of the Two Circle problem showed an opposite behavior. As long as students knew that no one had reached to a solution they were cooperative. Once Maya announced that she found a solution it changed student sense of achievements. As a consequence, they felt discouraged to solve the problem together. That sense was also reflected in an interview with one of the students when she said that a positive and constructive competition was created: on the one hand, they shared ideas and reasoned upon the work of their peers, but on the other hand they were motivated to be the first to come with a solution.

SUMMARY AND CONCLUDING REMARKS

Two case studies were presented in which secondary school students were engaged in collaborative problem solving activities in our two OLEs. In both case studies students were engaged in challenging tasks about two weeks during which the first 3-4 days were more intensive. In regards to past studies (see Theoretical Framework), it is quite long-term engagement. The findings showed autonomous and meaningful activities: students put effort to explore the problem situation, provided new mathematical claims and propositions and progressed beyond their initial results with a few external prompts. Specifically, students were engaged in both expository and exploratory participation. Exploratory participation was found as engaging students for longer and more productive participation. It was explained that students were engaged in exploratory participation as long as they did not sense they accomplished what they conceived as the mathematical goals of the task. Regarding this explanation and in line with previous results (Stahl, 2009b) it seems that highly-challenging proof tasks are effective in engaging students in exploratory participation but until someone finds a solution. In addition, it seems that the kind of intervention used in the first case study (see another case in Lachmy & Koichu, 2014) is effective in engaging students in exploratory participation in inquiry-based tasks. Further research is needed to explore the dynamic of student engagement in the two OLEs and to explain it by referring to social and motivational variables as well as to students' conceptual competence (Azevedo et al., 2012).

The two online learning environments presented here used free and available online technological tools and were implemented by the teachers during the school year on relevant curricular topics and with minimal intervention. Thus, this study encourages teachers to implement similar settings according to the principles described here for their pedagogical needs.

NOTES

1. The first OLE was developed at the Davidson Institute of Science Education. For a more detailed account on this environment see Lachmy et al., 2012 and Lachmy and Koichu, 2014. The development of the second OLE is part of a research project held at the Technion and directed by Prof. Boris Koichu.

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TEACHING DIFFERENTIAL EQUATIONS USING BLENDED INSTRUCTION

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This is an ongoing project, which is a Doctoral research at the Mathematics Education Department of the Center for Research and Advanced Studies of the National Polytechnic Institute. A group of students of undergraduate level of the School of Mechanical and Electrical Engineering of the National Polytechnic Institute in Mexico City, here in after referred as "the School" is receiving every week two sessions in a computer room of the School and one in a traditional classroom. Online teaching is based on the Moodle platform. The pedagogical design of the course is based on constructivism and behaviorism and it is supported with Mayer´s multimedia learning theory. The findings that have been observed so far show very good results whereas the progress is occurring according to learning plan; however some students are presenting some difficulties moving autonomously due to forgetfulness or lack of knowledge of some Integral Calculus concepts. The research will analyze the effectiveness of educational materials to promote learning.

Keywords: Face to face teaching, distance learning, blended learning, cognitive learning theory, learning with multimedia.

INTRODUCTION

This project is important for the School due to high level of learning deficit on the academic year 2012-2013. Approximately 32.5% of the students failed. It has been proved that the use of blended education is an efficient strategy to increase the quality of education and thus improves learning. (López-Pérez, Pérez-López and Rodríguez-Ariza, 2011). However, other researchers suggest that there are no different results between blended education and traditional face-to-face learning. We do believe that blended education works better.

To sustain the former argument it is necessary to use distance teaching with the best design of multimedia materials, for this reason, we will try to prove in this research that multimedia theory from Mayer is useful in Differential Equations teaching.

This project is focusing on the material's design, rather on the blended instruction. The research's main purpose is enhancing the mathematics teaching using the multimedia theory in order to design better materials to present in the Moodle platform.

For the above mentioned reasons, we designed a course on the Moodle platform to promote better learning of Differential Equations for the second semester students of mechanical engineering career. The students attend one and a half hours twice a week in the computer room and also a session of one and a half hours in a traditional classroom.

CONTEXT

Mathematics is taught in the School since its inception because it represents the backbone for sustaining engineering calculations. Since the National Polytechnic Institute was created by the General Lázaro Cárdenas del Rio, many Faculty members belonged to the military forces of México. Their instructional techniques were based on discipline, rote learning and behaviorism, instructional strategies that somehow remain to date.

Admission profile

Nowadays, the admitted students to the careers in the School should have the following basic knowledge, skills, attitudes and values:

- Theoretical and practical knowledge of physical and mathematical sciences.
- Fluency and reading comprehension and ability to express themselves through every day and scientific language, both oral and written.
- Capabilities on logical reasoning: analysis, synthesis and application of knowledge.
- Knowledge and use of scientific methodology.
- Understanding and application of information contained in different languages: graphic, symbolic and computational, in both Spanish and English.

Graduate profile

At the end of his career, the graduate will be able to justify and apply scientific and technological knowledge and skills, attitudes and values necessary for the exercise of their profession, to the benefit of society and development of the country.

The School currently has a total population of 4,849 students (2,448 in the morning shift, 1,686 in the afternoon shift, and 715 in mixed schedule). There are 489 members of the Faculty.

DIFFERENTIAL EQUATIONS COURSE

The pedagogical design of the course of differential equations use constructivism as seen applied in most research in Distance Education, (Bernard, Abrami, Lou, Borokhovski, Wade, Wozney, …Huang, 2004) and the Cognitive Theory of Multimedia Learning by Richard E. Mayer.

Theoretical support

Theories of learning in teaching sessions in the traditional classroom, and in the Moodle platform, practically apply the following:

- Incorporation of instructional activities located in the stadium student cognitive development (Piaget, 2006),
- Development of organizers and preprocessing of content with globalizing concepts (Ausubel, 2002),
- The application of rewarding stimuli (Skinner, 1948),
- Identification of the dominant intelligence on the learner (Gardner, 2011),
- Care that math skills are present close to possessing previous student´s knowledge as indicated in the scheme Vygotsky Zone of proximal development (Vygotsky, 1997).

As it was written, above, the first objective of the research is to prove the validity of the Multimedia Learning Theory (Mayer and Moreno 2002) used on mathematics teaching, and its implications.

> Mayer´s (2002) Multimedia Learning Theory establishes the following:
>
> The dual-channel assumption is that humans possess separate information processing channels for visually represented material and auditory represented material. The second

assumption is that humans are limited in the amount of information that can be processed in each channel at one time.

Multimedia Principle. The first principle is that students learn more deeply from animation and narration than from narration alone.

Spatial Contiguity Principle. Students learn more deeply when on-screen text is presented next to the portion of the animation that it describes than when on-screen text is presented far from the corresponding action in the animation.

Temporal Contiguity Principle. Students learn more deeply when corresponding portions of the narration and animation are presented at the same time than when they are separated in time.

Coherence Principle. The fourth principle is that students learn more deeply from animation and narration when extraneous words, sounds (including music), and video are excluded rather than included.

Modality Principle. The fifth principle is that students learn more deeply from animation and narration than from animation and on-screen text.

Redundancy Principle. The redundancy principle is that students learn more deeply from animation and narration than from animation, narration, and on-screen text.

Personalization Principle. The final principle is that students learn more deeply from animation and narration when the narration is in conversational rather than formal style. (p. 93-96).

In an article on the implementation of the cognitive theory of multimedia learning and its implications in the "E-Learnig" two researchers (Merriënboer and Ayres, 2005) suggest some reflections on the current trends of recent research in the field of this theory.

According to these authors three areas of research are presented:

a) methods to decrease the intrinsic cognitive load and processing of materials with high-interactivity; b) methods for increasing cognitive load directly relevant for learning c) methods for dealing with differences in levels of student experience and development of experience. This proposed research is located in the area b which is intended to identify the design elements of multimedia materials that promote better learning of Differential Equations engineering students. This time the research is aimed at verifying the principles of redundancy, spatial proximity, and temporal proximity. In other words the procedures to solve the equations presented with texts have simultaneous narratives (Principles of redundancy and temporal proximity); in other examples videos are presented in which other principles of the theory are applied as is the personalization or the familiar narrative rather than a formal conference.

It is important to notice that the research will not be focused on the blended instruction phenomenon.

Methodology

Each student uses computer equipment that has Internet access with multimedia through the Moodle platform, -as stated earlier- and also in each training session of an hour and a half in the computer room, it was simultaneously the presence of a teacher (this time the teacher is the researcher in this project). The teacher's interventions are performed when a student asks for help; also is involved when difficulty is observed to advance or when the student is using websites outside the course topic.

Blended learning strategy was chosen because several studies indicate greater benefit in learning, which when exposed separately distance education (DE) or face to face education (FFE). In the thesis "Finding the perfect blend: a comparative study of online, face to face, and blended instruction" the author (Pearcy 2009), suggests that the combination of DE with FFE, leverages strengths of both strategies to increase learning rate. Moreover, several studies show better learning, with complementing DE and FFE, that if are given separately; the combination of the two forms of imparting education leverages the potential of each of them, such as personal and immediate contact between student and professor at the FFE, whereas DE has simulations which facilitate learning, progress in the issues according to the rhythm of student learning, it presents individualized learning, the opportunity to submit any additional information on topics that are previous to differential equations and which do not correspond to the course, in this case -differential equations- and that not all students require to consult, only those who sometimes do not remember it or have not learned previously.

Learning is conducted involving students in three interaction options: student computer, student reflection, and communication with other students and teachers (Bañados, 2006).

The course design of differential equations both in DE mode and the FFE is based on a high interaction as well as a high content according to the model of Johann Engelbrecht and Ansie Harding (2005).

In this project qualitative research methods were applied and some of the principles of cognitive theory of multimedia learning (principles of redundancy, spatial proximity, and temporal proximity), which support of the following research questions:

a) Is there a relationship between the characteristics of the concepts expressed in the procedures for solving differential equations and generating meaningful learning?

b) Is there a relationship between the format of the images and animations and the acquisition of knowledge?

c) Is there a relationship between the format of image integration with the information and generating meaningful and transferable learning?

Therefore the null hypotheses are:

a) There is no relationship between the characteristics of the concepts expressed in the procedures for solving differential equations and generating meaningful learning.

b) There is no relationship between the format of the images and animations and knowledge acquisition relationship.

c) There is no relationship between the format of image integration with the information and generating meaningful and transferable learning.

The course considers the basis of mathematics knowledge and is designed for social management of applied mathematics. The course includes information display options and the students can select the one that best fits their learning style.

Course application

Pre-Ride the course was enabled on the Moodle platform, which took approximately four months. The course of differential equations with blended instruction strategy started to be taught on March the 25st of 2015 and will be ended on July the 11st of 2015. The group attended two sessions of one and a half hours in the computer room, using the Moodle platform of the School and a session also for one and a half hours in the traditional classroom. Each student uses a computer located in the computer room.

The course presents among others procedures for solving, separation of variables, homogeneous differential equations, exact differential equations, linear differential equations and Laplace Transform.

On this opportunity, research the analysis of the multimedia materials will be focused on the procedure for solve homogeneous differential equations.

The Moodle platform presents information including the topics of formal curriculum Differential Equations course, examples of application of the information, exercises, videos, tasks and links to various Internet sites containing related topics of the program.

During the first two sessions in the computer room the students were trained to use the platform. At both sessions they received personal assistance and support to resolve technical problems. The difficulties of previous mathematical knowledge, especially the concepts of integral calculus are still matters of additional guidance.

At the beginning of each session the students were instructed on what activity they should perform. It took four sessions to make the students comfortable and able to identify the activities they ought to perform.

The students could resolve their task using various tools. For example, math editors, manuscripts written subsequently photographed or in Word, to end any previous submissions must upload it to the platform.

Once the students uploaded their assignments to the platform the teacher reviewed them and gave them feedback. If necessary, some students were asked to repeat the task due to inconsistencies, lack of sequence in solving the problem or any other reasons.

At the beginning of the sessions the students could review the entire course material in order to have an overview of it. The issues were to be approached in the presented sequence. There was a list of the topics on the central part of the platform, which has the option to enter any of them.

The understanding of the issues was frequently accompanied by supplementary information obtained by online research, a practice highly recommended by the teacher.

An example of information used in the platform:

Resolution Procedure homogeneous second orders differential equations with constant coefficients.

1. $\underbrace{ay''}_{2^0\text{ order}} + by' + cy = \underbrace{0}_{\text{homogeneous.}}$

   a, b, c are constant coefficients.

   The following change of variable is applied.

   $y = e^{\lambda x} \qquad \lambda = constant$

   Deriving: $y' = \lambda e^{\lambda x} \qquad y'' = \lambda^2 e^{\lambda x}$ Substituting in equation 1

   $a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$ dividing by $e^{\lambda x}$

   $a\lambda^2 + b\lambda + c = 0$ Called Equation Assistant or characteristic

With the above operations the differential equation has been transformed into an algebraic equation of second degree.

λ values are obtained with the general formula for solving second degree equations.

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The face to face sessions were aimed to present issues discussed on the previous session on the platform. The mainly issues were the use of the blackboard, the use of the narrative technique, and questions about the items. The students were encouraged to practice those issues. They were invited to go in front of the class, to the board to explain subjects to their peers. This participation was taken into account to the final grade to be obtaining at the end of the course.

INITIAL FINDINGS

The course is being fluently conducted since the platform works continuously, in other words, there was no Internet outages or interruptions in the server that hosts the Moodle platform.

Some students do not focus on the use of the course by the platform and divert opening pages of entertainment. It is necessary to include in the next courses, interesting games and activities to catch their attention and avoid distraction.

The students have had nine sessions in the computer room and five sessions in the traditional classroom and the registered attendance at all sessions was 95% and have done seven tests so far (March-April 2015) and the average ratings they have obtained are as follows:

Test	1	2	3	4	5	6	7	Global Average
	85.69* (29 students)	70.81 (31 students)	56.67 (30 students)	74.19 (31 students)	83.90 (30 students)	94.70 (29 students)	98.90 (29 students)	72.82 (31 students)

Table 1 Ratings of the students. *the scale used is 0 – 100

The figures tend to show an increasing ability to solve the tests, a deeper analysis has to be done for determine causes for the increase in the grades obtained.

One clue in the increasing ability to solve the tests may be the principle of multimedia theory which mentions there is better learning without extraneous icons, words or figures in the teaching text (Mayer, 2005).

The seven tests are designed within Moodle platform, which has options to prepare them. In this occasion we have used: true or false, multiple options, numerical response, calculations and large responses. The first three, are verifying automatically by the platform, while the remainder have to be analyzed by the teacher.

Students have difficulty to evaluate themselves and resort to the teacher to check their learning, infrequently rely on the opinions of their classmates. They lack discipline for self-study, their study habits prevent them from concentrating, they are quite distracted, and part of this distraction involves access to personal e-mails and spend time with their friends on topics outside the course.

It was observed that students view Internet videos that explain the processes of solving differential equations to supplement the written information presented on the platform. It is evident the desirability of incorporating images according to the theoretical approach indicates the beginning of redundancy or greater depth of learning, when you have animation with narration instead of animation, narration and text on the screen. The principle of spatial proximity is also evident: People have better learning when portions of a figure and words are close rather than far apart on paper or screen.

Possible causes of distracted students they might be: lack of understanding, tedious information or too much information that is inaccessible at first. According to observations made most of the time the information did not understand, because they lacked the background and chose distracted by other activity either in the computer itself or with peers. It requires that computers only allow connections to the topics of interest in the course.

UPCOMING ACTIVITIES

At the end of the course it will be measured the efficiency of the teaching materials, the relevance of the exercises, the degree of participation and student learning.

It is necessary to include in the next courses, interesting games and activities to catch their attention and avoid distraction.

We will present learning games in which students assume group characters. They will interact with online fellows, solving problems that are presented to the group. In this way, mathematical knowledge is constructed on cyberspace (Rosa and Lerman, 2011).

Students will identify their learning style through a self-analysis tool that will be included in the course. The course design will facilitate the selection of options according to different dominant intelligences (Gardner, 2011) and learning styles (Canfield, 1987).

The course will be supplemented with videos, exercises in the form of games and will consider the possibility of dealing with issues synchronously.

Interested teachers are invited to join this project to complement its design and participate in running it online.

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THE USE OF HANDWRITING RECOGNITION TECHNOLOGY IN MATHEMATICS EDUCATION: A PEDAGOGICAL PERSPECTIVE

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In mathematics education, the lack of an intuitive means to enter mathematics expressions online has been a major barrier to effective communication, causing mathematics to be lagging behind in the development of online collaborative learning environments. This study evaluates the use of handwriting recognition technology as a potential solution from a pedagogical standpoint. With pedagogical needs in mind, a new handwriting recognition user-interface (MathPen) was developed as a research tool to investigate the teaching and learning perspectives through a) an expert review with practising teachers, and b) a usability study with undergraduate students.

Keywords: Handwriting recognition user-interface design, instrumental genesis, technology-induced distractions

INTRODUCTION

The benefits and effectiveness of collaborative learning in mathematics is well-established (Edwards, 2009). Through proposing, exploring and evaluating different ideas with their peers, students are better able to develop a deeper understanding (Wegerif, 2013). As students justify and defend their mathematical reasoning, underlying misconceptions are uncovered and addressed (Mercer, 2000). Additionally, since collaborative group work is common in the work place, this learning method prepares young people for future employment (Hoyles, Noss, Kent and Bakker, 2007; ACME, 2011). The ability to employ collaborative learning through the Web, thus transcending the limits of time and space, has already benefited many text-based subjects (Harasim, 2002). Yet, developments for mathematics education in this regard is reported to be lagging behind (Allen and Seaman, 2010). Researchers such as Catalin, Deyan, Kohlhase and Corneli (2010), Costello, Fox and Walsh (2009), and Reba and Weaver (2007) have alluded to the lack of intuitive input methods for mathematics expressions as the main cause of the problem. Although joint-editing whiteboards are now available in pictorial formats, these do not lend itself to integration with digital computational tools, which could enrich the collaborative discussions. As Lo, Edwards, Bokhove and Davis (2013) argued, "serious considerations should be given to online handwriting recognition systems as a means of opening the way to online collaborative learning for mathematics education" (p.173).

THEORETICAL FRAMEWORK

According to the theory of instrumental genesis, an educational tool in the hands of students who do not yet know how to utilise the tool for educational purposes has little value (Rabardel and Bourmaud, 2003). The theory argues that for the tool to become an educationally useful instrument (or *instrumentalised*), specific conditions must be met. Teachers need to bring the tool's affordance to the fore, while the students need to create personal concepts of the tool's behaviour and develop their own ways of leveraging the tools' capability for their own educational advantage (see theoretical overview in Drijvers, Godino, Font and Trouche, 2012). Typically, the onus of instrumentalisation rests with the teachers and the students, not with the tool design engineer.

However, depending on how the tool is designed, the process of instrumentalisation may be made is easier or more difficult. Therefore, instrumental genesis, when viewed from an engineering standpoint, can also be used as a guide leading to a more readily '*instrumentalisable*' tool. This study focuses on the investigation of the teaching perspective and the learning perspective in order to better understand the barriers to handwriting recognition technology being instrumentalised in mathematics education.

METHODOLOGY

This study is divided into three phases: 1) an engineering development phase, 2) an expert review phase with practising teachers, and 3) a usability study phase with university students. During phase 1, an online handwriting recognition system (MathPen) was designed and implemented. Although off-the-shelf software packages are available, these are prohibitively expensive for many and are not designed with education in mind (Lo et al, 2013). Since recognition algorithms are freely available through academic publications, an in-house development would a) provide greater engineering flexibility for research purposes, b) allow new-gained insight to be implemented in a reduced timescale, and c) eventually lead to a research-based tool which can be made publicly available free of charge. In order to take into account of users' needs, the design process, in line with a design-based research methodology (Reeves, 2006), began with findings from previous studies (Lo, 2012), which highlighted the need for multiline recognition (see Engineering Design section). Future designs will utilise findings from phases two and three, where the teachers' and learners' perspectives were explored, for future improvements.

Phase 2 was a three-part expert review with three practising teachers. First, since hardware capabilities and recognition accuracy are commonly thought to be a barrier to the instrumentalisation of the technology in the classroom (Lo, 2012), the teachers were given a range of devices (Android-based Samsung Galaxy Note 10.1, iOS-based 3rd generation iPad and Windows 7-based Tablet PC) to explore the capability of commercial handwriting recognition products as well as MathPen's modest implementation. Figure 1 shows an example of mathematics expressions supplied for testing.

$$\frac{\sin\theta+\cos\theta+\tan\theta}{x+y+z} \qquad \alpha_{n+1}-3\beta=\frac{2}{3}\alpha_n+\beta_1 \qquad \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4}}}} \qquad 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}$$

$$x^{\frac{1}{2}}\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{2^8}\right) \qquad \left[b^x\left\{\left(\frac{a}{b}\right)^x+1\right\}\right]^{\frac{1}{x}} \qquad A=\sqrt{a+\frac{1}{\sqrt{a+\frac{1}{\sqrt{a}}}}+\sqrt{b}} \qquad \lim_{x\to\infty}\int_0^x e^{-y^2}\,dy=\frac{\sqrt{\pi}}{2}$$

Fig 1: Sample expressions for exploring recognition accuracy

Next, three mathematics questions were posted online, to which the teachers were to respond with the supplied model answers. Question 1 involves very few steps with notations that are well within MathPen's recognition power. Question 2 has the same number of steps but with notations which are at the threshold of MathPen's recognition capability. Question 3 is also at the threshold of MathPen's recognition capability but with an increased number of steps (Fig 2). The session concluded with a focus group to reflect on their instrumentalisation process/ experience from a teaching viewpoint.

Q1. Given $y = 4x - 7$, find x.

$$y = 4x - 7$$
$$y + 7 = 4x$$
$$\frac{y+7}{4} = x$$
$$x = \frac{y+7}{4}$$

Q2. Given $y = \frac{\sqrt{2}x^2+1}{4}$, find x.

$$4y = \sqrt{2}x^2 + 1$$
$$4y - 1 = \sqrt{2}x^2$$
$$\frac{4y-1}{\sqrt{2}} = x^2$$
$$x = \pm\sqrt{\frac{4y-1}{\sqrt{2}}}$$

Q3. Evaluate $y = \int_0^2 (\frac{2x+1}{4})\ dx$

$$y = \int_0^2 (\frac{2x}{4} + \frac{1}{4})\ dx$$
$$y = [\frac{x^2}{4} + \frac{x}{4}]_0^2$$
$$y = \left[\frac{2^2}{4} + \frac{2}{4}\right] - \left[\frac{0^2}{4} + \frac{0}{4}\right]$$
$$y = \left[\frac{4}{4} + \frac{2}{4}\right]$$
$$y = \left[1 + \frac{1}{2}\right]$$
$$y = 1\frac{1}{2}$$

Fig 2: Online questions posted to the experts

Having established in phase 2 that the barrier to the instrumentalisation process is not the level of recognition accuracy as is commonly perceived, but the level of technology-induced distractions, phase 3 further investigates the distraction elements by inviting seven undergraduate students (studying engineering or mathematics) to complete two pieces of collaborative group work (ten minute each), using MathPen for one task and keyboard entry for the other. Since commercial products have already been shown to feature more technology-induced distractions, keyboard entry has been chosen as a comparator for phase 3. The students were split into two groups so that one group would complete one task with MathPen first and keyboard entry the second, while the other group performs the same tasks in the same order but with keyboard entry first and MathPen second. Throughout the exercise, all the keyboard and mouse interactions were recorded through screencast recordings. Additionally, students were also asked to think aloud throughout the process to externalise their thoughts. The session concluded with a 30-minute focus group discussion to reflect on their instrumentalisation process and experience from a learning standpoint.

ENGINEERING DESIGN

In terms of the engineering design, it is known that without handwriting recognition, the standard quadratic equation would have to be entered as "*[TEX] x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}[/TEX]*" in order to communicate online. While current technology is capable of recognising mathematics one line at a time, recognition of multiple lines of mathematics is not supported in any present systems (Lo, 2012). When users submit multiple lines of mathematics for recognition, present systems will merge these into a single line for recognition (Fig 3, 4). This is a pedagogical concern, because new lines of mathematics cannot be written until the recognition result is transferred to the communication medium. By then, without a visual point of reference to the ongoing mathematical argument, the mathematical argument for the next step is likely to have been forgotten.

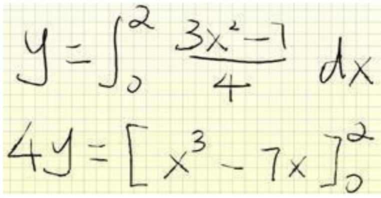

Fig 3: Multiple lines of mathematics for recognition

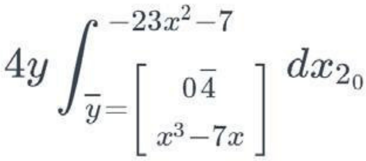

Fig 4: Erroneous recognition without multi-line recognition

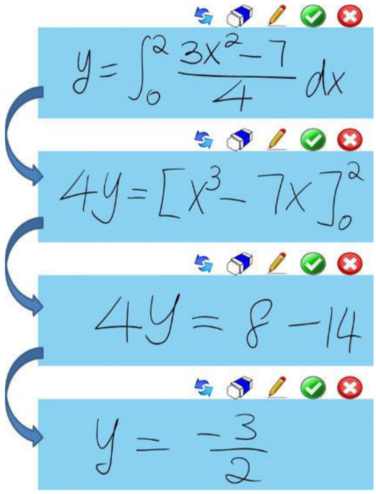

Fig 5: Multiple lines of mathematics in MathPen

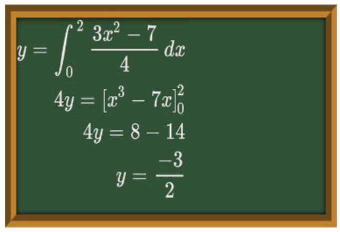

Fig 6: Floating blackboard

Latex Expression

```
y=\int^2_0\frac{3x^2-7}{4}dx
4y=[x^3-7x]^2_0
4y=8-14
y=\frac{-3}{2}
```

Fig 7: Complete set of Latex expressions

MathPen is designed to address this pedagogical issue (Fig 5-7). First, the users can submit unlimited lines of mathematics to the recognition engine so that they can concentrate on the mathematical reasoning from beginning to end. Figure 4 shows a 4-lined submission for an integral evaluation. Then, the formatted recognition result is displayed on a floating blackboard which ‘floats’ together with the users as the they scroll down the page to add more lines (Fig 6). Finally, the users receive a complete set of Latex code for copying and pasting into their choice of Web-based communication platforms (Fig 7).

PHASE 2 RESULTS: EXPERT REVIEW

During the technology exploration stage, all three experts were impressed with the standard of recognition currently achievable. As well as testing the technologies with the suggested mathematics expressions (Fig 1), they also created several of their own. Despite observing occasional recognition errors, the experts unanimously agreed that handwriting recognition for mathematics is quickly becoming a reality and should be given serious consideration. In terms of choice of equipment, the experts felt that only the Windows-based Tablet PC delivered the necessary processing power to keep up with the recognition needs. The experts also found the Tablet PCs’ palm rejection feature, which allowed them to rest their palm on the screen while writing, very helpful.

During the online interaction stage, all three experts adopted handwriting recognition without hesitation. Although initially inclining towards commercial packages for their superior recognition power, all three experts eventually adopted MathPen as their preferred recognition engine for question one. As they continued to the second question, two of the experts continued with MathPen while the remaining expert switched from MathPen to commercial products and then back to MathPen. Although all three experts attempted question three, they eventually abandoned the task and none progressed to the end.

At the focus group that followed, experts commented on their experience in handling question one and expressed their frustration at the commercial products’ lack of multi-line support; examples include:

"This going back and forth between the forum page and the recognition page one line at the time is driving me crazy."

"I'm a teacher and this maths is easy. Still, I can't remember the next step. One line at a time is too distracting."

"The kids using this would be trying to work things out. If we struggle with the answers in front of us, it would be impossible for the kids."

In the case of question two, when the mathematics involved complex notations and the number of erroneous recognition results increases, there was a battle between MathPen's multi-line recognition support and commercial products' superior recognition power. Finally, in question three, where there are many lines of complex notations, the experts gave up. Relevant comments include:

"Either way, you are distracted. Switching between pages at every line is distracting, but correcting recognition error has the same effect."

"However you look at it, I think we are talking about interruption to thinking. It doesn't matter what is distracting you. It almost seems the moment you stop thinking about the maths at hand, you lose your train of thought."

"I think this perfectly well explains why people don't discuss maths online."

During the focus group, the experts were asked to comment on the use of handwriting recognition from a teaching perspective, to which they answered:

"I think the most important thing for me is having sufficient recognition power for the task and the multi-line thing. I mean it was quite ok for the first question. Everything ran smoothly and MathPen was great."

"If I set a question for discussion, what I would like to see is the quality of discussion. So I guess if the kids are frustrated, they may be tempted to skip steps or even give up like us. You've probably hit the nail on the head there, there may be nothing to do with accuracy, but more to do with letting the kids focus on what they are doing. As MathPen's recognition power is up to the job for that level of mathematics, it doesn't really have to be the latest technology, does it?"

"There is a whole host of Web 2.0 stuff that my colleagues are using. They talk about how the kids work together online and correct their own mistakes, but it's just impossible for maths. It's just too much work involved. Software like MathPen really would be the answer."

"I teach in a boarding school with many international students, and there are always a few who have to return home during term time. Supporting these students through the Internet has been a real struggle, particularly when students get things wrong, there's always a reason for it. At the moment, it's a case of writing on a piece of paper and scanning. But that's dependent on them having a scanner on the other side of the world. We can kind of forget just how fundamental it is to be able to communicate. I mean, how is anyone supposed to teach without being able to communicate?"

"It boils down to being able to communicate at ease and without distraction. The whole point about paying attention in class is that you don't get distracted and you listen to what others have

> to say. I suppose, from a teaching point of view, what this does is to make this possible in an online environment."

Therefore, contrary to common opinions, the reason for handwriting recognition not having a major role in mathematics education is *not* because the latest recognition algorithms is not accurate enough for school use and it is *not* because the hardware is expensive. In fact, with the Tablet PCs costing around £150, it is the cheapest option available. The teachers' favourable comments towards MathPen's multi-line shows that, pedagogically speaking, it is the technology-induced distractions inherent in current user interface design that is preventing the instrumentalisation of handwriting recognition from happening in real life setting.

PHASE 3 RESULTS: USABILITY STUDY

During the first five minutes of both online discussion exercises (ten minutes each), contrary to expectation, both student groups spent more time discussing mathematics and were able to progress further in their discussion when using keyboard entry than when using MathPen. Reviewing the screencast recordings of the onscreen interactions and audio recordings of the think aloud commentaries reveals that students equipped with MathPen had spent a substantial amount of time with the software, switching between writing mathematics and responding to comments posted on the Web. However, during the last five minutes, where student groups were not equipped with MathPen, both groups shown less overall pedagogical progress, digressed and did not continue with the mathematical discussion. By contrast, where students were equipped with MathPen, the mathematical discussion continued to the end.

At the focus group, the students' comments regarding keyboard entry were as follows:

> "I think it's bearable, but then I have been typing maths at uni for some time now. It took a lot to get used to it, and [the maths] just doesn't look right. So you have to reinterpret the thing all the time. It gets tiring and makes you want to give up. It would be much better if handwriting can just get rid of that awkward stuff altogether."

> "For this exercise, it's kind of fine. You know, hat 2 instead of squared is kind of well-known to us. Not sure if I would have known that at school-age though. Also things would be very different if the maths have more difficult stuff. You know, things like square root and fractions. I think when you have loads of those stuff, typing becomes rather stupid. I mean you can't interpret those stuff without the proper formatting."

> "I hate the stuff, just can't get on with it. I know how to type using that funny hat thing, but I always have to copy that thing out by hand and read it that way. I mean, you know, that's how maths is meant to look like. Not a chance for me. Sorry I gave up in the end. At least the handwriting thing, I know it's slow and it's not great yet, but at least it gives you something that you can just read and understand."

With regards to keyboard-based communication, despite the students' engineering/ mathematics background, there is a consensus that not only is typing mathematics difficult, interpreting the unformatted mathematics is equally troublesome. This is even to the extent that none of these students managed to engage with the tasks for the entire ten-minute duration.

By contrast, when equipped with MathPen, all the students were able to engage with the mathematics throughout the ten-minute exercise, and their comments were as follows:

> "I really like MathPen's multi-line idea, but it would be even better if I don't need to switch pages at all. It's like I can hear the messages are coming through, and I want to just scan read the message. But you can't do that. It's better than typing on a laptop, but I think it needs to be one step further."
>
> "It is quite frustrating to see postings, and I can't just compare what they've said with what I am writing. They are on different pages. If the communication system and MathPen are integrated, then you can see things side by side."
>
> "I think MathPen is fine when it gets things right, but when you make a mistake, you can't just rub out one stroke. You have to start the whole lot again."
>
> "It's like you are busy thinking about the maths and what to do next, then you noticed MathPen doesn't always get it right. You'd want to just correct that bit that it gets wrong. You don't want to start that that line of maths again."
>
> "I like MathPen, at least you can say things without having to think 'Oh, how am I suppose to type this'. But when I am concentrating, ideas flow, and you want to capture that moment. When technology is so slow, it gets frustrating. I'd like it to be a bit quicker."
>
> "There's actually a lot going on at the same time. You are thinking about the maths, the questions, the solution, the way to express it and so on. And then on top of that, you are sort of bombarded with messages, you know, we are all talking at the same time. By the time you manage all that, you really haven't got that much patience to battle with technology. It's to do with the speed thing that [participant x] said. You want to be able to just write and not think about anything else."

Generally speaking, the consensus is that MathPen, when compared with traditional keyboard entry, is definitely a superior choice. There was also clear indications that the multi-line feature is much appreciated. However, it is also evident from the comments above that more needs to be done in order for the tool to be instrumentalisable.

SYNTHESIS

Overall, both teachers and students are supportive of the idea of using handwriting recognition in mathematics education. Concerning the online learning process, the students described it as a hectic time, trying to keep hold of their mathematical train of thought in their heads while simultaneously maintaining a channel of communication with their peers. In describing their experience, they use expressions such as, "*there's a lot going on*", "*it's information bombarding you all the time*", and "*by the time you've done all that*". However, despite the demands, the students' are generally favourable to this mode of learning. Their comments indicated a high level of interests in what their peers have to say and are keen to modify their thoughts accordingly. This is also reflected in the screencast recording where students are often seen to have paused from their ongoing activity to take note of every new message. There were also many occasions where students either deleted or modified their responses in view of what they had just read. As one student succinctly put it, "*Of course it feels intensive, cos you're thinking independently while listening and thinking about what* *they are thinking at the same time. But that's the whole point of teamwork: to bounce off each other, generate ideas and get things done quickly*". This corresponds well with one of the expert's comments regarding the quality of discussion and mathematical arguments. The learning process described here is precisely the mechanisms that make collaborative learning so effective (Harasim, 2002; Edwards, 2009), and the significance of this student's statement is that online collaborative learning can be just as effective for mathematics students too.

However, it is also clear that in order for the tool to be instrumentalisable in real-life settings, certain conditions have to be met. A recurring theme highlighted by both the teachers and the university students is the level of technology-induced distraction and its impact on the users' ability to maintain their mathematical train of thought. With regards to the impact of technology-induced distractions on the learning process, the students used expressions such as, "*you forget what you were doing*", "*lost track*", "*you'd want to keep the flow going, but can't*", "*you end up starring at what you've written and wondered what for*". As one of the experts pointed out: "*It boils down to being able to communicate at ease and without distraction*".

Sources of distraction can be many. For example, during phase 2, all participants identified the lack of multi-line support as a significant source of technology-induced distraction. However, results from phase 3 show that the provision multi-line support alone is not sufficient, hence all participants agreed that a more tightly integrated system where the online communication medium and MathPen can be viewed side-by-side simultaneously would be beneficial. Additionally, recognition error correction proved to be another source of distraction. From the students' comments, it appears that MathPen's accuracy level was sufficient for the task, but an intuitive means of error correction is lacking. Perhaps alluding to the pencil and paper experience, some of the students speak of "*rubbing out*" the mistakes instead of "*crossing out*" the entire line and starting again. Whatever the cause of the distraction, or whatever the engineering solution may be, it is clear from this study that, from a pedagogical viewpoint, the key to an instrumentalisable handwriting recognition tool is to facilitate online mathematics communication while keeping technology-induced distractions to the minimum.

CONCLUSIONS

This study provides additional evidence of limitations to entering mathematics online as a major barrier to online collaborative learning in mathematics. Evidence from this study shows that current handwriting recognition is already delivering sufficiently accurate results for school mathematics. Tablet PCs costing around £150 are providing sufficient computing power. Therefore, recognition accuracy, hardware demands and portability issues are not a cause for concern in usability.

Where previous studies led us to believe that the main issue is in "the lack of a natural and effective means of entering mathematical expressions online" (Lo et al, 2013, p.173), an even greater concern, from a pedagogical standpoint, is the impact that this is having on the students' ability to focus on their mathematics. In terms of students' progress in their mathematical understanding, the change of artefact from typewriting to handwriting means that students are progressing through collaborative discussions, instead of abandoning the exercise. Therefore, instead of focusing on vague terms such as "natural and effective", which are difficult to define, the results from this study suggest that identifying and reducing technology-induced distractions maybe a more fruitful line of research. Since this study has identified a number of technology-induced distractions, all of which are addressable through interface engineering, these will be the focus of our future studies.

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UNDERSTANDING AND QUANTIFYING AFFORDANCES OF THE MATHEMATICAL TASKS IN DYNAMIC AND INTERACTIVE MATHEMATICS LEARNING ENVIRONMENTS

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This paper describes our efforts to understand affordances of dynamic and interactive mathematics learning environments (DIMLE), and to develop a working rubric to quantify the degree of visual, dynamic, and explorative affordances of the mathematical tasks prepared in them. After providing background information and theoretical considerations for this study, we present our perspective on the DIMLE and their affordances. Following a brief review of literature, we outline the procedure on how the mathematical tasks were evaluated based on their degree of affordances. The paper concludes with a working rubric for further studies.

Keywords: Affordance, Visual Learning, Dynamic Learning, Explorative Learning, DIMLE

INTRODUCTION

Digital tools used in mathematics education have been grouped under two major titles: Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS) (Masalski & Elliott, 2005). CAS cover analytic software packages such as Maple, Derive, and Mathematica—suitable for working with algebraic expressions and functions; while DGS refer to software packages, such as GeoGebra, Geocadabra, Geometer's SketchPad, and Cabri—suitable for working on the interactive/ dynamic/continuous geometry objects and constructions (Leikin & Grossman, 2013; Leung, Baccaglini-Frank, & Mariotti, 2013; Prusak, Hershkowitz, & Schwartz, 2011; Scher, 2005). However, the latter group of tools also serve other mathematics disciplines, such as algebra, functions, calculus, data management, and even probability. The inclusion of data management and probability may bring other digital tools to the table, such as Fathom and ThinkerPlots.

In 2010, the first two authors of this proposal critiqued a limited educational viewpoint of the existing terms that did not include the types of mathematics learning software that were notably becoming increasingly heterogeneous and yet convergent (i.e., each package was becoming more diverse, and by doing so they became more alike, visually and operationally). For these reasons we proposed a new term: *Dynamic and Interactive Mathematics Learning Environments (DIMLE)*, which suggested some important shared features of the new generation of digital tools. To emphasize pedagogical use of digital tools and their connection to learning, we adopted Ulm's (2010) description of the learning environment as "the essential link between the teacher and the learner" which includes "the *tasks* for the learner's activities, the arrangement of *media* and the *method* for teaching and learning as well as the social situation with the teacher and other learners as *partners* for learning" (emphasis in original, p. 1284).

While working on the term *DIMLE*, our reference point was to emphasize the dynamic and interactive characteristics of these tools. In a working group for the 33rd PME-NA meeting, we led the group discussion with the inclusion of other scholars, such as Viktor Freiman (Karadag, Martinovic, & Freiman, 2011). Further discussions around these tools helped us to probe more deeply their common facets in which were the digitally-mediated mathematical practices rooted, namely: (a) moving images on the screen, (b) keeping interdependency between related objects, and (c) users having control (e.g., to develop, animate, and change objects on the screen). We noted that these tools allow the user to act upon the object that is not material, but it becomes material by being responsive to the user's actions. Similar to the situation in which one uses a virtual manipulative, the user moves between the material, the perceptual, and the conceptual domains, which may ultimately bring mathematical ideas and processes to the conscious level (Sarama & Clements, 2009). When the virtual object is designed so that the users can explore it (e.g., move it, change its features and parameters), the users may achieve more than when using physical manipulatives. According to Sarama and Clements, virtual manipulatives could be designed to have specific features and to allow for a more or less guided exploration (even with very young children), so in that way they are superior to physical manipulatives. This led us to highlight the three aspects of learning with digital tools, namely: visual, dynamic, and explorative, which we relate to affordances of (a) the digital tools, (b) the mathematical objects or artefacts implemented in them, and (c) the pedagogical ideas of the teacher.

This proposal reports on the task analysis stage of the first phase of a series of international projects exploring the affordances of *DIMLE*. In order to deeply understand the affordances of *DIMLE* and how students may benefit from these affordances, we needed to carefully design appropriate mathematical tasks and develop them in *DIMLE* before starting the data collection. However, we also had to evaluate the tasks to quantify their degree of affordances. This proposal discusses the effort put forward to develop and quantify the degree of affordances of the mathematical tasks.

The Affordances of DIMLE

American psychologist James Jerome Gibson (1997) developed the *Affordance Theory* where he considered that perception of the environment inevitably leads to some course of action. Affordances, or clues in the environment that indicate possibilities for action, are perceived in a direct, immediate way with no sensory processing. Here, affordances are characteristics of objects and their arrangements in the environment that contribute to interactive activity and, therefore, the characteristics of the environment that individual needs to perceive. While this view may be acceptable in case of natural affordances (those that emerge in the natural environment) which became directly perceivable to humans through the process of evolution and adaptation, this view is criticized when applied to artificial environments (e.g., buttons on a keyboard for pushing, computer mouse for rolling, cursor for pointing, point for dragging on the screen, etc.). Brown, Stillman and Herbert (2004) pose a question that we find very relevant for our study: "Is it that something— an object, tool, artefact, or instrument— affords users to do things in particular ways by constraining them to think or act in a specific way?" (p. 125). What Brown et al. suggest is that affordances and constraints go hand-in-hand.

Given that visualization, dynamism, and exploration emerged during various discussions about the affordances of the current digital tools for learning mathematics, we have decided to review the literature on these three concepts. It seems that visualization is the term that received most attention from scholars, while we have located limited literature on dynamism, and almost none about exploration. One reason for not finding the literature addressing specifically exploration or explorative learning is the use of other terms, such as investigation or inquiry (e.g., Leikin & Grossman, 2013).

Visualization

Different authors define visualization differently; for example Zimmermann and Cunningham (1991) use the term to describe "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated." Gattis and Holyoak (1996) consider visualization as a cognitive "tool for mathematical thinking [..., and] a type of representation that employs visuospatial relations in *making inferences* about corresponding conceptual relations" (p. 231). Rivera (2011) writes about "visual thinking in mathematics" which goes beyond "seeing [...] or having a visual or a sense experience [...]. It is, more importantly, a concept- or process-driven seeing with the mind's eye" (p. 36).

Dynamism

Jackiw and Sinclair (2009) suggest that the dynamism which was introduced through the DGS contributed to emergence of "Dynamic Mathematics" which includes dynamic statistics, graphing, and 3D geometry. The authors distinguish between the mathematical and pedagogical aspects of dynamism which are respectively: (a) "the powerful, temporalized representation of continuity and continuous change" and (b) "the sensory immediacy of direct interaction with mathematical representations" (p. 413). In our view, the mathematical and pedagogical aspects of dynamism discussed by Jackiw and Sinclair relate to affordances of (a) the digital tools, (b) the mathematical objects or artefacts implemented in them, and (c) the pedagogical ideas of the teacher.

Leikin and Grossman (2013) distinguish between dynamic and static changes in DGS. Dynamic are changes made by dragging, whereby the mathematical invariants remain intact (i.e., the square has all sides equal and all angles 90°, regardless of change in size or the position on the screen). Dragging object on the screen in essence forces one to notice the difference between the "moving image" (Mariotti, 2010, as cited in Leung et al., 2013), such as the object that changed its position or size on the screen, and its aspects that remained unchanged, such as the properties of the square. It seems that this description of dynamism in DGS addresses the question posed by Brown et al. (2004), since as affordance, dragging helps users to make distinctions between constrained and not constrained actions, which may lead them to think about related actions/concepts in specific ways.

Exploration

The mathematical experiments that could be performed in *DIMLE* provide opportunities to learn through exploration, discovery, and investigation (see e.g., Carter & Ferruci, 2009). Exploratory learning could be described as a cognitive process supported by the dynamism and interactivity as two pillars of the mathematics learning environment, such as GeoGebra (see Figure 1). Note that linguistically *exploration* is an action and a verb, while *dynamic* and *interactive* are the adverbs, describing the way of acting.

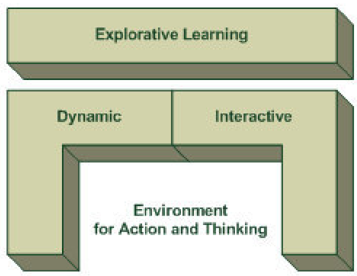


Figure 7. Relationship between the three concepts in the context of use of DIMLE.

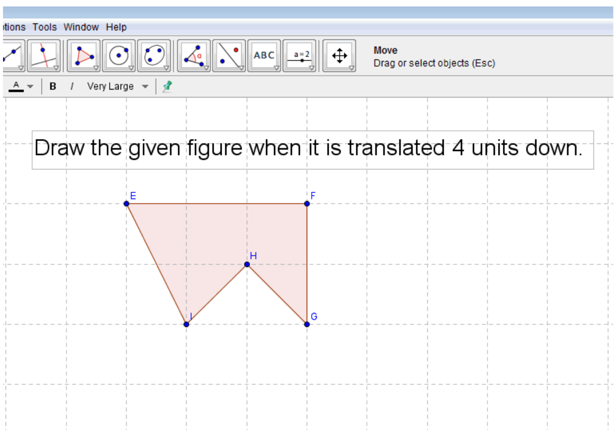
DIMLE provide teachers with opportunities to guide their students "to explore mathematics deeply" (Karadag & Aktumen, 2013). Despite many incidences of using the terms *to explore* and *exploration* in the context of learning, *explorative learning* has been used rarely. One reason could be domination of other terms, such as *investigation*, *inquiry*, and particularly *discovery learning*. The reason why we want to distinguish *explorative learning* from *discovery learning* is that the latter suggests a successful ending while the former emphasizes the process, regardless of its ending. Apparently, educators claim that learners build new knowledge as a result of exploration even if the result is not success, in other words nothing is discovered. In fact, a "fruitful geometric exploration" (Leung, Baccaglini-Frank, & Mariotti, 2013, p. 458) more often ends with a conjecture and a new question to investigate, then with a final answer.

THE STUDY

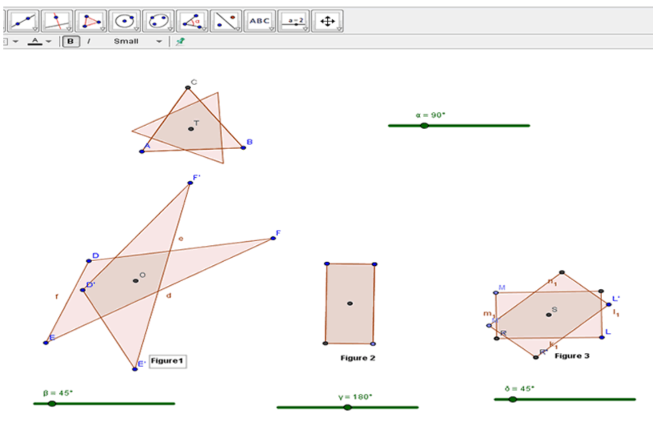
The project is a multi-step international project, whose aim is to explore the ways in which elementary school students interact with DIMLE while working on their mathematics problems. Aforementioned claims argue that DIMLE provide learners with opportunities to learn visually, dynamically, and exploratively. In order to understand to what extent these claims reflect the reality, a series of research projects has been planned. The first set of data was collected in Bayburt, Turkey in 2014, and the second set of data collection is being done in Windsor, Ontario, in 2015. There is a plan to collect third set of data in Turkey, possibly in 2016.

Data collection and analysis procedures are discussed in another paper presented at this conference. The scope of this paper is to describe our efforts in defining the affordances of DIMLE and creating a working rubric to evaluate the tasks—dynamic worksheets used in the study. First, we created 65 GeoGebra worksheets covering different curriculum topics, such as Fractions, Geometric Transformations, 2D/3D Geometry, Patterning, and Ratio/Proportion. Each worksheet described one task which involves various levels of visualization and dynamism, and a degree of exploration.

The Figure 2 illustrates geometric transformations tasks that use translation and rotation. Participant students working on the rotation task were given the following instructions: "When you rotate the ΔABC by 120º, it overlaps with itself. For which angle of rotation will each figure overlap?"



Translation 3



Rotation 6

Figure 8: A screenshot of the Task 3 of Translation and the Task 6 of Rotation tasks.

After using the GeoGebra file to explore the task and find a solution, the students typed their answers into the Word file. The expectation was that the students will realize that the scalene triangle has to rotate for 360º, while the two rectangles should rotate for 180º to overlap.

Each task, as well as the data collected from 17 participants in Turkey, was analysed by three raters, the authors of this paper. Through the constant comparison method we looked into the distinct features of the tasks and evaluated their level of visualization, dynamism, and exploration on the scale 0-4. About half way through the tasks, we noted that the categories repeat and that we have developed criteria to evaluate the three affordances of GeoGebra – or any other DIMLE tasks (see Table 1). Finally, we evaluated the remaining worksheets independently and rated them all.

Table 1. Rubrics for levels of visualization, dynamism, and exploration of the mathematics tasks

	None (0)	Low (1)	Medium (2)	High (3)	Very High (4)
Visualization	Since every task represented on the screen has a degree of visualization, we agreed not to vote this item by 0.	The task involves more than 2 non-visual representations (numeric or algebraic), and the non-visual one(s) surpluses visualization.	The task is performed by employing visual and non-visual approaches equally. No apparent surplus is anticipated.	The task demands mostly visual perception as well as visual approaches to get completed.	The task is completely visual so that no other representation or procedure is needed to get the task completed.
Dynamism	The task lacks dynamism completely. Could be performed by observation and mental consideration, using basic keyboard input.	The task has some degree of dynamism that is completely predetermined (e.g., using a slider), but static procedures and thinking are predominant.	It is hard to tell that either static or dynamic procedure(s) are dominant for this task.	The task is understood and performed in a series of dynamic actions, but still not completely. The user may perform dynamic actions to test the object OR search for a conjecture/answer.	The understanding and solution procedures demand dynamic actions only. The user must perform dynamic actions to test the object AND to search for a conjecture/answer.
Exploration	The task is completely straightforward, no exploration is necessary. There is a definite result that is based on the memorization, visual interpretation, or implementation of a basic calculation skill.	The task has some degree of exploration, but the majority of procedures are straightforward.	It is hard to define the predominant aspects of the task as either explorative or non-explorative.	The task is significantly explorative, however not completely. The user is still guided in some way.	The task is completely explorative such that the user has to explore the task while understanding and solving/performing it.

Results

The goal for the evaluation process was to understand to what extent the GeoGebra worksheets, which have been created for this study, afford users to do the mathematical tasks by constraining them to think or act visually, dynamically, and exploratively. Therefore, we separately evaluated each worksheet focusing on its visual, dynamic, and explorative potential. Reliability analyses for the evaluations from three raters were high: based on $N = 56$ cases (tasks with missing evaluations were excluded), for visual, dynamic, and exploration affordances Cronbach alpha was respectively .875, .862, and .901. We concluded that we had the high inter-rater agreement scores. After the scores from three raters were averaged, we calculated the descriptive statistics for each of the three task affordances (see Table 2).

Table 2. Descriptive statistics of affordance levels for $N = 65$ tasks (0 = "None" to 4 = "Very high")

Affordances	Min Level	Max Level	*M*	*SD*
Visual	1.00	4.00	2.43	1.01
Dynamic	0.00	3.33	1.05	0.96
Exploration	0.00	3.67	1.50	1.03

While the standard deviations for all three affordance levels was approximately around 1, their mean values differed: the average visual level of the tasks was medium to high, dynamic level was low, and exploration level was low to medium. One reason to get a high average on visualization could be the visual nature of the GeoGebra. We all agreed that whatever we create to use in this environment has to have some degree of visualization, therefore we rated the visualization of all tasks with a value of at least 1 (i.e., "Low"). Table 3 shows the list of GeoGebra tasks, organized by the topics covered in the first step of research, and how researchers evaluated them.

Table 3. Affordance levels of the tasks organized per topic

Topic	Affordance	*N*	*M*	*SD*
2D Geometry	Visual	7	3.23	.37
	Dynamic	7	1.19	.94
	Explorative	7	1.19	.86
3D Geometry	Visual	7	4.00	.00
	Dynamic	7	.43	.73
	Explorative	7	1.71	.99
Fractions	Visual	10	1.90	.80
	Dynamic	10	1.50	1.21
	Explorative	10	1.85	1.24
Miscellaneous	Visual	5	3.40	.43
	Dynamic	5	.73	.60
	Explorative	5	2.33	1.13
Patterning	Visual	8	1.50	.36
	Dynamic	8	.71	.76
	Explorative	8	1.25	.81
Problem Solving	Visual	6	1.00	.00
	Dynamic	6	1.56	.27
	Explorative	6	2.14	.58
Ratio and Proportion	Visual	6	1.83	.69
	Dynamic	6	1.44	.72
	Explorative	6	1.94	.57
Transformations	Visual	16	2.62	.47
	Dynamic	16	.92	1.13
	Explorative	16	.77	.90

DISCUSSION

After we evaluated the tasks and agreed upon their potential to afford visual, dynamic, and explorative thinking and acting, we developed definitions of the terms.

Visualization. In order to identify a task as completely visual, the task/problem should be perceived visually and every procedure should be based on visual perception. Geometry task may not be fully visual if it contains calculations.

Dynamism. The task should consist of some movement and demand some degree of thinking associated to this movement. The dynamic task asks for a co-action between the user and the software environment. Examples are: dragging objects, moving sliders, and using user-controlled simulations or animations, and transfer between different mathematical representations on the screen that are fundamental for the task.

Exploration. In exploration, neither result nor the procedure leading to the result should be obvious; therefore there must be some number(s) of unknown(s). The task should engage the user to put some degree of thought to understand and/or accomplish the task. Open-ended problems are highly explorative if not very high. Similarly, problems having more than one answer or no definite answer should be considered explorative because they have the potential to create some degree of cognitive conflict.

While the numbers of tasks in each topic category is too small to allow us to reach some generalizable conclusions, there are some outcomes of this research that may be of interest for the larger mathematics education community. First, our method enabled us to distinguish among the 100 possible permutations of levels (four levels for visual and five levels each for dynamism and exploration) the extremes—the tasks that are high in one affordance and low on another two. One example may be Task 3 in Translation, which is high in visual and very low in dynamic and exploration (Figure 2). Another important category would be the tasks that are very low on one and very high on the other two affordances, such as Task 6 in Rotation (see Figure 2), which is very high in Visual and Dynamic category, but low in Exploration. Given these distinctions, we have selected tasks for the next round of study, where we will observe and analyse how students use the tasks that we have found distinct on their level of visual, dynamic, and explorative affordances.

Moreover, there may be visible benefits of this type of analysis and discussion both in teaching profession and in teacher education. For teaching, it may help to restructure the curriculum to address the learning habits of net generation. For teacher education, it may help teacher to better understand their students. Reflecting on this taxonomy in teacher education course may engage teacher candidates to develop awareness on the suggested effects of technology and possible integration of DIMLE and Net Generation expectations.

ACKNOWLEDGEMENT

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GETTING MATHEMATICS TEACHING UP TO SPEED WITH THE BLOODHOUND SUPERSONIC CAR

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The Bloodhound Supersonic Car (SSC) attracted considerable attention as a show-piece of ICTMT10. Four years later the car is nearing completion ready for its initial trials in early 2015. Later in the year it will be taken to South Africa, where it is planned to reach supersonic speeds and break the existing world land speed record. By 2016 the intention is achieve a speed in excess of 1000mph. The primary objective of the Bloodhound project is to inspire the next generation with science, technology, engineering and mathematics. In this paper it will be explained how, just like the car itself, mathematical modelling of the car's motion can be progressively brought up to speed. Algebraic and numerical solutions to the equation of motion, with and without the use of technology, strongly motivate mathematical techniques studied at different university levels.

Keywords: engagement, applications, modelling, Maple, assessment

THE BLOODHOUND PROJECT

The Bloodhound Project (http://www.bloodhoundssc.com) is described as "a global engineering adventure, using a 1000 mph world land speed record attempt to inspire the next generation about science, technology, engineering and mathematics". During ICTMT10 at Portsmouth in the UK, the project director Richard Noble gave an invited talk about the quest for the land speed record and more significantly its underlying educational purpose. Appropriately it was near Portsmouth in 1956 that the world air speed record first exceeded 1000mph. A replica show-car was on display during ICTMT10 and, four years on, the car itself is now nearing completion in Bristol.

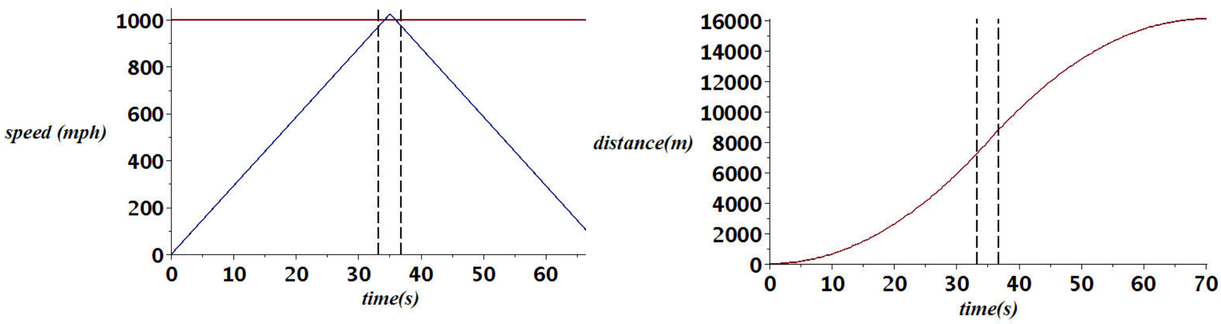
Figure 1. Bloodhound SSC Show Car at ICTMT10 with Project Students

Today Bloodhound can have the same impact as the Apollo space programme of the 60s and 70s on the enthusiasm for STEM subjects and the uptake of STEM-based careers. Early trials of the car will begin in early 2015, before it is taken out to Hakskeen Pan in South Africa later in the year to begin its initial attempt to break the sound barrier and the existing land speed record of 763 mph. The current plan is for the world record to be raised above 1000 mph in 2016, with education remaining the overarching goal.

Much of the educational activity has focused on primary and secondary schools with over 4000 schools registering within the first 18 months of the project. The Bloodhound@University Programme, led by the University of the West of England, Bristol in partnership with the University of Swansea and the University of Southampton, focuses on university education. The aim of the Bloodhound@University programme is "to enable academics, students and others working in universities to capitalize on the unparalleled access Bloodhound SSC offers to a live advanced STEM project". The K-Box (http://bloodhound.eprints.org) is an on-line repository for access to design data, teaching materials and sharing of ideas. Amongst these resources are project reports written by final year mathematics students at the University of Portsmouth and a past conference paper on the subject (Evans, Galloway and McCabe, 2012). In due course, test and actual car run data will become freely available for anyone wishing to use it. Students studying the first year Mathematical Models unit at Portsmouth can access associated Maple files from within Moodle.

CONSTANT ACCELERATION MODELS

University students are almost too familiar with the standard "suvat" equations for constant acceleration or deceleration. Nevertheless they do provide a good starting point and can be applied if constant thrust/resistance and mass are assumed. A typical problem is to find the maximum instantaneous speed, the acceleration/deceleration and time required for a Bloodhound run given a track length of 10 miles and an average speed of 1000 mph over the measured mile, assumed to be located between 4.5 and 5.5 miles. A calculator is sufficient to determine a maximum speed of 1026 (1050) mph, an acceleration/deceleration of ±13 m s^{-2} or ±1.3g (max 2.2g min -2.8g) and an identical time of 35 seconds for both acceleration (42s) and deceleration (58s) phases. The expected values based upon detailed models are shown in brackets, indicating a similar period of acceleration, but significantly different extreme values. When *calculations for several different average speeds or plots are needed, it is more convenient to use a CAS or other software. Figure 1 shows that* the maximum instantaneous speed exceeds the 1000 mph average for the measured mile. The times for the start and finish of the measured mile are shown as dashed lines



ation

Variants on the basic problem include:

- a difference between the magnitude of the acceleration and deceleration
- a period of constant speed, e.g. over the measured mile
- piece-wise constant acceleration/deceleration

Even at this simplest level, calculations become tricky and decisions already need to be made whether to use hand calculations or appropriate software. Furthermore students need to recognise that realistic models usually do not involve constant acceleration.

CONSTANT JERK MODELS

Suppose that the thrust of Bloodhound is not constant, but increases and decreases linearly at the same rate.

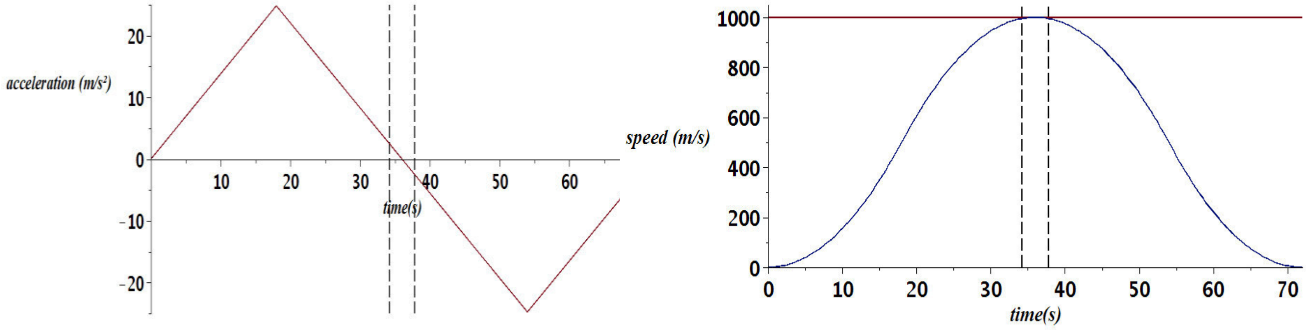


Figure 2. Maple Plots of Acceleration (left) and Speed (right) vs. Time for Constant Jerk

Figure 2 shows the Maple solution to the constant jerk problem. The solution method exploits the symmetry of the motion about the 5-mile halfway mark and solves for both the jerk (±1.38 m s^{-3}) and the times at which it changes sign (18s and 54s). The speed-time graph illustrates the very small (~1 mph) difference between the instantaneous maximum speed and the averaged speed over the measured mile. The distance s in metres travelled at time t is found as a piecewise function using the Maple dsolve command

$$s(t) = \begin{cases} 0 & t < 0 \\ \frac{23}{100}t^3 & t < 18 \\ -\frac{23}{100}t^3 + \frac{67068}{25} - \frac{11178}{25}t + \frac{621}{25}t^2 & t < 54 \\ \frac{23}{100}t^3 - \frac{1743768}{25} + \frac{89424}{25}t - \frac{1242}{25}t^2 & t < 72 \end{cases}$$

The problem can be further extended by allowing a period of constant speed or piecewise constant jerk. Other similar problems with non-constant jerk can be set up by assuming that the acceleration has different, easily represented profiles, e.g. sinusoidal. These idealised problems, although mathematically interesting, are physically unrealistic, so the next step is to consider some of the physical factors governing the motion.

WIND RESISTANCE MODELS AND JET PROPULSION

The equation of motion for the car including wind resistance f(v) can be written in the form

$$m\frac{dv}{dt} = T(t) - f(v)$$

where T(t) is the thrust at time t. Before developing a full numerical model for the Bloodhound car, it is useful to find simple analytic solutions. Apart from being of educational value, they can be used to validate more accurate models. At low speed the (viscous) wind resistance is roughly proportional to speed i.e. f(v) = cv . Assuming $v(0) = 0$ and a constant thrust.

$$v(t) = \frac{T}{c}(1 - e^{-ct/m})$$

Bloodhound quickly achieves high speeds, reaching 200 mph in around 20 seconds, so that wind resistance is generally better approximated as $f(v) = cv^2$. Again assuming $v(0) = 0$ and constant thrust, the solution is

$$v(t) = \sqrt{\frac{T}{c}}\, tanh\left(\frac{\sqrt{cT}t}{m}\right)$$

Both results, obtainable by separation of variables or integrating factor methods, are familiar to any 1st year university student. Assuming the mass of the Bloodhound car to be a constant 6422 kg and maximum jet thrust of 90 kN for 50 seconds, the acceleration, speed and distance can be calculated.

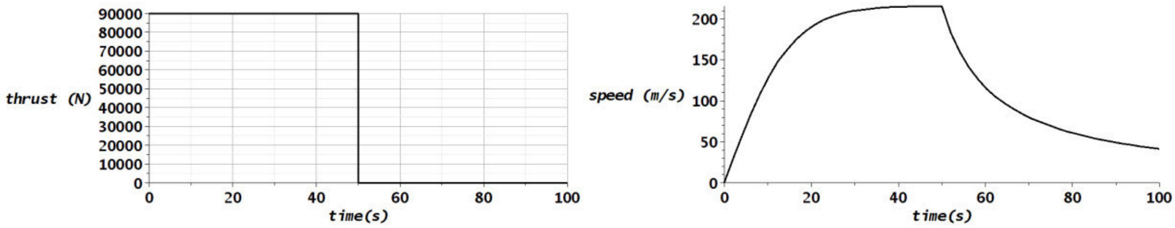


Figure 3. Maple Plots of Thrust (left) and Speed (right) vs. Time with Wind Resistance ∝ v^2

Simple calculations of this type can be used to help answer important questions. Why does Bloodhound not operate like Thrust 2, the car which holds the existing land speed record, and run on jet power alone? What speed could Bloodhound reach by jet power alone? What would happen to Bloodhound if its braking systems failed? Could the car overshoot its prepared track?

VARIABLE MASS MODELS AND ROCKET PROPULSION

Originally Bloodhound was to be purely rocket propelled, but a rocket cannot be controlled easily. Bloodhound therefore uses both jet and rocket propulsion to combine control and power. Unlike jet power, rocket power does not require air intake, since it includes its own oxidant. Hence it does not require such a large increase in the cross-section of the car which leads to lower air resistance. The rocket has a maximum thrust of 122 kN, with an average thrust of 111 kN over 20 seconds of firing. Suppose the car is propelled by rocket power alone, then the ideal rocket equation, arising from the conservation of linear momentum, can be written in the form

$$\mathrm{m(t)}\frac{\mathrm{dv}}{\mathrm{dt}} = -\mathrm{v_e}\frac{\mathrm{dm}}{\mathrm{dt}}$$

where m and v are the mass and velocity of the car. v_e is the velocity of the exhaust relative to the car. If v_e is constant, then this is integrated to give the Tsiolkovsky rocket equation

Increase in velocity $\Delta v = v_e ln\left(\frac{m_i}{m_f}\right)$ where m_i = 6422 kg and m_f = 5292 are the initial and final mass of a purely rocket powered Bloodhound car, based upon a lost propellant mass of 1130 kg. The Bloodhound rocket has a specific impulse of ~200 s, equivalent to v_e = 2000 m/s at sea level. Neglecting air resistance and friction, the top speed of Bloodhound would then be $2000 \ln\frac{6422}{5292}$ = 865 mph.

Despite its assumptions, such simple calculations are useful in understanding what is happening, for validating more detailed models and for posing further questions. What propellant mass fraction $1 - m_f/m_i$ would be required for a top speed of 1000 mph? How fast would Bloodhound go by rocket power alone, if air resistance and friction were not neglected? How high would Bloodhound travel if it was launched vertically like a conventional rocket?

Full understanding of Bloodhound rocket performance requires knowledge of both physics and chemistry, but the 1-D dynamics can be calculated by using estimates for the rocket thrust. A typical best estimate is a linear increase from 0 to 122kN over 6 seconds, then maintenance of full thrust for the next 14 seconds. Bloodhound rocket-alone calculations are most easily performed by setting the maximum jet thrust to zero in the detailed numerical model of the next section.

DETAILED NUMERICAL MODELS

We have developed a Maple model for the 1-D dynamic motion of Bloodhound described by the equation of motion

$T(t) - \frac{1}{2}\rho\, C_d(t)\, A(t)\, v(t)^2 - F_b(t) = M(t)\left(\frac{d}{dt} v(t)\right)$ where $x(t)$ is the position of the car at time t, $v(t) = \frac{d}{dt} x(t)$ is the velocity and $a(t) = \frac{d}{dt} v(t)$ is the acceleration.

The thrust $T(t)$, the mass $M(t)$, the cross-sectional area including the airbrake $A(t)$, the drag coefficient $C_d(t)$, and the foot brake force $F_b(t)$ are all dependent on the time t from the start of the run. Typical parameters used in the model are:

Initial car mass = 6422 kg Jet fuel mass =500 kg Rocket fuel mass = 1000 kg

Jet fuel burn rate = 12 kg s^{-1} Rocket fuel burn rate = 50 kg s^{-1}

Jet shutdown time = 40 s Rocket start time = 22 s Rocket shutdown time = 42 s

Maximum jet thrust = 90 kN Maximum rocket thrust = 122 kN

Carbon foot brake on time = 85s Carbon foot brake force = 100 kN

Initial X-section = 1.5 m^2 X-section with air brake = 5 m^2 X-section with 'chute 1 (2) = 10 (25) m^2

Airbrake start (finish) deployment time = 54 (59) s

'Chute 1/2 start (finish) deployment time = 59/64 (64/70) s

Air density at sea level = 1.225 kg m^{-3}

Minimum (maximum) drag coefficient C_d = 0.77 (0.88) Time at which C_d is maximum = 45 s

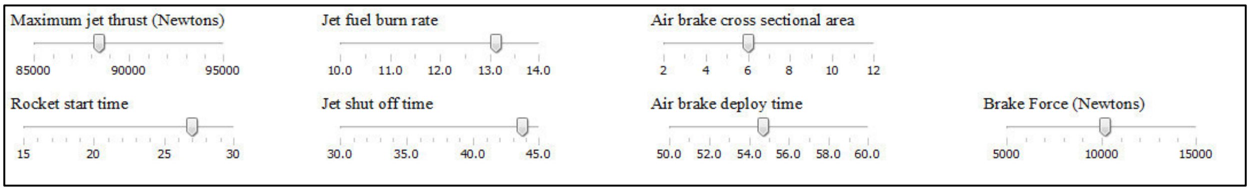


Figure 4. Sliders Used to Adjust Some of the Model Input Parameters

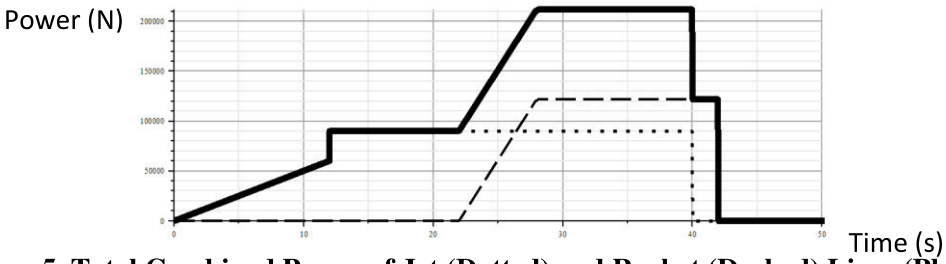


Figure 5. Total Combined Power of Jet (Dotted) and Rocket (Dashed) Lines (Philo, 2012)

The Maple command DynamicSystems[Simulate] is used to solve interactively for the motion of the car and a sample solution shown in the following Figure 6

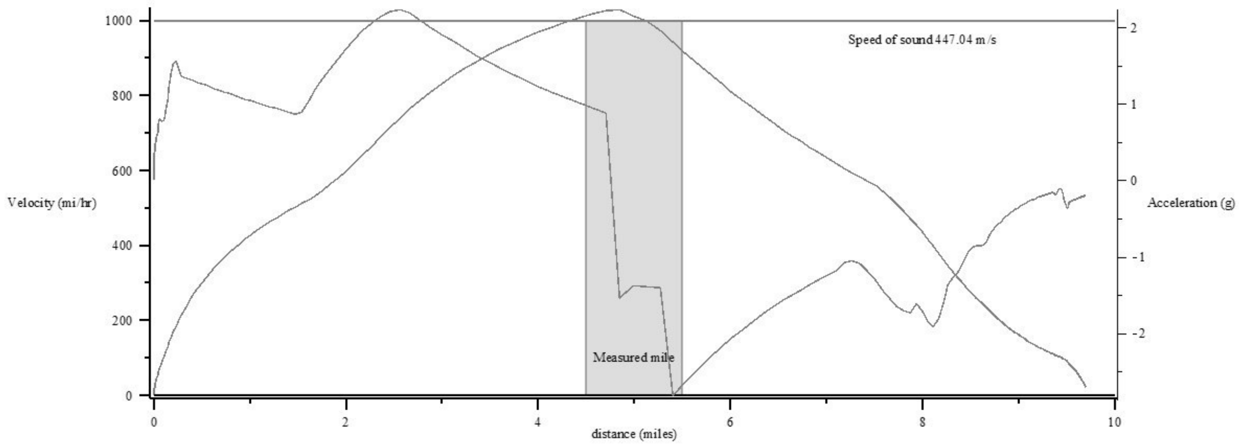


Figure 6. Bloodhound Velocity and Acceleration Calculated by the Maple Model

PEDAGOGICAL DEVELOPMENT

Applied mathematics courses can easily become outdated, if their applications are not interesting and topical. Lecturers and teachers can add or extend "bookwork" examples, but only if there is sufficient notice for their preparation. There are often predictable events, such as the Olympics, elections or eclipses, for which teaching can be planned in advance. The Bloodhound project provides a case study for how the preparation of teaching resources for topical events can progress:

2008	Bloodhound Project launched
2010 – 2015	Final year undergraduate projects including using Maple modeling
2014/15	Inclusion of learning and assessment resources within Year 1 Mathematical Models
Late 2015/16	Bloodhound runs in UK and South Africa expected

The initial preparation of resources, especially Maple solutions, was achieved by working with final year project students, who could both develop and trial the worksheets. A key element was in gradually building up the complexity of the model by relaxing assumptions about the motion:

application of basic 'suvat' equations ® hand solution of ODE by separation of variables ® analytic solution of ODE using Maple® numeric solution of ODE using Maple

The aim was to introduce the resources to first year students, who are often reluctant to engage with computer algebra systems or cannot see when they are necessary. They can quickly see the benefits of the technology when they learn that simple equations can lead to complicated solutions.

Figure 7 shows an algebraic Maple solution to an approximate equation of motion, which illustrates this point.

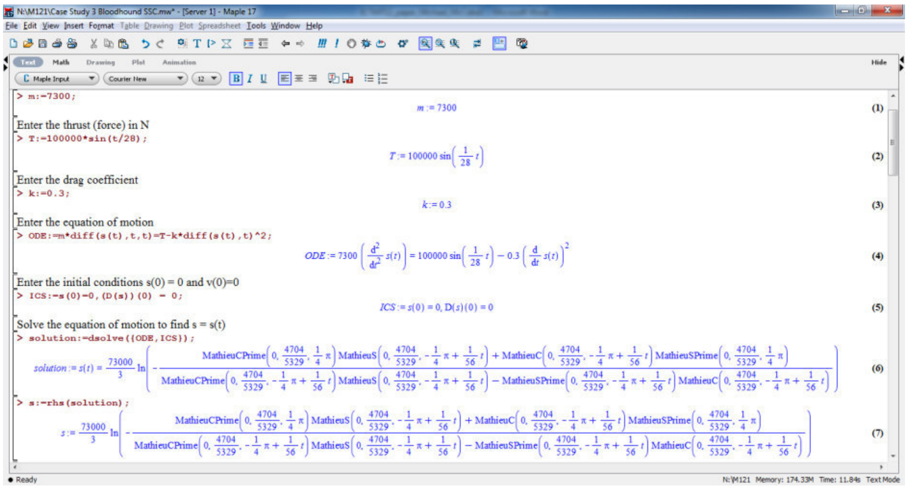


Figure 7. Analytic Solution to an Approximated Equation of Motion

The online testing system MapleTA has been used for assessment throughout the "Mathematical Models" unit. Students are expected to answer most of their weekly questions using pen and paper, such as the mechanics question on the left of Figure 8. Some questions though are deliberately made too difficult to solve by hand, such as the Bloodhound example of the right of Figure 8, which requires the Maple worksheet shown in Figure 7.

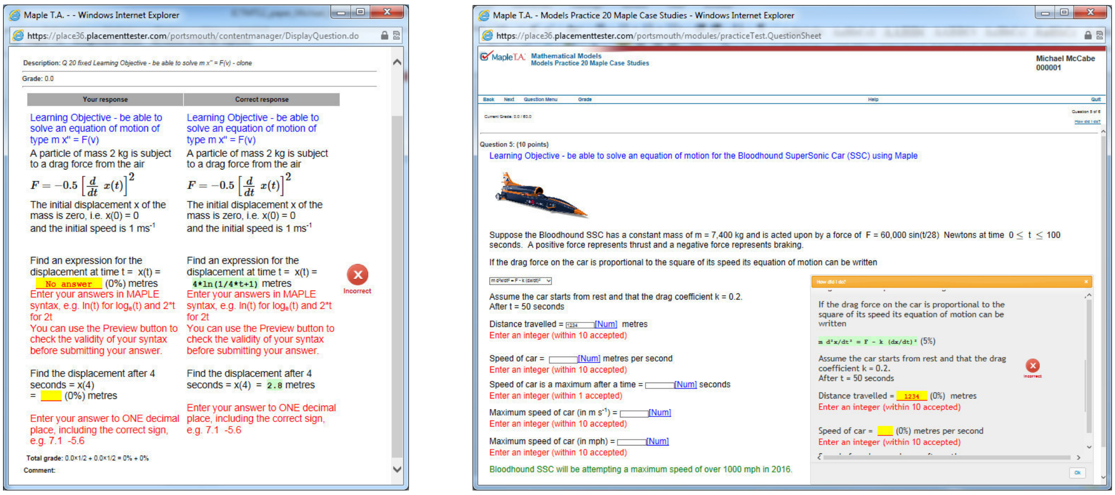


Figure 8. MapleTA Assessment: Written Question (left) Maple Question (right)

For the harder CAS questions, students are generally provided with a template Maple worksheet, which they are required to complete in order to generate a correct solution. A very useful feature of MapleTA is the "How Did I Do" link which can be seen at the top, right of the screen. The link allows students to check through their practice answers and get feedback as they work progressively through an extended problem. Another MapleTA feature is the ability to deliver randomly generated questions, which each student is required to complete as a coursework over a 24-hour period. By delivering tests in this mode, it is found that students are motivated to work harder, longer and gain higher marks than in a conventional, time restricted examination. While an exam is appropriate as summative assessment of written questions, an extended coursework period is much more suited to the answering of CAS problems.

FULL SPEED AHEAD!

Since Bloodhound SSC has not yet been completed, there is no data available yet from trial or record runs to compare with the model predictions. In 2015 the first real data from trial runs is expected to become publicly available, prior to the culmination of the project in 2016. This will allow the comparison of models with measurements and the better fitting of model parameters.

Without data, the models can still be used to explore a wide range of "what if?" problems. What if the air brakes, parachutes or friction brakes fail? What if the steering of the car wobbles or veers off course? What if the timing of jet or rocket power is adjusted for trial or record runs? What effect does wind have on the motion of the car? What if Bloodhound were launched upwards? What if Bloodhound were driven on another planet or moon?

This paper has demonstrated how Bloodhound run profiles can be calculated by solving simple equations assuming constant acceleration or jerk, by solving simple differential equations for its motion with variable air resistance or car mass and by developing a more complete numerical model. While simpler algebraic calculations can be performed by hand, we have used Maple to calculate, plot and animate the 1-D motion of the car. Just as Bloodhound will be tested at progressively higher speeds, its underlying mathematics can be built up gradually. Although the design of the car involves advanced mathematics, from the educational perspective it is valuable to use mathematics from all school and university levels. Whether Bloodhound succeeds in reaching 1000 mph or not, its ultimate legacy will be in STEM education (NFER, 2009)

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TEACHING MATHEMATICS WITH AN INTELLIGENT SUPPORT: A STUDY WITH PARAMETERIZED MODELING ACTIVITIES

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We report outcomes from a study that aims to investigate the role of feedback, by way of an intelligent support system, in parameterized modeling activities carried out by a group of tertiary education students. With such a system it is possible to simultaneously display on a computer screen a chat window and a window with a microworld, dynamically hot-linked to each other. While users work in the microworld, they can enter into dialogue with the system in natural language. In this paper we discuss the case of one pair of participant students, for whom the feedback provided by the intelligent support and by the microworld at key moments of the modeling activities were crucial for them to be able to build up a spreadsheet model and consequently, for their understanding of the long-term behavior of the phenomenon being modeled.

Keywords: Intelligent support, feedback, parameterized mathematical modeling,

INTRODUCTION

Over three decades of research have shown the potential of digital environments as a favorable means of learning and teaching mathematics. Yet, that research has also shown that in order for the potential to be effective, significant and sustained pedagogical support from the teacher is necessary. It is difficult to fully meet the latter requirement given that use of digital technology fosters an exploratory attitude among students, which leads to deployment of diverse strategies in class to carry out an activity. As a result, it becomes humanly impossible to provide timely and specific feedback in all cases. Moreover, whilst the majority of computer programs that are the basis of the so-called digital learning environments (micro worlds) comprise, in their very design, means of feedback aimed at helping students to understand mathematics concepts, methods or properties, said feedback does not always respond to the specific needs of the students. Recent research has dealt with this problem, incorporating an intelligent system into micro worlds. This is the case of the Migen project of the London Knowledge Lab (http://www.lkl.ac.uk), in which such a system for the *eXpresser* micro world [1] has been developed. In the study "Intelligent Dialogues with tertiary education and university students" [2] that we report here, an intelligent support was incorporated to provide feedback to the ideas and actions of students while they worked with parameterized mathematical modeling activities. In this research the feedback is based on dialogues (using natural language) between the system and its user(s). To accomplish the latter, we resorted to the software *Dialogue with Theo* (DiT) that was recently developed with the *Descartes* author system (http://recursostic.educacion.es/descartes/web/) [3]. With DiT, it is possible to simultaneously display on a computer screen (or another device) a chat window and a window with a micro world, both of which are dynamically hot-linked, as shown in Figures 1, 2 and 3 bellow. It is using this tool that students can work in the micro world on specific tasks and enter into dialogue with the system in natural language. The feedback is displayed in two ways: through the dialogue and through the micro world.

The main purpose of the study was to analyze the role of the feedback system while the pupils worked with the modeling activities in a spreadsheet micro world, in three different modalities, namely: *presential-introductory*, *presential with the support of a researcher*, and *remote*. In this article we report on the findings of the experimental work in the *presential with the support of a researcher* modality. Design principles and development of DiT were presented at the *2012 Constructionism* conference (Rojano and Abreu, 2012).

BACKGROUND AND THEORETICAL COMPONENTS OF THE STUDY

Design of the targeted feedback for parameterized modeling activities is based on the findings of the Anglo-Mexican project "The role of spreadsheets within the school-based mathematical practices in sciences" [4], in which students were enrolled in activities of that type, working in pairs on spreadsheets without intelligent support and with the participation of a researcher. The findings made it possible to identify, for each particular activity, critical modeling moments, such as moments of *prediction*, *validation*, *verification* and *generalization* (Molyneux et al., 1999). Taking identification of those moments into account, DiT feedback from the intelligent system was conceived to help students understand the phenomenon, verify their predictions (frequently formulated as of intuition), verify their responses and validate and generalize the models that they had built. It is important to make it clear that the DiT interaction does not correspond to the type of feedback that used to be provided in programmed instruction systems because the logic of the feedback tries to respond to the complexity of the logic of the answers or actions of the subjects who interact with dynamic representations of concepts and phenomena.

The micro world activities were designed by adopting the theoretical perspective of *learning with artificial worlds* proposed by Mellar & Bliss (1994), in which mathematical modeling consists of creating an artificial world with the characteristic that all of its components are known, given that when it is built one decides what its components (variables) should be. In each modeling activity, the students can, in an artificial world, express their own ideas (expressive modeling) or explore the ideas of others (exploratory modeling) about a phenomenon or situation.

In relation to the dialogue mode, the DiT design was inspired by what some authors have hypothesized about natural language, potentially being an important factor for feedback and learning (Litman, et al., 2009). These authors assert that use of interaction with this language contributes to improving the results of learning, specifically learning with computer systems.

With respect to the types of feedback provided during teaching and learning processes, different categorizations and taxonomies have been elaborated and used. Mavrikis et al., (2013) formalized three types in their support system (interruption, learner interactivity, system interactivity), which were the basis for the design of feedback strategies. Other authors (Segedy, 2014) constructed a taxonomy for adaptive scaffolds, which incorporates cognitive and metacognitive aspects. In our study we used open questions, multiple choice items and prompts to provide feedback within the chat window, whereas in the microworld window, we resorted to different mathematical representation systems to scaffold students learning. In both cases, feedback was provided according to the different modeling stages considered in the activity design.

THE STUDY WITH PARAMETRIZED MODELING IN DiT

Modeling activities using spreadsheets correspond to a version of the model in terms of the variation of parameters involved and no background in advanced mathematics is required for using them whilst analyzing modeling phenomena. In a spreadsheet it is possible to change the numerical values of the parameters and observe the immediate effect in the graphic and variation table representations, hence making it possible to analyze many cases with a simple numerical change in a cell and, consequently, the possibility of observing patterns of behavior and of generalizing properties (Molyneux, et al., 1999). In our study, two types of activities were analyzed in the intelligent dialogue modality: **a)** exploratory modeling activities, in which the model is provided (the Excel formulae have already been introduced to generate the variation tables) and the students explore different aspects of the phenomenon, changing a few parameters and analyzing graphs and variation tables; and **b)** expressive modeling activities, in which the students build the model by formulating their own Excel formulae and verifying their validity, in addition to using the model built to analyze the behavior of the phenomenon. So far the following modeling activities have been developed in DiT: *Pollution of a lake, Population growth, Molecular diffusion in a cell,* and *Non-crossing matchings* (arquimedes.matem.unam.mx/dialogues). Figure 1 depicts a scene of the expressive modeling activity in DiT *Molecular diffusion in a cell*, in which the screen displays a simulation of the phenomenon that consists of considering a simplified cell (in two dimensions) with six compartments, which outer walls are impermeable, although the internal membranes between each two compartments do allow molecules to enter and exit. The text in Figure 1 describes what we just said above, and the behavior of the phenomenon at t=0 and t=1.

At time t=0, there are 1200 molecules in the first compartment, and in each time unit the molecules move in the four directions with the same probability. The latter means that at t=1, one fourth of the molecules move to the second compartment and ¾ (i.e. 900 molecules) remain in the first one; at t=2, one fourth of the molecules move from the second to the third compartment and one fourth go back to the first, and from the first compartment, another fourth move into the second compartment, and so on and so forth. The students must learn the behavior of the phenomenon, build the Excel formulae that generate, for each unit of time, the number of molecules in each compartment and predict what the long-term situation will be.

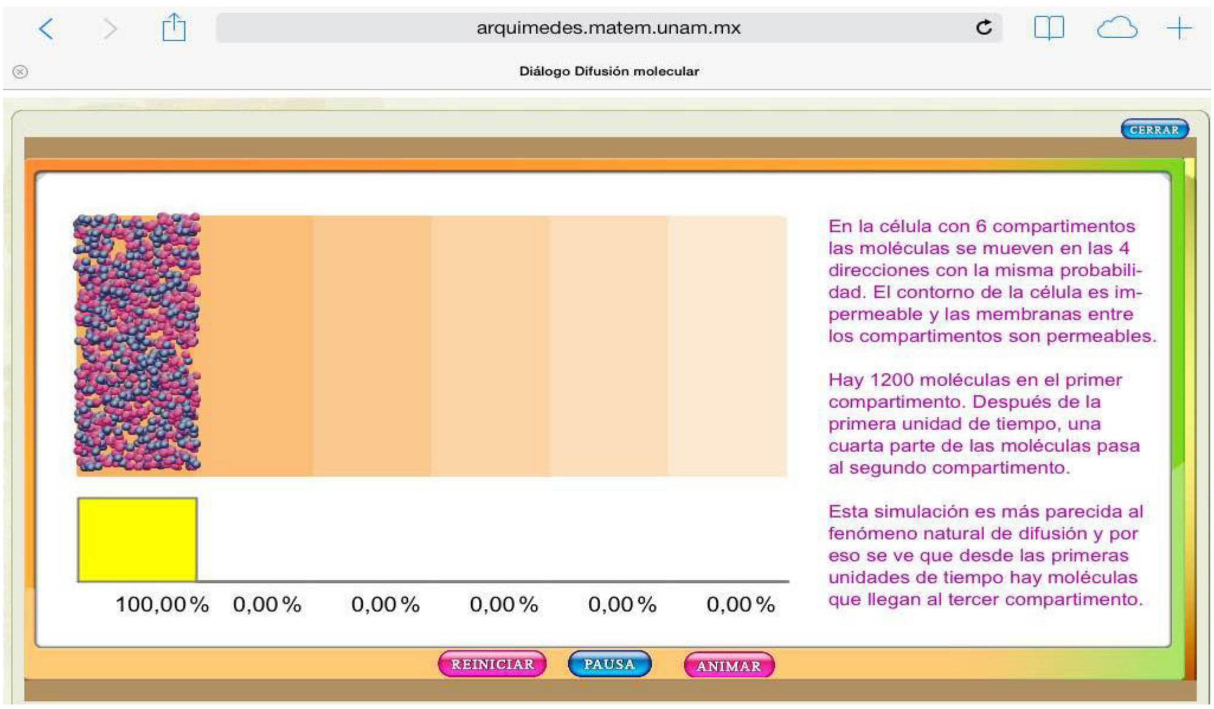


Figure 1: Screen with simulation of molecular diffusion in a cell.

METHODOLOGY

As we mentioned in a previous section, the general purpose of the study was to investigate the role of feedback, by way of a system of intelligent support, in parameterized modeling activities of phenomena of the physical world, carried out by tertiary education students in a spreadsheet environment. Specific interest lay in analyzing the role of said feedback at critical modeling moments, such as: *understanding the phenomenon* being modeled; *building up the model*; and *predicting the phenomenon's behavior*.

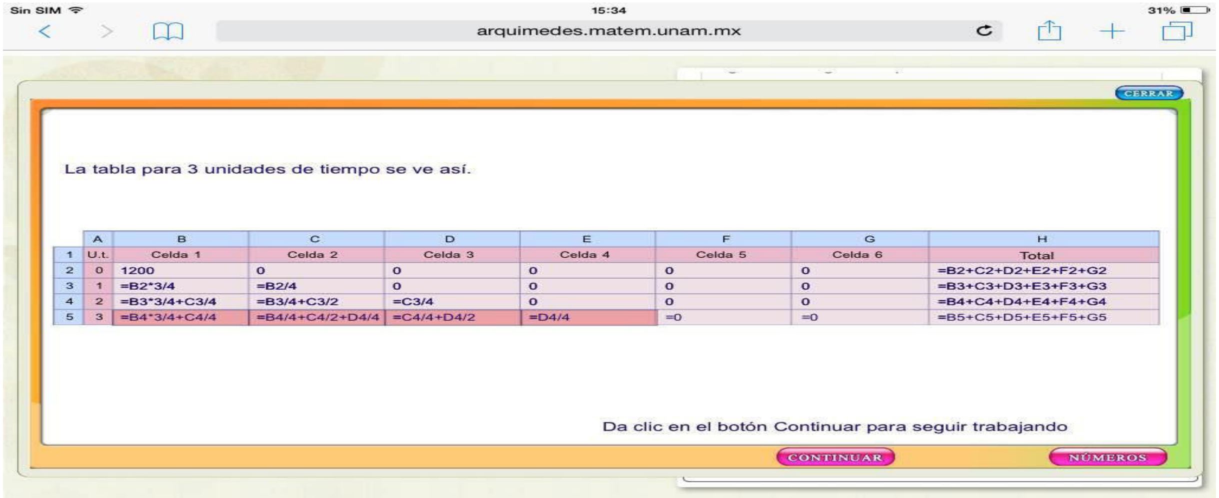
A group of six students of 16 and 17 years of age enrolled in the natural sciences school section voluntarily took part in the study, working on the modeling activities of *Pollution of a lake* and *Molecular diffusion in a cell*. The students were asked to answer a questionnaire before the experimental work in order to get information about their skills in reading and interpreting variation graphs and tables, given that both of the latter are requirements for being able to perform the activities. The sessions with the participant students were undertaken in the following modalities: 1) *Presential – introductory*, in a computer lab, in which the researcher explained the basic characteristics of the phenomenon of the exploratory activity of *Pollution of a lake* to the students and intervened when they so requested; the students worked in pairs on the activity on tablets with intelligent support. 2) *Presential –with support of the researcher*, in which the students took turns in pairs working on a PC with the expressive activity of *Molecular diffusion in a cell*, with interventions of the researcher at the request of the students. In these two modalities, the intelligent system and the researcher's interventions were complementary. The sessions were video-recorded and students' productions on paper were collected. The data of three pairs of pupils were analyzed, and in this article, outcomes from the second mode session with one pair of students are discussed. We chose to present the case of Monica and Leticia (M&L), since while they worked during this session, the three critical modeling moments emerged in a clear way and the relevance of the feedback provided by the DiT system at such moments became evident. In a subsequent stage of the project, students will work with a revised version of the foregoing activities in an individual remote work modality, with no researcher intervention. Moreover, an automatic record and data filtering system will be used.

THE CASE OF MONICA AND LETICIA (M&L)

In the list of questions and suggestion foreseen in the *Molecular diffusion in a cell* activity, those corresponding to three types of feedback were identified, as follows:

Helping to understand the phenomenon: 1) The question concerning what will happen to the 900 remaining molecules at time t=1 is asked for the purpose of drawing the user's attention to the fact that said amount remains in the first compartment, given that the outer walls of the cell are impermeable, that is that no molecules can either enter or leave through those walls. 2) The question that deals with what happens at time t=2 is asked for the purpose of having the user note that, in addition to ¼ of the molecules moving from compartment 2 to 3, ¼ also return to compartment 1. *Building the model*: The suggestion that users fill in the line of the table (see Figure 2) for the number of molecules at t=3 for the first three compartments is made for the purpose of having users write out the general formulae needed to calculate those quantities, be that by analyzing the behavior of the phenomenon or by observing the pattern of the formulae displayed in the line that corresponds to t=2 (see Figure 2).

Predicting: The (multiple choice) question concerning which graph a) or b) represents the long-term behavior of compartments 1 and 6 is for the purpose of having the user focus on the global and long-term behavior of the phenomenon (see Figure 3).



	A	B	C	D	E	F	G	H
1	U.t.	Celda 1	Celda 2	Celda 3	Celda 4	Celda 5	Celda 6	Total
2	0	1200	0	0	0	0	0	=B2+C2+D2+E2+F2+G2
3	1	=B2*3/4	=B2/4	0	0	0	0	=B3+C3+D3+E3+F3+G3
4	2	=B3*3/4+C3/4	=B3/4+C3/2	=C3/4	0	0	0	=B4+C4+D4+E4+F4+G4
5	3	=B4*3/4+C4/4	=B4/4+C4/2+D4/4	=C4/4+D4/2	=D4/4	=0	=0	=B5+C5+D5+E5+F5+G5

Figure 2. Excel formulae table for t=0, 1, 2 and 3 of the Molecular Diffusion model, displayed in the micro world window. Columns B to G represent 'compartment 1' to 'compartment 6'. And columns A and H represent the 'units of time' and 'the total amount of molecules' respectively.

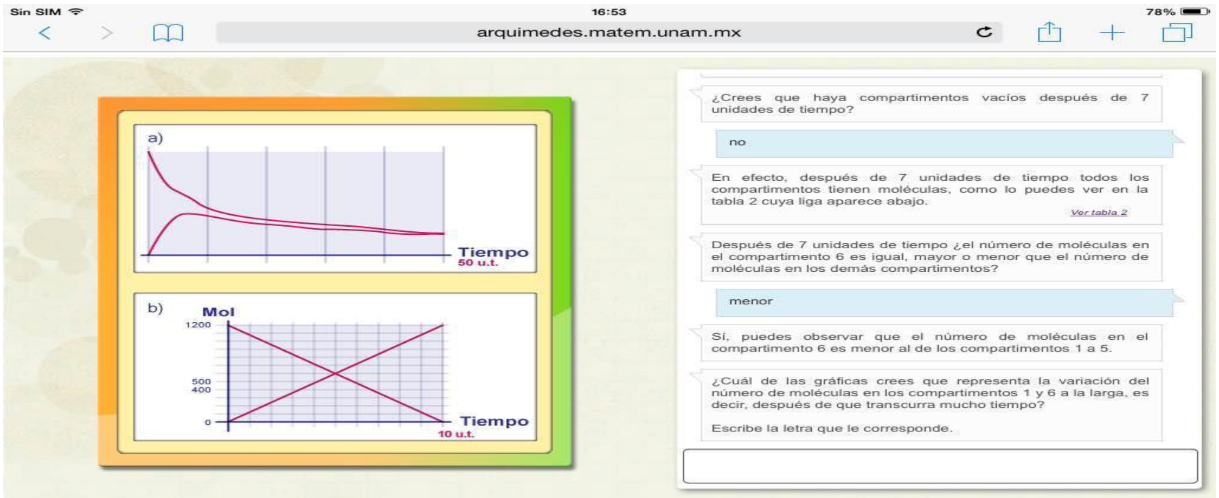


Figure 3: Chat and micro world windows (Choosing graphs for compartments 1 and 6).

In the analysis of M&L performance we make reference to the former three types of feedback.

Understanding the phenomenon. At the beginning, M&L had a very hard time understanding the behavior of the phenomenon, despite the fact that the system shows a simulation, an explanatory text and a dynamic bar chart (Figure 1). The latter was a distraction for M&L at this stage, specifically in terms of their ability to respond to the dialogue question of "what will happen [at t=1] with the remaining 900 molecules?" Afterwards, M&L had a very long discussion because they were unclear as to whether the question referred to t=1 or to later units of time. When they answered (mistakenly) 675 to the second question, namely: "After two units of time, how many molecules do you think will be left in the first compartment?" (their answer was a logical one, nevertheless ¼ of the 300 molecules return to compartment 1) the system feedback enabled M&L to realize that ¼ of the molecules of the second compartment returned to the first, hence 750 was the correct answer to the question. At this point M&L were able to better understand the behavior of the phenomenon, in qualitative terms. It would only be in the next stage (building the model) when they would also have to show their understanding in quantitative terms.

Building the model. M&L devoted close to 40 minutes to this stage, in which they were asked to enter the Excel formulae that correspond to t=3. For this task the students have the option of writing the formulae by bearing the displacement of the molecules in mind or by observing the pattern of the formulae for t=1 and t=2 in the table that is displayed on the screen in the micro world window (see Figure 2). M&L decided to work in Excel, outside of the DiT system, where they copied the formulae from Table 1. After that, they went through a combined reasoning, generating numerical data by doing calculations with a calculator and writing formulae that they entered in the cells. Spontaneously, M&L checked the validity of their formulae, going to the numerical version of the spreadsheet and verifying that for t=3 the total number of molecules (1,200) was kept; this automatically appears in the corresponding cell if the formulae are correct. It was this feedback in the micro world window that was effective in M&L's case at this stage of building the model.

Predicting. When the graphs as shown in Figure 3 are displayed in the micro world window, a question appears in the dialogue window, namely: "Which graph do you think represents the long-term variation of the number of molecules in compartments 1 and 6?" With their previous discussion and analysis of the Excel table that they had generated, M&L gave the correct answer: a). At this point, the system's only feedback consists of explaining why the correct option represents the long-term behavior of both compartments. After which another prediction question appears, that is: "What happens with the number of molecules in the other compartments after a long time has elapsed?" M&L typed a partially correct answer: "it drops", since it was clear that in the first compartments the number of molecules declines and it increases in the latter ones until the quantity stabilizes at around 200 in all of the compartments. The feedback invites the user to observe compartments 2, 3, 4 and 5 graphs. And finally, the following question appears in the dialogue window: "After how many units of time do the compartments begin to have approximately the same number of molecules?" At this point, the micro world window gives users the possibility of manipulating the graphs of all of the compartments so that they can visualize the tendency on the long-term. M&L answered "198" and the system gave a positive feedback. In this episode, the feedback in DiT is essentially of a graphic and visual nature. Although for some of the questions M&L preferred to resort to an analysis of the numerical variation tables, in general terms the feedback provided enabled them to verify their predictions and, where appropriate, rectify them.

DISCUSSION OF RESULTS

In short, in the case of M&L in the episodes of *understanding, building* and *predicting*, the feedback foreseen in the system for the students' answers was pertinent and crucial for them to be able to continue moving forward with the activity. In particular terms, in the *building* episode, the fact that the Excel table had a numerical verifier so that for each line (of each unit of time) the sum of all the molecules had to be 1,200, on the one hand, and that the right question was simultaneously posed in the dialogue window, on the other, were determining factors that enabled the students to draw up the formulae that describe the variation of the number of molecules for t=3 and to validate them in a recursive process (from the formulae to the numerical and vice versa).

The foregoing speaks of the powerful nature of specific and timely feedback during the most complex stage of the activity, to wit: the expressive stage or the stage at which M&L built the spreadsheets model. In general, this shows the power of having the two windows (the micro world and the dialogue windows) dynamically linked to one another in such a way that the feedback given to user actions or verbal responses appears in both worlds. This also make it possible to assert that the role of intelligent support in the form of dialogue was a meaningful complement to the feedback in the micro world and, hence, confirm what Litman and co-writers hypothesized. What is more, one can say that the case presented here is added to the evidence reported that states that a higher percentage of contentful talk is correlated with higher learning gain (Litman, et al., 2009) and that achieving self-explanation among students also significantly enhances learning (Chi, et al., 1994).

FINAL REMARKS

Although M&L could successfully complete the modeling activity with the help of the feedback provided by the intelligent support system DiT and by the micro world window, it was not the case of all the participant students. Other cases were analyzed in which researcher substantial intervention was required at different moments of the activity. Particularly, such an intervention was critical during the *build the model* stage. The foregoing represents a challenge for the research team with regards the need to improve the feedback system in preparation for the *remote* modality stage of the project, in which students will work without presential support from the researcher.

Notes

[1] The *eXpresser* micro world was designed to help students with mathematics generalization processes. Inspired by the robotics and adaptive systems methodology, the Migen project used a "layers" approach to develop and assess the intelligent support system for *eXpresser* (Gutierrez-Santos et al., 2010).

[2] *Intelligent Dialogues with tertiary and university education students* is an on-going 3-year research project, funded by the National Council of Science and Technology (Conacyt) in Mexico, Reference No. 168620.

[3] *Descartes* is an open source Authoring Tool for interactive mathematics resources developed by the Spanish Ministry of Education, with the participation since 2009 of the Institute of Mathematics of the National University of Mexico (UNAM) and the Laboratory of Innovation in Technology and Education (LITE) in Mexico.

[4] Anglo-Mexican Project developed in collaboration with the Institute of Education of the University of London and the Department of Mathematics Education of Center for Research and Advanced Studies (Cinvestav) in Mexico, funded by the Spencer Foundation of Chicago, Ill, Grant N. B-1493.

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TANGRAM, TEACHING AND TECHNOLOGY

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In this paper we will review how Tangram, as a classic puzzle with only three different basic shapes, can be used to teach a variety of mathematical concepts, such as similarity, self-similarity, fractal dimension and binary numbers. We describe activities for young teenagers as well as for adults that can be carried out on the internet as well as in the classroom. We will relate to a collaborative article written by the users of a math internet site and will mention how we used this puzzle in a computer course for maths teacher trainees.

Keywords: Web-based learning, collaborative exploration, tangram, self-similarity

INTRODUCTON

Tangram is a classic puzzle made from seven pieces which are sometimes also called "tans" (figure 1). Five of these seven tans are similar triangles, so that there are only three different basic shapes: an isosceles right triangle occurring in three different sizes, a parallelogram and a square. The various shapes show only three different angles: 90° (a right angle), 45° (half of a right angle) and 135° (the sum of the other two angles).

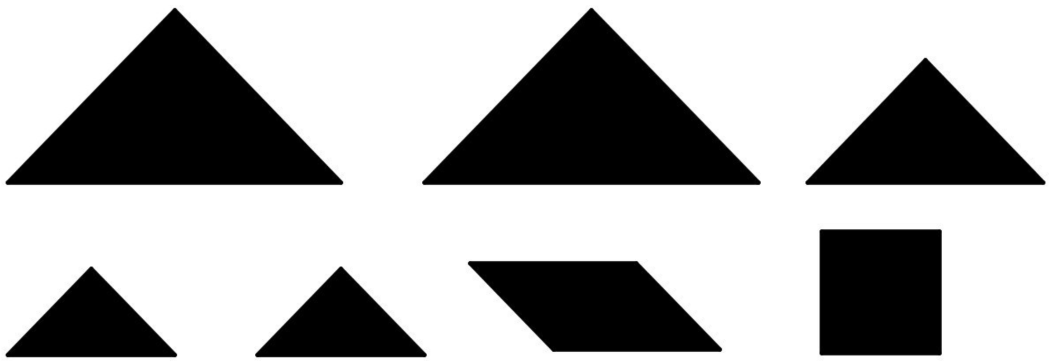

Figure 1: A full set of seven tangram pieces

It has been suggested that tangram can be used to guide students through the various levels of learning geometry as suggested by the 5th property of the van Hiele model: information or inquiry, guided or directed orientation, explicitation, free orientation and integration (Siew, Chong, & Abdullah, 2013).

The task to put together given shapes belongs to guided or directed orientation.

New questions that can be raised to trigger free orientation are:

Try to put together convex shapes. Which is the convex shape with the smallest number of meeting points between vertices? Which is the convex shape with the largest number of meeting points between vertices?

Which are the different numbers of tangram pieces of which you can shape a right, isosceles triangle? The answer "1" is obvious, "2" is easy, but there are more. What is interesting is the correct answer "7", since it means that a tan can be constructed from tans.

SIMILARITY AND SELF-SIMILARITY

The three different triangles of the tangram game are similar to each other: the hypotenuse of the largest triangle is by the factor $\sqrt{2}$ larger than the hypotenuse of the medium triangle. The hypotenuse of the medium triangle is by the same factor larger than the hypotenuse of the smallest triangle.

Indeed, each of the three different shapes of the tangram game can be reproduced from all pieces of the game as illustrated in figure 2. Continuing the process of replacing tangram pieces by appropriate arrangements of sets of smaller tangram pieces, we can create self-similar tans, at least theoretically. In practice we have to stop the process of replacing shapes by ever smaller shapes at a certain point. Strictly speaking, we should therefore refer to "approximations" of self-similar tangram shapes (or tans) rather than to self-similar tans.

Counting the number of times we subdivide shapes to appropriate arrangements of sets of smaller shapes, we define the approximation order of a self-similar tangram shape. The approximation order of a square made of seven tans is one, as well as the approximation order of the two other shapes in figure 2. Generalizing this concept, the n^{th} order approximation of a self-similar tangram shape is made of 7^n pieces. The approximation order of the square in figure 3 is 2, since the square is made from seven pieces (first approximation), and each of the seven pieces is also made from seven pieces (second order approximation). In the classroom we usually won't be able to go beyond second order approximations, using computers higher orders can be reached.

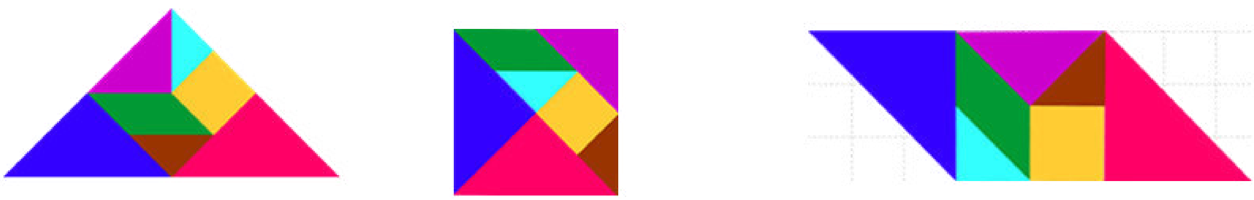

Figure 2: Three basic tangram shapes, the triangle, the square and the parallelogram, made from tangram shapes.

In the following, we suggest activities related to self-similarity that can be carried out by children and adults in a classroom or on the internet.

For the classroom activity each participant will receive the contour of a tangram shape (a triangle, a square or a parallelogram) and a classic set of seven tangram pieces printed on a sheet of paper. The task is to assemble the seven tans to fill the given contour (http://goo.gl/58pcsn). At this point, every participant will have a tangram shape made from smaller tangram shapes (for example a square as in figure 2).

In phase two every participant has to join six other participants so that the group's shapes will complete each other to a full tangram set. To give an example: a participant who assembled a square in phase one needs to find five participants holding triangles of appropriate sizes and an additional participant holding a parallelogram.

Each team of seven participants needs to assemble a tangram shape (a triangle, a square or a parallelogram). The result will be a tangram shape made from tangram shapes that are made from tangram shapes, i.e. a second-order approximation of a self-similar tan as in figure 3.

The winner will be the team which solves the task first.

Similarly, the task to construct one of the seven tangram pieces from tangram pieces can be distributed to user groups on the internet, providing appropriate applets (e.g. https://tube.geogebra.org/student/m579611 or http://tube.geogebra.org/material/show/id/648393). The various pieces can be carried together in a cloud (e.g. http://goo.gl/o1zCWA) and can be used to construct self-similar tangrams to be displayed in an internet album (http://goo.gl/WyDhhL). That way a collaborative or Wiki article about self-similar tangram can be written. The picture below shows a second-order approximation of a self-similar tangram square that was created as an example for a collaborative article about tangram. On the same internet site users had previously written a collaborative article about properties of Pythagorean triplets and were therefore used to the idea of collaborative articles (Stoecker-Segre & Users of the Math Circle, 2014).

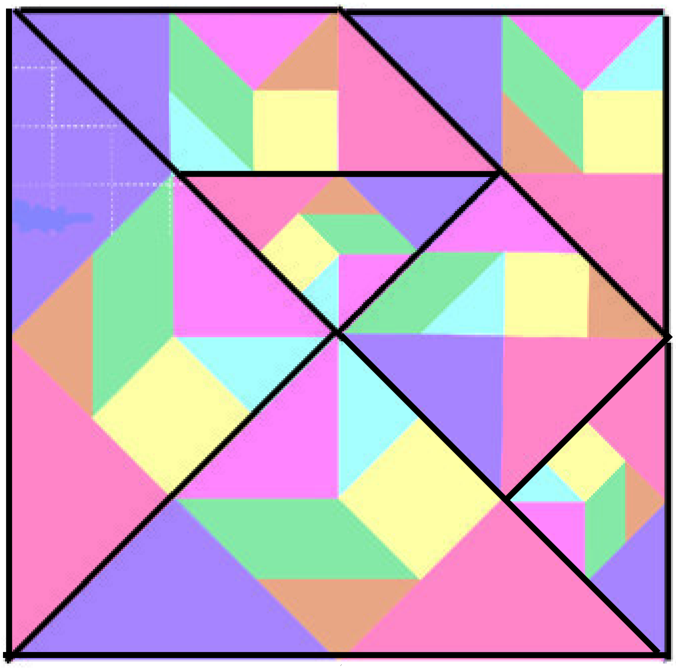

Figure 3: 2nd order approximation of a self-similar tangram square. Each of the seven pieces forming the square is assembled from seven tangram pieces.

BINARY NUMBERS

Assuming that the area of each of the smallest tangram triangles is 1 (choosing appropriate units), the area of the square, the parallelogram and the medium triangle will be 2, and the area of each of the large triangles will be 4. In this case, the area of the total game is 16.

The numbers 1, 2, and 4 are powers of two and form a binary basis for all natural numbers up to 7. Indeed, using only some of the tans, it is possible to form a series of convex shapes having areas from 1 to 7. Since there are two tans of area 1, three of area 2 and two of area 4, it is possible to assemble figures whose area can be each of the natural numbers between 1 and 16. It is not obvious but can be shown that for any area between 1 and 16 a *convex* shape can be assembled.

This task can be given to students in a combined lesson about properties of binary numbers and convex shapes. Furthermore, the example can be used to stress the interconnectivity of mathematical topics (Barabash, 2012).

To solve the task, puzzles or an applet should be provided (e.g. http://ggbtu.be/miG3ujll1). It should be investigated if solving puzzles is easier with physical shapes rather than with applets. For the collection and display of results the internet is the appropriate platform.

FRACTAL DIMENSION

After having reshaped tans from tans, we can remove parts of the tangram set (e.g. all squares on all levels). That way we create holes in the shape. If we'd repeat this process an infinite number of times, we'd construct a fractal.

There are several ways to define fractal dimension. The similarity dimension is defined as

$$D = {}^{\log N}\!/_{\log \varepsilon'}$$

where N is the number of shapes used to from the larger, similar shape and ε is the factor by which the smaller shapes are reduced. The similarity definition of fractal dimension is the easiest in our case, since it is not necessary to calculate the limit of a sequence of ongoing reductions of the basic shapes.

Since tangram is made from seven shapes of various areas, we need to go one step further to apply the above definition: each tan can be constructed from the smallest triangles.

To determine the dimension a fractal tangram we count the number of smallest triangles needed to cover the figure. This is N=16. The reduction factor ε is the ratio between the hypotenuse of the triangle made from tans and the hypotenuse of the smallest triangle (figure 2 on the left): ε=4⁄1. This method is similar to the box-counting method of determining the dimension of a fractal (Falconer, 2003).

Accordingly, the dimension of a normal tangram figure (without holes) is

$$D = {}^{\log 16}\!/_{\log 4} = 2,$$

as expected for a figure in two-dimensional Euclidean space.

The squares of the tangram set can be constructed from two of the smallest triangles. If we remove the squares, the remaining shapes' area is equal to the area of 14 smallest triangles. Therefore, the dimension of a fractal tangram from which all squares on all levels are removed is

$$D = {}^{\log 14}\!/_{\log 4} \approx 1.904.$$

We suggest fractal tangram as an activity to explore fractals and fractal dimensions.

Figure 4: Second order approximation of a fractal tangram triangle: the triangle is made from black tangram shapes of which the squares have been removed (on both levels).

USING TANGRAM IN A COMPUTER CLASS

Due to the above mentioned properties, the length of one edge of one piece of tangram is enough to reproduce the sizes of all other edges. Therefore tangram fits for teaching the use of spreadsheet programs, and in particular, of formulas in spreadsheet programs. Possible exercises are to calculate the length of all edges of all pieces, if the hypotenuse of the largest triangle is given, or calculating the area of all pieces.

We used tangram in a computer class of maths teacher trainees having very different backgrounds in the usage of computers. Since we chose examples from recreational mathematics (such as tangram) those students with a good background in using computers, knowing larger parts of the course contents were still able to learn something new.

In a survey carried out in the beginning of the course, about one third of the class claimed to have basic skills of using spreadsheet programs, almost half of the class considered their skills of being better than basic, and was worried of not being challenged by the course. A survey in the end of the semester revealed that however more than 90% of the students (93.75%) did learn something new in the context of using spreadsheets, and more than 80% (81.25%) had felt challenged by the course contents.

The spreadsheet program can be used to calculate the sizes of the tangram pieces of the various sets that are necessary to assemble all seven tangram pieces from (smaller) tangram pieces, i.e. to calculate the sizes of the tangram pieces that are needed to carry out the above described classroom activity. The activity itself found very good resonance at various age groups (5^{th} to 6^{th} graders and teacher students) and was completed with a web-based quiz about the geometrical properties of tangram (MasteryConnect, SOC # 15297070).

SUMMARY AND OUTLOOK

We presented a number of features of tangram that are different from the usual puzzle challenges and suitable for very different teaching environments. Even teenagers can be challenged by a contest to approximate self-similar tangram shapes where teams will be formed only during the game. Adults of very different backgrounds can participate in writing a collaborative article about new features of tangram. Web-based applets and clouds are necessary tools for such a project. Last but not least, tangram can be used as a challenging topic when teaching the use of spreadsheet programs.

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IMPROVING PROGRESS THROUGH FORMATIVE ASSESSMENT IN SCIENCE AND MATHEMATICS EDUCATION (FASMED)

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This paper will report on the ongoing work and progress of the FaSMEd project, which is a design research project, now in the second year of a three year programme. FaSMEd aims to develop the use of technology in formative assessment classroom practices in ways that allow teachers to respond to the emerging needs of low achieving learners in mathematics and science. This international project adapts and develops existing research-informed pedagogical interventions (developed by the partners), suited to implementation at scale, for working with low attaining pupils and transforming teaching. The project aims to: foster high quality interactions in classrooms that are instrumental in raising achievement for low achievers and expand our knowledge of technologically enhanced teaching and assessment methods addressing low achievement in mathematics and science. The project will be producing a toolkit for teachers to support the development of practice and a professional development resource to support it.

Key words: Formative assessment; mathematics; science; technology; design study

CONCEPT AND OBJECTIVES

The Rocard report (2007) identified widespread concern across the EU about the economic consequences and social impact of underachievement in mathematics, science and technology education and recommended the adoption of an inquiry based pedagogy. This project is a collaboration of international partners, all of whom are skilled in such pedagogies, to research the role of technologically enhanced formative assessment methods in raising the attainment levels of low-achieving students in science and mathematics. Outcomes are informing the development of a toolkit that informs teachers of emergent formative assessment pedagogies in mathematics and science.

This international project adapts and develops existing research-informed pedagogical interventions (developed by the partners), suited to implementation at scale, for working with low attaining pupils and transforming teaching. The intervention is cross-subject, focused on the development of technologically enhanced practices of formative interpretations of assessment [1] within day-to-day teaching approaches. The project focuses on upper primary and lower secondary age students (11-14), since this is an age group where teachers are actively shaping new norms of classroom participation and where it is relatively free from the 'backwash' effect of preparation for examination.

The partners in this project are:

- University of Newcastle upon Tyne, UK (Coordinator)
- The University of Nottingham, UK
- Ecole Normale Superieure De Lyon, France
- National University Of Ireland Maynooth
- University Of Duisburg-Essen, Germany

- University Of Turin, Italy
- Freudenthal Institute, University Of Utrecht, The Netherlands
- African Institute For Mathematical Sciences Schools Enrichment Centre, South Africa (Stellenbosch)
- University College Of Trondheim, Norway

The project draws on evidence from large scale systematic reviews of educational interventions which reveal that the effect size on achievement of interventions that focus on the development of teaching using formative interpretations of assessment in classrooms is significantly greater than most other intervention approaches (Hattie, 2009). A key element of this diagnostic approach to teaching using assessment and intervention relates to the quality of the information generated by the various feedback loops that exist in the classroom setting and the involvement of the students within this process. Hence, the introduction of innovative technological tools to create a digital environment which enhances connectivity and feedback to assist teachers in making more timely formative interpretations has the potential to amplify the quality of the evidence about student achievement in real-time for access by both students and teachers.

Objectives

The objectives for the project are to:

- produce a toolkit for teachers to support the development of practice. ***(NB. The expression 'toolkit' refers to a set of curriculum materials and methods for pedagogical intervention)***
- produce a professional development resource that exemplifies use of the toolkit.
- offer approaches for the use of new technologies to support the formative assessment of lower achieving students.
- develop sustainable assessment and feedback practices that improve attainment in mathematics and science for the targeted students.
- disseminate the outcomes of the project in the form of online resources, academic and professional publications, conference presentations as well as policy briefs to government agencies at a regional, National, European and International level.

RESEARCH QUESTIONS

In order to establish the educational context, the project seeks to: report the differences in the way that systemic structures influence the trajectories of lower achieving students within the participating countries; identify their typical pathways through the school system and reveal the educational opportunities that are open to these students. It has reported on the varying assessment tools that are used to identify lower achieving students and may determine these pathways, with attention paid to the different interpretations of low achievement in each country. The research has surveyed the current policies and practices in formative assessment and teaching in the partner countries and beyond. The research has also surveyed the technology currently available in classrooms to support formative assessment of students' understanding in mathematics and science. (See http://research.ncl.ac.uk/fasmed/deliverables/ for details).

Case studies will report on:

- How do teachers process formative assessment data from students using a range of technologies?
- How do teachers inform their future teaching using such data?
- How is formative assessment data used by students to inform their learning trajectories?
- When technology is positioned as a learning tool rather than a data logger for the teacher, what issues does this pose for the teacher in terms of their being able become more informed about student understanding?

SCIENTIFIC METHODOLOGY AND ASSOCIATED WORK PLAN

The scientific strategy for this project is design study. Shavelson *et al* (2003, p. 26) suggest that the key principles of design studies are that they are: a) iterative; b) process focused; c) interventionist; d) collaborative; e) multileveled; f) utility oriented and g) theory driven. Hence the design of the project is an iterative, collaborative, process-focused approach to the development of the toolkit for teachers, evaluation of technologies and professional development and builds on research evidence for approaches which have the greatest impact. However, pedagogical improvement at scale must take account of the existing state of the system and the resources and practices already in place. These constraints imply the adoption of a 'redesign' stance building on existing practices and research.

Evaluation is a constant theme in design study and this is aimed to be a 'learning and development project' where design does not cease after the first phase but is carried through by formative evaluation of the process of the project through reflection and evaluation by the participants.

The project is organised in three phases:

1. The first year began with the development of the theoretical and methodological framework for the project. The framework was then used to establish a baseline of current practice and achievement in mathematics and science education in the EU and internationally; research innovative practices and technologies for supporting formative assessment, develop a prototype toolkit and professional development protocol and select appropriate schools and students for the study. Dissemination and conferencing among the partners was an integral element of the project from the beginning with the development of a website a priority. A strategic guidance group consisting of representatives of technology companies and academic advisors has been appointed to provide input to the design process and quality control. The year finished with an event to launch the main intervention.
2. During the second year the main intervention in schools is iteratively initiating the approach(es) and professional development process, with frequent opportunities to evaluate and share progress among participants. Students' initial achievement and final achievement is measured using locally available instruments. A sub-contractor is filming the development process among a range of schools, teachers and students.
3. During the third year the final report will be compiled and the final version of the teachers' toolkit and the professional development package produced. A conference will be held to launch the final report.

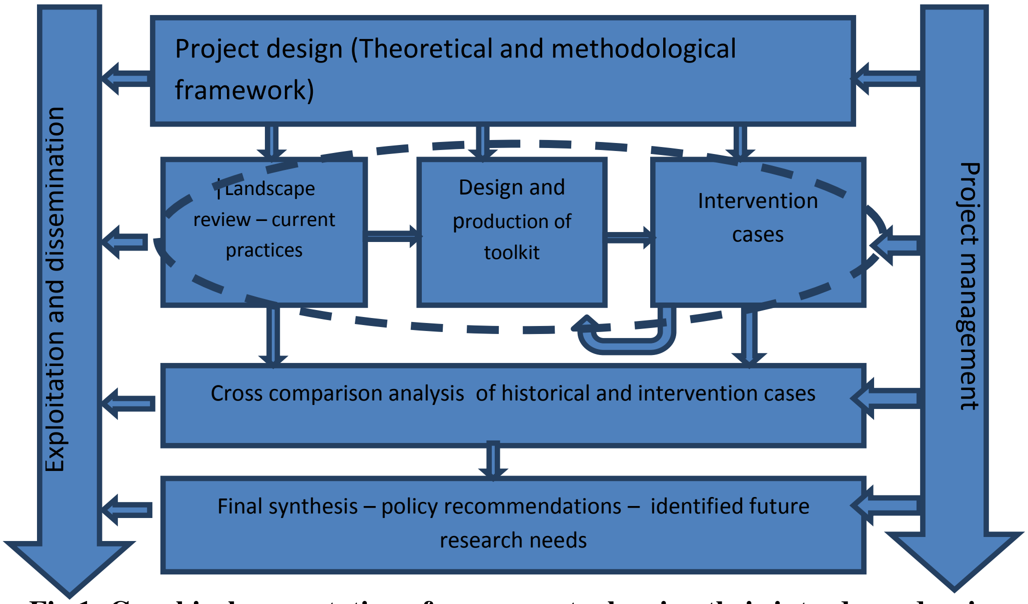

Fig 1: Graphical presentation of components showing their interdependencies

APPLICATION OF STATE OF THE ART RESEARCH

The project builds on the results of existing research studies concerning the raising of attainment levels within mathematics (Watson & De Geest, 2005) and science (Michael Shayer & Adey, 2002), low achievement (Ahmed, 1987), formative interpretation of assessment (Black, Harrison, Lee, Marshall, & Wiliam, 2003), the integration of technology (Hegedus & Moreno-Armella, 2009) and on best evidence syntheses of teacher learning (Timperley, Wilson, Barrar, & Fung, 2007). The project specifically includes South Africa in order to provide a contrast to the situation in more developed countries (Carnoy, Chishom, & Chilisa, 2012; Report, 2012) and to provide a robust test for the implementation of the strategies developed by the project.

The project also draws on the experience of colleagues at Nottingham University who, in partnership with the University of California, Berkeley and supported by the Gates Foundation, have developed a range of assessment materials[2] for US teachers and students to support US schools in implementing the Common Core State Standards for Mathematics (CCSSM). In science, several of the partners can draw on their work in inquiry based learning in science and mathematics as partners in the EU funded projects such as Compass, a Comenius project (Common problem solving strategies as links between mathematics and science), Primas, an FP7 project (to promote inquiry-based learning in mathematics and science at both primary and secondary levels across Europe).

This project also builds on the evidence of research from, for example, the LAMP (Ahmed, 1987), RAMP (Ahmed & Williams, 1991)and IAMP (Watson, De Geest, & Prestage, 2003) projects in mathematics teaching and the CASE (Michael Shayer & Adey, 2002) project in science teaching in the UK and elsewhere which adopted approaches focused on the proficiencies of the students rather than their deficiencies. The current best practice (Swan, 2006; Watson & De Geest, 2005) concerning teaching and assessment methods that address achievement in mathematics and science, uses an activating pedagogy and enhances the feedback loops (ie. formative interpretations of assessment) in the classroom between the teacher and student and between the students themselves in order to develop current understanding, stimulate debate and afford appropriate intervention by the teacher. This approach is based on the creation of a classroom environment in which there is clear, shared understanding of the value and functions of dialogue for learning (Alexander, 2004).

The impact of technology and low attaining students

The creation of a digital environment in the classroom has particular benefits for low achieving students. For example, the facility to respond 'anonymously' to questions from teachers or peers reduces the anxiety levels which research shows has a significant impact on participation. Also, the facility for teachers to carefully track individual responses supports a more focused diagnostic intervention with students, a key element in supporting the progress of these students who can often be lost in the wider mass of the classroom (Shirley, Irving, Vehbi, Pape, & Owens, 2011). The use of digital environments in classroom in recent years has changed from a more "private" to a "public" use that integrates private use (Hegedus & Moreno-Armella, 2009; Robutti, 2010) as predicted in Sinclair & Jackiw (2005). This shift, which echoes the historical shift from the use of individual handheld slate to blackboards, is recognised by recent literature about the relationships between the use of "private" activity (individual or in small groups) and "public" activity (to which all the students participate). The public screen not only displays the student work in real time, providing immediate feedback, it enables individual students to compare and connect their own work with that of others.

There has been widespread adoption of projective technology in the classroom in some countries. Although these technologies have the potential to afford a shared interactive space for teachers and students, the impact has been patchy, with many teachers using the technology to convey information rather than using it to stimulate more active learning. Indeed the students themselves have had diminishing opportunities to access the technology in the mainstream classroom. However, the rapid development of small mobile devices gives an opportunity for students to access technology as and when they need it in the classroom.

PROGRESS BEYOND STATE OF THE ART

This is a complex educational challenge, since there is no clear characteristic of low achievers in mathematics and science. While they share the common feature of underachievement, such groups typically contain a disproportionate number of those from disadvantaged social, cultural and ethnic groups, and in some countries without a good command of the first language of the classroom (Boaler, Wiliam, & Brown, 2000). Established approaches for working with such students are frequently characterised by a 'deficit' model of their potential which entails repeating material from earlier years, broken down into less and less challenging tasks, focused on areas of knowledge which they have previously failed and which involve step-by-step, simplified, procedural activities in trivial contexts. In contrast, the TIMSS seven-nation comparative study shows that high achieving countries (Hiebert et al., 2003) adopt approaches which preserve the complexity of concepts and methods, rather than simplifying them.

Reflective practice

One long-standing successful programme which developed a pedagogical intervention aimed at 'cognitive acceleration' (M Shayer & Adhami, 2007) suggests that this had been most successful where it had served not as a complement to conventional instruction, but as a constructive critique of it, leading teachers to incorporate elements of the new pedagogical model into their 'normal' teaching. Thus progress beyond the state of the art in this educational context will depend on the teachers having the opportunity to engage in a process of development where they can reflect on and contrast their experience in using this approach. This process will be built into the iterative, collaborative methodology for this study. Hence the approach of the intervention will be to engage teachers as practitioner researchers in 'teacher learning communities' using a 'lesson study' method for professional development (Lewis, Perry, & Murata, 2006).

IMPLEMENTATION AND PROGRESS SO FAR

In a project of this size and complexity each component has an associated 'work package' with specific objectives and partners agree to lead or participate in its implementation. For example, this project has ten work packages as follows:

Work package	Description
1	Project design
2	Landscape collection of data and review of literature and systemic practices
3	Design and production of toolkit for teaching and assessment
4	Intervention cases
5	Cross comparison analysis of historical and intervention cases
6	Final synthesis – policy recommendations – identified future research needs
7	Exploitation and Dissemination
8	Scientific Coordination
9	Evaluation
10	Project management and administration

Co-ordination and collaboration

As might be expected, Newcastle, as co-ordinator, has the major role in leading the project and ensuring that the outcomes are delivered.

Co-ordination is achieved through an agreed schedule of meetings, deliverables, effort in person-months per work package and deadlines. A Gantt chart provides way of mapping the progress of the project (see http://research.ncl.ac.uk/fasmed/deliverables/ D1.1 for more detail). It is Newcastle's responsibility to ensure that the outcomes and timetable are adhered to.

However, collaboration is more difficult to achieve. A series of co-ordination meetings were planned but it has been found necessary to add further meetings in order to achieve some mutual understanding of the aims and objectives for the project. Unsurprisingly the work package which has so far caused the most discussion is the 'core' work package three – the 'Design and production of toolkit for teaching and assessment' which has required two extra meetings in order that participants develop a mutual understanding of the concepts of design research, formative assessment, 'toolkit', professional development and what sort of classroom activities might best support the aims of the project. The discussion has also been informed by the production of 'position papers' commissioned from the partners as part of work package one in order to clarify some of the issues under discussion. The outcome of this process is that there has been a divergence on the issue of working with teachers and the nature and range of activities, although this is not necessarily a problem and the project may benefit from this diversity of approaches.

Progress and achievements

The project is now in its second year and work packages one and two have been completed so that we have agreed on the research protocols to be used (case study research); produced a glossary of the main concepts drawn on and completed surveys of the landscape for low achieving students across Europe and South Africa in science and mathematics and surveyed the use of tools and technologies to support learning in science and mathematics (see http://research.ncl.ac.uk/fasmed/deliverables/)

Prototype toolkits and professional development approaches have been produced – for example, see https://toolkitfasmed.wordpress.com/ where a website exemplifies some of the activities and approaches adopted.

Schools

Partners are now working with clusters of schools to trial the activities and provide the feedback necessary to develop the materials, approaches and guidance which comprise the toolkit. Each partner necessarily negotiates their own arrangements with the schools and teachers but there are agreed processes such as cluster meetings and reports that all partners use. The interventions are expected to be carried out between January and July 2015, although some may continue after the summer break. Case studies will be completed for each school with a selected teacher as a focus in order to provide exemplars for the toolkit and data for further research studies to be carried out in the third year of the project.

Approaches adopted

We are aware that we necessarily build on existing practices and research. Hence in the UK, the approach to mathematics activities adopted those developed at Nottingham University and exemplified on the MAPS website (http://map.mathshell.org/) and each school has been invited to adopt a technology which they felt able to implement with a relative minimum of effort. In the UK we have a range of technologies being trialled including Chromebooks with googledocs software, ipads with socrative and Classflow software and several other combinations of software and hardware. As a contrast, in South Africa, where access to any sort of technology is limited, a number of low tech 'tools' are being trialled to provide the feedback necessary for formative assessment in the mathematics classroom, since both the concept and practice of formative assessment are relatively new.

Other partners, such as Utrecht and Duisburg-Essen, are developing digital environments where the activities provide tools for classes or individuals to diagnose understanding and obtain feedback.

A benefit of the project's funding coming from the European Commission is the opportunity to draw on and work with colleagues engaged in complementary research. For example, FaSMEd has linked with SAILS (Strategies for Assessment of Inquiry Learning in Science http://www.sails-project.eu/) and ASSIST-ME (Assess Inquiry in Science, Technology and Mathematics Education http://assistme.ku.dk/) to share activities (SAILS) and draw on research (ASSIST-ME) to inform our research.

Crossing boundaries

Akkermann and Bakker (2011) provide an interesting study of the main issues arising when boundaries (defined as: "a sociocultural difference leading to discontinuity in action or interaction." p.132) are encountered or crossed. They point out that although these situations can be uncomfortable, they also provide opportunities for learning – particularly through dialogue and identify four mechanisms through which this can be achieved: identity, co-ordination, reflection and transformation. This project is certainly engaged in crossing boundaries and our experiences so far provide ample opportunities for dialogic learning in questions of identity, achieving co-ordination, reflecting on issues and transforming practice. We are producing a film which, we hope, will capture some of these moments and tell more of the story of FaSMEd.

NOTES

1. Defined by Black, P., & Wiliam, D. as 'Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited' *in* (2009). Developing the theory of formative assessment. *Educational Assessment Evaluation and Accountability, 21*(1), pp. 5-31.
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Theme: Resources

GUMMI(I)NG UP THE WORKS? LESSONS LEARNED THROUGH DESIGNING A RESEARCH-BASED "APP-TUTOR"

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Like many information technologies in recent history, including motion pictures and television, tablets and other "smart" interfaces are regarded by come commentators as solutions to persistent difficulties of school mathematics. These difficulties include personalization of learning, continuous formative assessment, genuine problem solving, bodily engagement, and simultaneous presentation of multiple interpretations of concepts. I report on such matters in the context of my own on-going involvement in the development of an app tutor intended to be attentive to these and other difficulties. I highlight that such technologies may not so much "solve" problems as "reconfigure" them.

Keywords: app-based mathematics tutors; concept development; curriculum innovation

THE TUTORING INDUSTRY AND THE PROMISE OF APPS

Extracurricular tutoring in school mathematics is a massive and growing business. While precise data are hard to gather, in large part because so much of the activity is unreported, most estimates place the worth of the enterprise in North America alone well into the billions of dollars (Sullivan, 2010). It is no surprise, then, that apps designed to support procedural fluency in arithmetic were among the first to emerge with the introduction of tablets and smart phones over a decade ago. Such "math" apps now number in the hundreds, perhaps thousands.

As a quick search through the App Store will confirm, the vast majority of these focus on basic operations, with most serving as little more than digitized flash cards. Strikingly little attention tends to be given to the sorts of emphases and strategies that mathematics education researchers have advocated for several decades, such as presenting diverse interpretations of key concepts and engaging users in genuine problem-solving challenges. If anything, then, the reduction of "math" to "calculation" is amplified in most cases – particularly for those apps intended for younger learners. It was for this reason that, when asked to participate in a project to produce an app tutor that incorporated insights from research, I found myself unable to refuse. Late in 2012, I was invited to into a partnership with a local entrepreneur and a local software developer, who were seeking someone with expertise in mathematics learning and teaching. They found me through Google.

My new partners were motivated by the struggles of their own elementary-aged children. One had enrolled his 10-year-old daughter in a well-known tutoring service, and was horrified to learn that it was little more than drill-based, reward-driven practice with no attention at all to conceptual development. The other had attempted to tutor his daughters himself, but quickly realized there was an unbridgeable gulf between his refined expertise and their fragmented understandings. Like me, both of my new partners had scanned the available offerings in the App Store, and neither was impressed. As well, and fortuitously, we had all recently read McGonigal's (2011) *Reality is Broken* and shared a conviction that gamification is an important and underutilized tool. A partnership was thus forged to produce an app embodying the research into the structures of mathematical concepts, the complexities of learning, and the evolving disciplinary needs of society. Unfortunately, in spite of intense and sustained effort, we continue to fall short. In this brief report, I speak to some of the many issues – technical, conceptual, and perceptual – that have frustrated our efforts.

We call the app "Gummii." For ease of reference, I'll be using that title for the remainder of the writing.

SOME DESIGN PRINCIPLES

My first task as the educational "expert" in the Gummii partnership was to develop a preliminary list of design principles, informed by current research. My initial list had four entries:

- micro discernments – Incorporating notions from Variation Theory (Marton, 2014), app builders should be attentive to small steps in the development of concepts, ensuring that deliberate elaborations and extensions to ideas arrive in measured and controlled ways.
- multiple instantiations (Lakoff & Núñez, 1999; Davis, 2011) – The app should offer deliberate and structured encounters with diverse interpretations of core concepts – in particular, of *number* (as count, as measure, as position in space, etc.) and of *basic operations* (e.g., multiplication as repeated addition, scaling, area-making, etc.). Users should be afforded opportunities to blend multiple instantiations into more sophisticated constructs (Fauconnier & Turner, 2002) – through, for example, juxtaposition of interpretations.
- recursive movement through enactive → iconic → symbolic representations – Following the distinctions offered by Bruner (1966) and drawing on literatures on embodiment (e.g., Gerofsky, in press), gesture (e.g., Roth, 2001), and spatial reasoning (e.g., Davis et al., 2015), designers should ensure that activities are rich with actions that correlate to concepts, such as "pinching" to combine, "stretching" to multiply, and so on.
- a fractal trajectory space through the curriculum – Mathematics curriculum is often characterized in terms of pre-scripted linear trajectories through a well-engineered program of study – a conceptualization that is frequently experienced as fragmented and disconnected. We aim at a very different way of laying out and linking topics.

For ease of discussion, I will use these principles as headers of sections as I highlight successes and frustrations.

MICRODISCERNMENTS

There has been a great deal of research into teachers' disciplinary knowledge over recent years (see Davis & Renert, 2013, for a review). It is impossible to offer a concise summary of this work, but there is one statement that I believe captures its gist: *Teachers are experts who are able to think like novices.* That is, effective teachers have refined abilities to analyse concepts and tasks in ways that afford them insight into the multiple discernments and associations that are needed to render those concepts or tasks coherent.

Consider, for instance, a topic identified in our local 1st-grade curriculum: "sums within ten." For most non-educators (including most pre-service teacher candidates), this descriptor collects a small range of largely undifferentiated possibilities, such as "1 + 7," "7 + 1," "2 + 6," and "4 + 4." In contrast, the descriptor typically evokes something quite different for more experienced teachers, who see it as an imperative to differentiate, organize, and supplement. For instance, as reported elsewhere (Davis & Renert, 2013), when practiced teachers are presented with the four items presented a few sentences ago, a common action is to move the "7 + 1" to the start of the list because (1) adding by 1 is an earlier competency and, (2) this item lends itself to the readily mastered strategy of counting up. Typically, that advice is followed by a suggestion that addends of 2 be included after some additional practice with addends of 1. Then addends of 3. Then perhaps playing with the commutative property, depending on emerging facility. Eventually doubles would be explored as special cases. And so on.

What is going on here is something subtler and more refined than making discernments among practice exercises; it involves fine-grained distinctions – micro discernments. Teachers who are aware of these micro discernments are more likely both to support the conceptual growth and to facilitate the process of isolating the junctures at which conceptual breakdowns might occur. That is, they are able to structure sequences of tasks that are both developmental and diagnostic.

A strong case has been made that such micro discernments must be brought to conscious awareness in order for them to be learned (cf., Donald, 2001). It is also the case these micro discernments must typically be "forgotten" – that is, automated and non-conscious – for them to be useful to expert knowers (Kahneman, 2011). Therein rests our most persistent challenge in designing and building Gummii. It turns out that persons with the skill sets necessary to build such tutor apps and/or to secure financing for such projects tend to have such automated skills in arithmetic that summoning long-"forgotten" micro discernments is all but impossible. It is not always easy to convince these expert knowers that they are glossing over key distinctions or moving too quickly.

This part of the work has been filled with surprises. Three were notable to us: Firstly, we have a list of roughly 20 micro discernments that are necessary to move from 1 + 1 = 2 to all sums within 20. Secondly, a similar number of micro discernments are needed to extend competence to sums within 100. Thirdly, a similar number are needed to move to the sum of any two whole numbers. These sorts of realizations bring with them additional design challenges – most notably the need for a pacing mechanism that neither moves the learner too quickly nor forces them into tedium.

MULTIPLE INSTANTIATIONS

There are actually at least distinct two categories of micro discernments that are important in discussions of the learning arithmetic. The first – and arguably the most familiar – is of the sort just described, which is focused on the development of procedural competence through alerting learners to many of the small procedural elements that lurk inside "elementary" operations.

The second includes the many instantiations – that is, gestures, images, metaphors, algorithms, metonymies, exemplars, and other interpretive elements – that might be associated with a mathematical concept. As with the first category, the sheer number of entries in this one can often be surprising. For instance, in their work with elementary-school teachers, Davis and Renert (2014) have identified more than two dozen instantiations of multiplication that are typically invoked by the 6th grade. These include such frequently noted instances as "repeated addition," "repeated grouping," and "grid-making," such less commonly noted instances as "scaling," "combination-making," "folding," and "branching," and the important-but-rarely-mentioned-explicitly instances of "number-line compression" and "slope."

The importance of being exposed to and aware of multiple instantiations of concepts has been foregrounded in the mathematics education literature since the 1980s (e.g., National Council of

Teachers of Mathematics, 1989). The emphasis is now a predictable feature in classroom resources, teacher manuals, and teacher education programs. Unfortunately, varied interpretations are often treated as *instances* rather than *instantiations*. That is, they tend to be handled as *examples* of, say, multiplication, rather than *constitutional elements* that might contribute to a grander, emergent construct. Doubtless, part of the issue here is the static nature of most classroom resources. Staying with the example of multiplication, it is one thing to be to presented with two paper-based images of "2 × 3" – such as "three hops of length two" and "three groups of count three" – and it is quite another dynamically gesture through hops while groups form and grids self-organize on a tablet. In the first case, the observer must undertake a point-by-point comparison of the two instantiations in an effort to find their relationship; in the latter, the simultaneity of the actions capitalizes on the irrepressible human tendency to link and blend experiences that overlap. Structured well, then, overlapping and dynamic instantiations should support robust conceptual blending.

The interactive surface of a tablet invites such dynamic juxtapositions. As illustrated in Figure 1, for example, four distinct instantiations of multiplication are deliberately represented on the same screen in one of Gummii's factoring exercises. Users are asked to identify factors of an assigned a number ("Make 6."), which can be undertaken in one of four ways:

- making equal-sized groups by specifying the number of groups and the count in each group;
- splitting the blocks on the conveyer belt into subgroups, and sweeping an appropriate number of those subgroups off the belt;
- making a grid-arrangement of blocks; or
- writing a multiplication sentence.

In the design of this particular activity, when one instantiation is "performed," the other three unfold in unison. (It is not obvious from the screen shot, but the user is free to act on any of the instantiations. The others will respond automatically.)

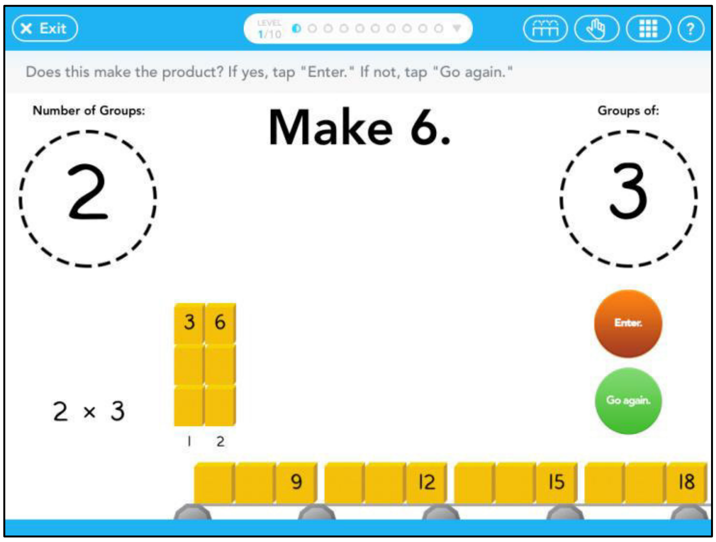

Figure 1. Simultaneous instantiations of multiplication (© Gummii, Inc., 2012; used with permission)

The potential of such activities for supporting sophisticated construal's of key concepts is evident. However, a host of issues emerges almost immediately. For example, in spite of general consensus that encountering multiple simultaneous instantiations is a good thing, there is not a great deal of research into such basic questions as:

- Following principles of variation theory, it makes sense to build up gradually as multiple instantiations are introduced. Does that mean one-at-a-time, or does a dynamic tablet interface permit/enable more. Is there a threshold? How many instantiations is too many?
- Given the dramatic variations that can appear among instantiations, what sorts of sequences make sense for introducing them? (I revisit this question in the next section.)
- How might one make decisions on relative values of instantiations? For example, a multiplication-as-number-line-compression metaphor is arguably the most valuable for tertiary-level mathematics (Mazur, 2003), but its virtual absence in the elementary-level grades suggests either that it is not particularly useful in the early years, it is too complicated for young learners, or teaching resources to date have been insufficient to support learners appropriately. Which is it? And might a tablet-based interface change things?

These issues are anything but trivial. They cut to and through the foundations of entrenched pedagogical sensibilities. It is at this juncture that I find myself unsurprised that most math tutor apps are little more that digitized flashcards. Although the literature has told us that we can and should go beyond that, only a handful of researchers seem to have grappled with the realization that tablet-like interfaces are more than additional tools (cf., Sinclair, 2014). They not only afford new sorts of learning activities; they are transformative of the concepts and sequences of learning.

RECURSIVE MOVEMENT THROUGH ENACTIVE → ICONIC → SYMBOLIC REPRESENTATIONS

Of course, at least one of the above questions has been the subject of intense analysis for a half century – namely the one on sequencing. Jerome Bruner's (1966) model is the basis of the most frequently heard advice: learners should first encounter experience-based ("enactive") representations, and then interpret those through image-based ("iconic") representations. "Symbolic" representations should only come later, after opportunity to develop and consolidate a bodily basis for meaning.

This model comes with many qualifications, one of which is particularly important for app development. First, the sequence is not intended to be strictly linear, but is typically presented in terms of a more recursive unfolding. That is, for example, an iconic representation is not intended as an image a of specific physical instantiation, but as an abstraction that highlights salient aspects of an instantiation – and that, in the process, might foreground relationships among multiple enactive representations. Such an iconic representation would thus invite learners to fold back on and reformulate prior understandings. Similar might be said of symbolic representations. So, to re-emphasize, the model is not one of strict ordering of specific representations, but of general advice on moving through clusters of related experiences.

Unfortunately, the necessarily iconic-heavy nature of a tablet interface greatly complicates Bruner's advice. To elaborate, our initial designs for activities were based on an assumption that children would arrive with a repertoire of ready-to-generalize inactions, including gestures for increase (e.g., leftward, upward, or spreading), combining (e.g., sliding, encircling, or linking), and so on. So far that assumption has proven valid, but it has also turned out to be somewhat naïve. As we track and analyse children's engagements with the interfaces, it is apparent that we (1) over-estimated the toolbox of meaningful gestures that many young children bring, but (2) under-estimated that pace at which they might develop gestural complexity (i.e., combining gestures and clusters of gestures).

Phrased differently, based on analyses of users' first experiences with some Gummii's designs, they can make a surprising number of initial errors. Likely some of these are not mistakes at all, but explorations (i.e., asking, "What happens if I …"). However, many are systematically repeated, suggesting a disconnection between a learner's expectation and a designer's intention. For the most part, these breakdowns occur in places we assumed what we might call "ubiquitous cultural knowledge" – that is, competencies and/or habits of interpretation that are so common that there is a risk of assuming them to be universal. For example, the idea that "more is up" appears to be far from consolidated among many primary-school learners. In a few instances, the frustrations associated with such over-assumptions have been amplified by errors of under-assumption, specifically around the pace at which learners can cobble together simple gestures into sophisticated constellations of action. The activity illustrated in Figure 1 is a good case in point. Gestures intended to be associated with multiplication within that activity include, for example, "parsing of subgroups combined with repeated sweeping" and "grid creation through stretching out height and width." A good deal of programming time was devoted to these, on the assumption that learners would require time and practice to cluster more primary gestures into meaningful composites.

We were wrong. In fact, emerging evidence suggests that this sort of rapid compiling of gestural knowing's into more complex clusters of action is the norm. Indeed, across a half million users, we have not witnessed a single case of a learner struggling to chain gestures together. For us, this renders the enactive→iconic→symbolic sequence more complicated in some interesting ways. In particular, we would posit that the categories of enactive and iconic are prone to blurring within a tablet environment, as images become the objects that are physically manipulated. Consider, for example, the multiplication activity presented in Figure 2, which is blend of familiar grid-based multiplication using Base-10 blocks and the misnamed "Japanese multiplication." The actual interface is clearly an iconic representation, but the mode of interaction is highly enactive, involving building numbers (e.g., 35 is created with 3 wide bars and 5 narrow ones), crossing those numbers, and manipulating an emergent type of number (i.e., one based on areas, not widths) by regrouping, sliding ("carrying"), and so on. Users who are new to multi-digit multiplication tend to master these constellations of actions quickly and, typically, have little trouble making the transition to interfaces that are more symbol-heavy. That is, most users seem to move at a pace that is much faster than the many-lesson trajectory that is built into common text-based classroom resources.

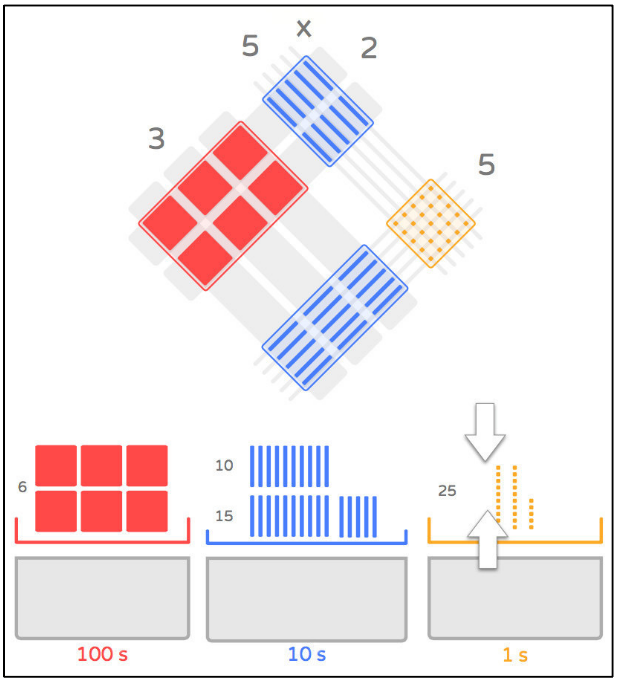

Figure 2. A multi-digit multiplication activity (© Gummii, Inc., 2014; used with permission)

Admittedly, we are still struggling with how to interpret the data that we are gathering. However, one point is unambiguously clear: the all-at-once nature of a tablet interface can transform learners' experiences of concepts – allowing, for example, multi-digit multiplication to be encountered simultaneously as a coherent whole and a sequence of steps, rather than merely the latter.

A FRACTAL TRAJECTORY THROUGH CURRICULUM

Such realizations regularly prompt us to revisit research on "learning trajectories." As described by Confrey et al. (2009), a learning trajectory is:

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

This description would seem to touch on every issue associated with designing an app tutor. However, the notion was not particularly enabling when we began the work. On the contrary, more than anything, it seemed to constrain the work. As it turned out, the issue lay not with the description but with our own assumptions of linearity, which were bolstered rather than challenged by the notion of "trajectory." Specifically, we have struggled with breaking free of the popular characterization of mathematics curricula as linear, incremental, and ill-fitted the fits-and-starts manner in which humans actually learn. An important break came when we attempted a two-dimensional map of the K–8 arithmetic and algebra curricula (see Figure 3).

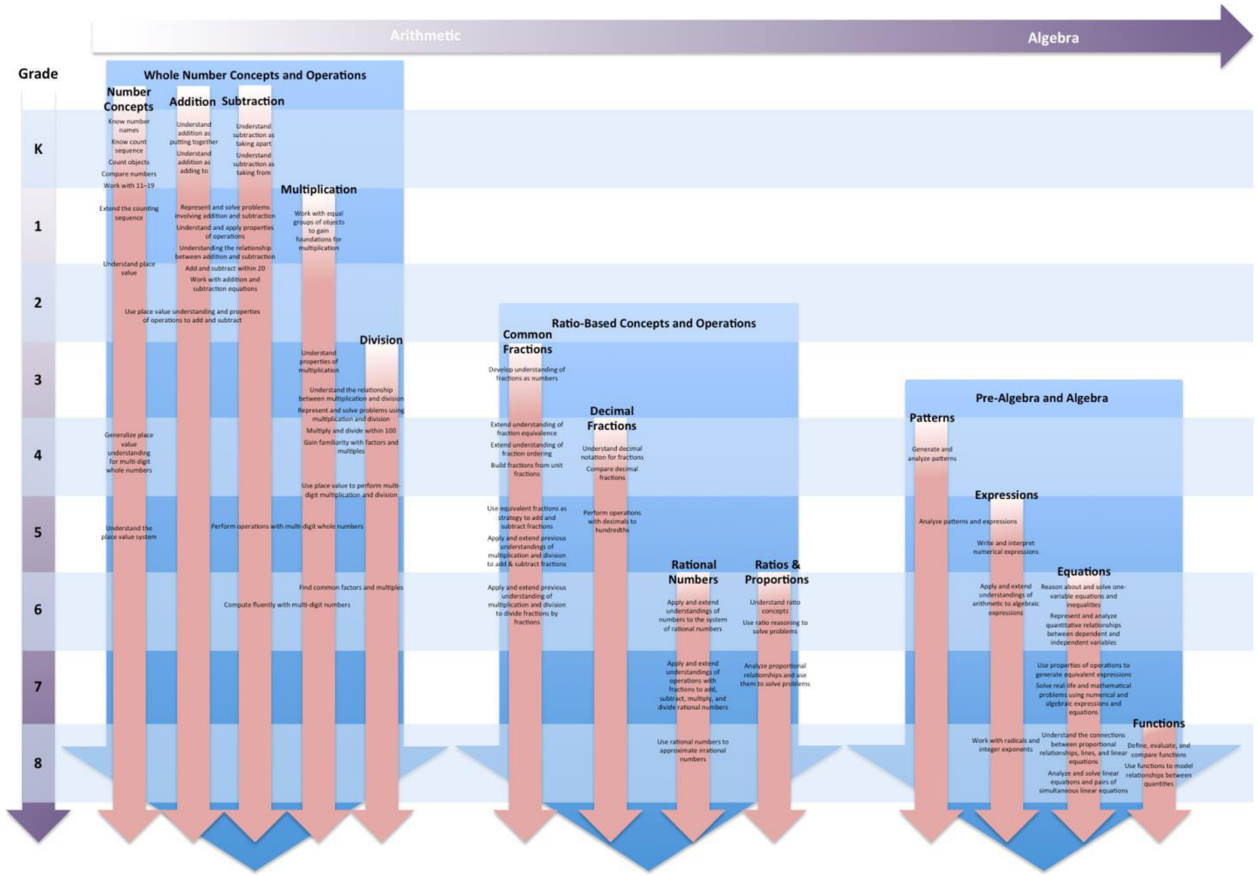

Figure 3. A "fractal" perspective on curriculum (© Gummii, Inc., 2014; used with permission)

This map is based on several North American programs of study, including the US Common Core Curriculum. It was assembled by identifying how and when key concepts are developed ("threads," in pink), identifying how those concepts/threads are wound together into larger "strands" (in blue), and recognizing how those strands are wound together into one's school mathematics experience.

Typically, a grade school teacher would regard only a single horizontal slice of this grid. Viewed in that way, the curriculum does indeed feel linear, incremental, and fragmented – and, in the process, is more likely to be experienced as incoherent and overwhelming. An app-tutor developer, in contrast, must be attentive to both horizontal and vertical slices – which, when considered together, presents a fractal-esque image of strands weaving into grander strands, which weave into grander strands. In terms of trajectories, this image suggests a host of pathways that are perhaps better conceived in terms of simultaneous potentialities than a collection of discrete routes.

This realization has proven critical to us, as it distinguishes the project of app design from the usual preoccupations of teaching – and, for that matter, from the project of grade-level-focused program development. In particular, whereas a focus on learning trajectories seems fitted to the immediacy of teaching and the design demands of programs of study, for us it now feels inadequate for the emergent possibilities presented by tablet technologies. For that reason, we are thinking more in terms of "learning landscapes" than "learning trajectories."

FINAL REMARKS

I close where I began, grappling with the realization that tablet-based technologies do as much to transform the enterprise of mathematics education as they do to address frustrations and limitations of print-based and other entrenched pedagogical technologies. Moreover, owing to the much broader range of gestural interactions supported by tablets, the emergent differences are in a different category from the transformations associated with motion pictures and televisions, and more recently with calculators and personal computers.

When this realization is coupled to the voluminous and immediate feedback from users that can be constantly fed to app builders, it would seem that mathematics educators and mathematics education researchers are presented with a whole new world of pedagogical and investigative possibilities. Those possibilities, however, will be contingent on teachers' and researchers' abilities to think differently about the project of school mathematics.

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TEACHING MATHEMATICS WITH AUGMENTED REALITY

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Low achievement in mathematics education has been an increasing problem in the recent years in Portugal. According to a 2010 study from the U.S. Department of Education, blended learning classes produce statistically better results than their face-to-face. There is also an increasing number of students using smartphones and tablets in schools. Mobile devices gained popularity as an educational tool and there are many schools that used them frequently in educational activities to improve learning. In this paper, we introduce the use of Augmented Reality for providing activities that students can do at home and increase the time they spend learning and practicing mathematics. We present teaching activities that use different augmented reality technologies for presenting solutions to practical problems by multiple types of media, including videos, to be shown on top of interactive documents.

Keywords: e-learning, b-learning, augmented reality.

INTRODUCTION

Low achievement in mathematics education has been an increasing problem in the recent years in Portugal.

In 2014, the average classification in the 12th grade exam, from 0-20, was of 7.8. Mathematics exams in the 1st cycle, 2nd cycle and 3rd cycle had an excessive percentage of negatives (levels 1 or 2), 36%, 54% and 47%, respectively.

According to a 2010 study from the U.S. Department of Education, blended learning classes produce statistically better results than their face-to-face. B-learning combines face-to-face instruction with online learning and has yielded strong results since officially being researched as an education model. An advantage of this approach is that it increases the flexibility and individualization of student learning experiences, and it also allows teachers to expand the time they spend as facilitators of learning.

The recent availability of smartphones and tablets with increased processing power and usability, accessible on a large scale, allow an exponential expansion of social and participative web technologies.

It is also important to note that these students are the generation of digital games and social networks. We cannot ignore that they are no longer the same for which the education system was designed a few decades ago. See, for example, the prospect of Heide and Stilborne (2000), for whom "the technological revolution has produced a generation of students who grew up with multidimensional and interactive media sources. A generation whose expectations and world views are different from those that preceded it" (p. 27).

In this context it is wise to consider the integration of digital media and mobile devices (iPad, iPod, tablets, smartphones), allowing students to set personal goals, to manage educational content and to communicate with others in the right context.

The increased availability of smartphones and tablets with Internet connectivity and increasing power computing makes possible the use of augmented reality (AR) applications in these mobile devices.

In the near future, eventually everyone has a smartphone or a tablet that is capable of displaying augmented information. This makes it possible for a teacher to develop educational activities that can take advantage of the augmented reality technologies for improving learning activities.

According to Fernandes and Ferreira (2012), the use of information technology made many changes in the way of teaching and learning. We believe that the use of augmented reality will change significantly the teaching activities by enabling the addition of supplementary information that is seen on a mobile device. Several examples are already showing that this is happening. For example, the recent work of Restivo et al. (2014) with Augmented Reality involving STEM students, using markers for teaching DC circuit fundamentals, revealed very good student perceptions and satisfaction.

In this paper, we want to expand the use of Augmented Reality for STEM teaching and learning by describing several educational activities created for teaching mathematics using augmented reality tools that do not require programming knowledge to be used by any teacher. With the produced material students can do their home works and see the complete solutions for problems making learning more interactive.

We describe educational activities using several types of augmented reality technologies. Examples presented cover the marker and marker less based augmented reality technologies to show how to create learning activities to visualize augmented information like text and video that help students understand the educational content.

AUGMENTED REALITY

Augmented Reality applications combine 3-D virtual objects with a 3-D real environment in real time. Virtual and real objects appear together in a real time system in a way that the user sees the real world and the virtual objects superimposed with the real objects. The user's perception of the real world is enhanced and the user interacts in a more natural way. The virtual objects can be used to display additional information about the real world that are not directly perceived.

Milgram and Kishino (1994) introduced the concept of a Virtuality Continuum classifying the different ways that virtual and real objects can be realized. In this taxonomy scheme Augmented Reality is closer to the real world.

Azuma (1997) defines augmented reality systems as those that have three characteristics: 1) combines real and virtual; 2) interactive in real time; 3) registered in 3-D. In general, augmented reality applications fall in two categories: geo-base and computer vision based.

Geo-based applications use the mobile's GPS, accelerometer, gyroscope, and other technology to determine the location, heading, and direction of the mobile device. The user can see 3D objects that are superimposed to the world in the direction he is looking at. However, this technology has some problems. The major problem is imprecise location which makes difficult for example the creation of photo overlays.

Computer vision based applications use image recognition capabilities to recognize images and overlay information on top of this image. These can be based on markers, such as QR (Quick Response), Microsoft tags or LLA (latitude/longitude/altitude), or marker less that recognize an image that triggers the overlay data.

There are currently many augmented reality applications and development systems for Android and iOS (iPhone Operating System) smartphones and tablets.

CREATING LEARNING ACTIVITIES USING MARKER BASED AR TECHNOLOGIES

In this section, we use marker based augmented reality technologies to provide the solutions of problems so that students at home solve math problems and know if it is correct.

We would prefer to use the Microsoft tags because we want to use smaller codes that become less intrusive. Reading smaller Microsoft tags are more reliable then the equivalent QR codes. Unfortunately Microsoft selected Scanbuy to support Microsoft Tag Technology and it will be not free.

In this way, figure 1 shows the use of marker based augmented reality to present problems solutions for students.

1. Na Figura 1, está representado um tabuleiro quadrado dividido em dezasseis quadrados iguais, cujas linhas são A, B, C e D e cujas colunas são 1, 2, 3 e 4. O João tem doze discos, nove brancos e três pretos, só distinguíveis pela cor, que pretende colocar no tabuleiro, não mais do que um em cada quadrado.

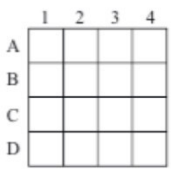

Figura 1

De quantas maneiras diferentes pode o João colocar os doze discos nos dezasseis quadrados do tabuleiro?

(A) ${}^{16}C_{12}$ **(B)** ${}^{16}C_9 \times {}^7C_3$ **(C)** ${}^{16}A_{12}$ **(D)** ${}^{16}A_9 \times {}^7A_3$

2. Considere a linha do triângulo de Pascal em que o produto do segundo elemento pelo penúltimo elemento é 484.

Qual é a probabilidade de escolher, ao acaso, um elemento dessa linha que seja superior a 1000?

(A) $\frac{15}{23}$ **(B)** $\frac{6}{11}$ **(C)** $\frac{17}{23}$ **(D)** $\frac{8}{11}$

3. Sejam a e b dois números reais tais que $1 < a < b$ e $\log_a b = 3$

Qual é, para esses valores de a e de b, o valor de $\log_a\left(a^5 \times \sqrt[3]{b}\right) + a^{\log_a b}$?

(A) $6 + b$ **(B)** $8 + b$ **(C)** $6 + a^b$ **(D)** $8 + a^b$

Figure 1: Math test with QR codes to provide solutions to the problems.

Using marker based codes for presenting additional information in a mobile device is very simple to use and straightforward. The teacher can use simple QR (Quick Response) two dimensional codes for associating information such as text, URL or any other data. Quick response codes are much more popular than the other code formats and there are several sites where the teacher can easily create such codes.

The example of figure 2 uses a QR code to show the correct answer to problem 1. Figure 3 shows the use of a QR code to show a pdf file that is stored in dropbox and the QR code encodes the URL of the pdf file that is shown with the resolution of the problem. Figure 4 presents a video with the resolution of problem 3 that is activated with the QR code that links to the dropbox where the video is stored.

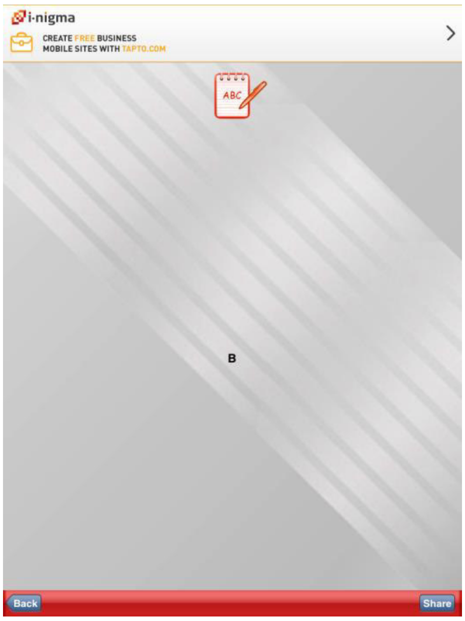

Figure 2: The first QR code was activated using the i-nigma app to show the letter of the correct option.

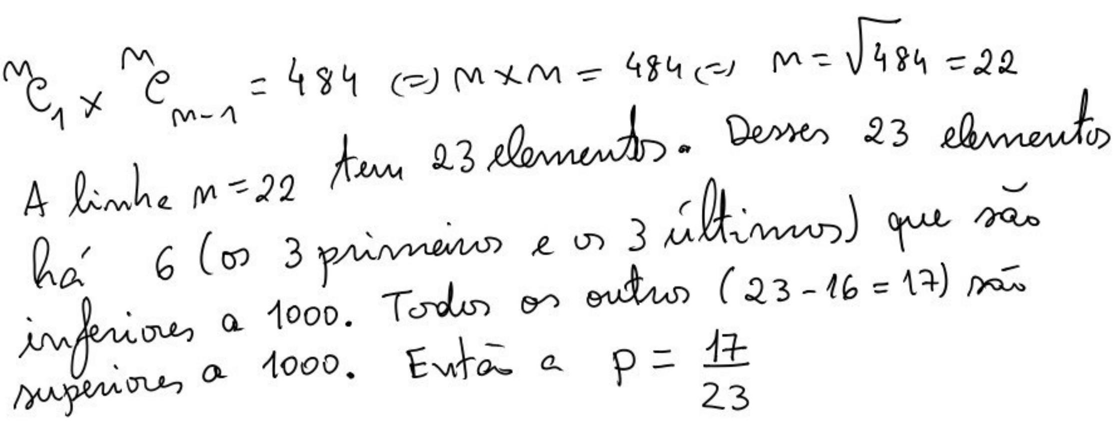

Figure 3: The QR code of problem 2 shows the resolution.

CREATING LEARNING ACTIVITIES USING MARKER LESS AR TECHNOLOGIES

This section presents interactive mathematics activities that students can do at home based on marker less augmented reality technologies.

The teacher gives students the printed augmented reality maths activities or a PDF file. Students using augmented reality technologies and point a tablet or smartphone on top of the paper or the PDF file in the computer to see the videos about the theoretical materials and the problems resolution step by step (Figures 5 and 6).

In this way, teachers can extend the class into a virtual class in a form of blended learning in which students can view video lectures and home works at home. This can be especially interesting for learning mathematics. If students can learn at home from watching video lectures and solving problems, time in-class can be dedicated to explore more motivating problem solving. Math teachers have a difficult situation. Studying math is many times a cumbersome task. But this can be changed if the teacher takes advantage of the technology that is currently available in the classroom. Students are surrounded by multiple devices, such as smartphones and tablets, that give them access to multiple media that is easily available. This is an opportunity for the teacher. The technology related to teaching/learning will have a vital role in the coming years in the education field.

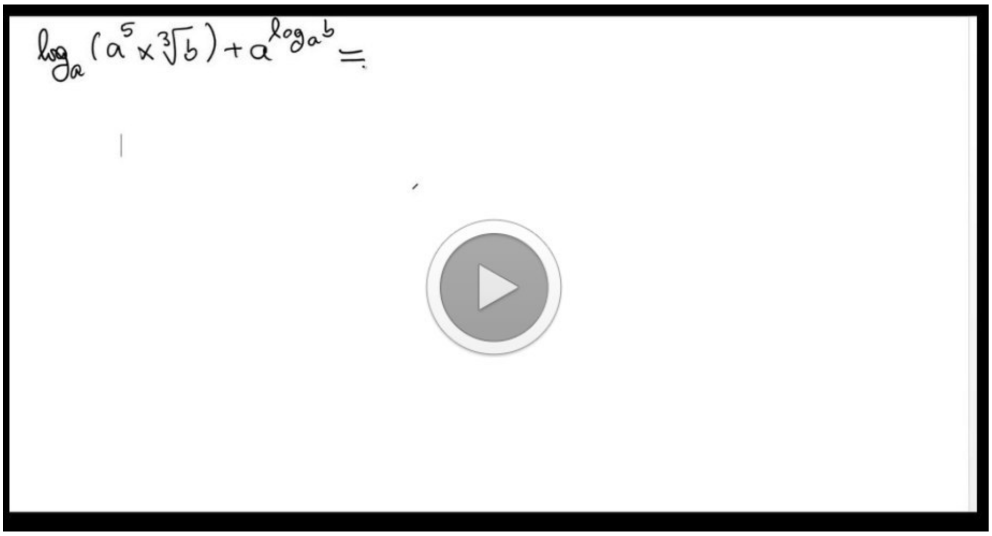

Figure 4: The QR code of problem 3 shows a video explaining the problem resolution.

One of the problems identified by in the re-design of mathematical programs was that students had difficulties due to the fact that they were not familiar to the new room setting (Yi and Mogilski, 2014). To overcome this problem we decided to have always the same layout for the activities based in augmented reality. In this way, all the activities follow always the same layout (Figure 5).

There is an *A* part that is outlined with an orange box and the student knows that there is a corresponding video about an introduction to theoretical concepts. In this example, there are two *B* parts that are outlined with blue boxes that let the student know that there are videos about an introduction to more practical problems that help understand the new concepts.

There are many applications that can be used to capture the teacher actions on computer screen, accompanied by audio narration, to produce the videos for the virtual classroom. Video recording is well suited for demonstrating basic concepts. It allows students to learn at their own pace and in their own learning style. Video lectures are well adapted for classes with students who have different levels of knowledge of the subject. There are students that can view the materials once and have a good understanding of the subject. Other students can view the videos several times to better understand the subject. This is an advantage over the traditional classroom where many times the students do not understand and do not ask to repeat the subject until they are able to understand. The use of videos for teaching and learning is effective for both visual and auditory learners as there is video and narration that is less complicated than written explanations (Spilka and Manenova, 2013).

With the number of students increasing in the class this is an important tool to enable students to work at home and leave classroom time to implement problem based learning methodologies together with virtual learning classrooms.

For this project we are developing an augmented reality application that can integrate also gaming technologies (Figure 7). We want to bring together augmented reality technologies and gaming to study how motivation can be improved through these methods. The application has a tracker manager that recognizes trigger images seen by the smartphone or tablet and fetches the corresponding videos from the database. The game manager integrates augmented reality and game-based activities to enhance the learning process. The student specific achievements are stored in the user database.

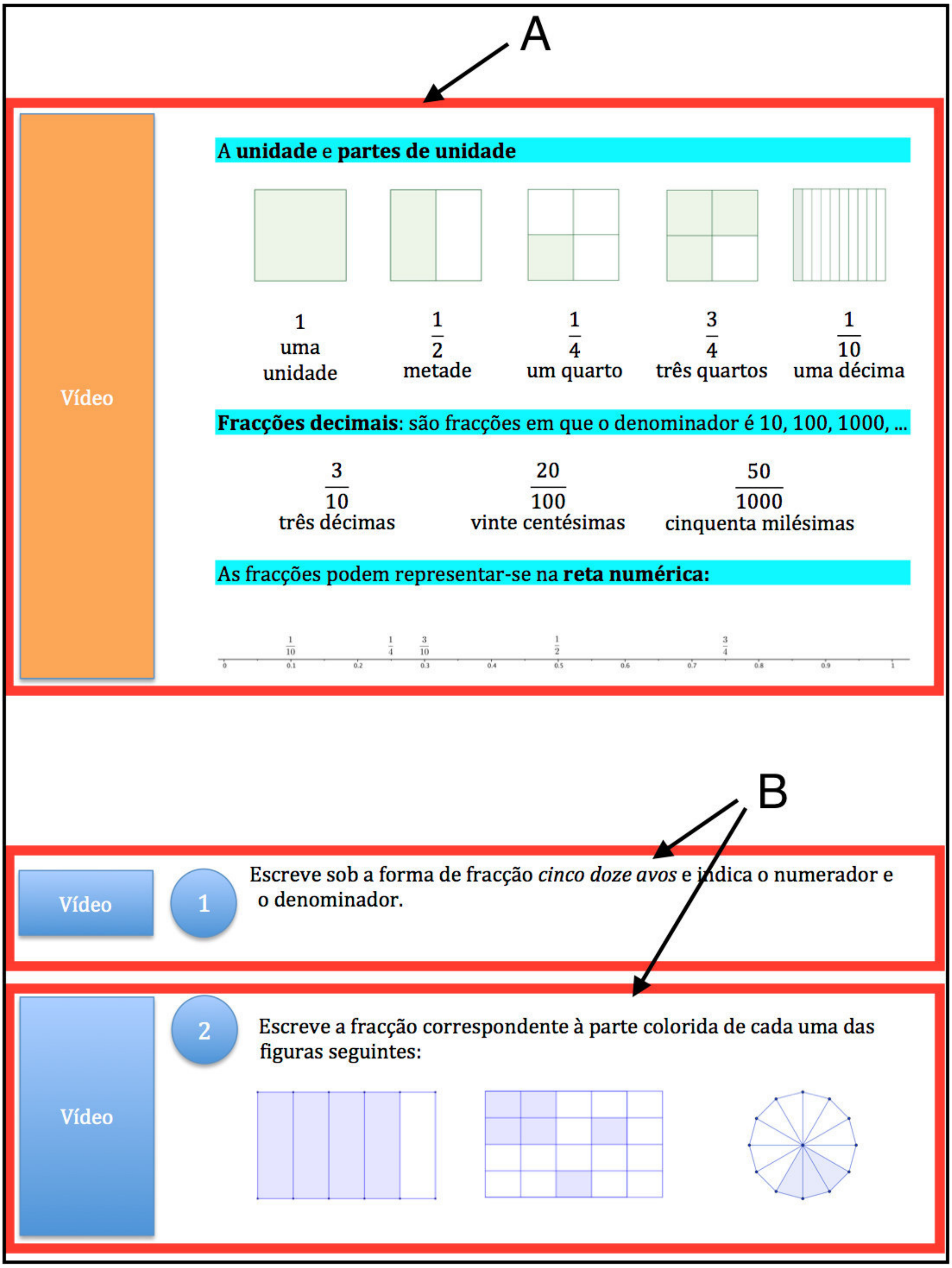

Figure 5. Page layout for the activities to study at home that is used to trigger Augmented Reality contents. A - Theoretical video lectures. B - Practical video lectures.

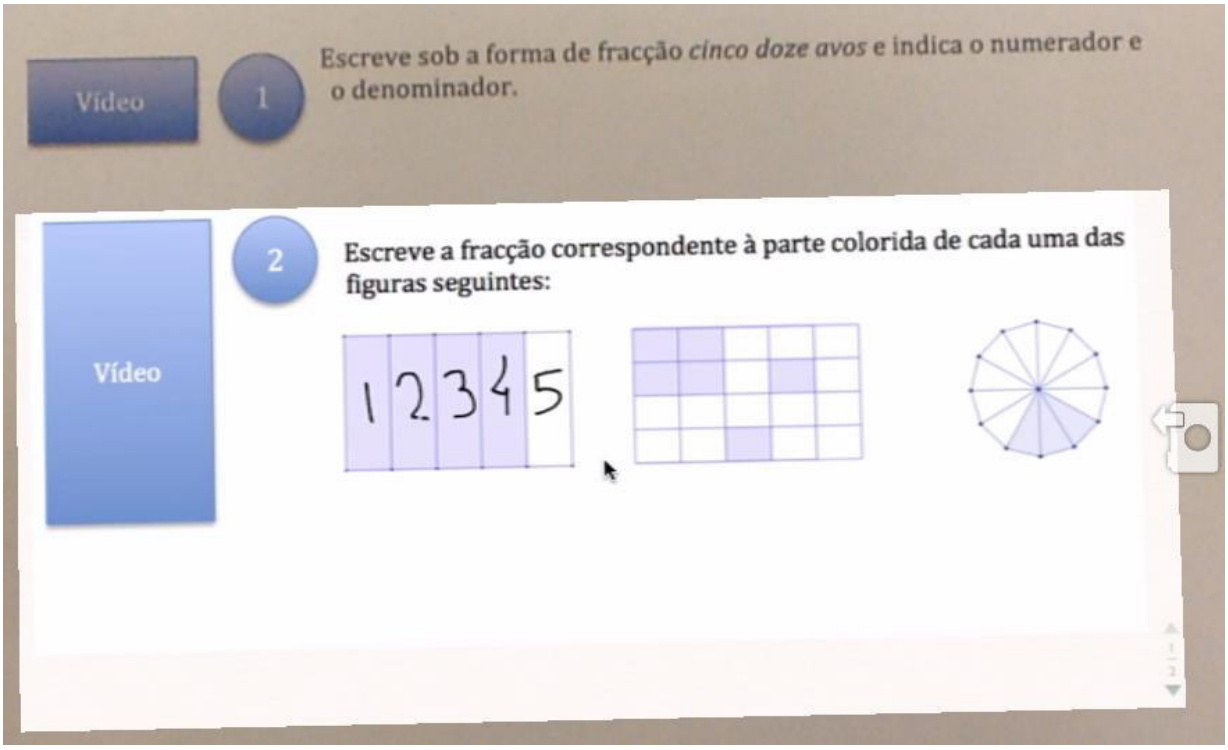

Figure 6. The student can study the lectures at home using Augmented Reality.

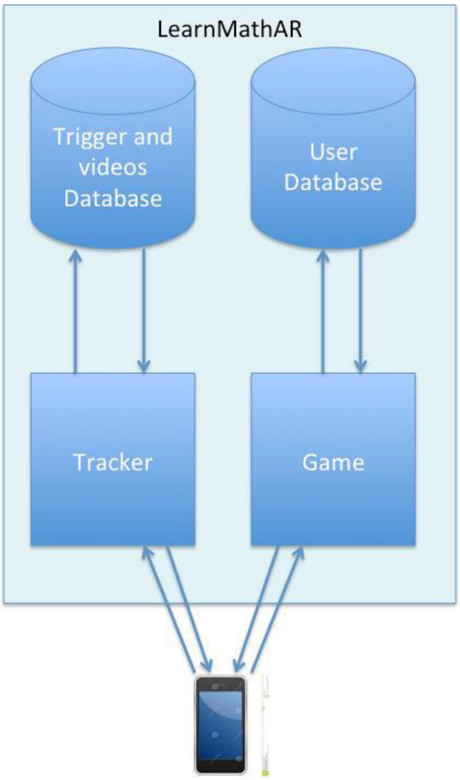

Figure 7. The application architecture includes a tracker and a game manager.

Producing course contents requires too much time and cost. An alternate way to produce course contents is to use the textbook companies provided materials, such as PowerPoint files, lecture videos and exercises that are readily available for teachers to create augmented reality activities for students.

In this way, teachers can use their own produced materials or textbook companies contents to create their own activities targeted to the particular needs of each class and individual student. We believe that this can be very motivating for students and it also helps in delivering lectures, hands-on activities and customized study modules. This is a main advantage of using augmented reality for education because teachers can tailor activities to each student.

CONCLUSIONS

The increasing processing power of mobile devices, the increasing number of augmented reality applications and the increasing number of mobile devices, makes possible the use of augmented reality in the classroom.

Math teachers have a difficult situation. Studying math is many times a cumbersome task. Low achievement in mathematics education has been an increasing problem in the recent years in Portugal.

In this paper, we show that using Augmented Reality to produce interactive materials can be motivating for students and contribute to extend the class into a virtual space where students can have more time practicing problem solving.

We show that technology is accessible and easy to use by math teachers and students. In this paper it is explored the creation of educational activities supported on marker based and marker less augmented reality technologies for teaching and learning mathematics.

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LEARNING MATHEMATICS FROM MULTIPLE REPRESENTATIONS: TWO DESIGN PRINCIPLES

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This paper describes two design principles for designing mathematics tasks using technology. These are: The parallel instantiations principle. *Presenting students with a large number of non-prototypical instantiations simultaneously and non-transiently perturbs their thinking and supports thinking-in-change.* The discriminating tools principle: *Discriminating tools enable children to differentiate between the tools' feedback to acquire knowledge from their use. Students will learn within the task rather than merely from the task. These principles were first developed in a study on 9-11 year olds' geometric defining and later applied and amended in a larger study that encouraged 9-11 year olds' conceptual understanding of fractions. The paper presents how the principles were applied within the two studies.*

Design-based research, design principles, task design, exploratory learning environment, multiple representations

INTRODUCTION

Despite the plethora of technology-based applets, games and websites available for students and teachers, there remains a significant proportion that are developed by those with technological understanding but not necessarily with knowledge of mathematics education. We can generally consider two ends of the spectrum. At one end, perhaps due to lack of imagination or due to lack of resources, there are the 'low hanging' fruits of simple flashcard-like arithmetic, matching games, drag-and-drop onto number lines or simply inputing a number and receiving basic feedback. At the other end those with often very complex and difficult-to-access or use in class. We see a gap in those more interactive pieces of software that involve virtual manipulatives, perhaps because it may not be clear how to design these. One of the aims behind the design-based research that we have been undertaking is to seek, in the context of different projects, principles for designing this type of mathematics software.

In this paper we discuss two principles that originally evolved from a PhD study (Hansen, 2008) focusing on the geometric defining of 9-11 year olds. The principles were later implemented in a larger study (www.iTalk2Learn.eu) that aims to develop an open-source intelligent tutoring platform that supports fractions learning for students aged 5 to 11. We offer these design principles as a contribution to the design-based research community. We reflect upon how the principles worked within two studies. The paper is outlined as follows. The remainder of the introduction provides the wider methodological context of both studies, design-based research, outlining how design principles are a common outcome from this process. The second and third sections introduce *Quads* and *Fractions Lab*, the environments from each study utilising virtual manipulatives, and a selection of the tasks students completed within them. The principles themselves are introduced in the fourth and fifth sections. Within each section a justification from the literature related to the principles and data or findings are shared. The conclusion brings the two principles together, demonstrating their symbiotic relationship.

Educational design research involves the development of a tangible outcome that could be an educational product, process, programme (of CPD) or policy (McKenney & Reeves, 2014) using interventionist, iterative, process-orientated, utility-orientated and theory-orientated methods (van den Akker, Gravemeijer, McKenney & Nieveen, 2007). Because design research is situated within numerous domains, methods vary, and the outcomes are unique and often descriptive because the designer determines the next steps from what the specific context dictates (Visscher-Voerman & Plomp, 1996). However, design experiments typically include a phase that produce trustworthy resulting claims (Cobb et al, 2003:12), some of which can be presented as design principles (McKenney, Nieveen & van den Akker, 2006; Wang & Hannafin, 2005). Such principles are situated between "scientific findings, which must be generalized and replicable, and local experiences or examples that come up in practice" (Bell, Hoadley, and Linn, 2004:83) and are often presented as heuristic guidelines (van den Akker, 1999) which are intended to "help others select and apply the most appropriate substantive and procedural knowledge for specific design and development tasks in their own settings" (McKenney, Nieveen & van den Akker, 2006). As design principles are refined by others adapting them to their own experiences (Bell, Hoadley, and Linn, 2004), they become more fine-tuned.

Here we present two principles for designing environments using virtual manipulatives and their associated tasks. The principles were identified after observations of how students were interacting with *Quads* while following three different tasks. They later acted as guiding criteria for design decisions and were fine-tuned when designing *Fractions Lab*. These design principles have their roots in cognitive load and instructional multimedia aids theories and a selection of the literature informing the original design decisions that led to these principles is discussed here. *Quads* and its three games, and *Fractions Lab* with its related tasks are discussed below; the principles follow.

QUADS

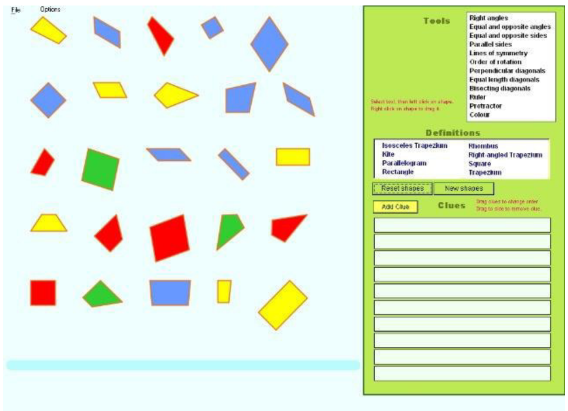

Fig. 1: Quads (Hansen, 2008)

After three previous design iterations, Hansen (2008) designed *Quads*, a virtual environment inspired by the popular board game, "Guess Who?" (Milton Bradley Company, 1987) that enables students to play three geometry-definition games involving quadrilaterals. Essentially, 25 instantiations (referred to as *mightbes*) are positioned on the screen (see Figure 1). In pairs, students (the "clue-setters") select one figure/definition and generate property-related clues for a second pair (the "clue-followers") to follow in order to identify the chosen figure/definition. Clues are generated using *musthaves*. These are inclusive statements that refer to the properties those particular figures 'must have' to belong to a definition. A further game explores notions of necessary and sufficient properties. *Quads* were trialled by 32 students aged 9-11 years from four schools.

FRACTIONS LAB

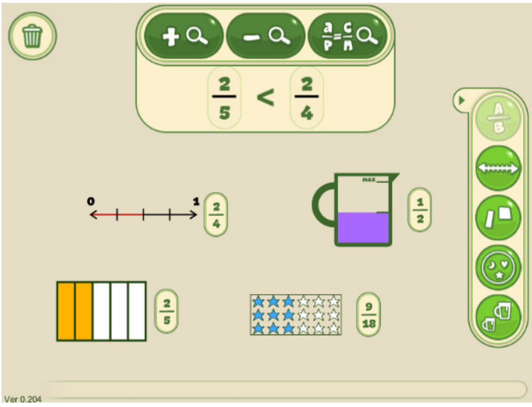

Figure 2: Fractions Lab (www.iTalk2Learn.eu)

Fractions Lab is an exploratory learning environment that acts as a stand-alone program or as a component of the iTalk2Learn project's (www.italk2learn.eu) intelligent tutoring system. Students are given tasks that require them to construct models from a range of representations (see Figure 2) and act upon them to challenge common fraction misconceptions (Hansen, 2014). Tools enable a student to change the numerator and denominator, partition the models, change the colour or copy a fraction. The addition, subtraction and comparison tools (at the top of the screen in Figure 2) allow students to check their hypotheses. The iteration of *Fractions Lab* used in this discussion involved 32 students aged 9-10 and 37 students aged 10-11 years from one school.

THE PARALLEL INSTANTIATIONS PRINCIPLE

When presenting students with non-prototypical instantiations simultaneously and non-transiently, children's thinking is perturbed and this supports thinking-in-change.

When students create their own representations of a concept, it is limited to their own understanding and often reflects misconceptions. However, presenting representations to students can lead to inappropriate interpretations (Handscomb, 2005). There is a wide range of factors that influence learning using multiple representations, including the number of representations that are presented, either at any one time or at some point during the session (Ainsworth, 2006), as well as the mode of presentation. Multi-modal representations support the dual processing in the working memory (Mayer, 1999; Sweller, 1999). In our work we have used static and dynamic representations with symbolic text because "pictures have more features available for processing than do words, and pictures may help access meaning more quickly and completely than words … text conditions also allow the learner to process the verbal information at the learner's pace" (Najjar, 1998, p.312).

The parallel instantiations principle in Quads

Great care was taken to design 25 instantiations (*mightbes*) (see Figure 1) ensuring a range of *musthaves* within each definition and a variety of prototypical and non-prototypical representations that can be refreshed. Each time the game starts over, or the 'new shapes' button is pressed, the instantiations change within programmed constraints. The instantiations were designed to challenge students' 1) visual perceptions of individual instantiations and 2) understanding of what figures constitute a definition. For example a figure may appear as if it has right angles, but on closer inspection with the protractor tool it has none. In relation to 2), the highlighting function shows the figures that sit within a particular definition.

Although an "excessive number of representations rarely helps learning" (Ainsworth, 2006) the number of instantiations (25) was settled upon through consideration of the range of definitions and *musthaves* to be covered and the physical space on the screen, as well as needing to provide counter-examples (Fischbein, 1987).

The highlighting function was used by students to identify the instantiations that exemplified a given definition. This often challenged students' pre-conceptions. For example, in every instance that a pair of students (n=5) selected the kite definition they were surprised that the set included instantiations of squares and they made comments such as "That doesn't look like a kite, that's a square" or that they "don't look right." Many of the other children tended to initially express surprise or admitted they were challenged by the figures presented to them within the highlighted sets. This was seen within a number of definitions, for example squares in the rhombus set (2 pairs), squares in the right angled trapezium set (3 pairs), squares in the isosceles trapezium set (1 pair), squares in the trapezium set (1 pair). However, by the end of the task students were able to define each set and competently explain why all the instantiations sat legitimately within the definition (Hansen, 2008).

The parallel instantiations principle in Fractions Lab

In *Fractions Lab* it is possible to present students with tasks that utilise four different graphical models (area, number line, sets and liquid measures), each with its associated fraction symbol (see Figure 2). One task provides four fractions and asks students to find the odd one out. Another task asks students to make a fraction using each of the models. In tasks where students are asked to manipulate existing fraction models, they are able to do so while leaving the original intact, something that is not easily achievable using physical manipulatives (Olive & Lobato, 2007).

We have emerging evidence that parallel instantiations have an impact on students' understanding of how fractions may be represented (Hansen, Geraniou & Mavrikis, 2014). For example, when we asked 35 10-11 year old students to draw as many ways as possible to show ¼ (before using Factions Lab), none drew a number line or in a jug. Yet, after using *Fractions Lab* for a relatively short time (10-15 minutes), 89% of students later added a measuring jug and 77% a number line. Furthermore, in an open-ended question about what they had learnt when using *Fractions Lab*, 18 of the 35 students stated that they learnt more about the way fractions were represented. Their written comments included statements such as, "Fractions can be presented in many ways", "You have a variety of choices to represent fractions", and "You can show fractions with liquid". This is worthy of note when most instruction materials and teachers, rely on the limited part/whole representation (Alajmi, 2012; Baturo, 2004; Pantziara & Philippou, 2012) and our personal experience suggests that students would not provide such reflective statements around multiple fraction representations unprompted.

THE DISCRIMINATING TOOLS PRINCIPLE

By providing tools to carry out the tasks that require pre-requisite knowledge to achieve the objective (but if undertaken manually would detract from it) students will learn within the task rather than merely from the task. Discriminating tools enable children to differentiate between the tools' feedback to acquire knowledge from their use.

Novices (in this case, students) often fall back on weak problem-solving strategies because they do not possess the schemata to support the work they are undertaking (Sweller, 1999). This is an issue when designing for challenging or complex mathematical concepts such as geometric defining and fractions because of the multiple facets to their nature. In light of this Hansen (2008) developed tools to free up the working memory to focus on achieving the objective of the task (Kalyuga, Chandler & Sweller, 1999) and provide procedural information that is prerequisite for learning to take place in a complex task (Van Merrienboer & Kirschner, 2007). Students consider feedback from tools, discriminating between instantiations and their built-in epistemic constraints. In doing so the tools are catalyst for learning within the task. For example, a protractor tool providing a figure's interior angles enables students to focus on their line of thought regarding the property of a 'number of equal and opposite angles' while they investigate if a figure contains them without having to manually carry out measuring procedures. In *Fractions Lab* a 'find equivalent' tool showing how a fraction symbol changes while a model is partitioned enables students to think about the relationships within and between equivalent fractions rather than carrying out rote multiplication procedures on the numerator and denominator.

The discriminating tools principle in Quads

The discriminating tools in *Quads* requires students to be attentive to the properties of instantiations. A tool can be selected from a menu on the screen (see Figure 3) and data about figures are displayed in static form (e.g. the internal angles are given for the protractor) or dynamic form (e.g. order of rotation, see Figure 4).

Figure 3: Quads discriminating tools

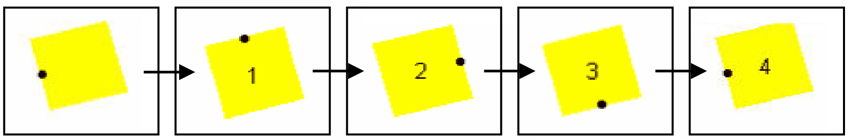

Figure 4: Five screen shots from the Order of Rotation animation showing a square with Order 4

When clue-setters or clue-followers, students were able to efficiently work through the properties of the *mightbes* to make clues or to identify the shapes their partners had selected. In an earlier iteration with a physical manipulatives task, Hansen (2008) noted that students did not engage with properties to the same depth, often using incorrect visual cues. Using the tools, the students worked through many more instantiations than they could have using a real protractor or ruler and at the same time.

The discriminating tools principle in Fractions Lab

Fractions Lab uses tools to manipulate fraction representations by partitioning to make equivalents), adding and subtracting (see Figure 2, top of screen). Students are able to make equivalent fractions using the partition tool. As the models are manipulated, their corresponding symbols change. The addition and subtraction tools use animation to show the results of fractions being joined or taken away. These three tools provide feedback to the students to check their activity.

When asking 35 10-11 year old students an open-ended question about what they had learnt after using *Fractions Lab* for a short duration (10-15 mins), 60% referred to an aspect related to the discriminating tools (15 referred to addition or subtraction and six to equivalence, others referred to the size of fractions or representations). For example, "Before you add two fractions together you need to make sure that both denominators are the same", "It has made me more confident at adding and subtracting fractions" and "You can double the numerator and denominator and it equals the same."

The equivalence tool is discussed in detail in Hansen, Mavrikis, Holmes & Geraniou (submitted). Here we briefly discuss the addition tool. In *Fractions Lab*, if a student attempts to add two fractions with unlike denominators, *Fractions Lab* will refrain from providing the answer. It is only when the denominators are the same that an answer will be given. In this case, when students attempt to add two fractions with unlike denominators the feedback they receive typically does not give students their expected or desired outcome (i.e. the answer). As a result, the system encourages them to use the equivalence tool to make fractions with like denominators before adding them together. The feedback from the addition tool appears to have supported students to learn within the task rather than merely from it and this situated abstraction is a stepping stone to adding fractions with unlike denominators procedurally with understanding.

21 9-10 year old students who had not been introduced to addition and subtraction of fractions with unlike denominators were asked to consider the work of a fictitious student (2/3 + 1/6 = 3/9) while using *Fractions Lab*. When asked to provide an explanation for why they thought the student was correct or incorrect, 19 out of the 21 stated the student was incorrect and gave a plausible justification (see Table 1).

Comparing the size of various fractions, e.g. “2/3 and 1/6 put together are bigger than 3/9 and 2/3 is bigger than 3/9 to start with.”	2
Identification of fictitious students’ misconception, e.g. “he has added the numbers together.”	4
Refers to need to change denominator / denominators need to be the same, e.g. “I tried it on *Fractions Lab* so the denominator needs to be the same.”	5
Refers to partitioning / equivalence, e.g. “because if you partition 2/3 it will go to 4/6 which will make it easier.”	8

Table 1. Types of response given

Of the two who did not provide a clear, correct explanation one student wrote, “It looks right but it isn’t” and the other wrote “I think it is because you don’t add up the denominator. I am not so sure though.”

CONCLUSION

The presentation of multiple parallel instantiations and discriminating tools that enable students to act upon the instantiations are very difficult or even impossible to re-enact with physical manipulatives. We, therefore, offer these two design principles as guiding criteria for designing affordances that have the potential to support students’ conceptual understanding in mathematics.

There is a symbiotic relationship between the two principles we have presented. Students’ thinking is initially perturbed by being presented with non-prototypical instantiations, yet feedback provided by the discriminating tools enables students to acquire knowledge within the activities rather than from them. For example, in *Quads* the non-prototypical instantiations forced students to use the discriminating tools to check properties more than earlier iterations of Hansen’s (2008) work had elicited. In *Fractions Lab* the constraints built into the addition tool encouraged students to find equivalent fractions as a step towards adding fractions with unlike denominators successfully. We claim therefore that the students used the feedback to acquire knowledge from the use of the discriminating tools.

Acknowledgment

We would like to thank Prof Dave Pratt for his doctoral research supervision in which Quads was developed and the principles evolved. Part of the work described here has received funding by the EU in FP7 in the iTalk2Learn project (318051). Thanks to all our iTalk2Learn colleagues and particularly Testaluna s.r.l. for their support and ideas and implementing Fractions Lab.

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STUDYING THE PROCESS OF DESIGNING DIGITAL EDUCATIONAL RESOURCES WITH THE AIM TO FOSTER STUDENTS' CREATIVE MATHEMATICAL THINKING

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The present paper studies the process by which members of a collective with diverse expertise and experiences, engaged in a joint enterprise to design an e-book targeting the stimulation of students' creative mathematical thinking (CMT). Our main aim was to investigate the role of this digital artifact (c-book: c for creativity) in helping these members to gain some understandings of each other's perspectives and knowledge. Our findings showed that the creation of the c-book evolved in four phases incorporating learning mechanisms and challenged the co-designers to make crossings between the boundaries of diverse expertise.

Keywords: Social creativity, c-book, creative mathematical thinking

INTRODUCTION

In this paper we report a research aiming at shedding light on the process by which professionals representing different communities of practice (mathematics teachers from secondary education, teachers from engineering / vocational education, researchers in online teacher communication and collaboration or environmental education as well as academics) co-designed digital educational resources (c-books), with the aim to challenge and foster its users' creativity through constructionist and investigational activity. Since creativity is expected to be at the core of both the process of jointly developing the c-book and the product itself we were particularly interested in identifying their characteristics and their potential for the teaching and learning of mathematics.

The designers' online discussions and outcomes that we analyzed here, took place in the context of a European R&D project called 'M C Squared' focusing on technologies affording creativity both in collaborative designs and in using the media to engage in mathematical thinking. The project aims at developing the 'c-book' ('c' for creativity), a technology supporting collaborative authoring, diverse constructionist widgets and data-analytics configurable by its authors for the design of creative educational resources for CMT (the c-book units). This technology has been thought to go along with the generation of some specifically generated collectives of designers characterised by their diverse disciplinary backgrounds, expertise, history and membership in different communities of practice, which are defined in literature as Communities of Interest (CoI) (Fisher, 2011). Four such CoIs were formed in the mc2 project whose members were as diverse as developers of mathematical digital media, publishers of mathematics educational materials, researchers specialized in creativity or in math education, creative math teachers and students. In this paper we discuss a group within one of these CoI from Greece.

THEORETICAL FRAMEWORK

The main theoretical construct that framed this study and helped us to understand the collective design of communities with diverse expertise working together to create joint constructions in the form of c-book units was 'Boundary Crossing' (Akkermann & Bakker, 2011), in which the design, sharing and modification of resources are considered as mediations and opportunities for learning.

Simultaneously, the wider theory and research of creativity informed our knowledge as regards Social Creativity and Creative Mathematical Thinking.

We perceive mathematical creativity at the school level as a process which results in novel or insightful solutions to a given problem and allows an old problem to be regarded from a new angle (Spiraman et al, 2011). Novelty is interpreted as having a local character, i.e. something that may not be novel to experienced mathematicians but from the perspective of the learners it can be judged as novel and therefor as creative (Askew, 2013). This issue brings to the fore the distinction (Craft, 2000, 2001) between 'high' creativity and 'ordinary' creativity. The former, (big C) describes 'great works' by experts or gifted persons which change knowledge and/or our perspective of the world. The latter (small c) recognizes that all pupils can be creative and arises - for example - when a student creates a solution to a novel problem or connects together two seemingly disparate ideas. At the level of educational designers of mathematical resources our view on creativity is in accordance with Ervynck's (1991) who considers as creative mathematical activity every designer's attempt aiming to reform or improve the network of concepts of a mathematics curriculum for pedagogical reasons, even if new mathematics is not generated. In this paper we sought for creativity in both the process and the products of collaboration among teachers so as to produce material which is expected to offer students some of the above creative opportunities. The kind of creativity which is developed in collectivities through joint enterprises is described as 'middle c' creativity and is needed to create strategies, to find ways to make the differing views of individuals capable of existing together and to produce collective learning outcomes, including an elaborated understanding of the learning topics addressed (Eteläpelto & Lahti, 2008).

In MC2, the engagement of teachers in this kind of collaborative design targeted the empowerment of students' CMT. Although the use of digital media can facilitate the engagement with CMT in unprecedented ways (Healy & Kynigos, 2010), there is a lack of both pedagogical designs targeting CMT and corresponding technologies supporting them. Even in the case of digital tools with great potential for enhancing CMT, such as e-books, the pedagogies that accompany those tools are often outdated, following the traditional teaching and learning models. Moreover, the process of designing them is rather restricted and limited to the authors, instead of being open to collective design that leaves space to the designers for sharing creative ideas. In the context of MC2, learners' engagement with CMT was planned to be designed in collectives with the use of digital media, resulting in a new genre of authorable e-book, the **'c-book'**. The technology of a c-book differs from the one of an e-book as it includes dynamic widgets and interoperability, anticipates collective design incorporating an authorable data analytics engine with appropriate interface, drawing on end-users' and resource designers' interactions. Designing digital educational resources for CMT can be therefore viewed as a 'squared' creativity challenge, since it requires not only fostering students' mathematical creativity but also situating the design process itself within a socio-technical environment that can boost educational designers' creative potential (Kynigos, 2014).

This environment is aiming to enhance and stimulate ***social creativity*** in designing for CMT through the generation of ***Communities of Interest*** (CoI) (Fischer, 2001). A CoI has a heterogeneous character, as each of the participants of a CoI represents a group of practitioners from different domains or communities while all of them target to resolve collectively a problem and achieve common understandings overcoming their cultural differences. The "symmetry of ignorance" in the process of framing/solving design problems and creating new artifacts/understandings, triggers the emergence of social creativity (ibid). In this context, such CoI is anticipated to operate as a ***socio-technical environment***, i.e. a living entity where everyone might be, at the same time, "designer" and/or "consumer", in the process of co-designing dynamic re-useable and re-constructible educational materials for CMT. Interconnecteness, different perspective-taking, knowledge exchange and integration between diverse domains are features of this environment expected to provide more opportunities for creative thinking and learning. The members of a CoI in order to integrate these features in their communication need to cross the boundaries between the different sites and the c-book is anticipated to operate as boundary object by fulfilling a bridging function (Star, 2010). The process of ***boundary crossing*** entails four learning mechanisms (Akkerman and Bakker, 2011): a) identification of the intersecting practices, b) coordination of practices through establishing routinized exchanges to facilitate transitions, c) reflection leading to perspective-making and perspective taking and d) transformation that provoke changes in practices or even the creation of a new in-between practice. We perceive these mechanisms as indicators for teachers' creativity, thus in our study we sought for them during the design process.

The present study is a first attempt to explore this kind of innovative design and identify its characteristics, as well as the role of the c-book environment in the design process. More specifically, we were interested to investigate its potential for helping the CoI members to gain some understandings of each other's perspectives and knowledge on how to design for creativity.

THE C-BOOK ENVIRONMENT

The c-book provides ***space for organized discussions (CoICode)*** in two parallel interfaces: a threaded forum discussion view and a mind-map view (fig.1) and gives designers the possibility of switching from one to the other with a toggle button. They can attach and refer to widget instances which reside in the c-book unit under construction. Users can characterize the nature of their contribution by selecting between different semantics designed to promote social creativity in the design process: (a) Alternative: Expressing opinions, statements, arguments, initiating design process (b) Contributory: Adding, cumulating, building to an existing alternative, questioning, refining, focusing, narrowing, expanding it. (c) Objecting: Expressing objection to alternative, either by directly contradicting an idea, using disputational style, or by proposing another alternative (d) Off task: Social interactions not associated with the task at hand: greetings, expression of humor, emoticons, phatic elements. (e) Management: Management of the progression of the task itself: planning what is to be discussed, who does what, if a problem is solved or not. In these semantics, the designers can write text, links, attach files or widget instances (software such as Geogebra is a widget factory and a microworld of this factory is a widget instance).

The ***platform*** is the space for authoring where the students interact with the c-book. It incorporates pages with dynamically manipulated widget instances accompanied by corresponding narratives.

METHOD

In this paper we study the development of the c-book unit 'Windmills' which was meant by the researchers to operate as a sparker for social creativity. It was planned to be jointly created by the following 9 members of the Greek CoI (each one of them belongs to more than one communities of practice): Tom (Computer Science developer), Dimitris (experienced mathematics teacher, designer of math resources with digital tools, master degree in the didactics of mathematics), Katerina (master degree in the didactics of mathematics) and Areti (master degree in pedagogical use of technology) - both of them with limited teaching experience in mathematics classrooms - Popi (experienced mathematics teacher and teacher educator for the pedagogical use of digital tools in mathematics), Marios (phd-student and Informatics teacher), Foteini (teacher in engineering education, phd in the pedagogical use of digital tools for the vocational education), Yannis (university teacher) and Elissavet (moderator of this discussion, experienced mathematics teacher, phd in the domain of mathematics teachers' education with the use of digital tools). Our data were: (a) 75 contributions in the CoICode, (b) the files attached in the CoICode (c) the pages in the platform of the c-book (the widget instances and the respective narratives). For analysing the content of CoICode, we adopted the data grounded approach (Strauss & Corbin, 1998).

RESULTS

The analysis of the 75 contributions posted in CoICode during the first cycle of MC2 showed that 16 of them were related to technical and organizational issues, 4 were off-topic and the majority (55) concerned the development of the c-book unit. It should be noted that Tom and Marios did not participate in this cycle of the design. From the analysis of our data we identified that the process of developing the c-book unit was evolved in four phases incorporated different characteristics: 1) Searching for mathematical ideas and concepts, 2) Implementation of the raw ideas 3) Pedagogical and didactical contextualization 4) Transition to the c-book: texts accompanying the constructions. These phases are interconnected and sometimes not clearly limited, as one of them may penetrate or overlap the previous and evolve together overtime. Below, we describe their characteristics in terms of both processes and products.

Phase 1: Searching for mathematical ideas and concepts

This phase starts with the exchange of files depicting different types of windmills all over Greece (Elissavet and Katerina) or the way they function (Foteini), aiming to operate as starting point and trigger the subsequent expression of the first ideas. The shapes of the windmills and their operation turn the participants' attention on the kind of software that should better integrate those characteristics. Their first ideas are related to Turtlewords (Dimitris, Elissavet and Areti), as they were all fully aware of its functionalities for rotation with the variation tool. Then, Elissavet (Fig.1, 27/3) proposes a brainstorming concerning the concepts embedded in a windmill and mentions some of them: regular polygons, isosceles triangle and rotational symmetry of k grade (depending on the number of sails) for the wheel as well as cone and cylinder for the main building. Dimitris (Fig.1, 28/3) adds some more complicated ideas, for example when an ant is walking on a wing of a windmill as it turns the ant's orbit might looks like a helix. Katerina (fig.1, 28/3) also adds concepts such as parallelograms, rectangles, angles, turns and curves in 3d space. She proposes ways of connecting them and the use of Turtlwords in order to investigate the properties of parallelograms and of different kinds of triangles as sails of the windmill. Popi (fig.1, 30/3) raises some questions

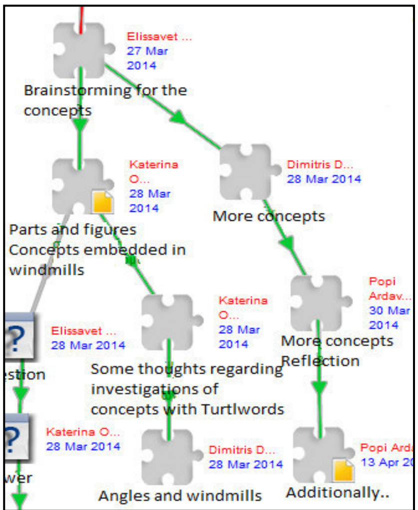

Figure 1

about the mechanics and architecture of windmills, the locations which favor their operation, the time required for a complete rotation of two similar windmills with similarity ratio ½, the way the sprockets work etc.

In this part of the discussion we observe that most of the concepts come from geometry and have to do with the perceptions of windmill and its parts as geometrical figures. Popi, who is experienced in both mathematics teaching and technology introduces the time parameter and searches for a context where a windmill would be integrated. At the same time, the participants start thinking which of the available software should better be related to those concepts. The discussions of this phase seem to be focused rather on ideas than on actions.

Phase 2: Implementation of the raw ideas

At this point the CoI members start implementing the ideas expressed during the 1st phase. The constructions are individual and we note a lack of intervention in the other members' constructions. Simultaneously new ideas are invented concerning mathematical concepts, technological tools and problems. Elissavet gives a logo-code in Turtleworlds that initially constructs an equilateral triangle and then rotates it in order to take the shape of a six sails wheel. Taking this perspective, Dimitris and Katerina create half-baked logo-codes constructing an equilateral triangle using trigonometry and a parallelogram for representing the sails respectively. Areti also attaches a file with a half-baked code in Turtlwords (variables a, b, c) that initially constructs an open jagged line of two equal sides of length a, turns depended on c and the third side of length b. The dynamic manipulation of variables with the variation tool, constructs an isosceles triangle. Subsequently, the triangle turns and it is rotated n times with the variation tool, so as to give the impression of the sails of the wheel. At the same time, Dimitris and Popi start expressing their ideas with the use of Geogebra. Dimitris constructs a dynamic figure of a sail and Popi a simple model of the wheel in which she has incorporated the time parameter and the possibility of changing the number of sails and the radius of the wheel by dynamically manipulating the corresponding sliders (fig.2). The investigation of the model brought to the fore algebraic concepts and relationships and led to direct proportional amounts, linear, quadratic and multiple branch functions, since the production of flour over time is depended on the number of sails and on the radius of the wheel. Dimitris makes a further refinement of the concepts and classifies them into: (1) structural relationships and construction of a windmill as a 2d shape and (2) movement of the windmill with the slider. Elissavet adds to this distinction a third group: the view of the wheel from different perspectives as a solid 3d shape, i.e. its transformation from a plane to a solid shape. She also emphasizes on the complementarity of the activities with Turtlewolds and Geogebra: the former needs a constructionist activity from simple mathematical concepts as structural units to more complicated ones, while the latter requires a de-construction of the model, resulting in the embedded concepts.

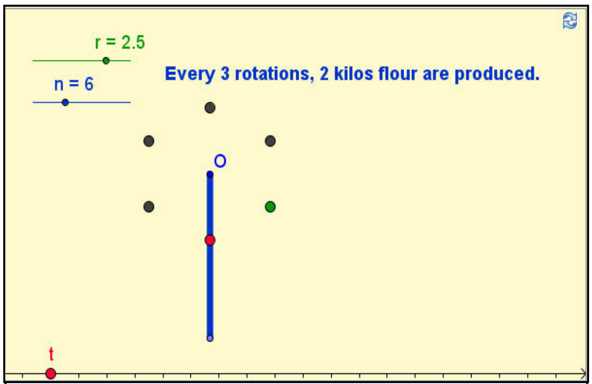

Figure 2

The discussions of this phase include both the expression of ideas and their implementation individually. The mathematical concepts are classified according to the way they are used, while they are also related to the functionalities of specific software tools. As the CoI members exchange their constructions we observe their smooth transition from one practice to another, effortlessly, just crossing the boundaries, without reconstructing them.

Phase 3: Pedagogical and didactical contextualization

In this phase the majority of constructions in the form of widget instances has already been completed and the next step was to put them on the platform as different pages. The CoI members discussed the optimal sequence of the activities encompassing their constructions, the targeted students' ages, the relation to the official curriculum and the openness (or not) of activities.

Foteini believes that this c-book unit should be addressed to specific school grades, while Dimitris' opinion is to put 'neither floor nor ceiling' in the targeted ages. From this disagreement the following dilemmas emerged: "*How can we expect our students to do creative mathematics with the use of the c-book, without having previously defined their ages?*" or "*Did defining in advance the ages of students limit our own creativity?*" Finally, they decided to include in the c-book a wide range of activities, starting from the simpler and gradually address them to students 12-17 years old, with the structure of the concepts not aligned to the official curriculum. Then, they had to choose which of the constructions should be incorporated in the c-book as well as their sequence on the different pages. They all agreed to start from the constructive activities with Turtlewords, giving students the opportunity to explore separately the figures – and the concepts of isosceles, equilateral triangle, parallelogram - which represent the sails of the windmill and then to continue towards more complicated concepts (for example rotational symmetry), using the initially investigated ones as structural units. The investigations of the model with Geogebra were decided to be posted after the 6th page, with the aim to offer students the experience of understanding functional relationships through their use (for example multiple branch functions, representing the operation of the windmill in different time intervals). We note that the multiple branch functions are taught in the 4th grade of secondary education; however, the investigation of the model could lead students of lower grades to understand this notion through its use (a typical example of mathematics-in-use). At the end of this phase Dimitris expressed the following alternative opinion: "*It does not really matter which activities we'll choose from Turtlewords or Geogebra. The widget will change when we start writing the texts. I believe that text and widget are a unit and not two different pieces*". The interrelation of widgets and texts was an issue of less importance till then and Dimitris' contribution revealed his conception of the c-book as a whole, integrating both instances and texts. Another issue was the form of the activities around these concepts, i.e. their openness and the level of instruction. Since most of the constructions involved activities on given micro worlds the participants expressed the need for thinking up ideas that would give students the opportunity to create their own constructions around windmills. Thus, for the last page of the c-book unit they chose to ask students to construct their own windmill.

Summarizing the content of this phase, we observe that the conversations are rather oriented to pedagogical and didactical ideas than in action, which was prominent in the previous phase. The CoI members are facing dilemmas regarding the pedagogy of their constructions (widget instances) and have to think of ways to stimulate students' creativity without decreasing their own creativity in design. The coherence of the activities had also provoked a number of contributions, since till this phase the constructions were products of individual work and, at this point, the need to be synthesized emerged. These constructions expressed different pedagogical and didactical approaches of the embedded mathematical concepts and the participants had to deepen on these concepts, their interrelations with the available tools and the way they should be communicated through corresponding activities. The CoI members made clear their perspectives (perspective making) and extended or synthesized the ideas and perspectives of the others (perspective taking). Now, the artifacts are accompanied with pedagogy and the participants start thinking about the form of the c-book, expressing ideas alternative to their traditional way of designing.

Phase 4: Transition to the c-book

This phase begins when the CoI members' constructions have just been put on the platform as widget instances, with the sequence which had already been discussed in phase 3. Now the participants are discussing the number of available tools and the kind of functionalities they should provide to students. They are also trying to connect them through narratives and real life situations, so as to make more attractive the students' involvement with the c-book. 'Eva's adventures at Cyclades!' was the title of the c-book unit, in which Eva, her father and a windmill owner were some of the heroes of the story. The difficulty in connecting widgets and texts is evident below:

Areti: Dimitris, how would you find a story, such as 'a boy trying to construct with a logo-code the windmill's wheel doesn't finally manage it, can the students help him?'

Dimitris: I believe that we don't need to ask something. If you have a story and a widget connected to it, the text needs to be challenging, otherwise the student would not use it, even in case he is asked to. I don't know if it's better to ask clearly students what we want them to do or not. If not, do we lose the story's coherence, as we continuously interrupt it with questions?

Areti: I'm confused. Do you disagree on the way me and Katerina changed the texts?

Dimitris: Texts resulting in a question constitute a safe teaching method. My objection is that I imagine the c-book unit as a book with texts, which is also a tool, something like the interactive books in Harry Potter. This kind of books does not need any questions in order to attract you…. Trying to correct the texts of Popi's model I inspired a new activity (page 12), in the form 'text-widget' (fig.3).

Foteini: It seems that the texts have the potential to re-formulate the widgets.

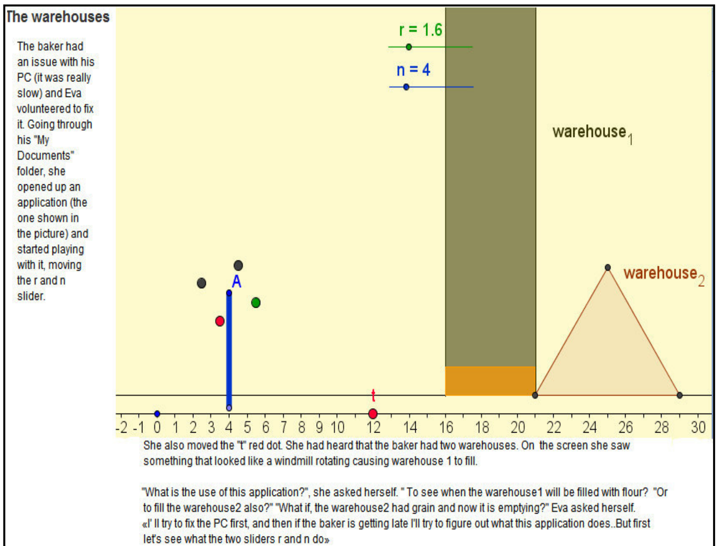

Figure 3

In this part of the discussions, the CoI members seem to reconceptualise the use of the c-book unit: Areti expresses the need for framing the widget with a real problem in order to stimulate students' interest. This approach differentiates the c-book from a micro world accompanied by a formal mathematical problem. The development of the c-book unit was an unknown and unfamiliar territory for all and the perspective of the experienced Dimitris favoured the development of new understandings and practices around its use. In this process, the c-book was the boundary object around which the participants interacted, reflected and finally transformed their knowledge. At the same time, new ideas incorporating both a story and a widget instance are generated: Yiannis proposes to give students a photo of a ruined windmill (without sails or roof) and ask them to repair or complete it or to put inside its conic roof a rectangular parallelepiped reservoir of maximum volume, for water storage. Furthermore new widget factories are introduced: Katerina exploits DME widgets and creates the respective story in which Eva starting from a 2d plan is challenged to create a 3d building with a windmill. These ideas reflect a new holistic perception of the c-book and the need for making their constructions more attractive using 'non-mathematical' material (photos, plans) in challenging situations.

RESULTS

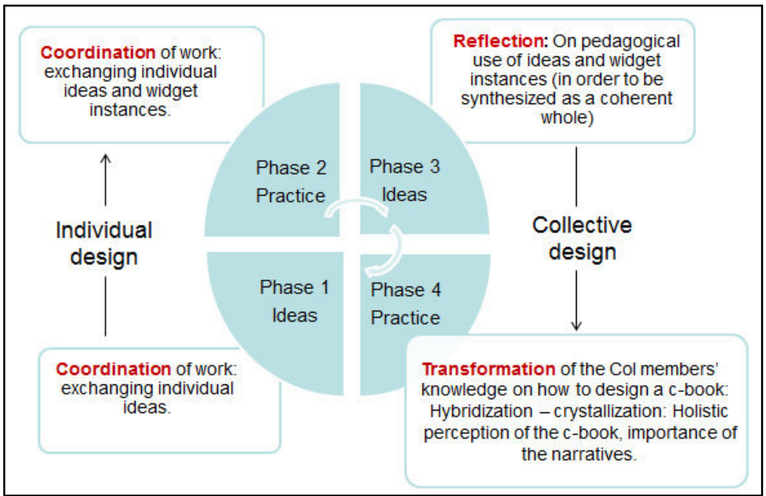

Figure 4

The results of our research reveal that co-designing the c-book unit was a process evolved in four phases that incorporated different characteristics and underlying learning mechanisms for the CoI members. The ideas about the product and the product itself were modified over time. Phases 1 and 3 included discussions rather concentrated on ideas, while phases 2 and 4 were oriented to practice (fig.4). In phases 1 and 2 the CoI members coordinated their work, collecting individual ideas and widget instances respectively, which became objects of pedagogical and didactical reflection in phase 3, in order to be synthesized as a coherent whole. Before the end of phase 3 and at the beginning of phase 4, the form of the c-book was outlined and the participants gradually acquired a more precise picture of what the c-book unit might be. Holistic perception of the c-book and the importance of the narratives were the outstanding characteristics of phase 4. At the end of this phase the c-book hybrid form was progressively crystallized and the transformation of the CoI members' knowledge on how to design a c-book aiming to stimulate students' CMT was evident, due to the contributions of the more experienced members who acted as "brokers" introducing new elements in the design process.

In conclusion, our findings showed that the creation of the c-book challenged the CoI members to express exchange and discuss creative ideas and became the boundary object around which they coordinated their activities, reflected on their pedagogy and transformed their knowledge about how to design for creativity.

Acknowledgment

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made of the information contained therein.The c-book technology is based on the widely used Freudenthal Institute's DME portal and is being developed by a consortium of nine partner organisations, led by CTI&Press 'Diophantus'.

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CONCEPTUALISING AXIAL SYMMETRY THROUGH THE USE OF CABRI ELEM WITHIN AN INTEGRATED LABORATORY APPROACH

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The main idea of the study herein presented is to verify if an integrated laboratory approach, exploiting the potentialities of both physical and technological instruments, can afford the construction of a geometric concept. The specific aim is to investigate the potential and the effectiveness of the interactive activity books of Cabri Elem within an integrated laboratory approach. This paper presents an on-going research project which involves pupils at 3rd and 4th grades dealing with axial symmetry. According to the early results of the project it seems that the integrated use of concrete manipulatives together with the activity books could guide pupils towards the progressive geometric conceptualisation of axial symmetry and its properties.

Keywords: Physical and technological manipulatives, Integrated laboratory approach, Cabri Elem, Axial symmetry

INTRODUCTION

Geometrical work at primary school is mainly about identifying geometrical properties in a drawing, constructing geometrical diagrams (which actually means realising a specific drawing), and using instruments to verify properties in a drawing. Unfortunately, the traditional approach to geometry often involves pupils in solving artificial problems (such as the ones concerning a gardener's troubles about a lawn in the shape of a diamond) and is thus based on calculations of lengths, angles, etc., and application of formulae for perimeters, areas, etc. On the other hand, dynamic geometry research has underlined that the dragging function appears to be particularly important for experiencing the necessity of relevant geometrical facts (Hoyles & Jones, 1998). The dragging function can also open up some kinds of semantic space (meaning potential) for the construction of a mathematical concept in which dragging modalities (strategies) are temporal-dynamic semiotic instruments that can create mathematical meanings.

Today scholars generally agree that technological instruments can play a crucial role in the processes of teaching and learning. Our research focuses on the impact of technology on the construction of geometric concepts at primary school, and aims at fostering an effective integration of new technologies in teaching practices.

This paper presents the analysis of an on-going teaching project involving 3rd and 4th graders. The sequence of learning activities (using both physical and technological instruments) will be described and studied in terms of the construction of mathematical meanings.

As far as technology is concerned, the focus is not just on the choice of the technology itself but rather on the integration of the technologies in teaching practices. The environment Cabri Elem has been chosen as it seems to meet the needs of primary education, offering ready-made resources to help teachers take more advantage of dynamic mathematics environments. Its "activity books" consist of a succession of pages incorporating a sequence of tasks. These are created in the Cabri Elem Creator task design environment and can be used directly (like an "App") by teachers and students in the more restricted environment of Cabri Elem Player. A sequence of tasks can be created changing the values of the various didactical variables, thus provoking an evolution in student strategies.

The specific aim of this research is to investigate the potential and the effectiveness of the use of activity books from the collection *123... Cabri* (http://www.cabri.com/special-pages/bett2010/) within an integrated laboratory approach, in which pupils manipulate both physical and technological instruments, for fostering the conceptualisation of axial symmetry.

THEORETICAL FRAMEWORK

Herein, attempting to emphasise the crucial role of the integration between the physical/real and the technological/virtual environments, we focus on two theoretical aspects, both aimed at the development of geometrical thinking. The first concerns the use of manipulatives, and the second refers to the use of technological instruments.

According to Arzarello (2014), the exclusive use of one of the environments, the physical/real or the technological/virtual ones, does not generate the same cognitive enhancement. It is the integration between the two interlaced environments that creates, in a natural way, a sort of dialectic among different types of knowing. A first type of knowing is declarative knowledge, typically expressed in verbal assertions and theory-like elaborations. The second is applicative knowledge that is performative and gets expressed in actual performances. Broudy (1977) argues that we must go beyond the "knowing that" (declarative knowledge) and the "knowing how" (applicative knowledge). There is a type of knowing, different from the two other types, that he called "knowing-with", which furnishes "a context within which a certain situation is perceived, interpreted and judged" (p.12). The "knowing-with", an essential and not always recognised aspect of developing fluencies with tools and techniques in mathematics education, is the most difficult type of knowing to obtain.

Using manipulatives to construct geometric concepts

Manipulatives are defined as concrete objects used to help students understand abstract concepts in the domain of mathematics (McNeil & Jarvin, 2007). Recognition of the importance of children's manipulative experiences is not new; many researchers have highlighted the way in which children whose mathematical learning is firmly grounded in manipulative experiences will be more likely to bridge the gap between the world in which they live and the abstract world of mathematics. Sowell (1989) found that the long–term use of manipulatives had a positive effect on students' achievement by allowing students to use concrete objects to observe, model, and internalise abstract concepts.

In the past decades some researchers have acknowledged that manipulatives allow student to construct their own cognitive models for abstract mathematical ideas and processes, thus providing an additional resource in learning mathematics. Moreover, manipulatives with technological interaction seem to have the additional advantage of engaging students and increasing both interest in and enjoyment of mathematics. In their most common use, these kinds of manipulatives allow students to directly interact with a computer (or an Interactive Whiteboard) that reinforces the same concepts being taught in class, allowing for accommodations and differentiations for students at various levels of learning. However, as suggested by Moyer (2001), manipulatives do not guarantee success if teachers use them primarily for fun and fail to use them effectively: children cannot learn mathematics simply by manipulating physical or technological objects. Describing Cabri Elem design principles, indeed, Colette and Jean-Marie Laborde (Laborde and Laborde, 2011), pointed out that:

> "authentic direct manipulation software are mainly not driven by the press of buttons, or by the filling of dialogs (or forms) or by typing command lines. They offer an interface where the user is invited to directly act on the mathematical objects. Actually, the action is on the representation of an object or an abstract entity" (p.1)

Hence, when using manipulatives, teachers should closely monitor students to help them discover and focus on the mathematical concepts involved and help them build bridges from concrete work to corresponding work with symbols (Kamina & Iyer, 2009).

Cabri Elem and the collection of dynamic resources "123… Cabri"

Cabri Elem has the affordances of earlier Cabri technology for direct manipulation of geometrical objects and numbers, together with some additional features. The most important design principle in the development of Cabri Elem is that of direct manipulation, which involves both action and feedback on action. The affordances for interaction with objects have been extended both by enabling restrictions on default behaviours (objects may be locked to prevent changes or feedback on student actions may be delayed), and by enabling new actions, such as feedback given at the click of a button. The user interface of the task performance environment is under the control of the activity book designer, who must decide which objects (tool icons, images, text, geometric figures, etc.) to arrange on initially empty pages, and who may program control actions on these objects (Mackrell, Maschietto & Soury-Lavergne, 2013).

The collection of dynamic resources called "123… Cabri", recently developed using the Cabri Elem technology, is a collection of activity books designed for primary school and the beginning of middle school. Each activity, carefully designed, contains a progression in the questions given to the student (in order to foster the evolution of solving strategies) and some important comments for the teacher about the proposed tasks. While the student is provided with tools for exploring, building, computing and solving, as in the logic of Cabri Elem, the objects react to the actions of the student providing feedback of various types. Feedback coming from the situation can be very rich in that it allows an interaction between the visual and the theoretical aspects of geometry and it may favour an evolution of solution strategies more than a judgement coming from the teacher (Laborde and Laborde, 2011).

RESEARCH QUESTION

The research hypothesis is that through the use of Cabri Elem activity books within a laboratory approach, in which pupils manipulate both physical and technological instruments, they can effectively construct a geometric concept, such as axial symmetry. In particular, we aim at answering the following research question: can an appropriate integration of physical and technological instruments foster the evolution of strategies for mathematical conceptualisation?

METHODOLOGY

The study is based on the results of a sequence of activities centered on the use of different manipulatives (such as paper, pins, carbon paper and rulers, Cabri Elem activity books) carried out in a public school in southern Italy over a time period of 2 weeks.

The results were analysed through video analysis. However, it is worth noting that, at this point of the research, the aim of the analysis was to identify and describe the situations that the pupils faced while interacting with both physical and technological instruments. Observation of the strategies which pupils enacted in those situations will be the starting point for the next step of the research: a large number of pupils will be involved with the additional aim of identifying possible regularities in pupils' behaviour.

The sequence of activities was carried out with small groups of pupils at 3rd and 4th grades. The pupils were introduced to the activities with physical manipulatives during 2 initial sessions (2h), then they had 3 sessions of about 2 hours, working in groups with an Interactive Whiteboard. Moreover, during the activities pupils were free to move around and had at their disposal a box in which there were tools of various types (such as a mirror, a ruler, some pins, some pencils and crayons, a string, a pair of scissors, some straws, etc).

SESSION 1. Pupils were given a multiple-choice question taken from the national standardised assessment test in which a word in the mirror was given as a model and they were asked to find the correct reflection of another word in the mirror. They then received a sheet of squared paper on which a half-figure of a mouse was drawn with the line of symmetry without any instructions.

SESSION 2. Pupils had to reproduce the symmetric figure of a small rabbit, on a white sheet where only a line of symmetry was given. They could use several tools, but couldn't use a ruler. They were then given a question taken from the national standardised assessment test in which the question was to find and enumerate the axes of symmetry of a figure.

SESSIONS 3-5. Pupils were first involved in the use of Cabri Elem on the Interactive Whiteboard, in order to become confident with some basic dynamic geometry tools. For the main part of the teaching experiment, they then interacted with the activity book "Reflection".

The activity book "Reflection" of the collection "123… cabri"

In the present research the following interactive activity books have been used:

- Points, straight lines, segments: Introduction to the construction of segments, straight lines and points with dynamic geometry tools;
- Reflection: Constructing the reflected image of a figure with dynamic geometry tools and identifying properties of a reflection.

The former concerns the dynamic geometry tools whose use is necessary in the latter. In particular, in what follows, we will look at the activity book "Reflection".

The title page contains text and images aimed at enabling the student to connect to a familiar activity in a real-world environment. The aim of the first two pages is to allow students to explore interactions with the elements of the activity.

In the first page there is an introductive activity in which students can try to move some vertices of a stylised butterfly. The butterfly is symmetrically built and students can move only two vertices; they can start observing what happens on the other side of the axis of symmetry.

In the following two pages there are two activities concerning the construction of the mirror image of a given figure. In the first there is a cherry on the left side of a line (which is the symmetry line) and a banana on the right side. Students can construct the reflected figures by clicking on a special button, clicking on the symmetry line and then clicking on the figure to reproduce it symmetrically. The reflected figure will appear on the other side of the line. Students can do the same procedure to construct the reflection of the banana. Once they create the reflections, students can try to move the objects on the screen and observe the behaviour of the entire system of figures (cherry, banana, axis and reflections).

The next page deals with a similar activity, but now the figure to reproduce is a polygonal fish (Fig. 1). Moreover, students, once they have created the reflection, can also try to move the axis of symmetry and to rotate it in order to observe what happens.

The aim of the next pages is to let students discover the properties of the axis of symmetry, observing a triangle and its reflection. The first guided activity helps them to understand that the axis passes through all three of the midpoints of the segments connecting the corresponding vertices of the triangles. Now students can use not only the reflection button, but also other new tools: straight line passing through two points, midpoint of a segment, and straight line perpendicular to a given line passing through a point. Once students have found the three midpoints and traced the straight line passing through them, they can check if that line is the axis of symmetry.

In the second activity (Fig. 2), on the contrary, the perpendicular bisector of the segment connecting two corresponding vertices of the two triangles is given. A new tool appears: segment between two points. Students have to construct the segments connecting the other corresponding vertices and to check that the given line also passes through their midpoints. Then they have to verify that this line is also perpendicular to the traced segments. Finally students are encouraged to check if the two triangles are symmetric with respect to the given line.

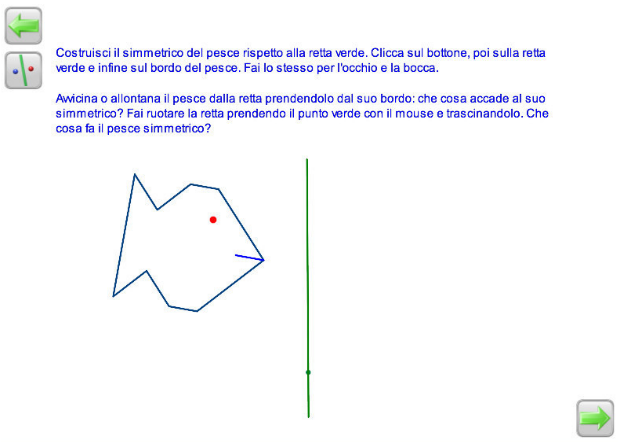

Figure 1

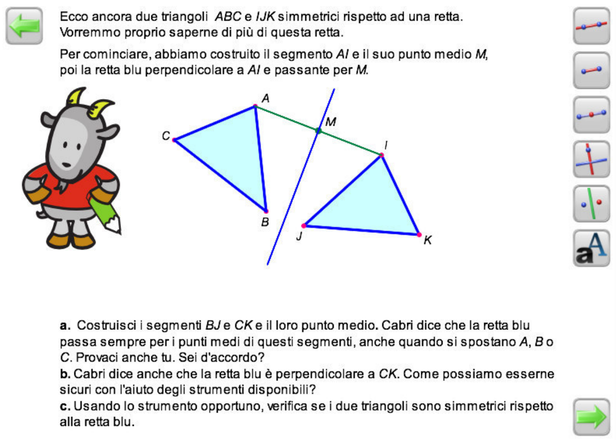

Figure 2

In the last three pages there are three activities: find the axis of symmetry of a given figure; construct the symmetry axis of symmetry between two polygons; complete the construction of the mirror image of a given triangle.

RESULTS OF THE TEACHING EXPERIMENT

In general, all pupils exhibited interest and conscious involvement during the activities. In particular, they all found the use of the interactive activities to be extremely challenging. Moreover, pupils gave evidence of being able to link the technological interactive activities with those which involved experimenting with physical manipulatives. This seems to confirm our hypothesis concerning the integrated laboratory approach. Furthermore, nobody was ever discouraged and they always collaborated in order to develop effective strategies and to find shared conclusions. As the video recordings show, while engaging in the activities given, pupils adopted different strategies, discussed them and finally reached a conceptualisation of axial symmetry. The pupils' interactions with both physical and technological manipulatives are presented and analysed below.

Pupils' interaction with physical manipulatives

In the question taken from the national standardised assessment test, the reflection of the word "CHIARA" was given and pupils were asked to find the correct reflection in the mirror of the word "PIERO" (Fig. 3).

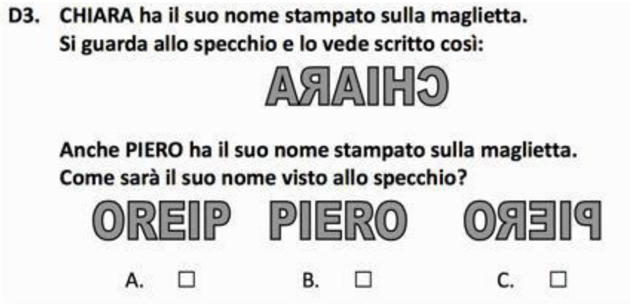

Figure 3

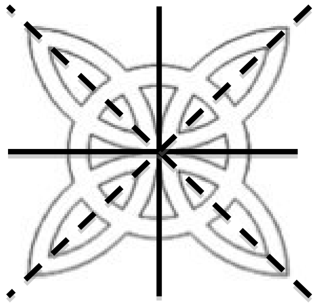

Figure 4

While all pupils succeeded in finding the correct reflection, thus exhibiting at least an intuitive knowledge of reflection symmetry, it is in their justifications that different approaches and strategies arose. Michele said that the reason is "in the R"; comparing the two words he noticed that "R" was the only letter that could give him information about the correct answer. He also said: "I can't understand anything only by looking at the letter I… it does not change!". Michele and the other pupils who developed this strategy seemed to recognize the symmetry of each of the capital letters of the alphabet. Gaia, instead, took the mirror from the box and used it to verify her hypothesis. The tool here became the means of her justification, in fact she said: "Yes, it's true! …Look! I was right". Angelo seemed to understand what Gaia had done but he felt the need to observe his hand when reflected in the mirror. Angelo's strategy started with the recognition of the situation of his hand being reflected in the mirror. Its enactment, even though not so sophisticated, can be considered a sort of bodily counterpart of the reflective principle and it seemed to make him aware of the basic symmetry concepts.

When pupils received the second task, they immediately reproduced the other half of the figure even without any explicit request. We interpret behaviour like this as being representative of the effects of Brousseau's "didactical contract". A common strategy consisted in reproducing the figure exploiting the equidistance of each part of the figure from the given axis. In fact they counted the small squares on both side of the given line and drew piecemeal all the missing segments. However it seemed that they did not effectively perceive the link between the concepts of equidistance from a point and midpoint of a segment.

We also consider worthy of note that Melissa said: "We have to draw on the contrary"; then, turning the sheet, she added "if you turn the sheet it's easier, … the figure is up and you have to draw down”. This seemed to reveal the importance traditionally given to the terms “vertical” and “horizontal”. We found this aspect also in the activities that followed.

Particularly challenging for the pupils was the activity concerning the reproduction of the small rabbit on paper without squares. Vincenzo suddenly exclaimed: “I can’t do it without a ruler!” Then pupils attempted strategies in which they would use different tools in an apparently random way. (By writing “random” we mean that, as observers, we couldn’t recognise any clear relations between their actions and their intentions). A long discussion was necessary in order to draw the desired figure. At the end, two strategies resulted in success: the use of the mirrors to reproduce the symmetric rabbit, once the paper had been folded on the symmetry line, and the use of pins to mark the points of the figure. In particular, the concept of vertices of a polygon arose when Melissa noticed that it was not necessary to punch all the points but that an appropriate choice was enough.

Concerning the request to find and enumerate the axes of symmetry of Figure 4, all pupils succeeded in discovering the four axes, even though the task turned out to be easier for younger pupils. We suppose that this result could depend on how far the task was from curricular activities but this needs further investigation. Also in this activity, as already happened in the previous ones, turning the sheet had been essential in order to find the “dotted” axes (Fig. 4). Alessandro needed the mirror to verify that the other two lines, proposed by Melissa were axes of symmetry; putting the mirror on the axes he exclaimed “Now I see, that’s true!”.

Pupils’ interaction with technological manipulatives

The last activity was the key one in order to reach the conceptualization of symmetry. Everything done previously converged here in a natural way. The activity book “Reflection” needed pupils to become confident with Cabri tools. For this reason it had been necessary to give them the introductory activity book “Points, straight lines, segments”. With this activity pupils were introduced to dynamic geometry for the first time. They discovered how to draw, construct, and drag and observed the feedback on their actions.

During the course of using the activity book “Reflection”, the students progressively developed the instrument of “dragging to verify the symmetry”. Worthy of note in this sense was what Melissa said while verifying the symmetry between two triangles: “It should be folded on the line but the Whiteboard can’t be folded!”. Michele added: “Our mirror is too small…” Finally Angelo said: “We now have new tools. Let’s try to use them”. By doing so, the students proved to have developed understanding of the way in which dynamic geometry tools constitute a means of action on the representation of symmetric situations and also an understanding of the properties of symmetry. For instance, the interactive activity gave another opportunity to focus on the link between the concepts of equidistance from a point and the midpoint of a segment. Thanks to the Cabri tools and the proposed activity, it seemed that students now understood that the midpoint of a segment is the “unique” point of the segment that has the same distance from each of the endpoints. The final activity of constructing the symmetry axis between two polygons was solved without any problems, thus seeming to confirm our hypothesis.

CONCLUSIONS AND FUTURE WORK

The analysis of results seemed to confirm the hypothesis that when an interactive technology is used within an integrated laboratory approach, the technology is not only an artefact to be instrumented, but consists of an environment which contains objects, such as representations, which are acted upon and can become means of action. In this way students can manipulate objects, and act on these objects, thus evolving their strategies and generating a mathematical conceptualisation.

In particular our analysis outlines a crucial moment in the evolution of strategies when students attempted to transfer the paper folding action to the use of the appropriate technological tool. It seems here that the dialectic thus created within the integrated activities has fostered, in a natural way, the development of knowing-with.

Further development of this research will be devoted to: analysing the teaching experiment in greater depth, extending it to the whole class and, finally, focusing on the potential impact of this research on everyday teaching practices (in terms of effectiveness for mathematical conceptualisation) and on teachers' professional development.

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GUIDING STUDENTS INSTRUCTION WITH AN INTERACTIVE DIAGRAM: THE CASE OF EQUATIONS

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The present study focuses on a specific class of interactive text called Guiding Interactive Diagrams (GID) and specifically on the functions of the boundaries designed to guide student's explorations. We report on an experiment in which 14- and 15-year-old students were challenged by an interactive task. The study provides evidence about the guidance's role and the control over the exploration that guiding interactive text grant to students engaged. Following Vygotsky's and Piaget's cognitive and social constructivism, we argue that in forming mathematical concepts students' exploration begins with spontaneous ideas relating to the context of the task, which they then develop with guidance supported by the design of the interactive text. To educators, who are challenged by the design and the implementation of interactive mathematics instructional materials, the study offers ways and terms to think about designs that limit the student's action and so support guidance, and at the same time remain an open space for student ideas.

RATIONALE

The availability of interactive web texts and the search for interactions that would better involve students in investigative activities and guided explorations have created expectations for a new type of book, sometimes referred to as an interactive book. We define an interactive text as any representation, visual and other, containing clarifying or demonstrating information. The components of interactive text are the given example, its representations, and a set of interaction tools.

The present study focuses on a specific class of interactive text called Guiding Interactive Diagrams GID. Current interactive reading is characterized by multi-modality, and we explore the roles that GID assumes in this sense. The GID designed to provide means for student explorations. It designed to set boundaries for the available exploration options in a way that narrates the story to be learned by working on the task. We use the term GID in relation to guided inquiry. Although GIDs provide tools that promote inquiry, they also set the boundaries that can guide inquiry and provide a framework in the process of working on the task. The guided inquiry approach as a pedagogic strategy calls for students to integrate empirical with conceptual work as they take an active and responsible role in the learning process (Yerushalmy & Chazan, 1992). As a part of our broader research we explore the guidance that the boundaries designed in GIDs provided to the learning process (Naftaliev & Yerushalmy, 2011, 2013).

THEORETICAL FRAMEWORK

Vygotsky (1978) emphasized the effect of cultural artefacts on the development of new knowledge and introduced the concept of the ZPD. He defined ZPD as the difference between the actual developmental level of an individual, as determined by independent problem solving, and the level of potential development, determined by problem solving under appropriate guidance, with cultural artifacts capable of bridging the gap between the actual and the potential levels of development. According to Vygotsky (1978, 1986), the appropriation of cultural artefacts liberates students from direct stimulus control and creates an intrinsic link between cognitive development and culture

(Saxe, 1991). Psychological tools, assistance from others, and scientific concepts are all cultural artifacts. According to Vygotsky (1930/1981), psychological tools modify the structure of mental functions by determining the structure of the new instrumental act. The instrumental method distinguishes a dual relation between behaviour and an external phenomenon: an external phenomenon can play the role (a) of the object toward which the act or behaviour is directed, and (b) of a means by which one directs and realizes the psychological operations needed to solve the problem. Recent studies have adapted Vygotsky's general framework to the specificity of mathematics education and to learning-teaching mathematics in computer environments (e.g., Trouche, 2004; Moreno-Armella & Sriraman, 2010). Scientific concepts (e.g., quadratic equations, algebraic manipulations) are missing from the "natural" cognitive baggage of children, who usually meet scientific concepts for the first time in school. Vygotsky emphasized that the awareness of scientific concepts requires the presence of spontaneous concepts in a child, and argued that “reflective consciousness comes to the child through the portals of scientific concepts” (1986, p. 171). Vygotsky (1986, 1994) argued that a process of teaching-learning ("obuchenie") is necessary for the development of scientific concepts. Piaget (1962/1995) maintained that to allow the development of the new scientific knowledge, it is necessary to present the problem in a way that matches the structures already formed by the student and creates situations, which, while not "spontaneous" in themselves, evoke spontaneous elaboration on the part of the student. The frameworks of Piaget and Vygotsky complement each other (Shayer, 2003) and offer lenses for understanding the processes of development of new scientific concepts derived simultaneously from the students’ “spontaneous” activity and from social practice with cultural artefacts.

In the present study we follow students’ engagement with meaningful ways of solving and manipulating quadratic equations while learning algebra with a guiding interactive text.

METHOD

We analyse an experiment in which students were presented with two comparative challenging activities: one on paper and another in an interactive textbook. The first challenge asked students to perform the activity using a paper diagram (Figure 2, Table 1). Problem solving with the paper diagram served as a baseline situation designed to evaluate the students' knowledge and solution techniques. After the students appeared to have exhausted the activity, the interviewer suggested continuing working on a similar activity that was presented by means of a GID (Figure 1, Table 1). Problem solving with the GID served to explore whether and how the GID functions as a form of instruction for the development of scientific concepts.

Cultural Artefacts Addressed in the Study

The mathematical concept

The function-based approach to algebra that we take offers viewing equations as a comparison between the two functions on the two sides of the equation makes it possible to (a) find a solution of the equation in ways that are not analytical, such as reading the solution on a graph or in a table of values; (b) perform qualitative analysis of the reciprocal relation between graphs in a two-function graphic representation of equations, and predict the number and the type of solutions of the equation; and (c) reference the letter in an equation as a variable rather than an unknown.

The GID

The constraints we used in the design of the GID were: (a) a small number of representations; (b) presentations of certain values of x; (c) disallowing free typing of equation expressions when performing operations linked to the graph; and (d) sketchy presentation of the graph (without grid and with hidden numeric values).

<u>The task and the diagrams</u>

$2x(x+3) = x+3$

If possible, change the right side of the given equation and leave the left side as is to obtainquations that comply with the following constrains:

1. The equations have no solution

2. Two of the equation solutions are negative

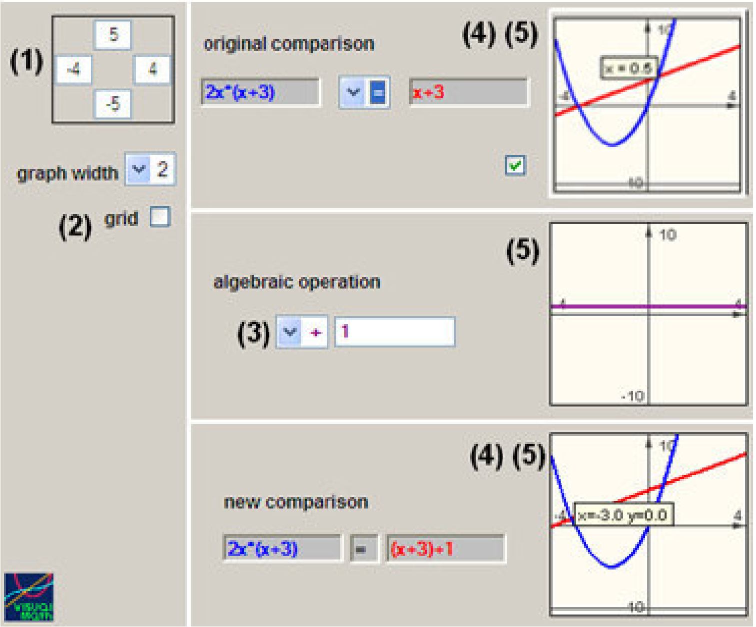

Figure 1. GID. (1) changing the scale; (2) adding the grid; (3) performing operations on the right side of the equation, starting with the "+1" operation; (4) observing the solutions of the equation; (5) observing coordinates of points.

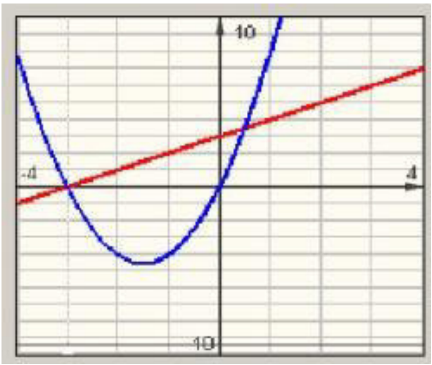

Figure 2. The paper diagram

	Paper diagram	Guiding ID
Graph	Neat graph. Student can read the coordinates of points using the marks on the paper.	The graph is a scalable sketch that can be made accurate by revealing the coordinates of the points. To do so, it is possible to slide the mouse and to add a grid. Students can observe the numeric value of the solution in a graphic representation when the mouse is close to the intersection points.
Algebraic expression	Student can perform algebraic manipulations on paper.	Performing various operations on the right side of the equation, linked to the graph.

Table 1. Comparative view of the components of the diagrams

Participants and Procedure

We report on the work of four 15-year old 9th-grade students, all of whom have been studying in the same urban public school since the 7th grade. As part of their algebra curriculum, students first studied various functions and related situations mostly in graphic and numeric representations, occasionally supported by software applications similar to the ones used in the study. The students have already studied linear and quadratic functions in graphic, numeric, and symbolic representations. They also studied linear equations: initially they studied the function perspective of linear equations, then conventional algebraic algorithms to solve the equations. The research tools were task-based interviews conducted with individual students in school, outside the classroom. The interviewer provided each student with a paper diagram and later with a GID. The students' main interaction was with the task and not with the interviewer. Interviewees were encouraged to solve the problems freely and to try out new as well as familiar ways, to raise conjectures, and to describe their line of reasoning. The interviews were video-recorded and used to track the students' development of concepts.

DATA ANALYSIS

Four steps data analysis focused on producing evidence about new scientific concept (meaningful ways of solving quadratic equations and performing algebraic manipulations) developed while engaged printed text and GID: Preliminary analysis traced indicators of student concepts; Listing of indicators of student concepts following the analysis; Re-analysis of the interviews, viewed through the lens of the list of indicators (Table 2); Identifying and describing key episodes in the interviews that contain evidence about student concepts and the students' use of GID components to achieve the goals of the task or goals that emerged during the activity.

Indicators of student knowledge	Mathematical scientific knowledge
Finding the solution of a linear equation as a value of x that is obtained by algebraic manipulation of the equation	Linear equation: solving process and solutions in algebraic representation
Describing or finding the solution of a quadratic equation as the x coordinate of the point of intersection between the functions on both sides of the equation	Solution of a quadratic equation in graphical representation
Describing or finding the solution of a quadratic equation as the point of intersection between the functions on both sides of the equation (including a solution involving 2 coordinates (x, y))	
Describing or using the condition of a quadratic equation with no solution (or with an infinite number of solutions) as a contradiction (or equality) that is the result of substituting a certain value of x in the equation	No solution exists (or infinite solutions exist) for a quadratic equation explained by algebraic representation
Describing or using the condition of a quadratic equation with no solution in a graph as 2 functions that do not intersect	No solution exists for a quadratic equation explained by graphical representation
Describing or using the relation between manipulations of an expression and a transformation of the graph	Algebraic manipulations linked to actions with the function graphs.

Table 2. Indicators of student concepts

MATHEMATICAL ENGAGEMENT WITHIN THE "BOUNDARIES" OF THE GID (SHAY'S CASE)

Below we illustrate students' mathematical engagements with GID and role of the resources and constraints in GIDs' design in the processes by analysing an interview with one of the four students (Shay). The processes followed by all four students that were interviewed were similar.

Working on the paper task, Shay stated that in order for the equation to have no solution it is necessary to arrive at a contradiction.

> Shay: In order for the equation to have no-solution we need… that… That means, that x couldn't equal anything, we'll have to have zero equals some number and in that condition we'll have no solution. That means... We need to make one of the sides zero and the other some number, and then I think that x will have to equal zero in order for the equation to have no solution.

> Interviewer: What does that mean?

> Shay: Because if we substitute here zero [2x(x+3)=x+3]… so there will be no solutions. Because if… If we substitute zero, we'll receive the value of three and if we substitute here zero it'll be zero.

Shay substituted a certain value for x, and as a result reached a condition in which the left side of the equation was zero and the right side 3. Shay described the situation as an equation with no solution. Two explanations can be found for this idea: (a) considering a variable to be an unknown, and searching for a certain (unknown) value as the solution to the equation, so that when there is no solution one value can be satisfactory; and (b) attempt to apply a known algorithm about linear equations to quadratic equations (the algorithm for solving quadratic equations has not yet been learned), in order to reach a solution: performing algebraic manipulations on the equation, and if in the end a contradiction emerges (one side equals zero and the other side equals a non-zero number), the equation has no solution. Shay began the GID task by familiarizing himself with the various options provided by the GID. He saw the solution of the original equation displayed in the GID, added and subtracted a number to the expression in the equation, and observed the changes in the function graph and in the displayed solution of the new equation. At first the objective of these changes was to reach the state of contradiction that he created on paper:

> Shay: Umm… [Clicks various spots on the GID. Sees the display of both solutions to the original equation, and examines the options of changing the expression in the GID.]

> Interviewer: What do you see? What are you doing, Shay?

> Shay: Umm… I'm trying to find here a way to change the right side of the equation. So the left side… So it'll be… Unequal to the solutions [to reach the state of contradiction that was achieved on the paper diagram].

> Shay: I need to change, I need to add a number so it'll turn out that there are no solutions. [Using the change expression function in the GID, he adds the number 3 and observes the graph moving upward, then changes the operation to subtraction and observes the graph moving downward.]

> Shay: Now I understand. In order for the equation not to have a solution, I need to make it so that they could never meet. [Moves the cursor over the graph of the new equation. Moves near the intersections and observes the display of the solutions.] Ah! I need to increase the number. [Using the change expression function in the GID, he types in the number 12 and retains the subtraction operation; the graph moves downward.] Basically I inserted a

number so the two equations never intersect and then… And then actually there is no solution, that way we have completed the task.

While changing the expression, Shay observed the displayed solution of the new equations and followed the changes in the graph. As a result, a development occurred in his knowledge about a no-solution equation in graphical representation and in the interpretation of the required condition, as it appears in the graph. Shay articulated the no-solution task in the graphical representation, and later he did so with reference to the tools of the GID, as we have seen in case of other students as well. It is possible that following the manipulations of the expression, adding 1 (when he first began to work with the GID) and examining the addition and subtraction of 3 to the right side of the equation, Shay reached the conclusion that adding/subtracting a number to/from the expression causes a vertical transformation of the function graph. Shay translated the graph by subtracting 12 from the right side of the equation and in this way solved the task. He used algebraic manipulation to change the graph in the next episode as well, when he performed an algebraic manipulation of the expression to achieve the required condition in the graph:

Shay: …So the two solutions of the equations will be negative… [Inserts 4 in the GID, using a subtraction operation.] There. Basically now I solved B as well. I made it so that the two solutions are negative, in the way that they intersect at a negative and not at a positive. And that makes them negative. … The first solution where it meets is minus *2.280* [negative *2.280*] [moves the cursor towards the left intersection and looks at the display of the solution] and one of those is negative [moves toward the right intersection and looks at the display]… Negative… Here the x axis is zero point… Minus 0.2 [negative 0.2].

To complete the two negative-solution tasks, Shay continued to combine the graphical representations of the equation and described the condition that must be reached in the graph in order to complete the task. He then changed the expression to achieve the required condition. Shay appeared to know the required changes to the expression, and did not try different manipulations. He made the connection between algebraic manipulation and changes in the condition of the graph, thus solving the equation.

DISCUSSION

Working with paper task, students tried to implement the known algorithm for solving a linear equation to a quadratic equation and we can identify a spontaneous concept developed by them: if you can substitute a number that produces a state of contradiction (or equality), the equation has no solution (or with infinite solutions) (Table 4). There could be two explanations for the development of this idea: (1) An attempt to apply the known algorithm (Table 3) for solving a linear no-solution (infinite solutions) equation to the quadratic equation, which is a new type for the students. This explanation is based on the fact that students used algebraic manipulations of the equation, and if a contradiction was reached (one side equals zero, the other side is a non-zero number) they concluded that the equation has no solution. (2) It is possible to treat x in the equation as a variable or as an unknown. If x is regarded as a variable, it is necessary to analyse the relations between the two sides of the equation for any x in order to reach a solution. If x is regarded as an unknown, a single value for x is sufficient to solve the equation. In the conventional approach, x is the value of an unknown, and students find its value by manipulating the equation. We can see that the students broadened their notion of x as an unknown to cover also cases of no-solution and infinite solutions.

No-solution linear equation	*Infinite*-solutions equation
2x = 2x-1 0 = -1	2x-1 = 2x-1 0 = 0

Table 3. Solving a linear no-solution (infinite solutions) equation

No-solution equation (solved by Shay)	*Infinite*-solutions equation (solved by Lin)
2x(x+3)=x+3, substitutes *x=0*	*2x(x+3)=x+3,* substitutes *x=-3*
0=3	*0=0*
therefore the equation has no solution.	therefore the equation has infinite solutions.

Table 4. Implement the algorithm for solving a linear no-solution (infinite solutions) equation for the quadratic equation

Learning the "boundaries" of the GID is learning the task's content

The students started with a situation in which there was a mismatch between their knowledge and the components of the example displayed in the GID. The mismatch related both to the mathematical topics displayed in the GID and to the resources and boundaries of the GID. The students spent time learning the various options of the GID, and during their exploration also learned about the content of the task. As the limited input prevent to type in a free expression they changed the expression repeatedly to reach their goal, and while doing so they attend to the effect of the change on the graph and on the relations between the graphs. Tracking these changes helped build the concept of the no-solution equation in a graphic representation.

Manipulations of expressions: generating new examples as the basis for inquiry, conjectures, and generalizations

Performing the changes described above, the students created new examples of equations without intending to do so. In each equation initially created by the students there is a solution, and the graphs intersect in each example. Following these examples, students in the reached the generalization that for the equation to have no solutions the graphs must not intersect. Based on this generalization, all four students started looking for a proper example. Manipulations of the expression, together with addressing the goal they set themselves, caused students to identify and grasp the links between the manipulations and the graph transformations, which enabled them to complete the task. The GID appears to organize a path for students to create and test assumptions about subtracting or adding numbers to an expression and about translations of the graphs.

Summary: Concept Development Using GIDs

Concept development using GID was mainly the result of manipulating the expressions as directed by the design of the GID and of reflecting on the changes in the graph, in conjunction with two ways of treating the graph: as a sketch and as a precise graph. There was mutual dependence between actions that students performed with the GID and the development of the mathematical concepts. The concept development process was rooted in spontaneous concepts and in the context of the artefact. The solving process with the GID included matching the formulation of the task with progress in mathematical knowledge and in conferring meaning upon the components of the GID.

The students articulated and solved the task in the graphic representation when the latter acquired meaning as a representation of the equation and of its solution. They also performed algebraic operations on the equation expression, which was supported by the GID, in order to formulate and conclude the task (first assigning meaning to the equation solution in graph, then translating the task to the terms of the GID, and finally reaching conclusions about translating the graph and applying these conclusions). The processes of conferring meaning upon the components of the GID, of understanding and rephrasing the tasks, and of developing concepts affect each other, and a change in one process causes the others to change as well.

The task design described in this study limits the student's actions and at the same time provides space for student ideas. On one hand, the design of the GID organized and directed the process of development of the students' knowledge; on the other hand, the students controlled the task and were empowered by the changes they chose to make in the presentation of the task and by formulating new questions.

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REAL AND VIRTUAL HELIXES FOR THE INTRODUCTION OF TRIGONOMETRIC FUNCTIONS

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The helix as a three-dimensional (3D) structure unites two different visual representations, a vector rotating around a unit circle and a sinusoidal wave; it has the potential to guide slow learners to understand the relation of trigonometric functions and their graphs, providing those learners with more intuitive and meaningful learning. However, the 3D object was difficult to draw on paper or chalkboard, thus difficult to use as learning material in actual lessons. In this study, the authors tried to examine the effect of using real and virtual 3D objects, which could be observed from different viewpoints and manipulated by students, especially on understanding phase shift of a sine curve. A student group scored higher on a paper test of drawing a graph of sine curve $v(x) = \sin(x + \quad)$ *after the introductory activities of observing and handling real and virtual helixes. Further studies would be needed to confirm the effectiveness.*

Keywords: Helix, 3D object, trigonometric functions, interactive worksheet

INTRODUCTION

Trigonometric functions are foundation of electrical engineering because alternating-current (AC) voltage in an electrical circuit is expressed as a trigonometric function of time, and the time-dependent voltage graph becomes a sinusoidal wave. Especially, understanding the relationship between the graph and the symbolic expression of a trigonometric function is the starting point for learning electrical circuit theory. However, some novice learners struggle to learn the relationship and have hard time. Slow learners' typical pitfall is to start memorizing as many symbolic formulas and their manipulating procedures as possible for the examinations but to ignore their links to graphic representations. Knowledge expressed symbolically, but disconnected from graphical image, tended to be shallow, so, as a result, would be lost quickly just after the examination. Their knowledge tended to stay in instrumental understanding but did not extend to relational understanding (Skemp, 1987: 152). The struggling slow learners need some good guidance to visualize the relationship.

Visualization (Zazkis, Dubinsky and Dautermann, 1996), visual thinking (Rivera, 2011: 89), or visual reasoning (Viholainen, 2008) played an important role in the learning of mathematics. Visual reasoning (Viholainen, 2008) was based on visual interpretations of mathematical concepts. Visualization made it possible to perceive abstract mathematical objects through our senses. Visual representations could be considered more concrete than analytic ones, because they are based on external objects. Analytic reasoning was often exact and detailed, but visual reasoning was needed to reveal wider trends of the whole problem solving process and holistic features of the problem situation. The inclusion of visualization into the teaching process by increasing geometrical rather than algebraic presentation enhanced university students' conceptual learning of vector spaces (Konyalioglu, Ipek and Isik, 2003; Konyalioglu, Konyalioglu, Ipek and Isik, 2005). Visualization also helped students to learn the rather complicated concept of complex integrals intuitively (Miki, 2004). The research suggested that geometrical structures helped the students' meaningful and conceptual learning.

However, many students had difficulties in analysing visual representations, and therefore, they could not utilize them in problem solving (Stylianou & Dubinsky, 1999). Zazkis, Dubinsky and Dautermann (1996) observed the value of the interrelationship between the visual and analytic approaches from their interviews with 32 university students in their first abstract algebra course. In their proposed model of students' development, novice students' acts of visualization and analysis were initially separate and quite different. Both kinds of thinking gradually became more and more deeply interiorized within the mind of the individual until, finally, analysis and visual understandings were synthesized so that it could become very hard for the individual to distinguish between them. There was a need therefore for consistent visual conventions and practices, from a young age, including plane representations of three-dimensions (3D) and a wide range of transformations (Whiteley, 2000).

The authors also confirmed that it was easier to introduce new mathematical concepts from actual examples. For example, we started our lesson of linear algebra by showing actual applications, let students handle real objects, and let them use graphic objects in 3D virtual space, before we discussed the relation between graphic and symbolic representations (Nishizawa et al., 2010). Applications, especially ones from everyday life, were good starting point for the lesson. Real objects, which the students could touch and handle directly, helped struggling students to imagine 3D graphic objects in their minds, and interactive worksheets, on which students could manipulate and observe the changing shapes or positions of the 3D objects interactively, helped students to start considering the connection of the graphic objects and symbolic representations (Nishizawa et al., 2014).

In learning trigonometric functions, the horizontal shift of a sine curve $y(x)=\sin(x+\quad)$ by phase angle   was found to be the first barrier for novice students. Quick learners used the earlier learned analogy of shifting the graph of a quadratic function horizontally and easily confirmed the correctness by substituting *x* with actual number, for example $x=0$, but slow learners needed a longer period of learning and more paper-and-pencil exercises to understand the relation. Many slow learners grasped the concept of a trigonometric function just as the ratio of two sides of a right-angled triangle, and could not visualize how the phase angle   shifted the sine wave horizontally. For example, typical 2D visualization of a unit circle and sine or cosine curves (Figure 1) did not help slow learners so much when trying to understand the phase shift. They needed more direct and intuitive visualization.

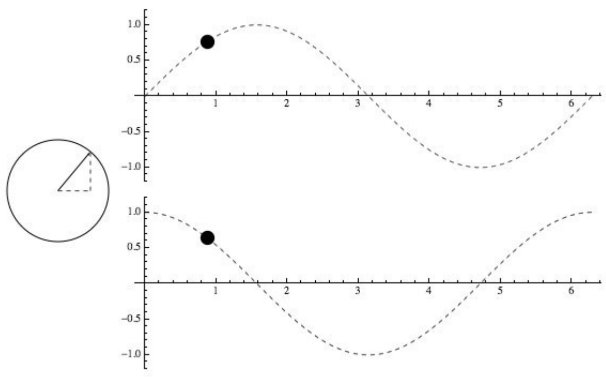

Figure 1. 2D visual explanation of sinusoidal waves

The aim of this paper is to find if a helix, which is closely observable from different viewpoints and can be manipulated, could ease the slow learners' difficulty in visualizing the phase shift of a sine wave. The helix as a 3D object had the advantageous structure of showing two aspects, a unit circle and a sinusoidal wave simultaneously. However, merely mentioning it, or showing the pictures of it as a real world example, were not enough for struggling students. They benefited from the richer experience of handling the object by themselves.

We also assumed that an element of surprise, along with some level of interaction, gave students a deeper impression. So we provided unexpectedness for the demonstration, and prepared real and virtual objects, both of which could be handled and observed from different angles by the students. The effectiveness was measured by a pencil-and-paper test of students' ability to draw a graph of $y(x) = \sin(x + \quad)$, and the test score was compared with test scores of the students who had learned in previous years.

METHOD

Learning Materials

We selected a helix for visualizing the relationship between graphic and symbolic representations of trigonometric functions. A helix is a 3D structure of a curved line wound spirally around a cylinder. If we looked at a helix from the mouth of the cylinder, it looked like a unit circle, which we often used to visually define a value of trigonometric function as the coordinates of a point on the unit circle such as $(\cos x, \sin x)$. But if we looked at the same helix from the side of the cylinder, it looked like a sine wave such as $\sin(x + \quad)$. A helix could unite those two visual explanations, a unit circle and a sine wave, together.

We used three types of helixes as the learning materials: pictures of an electronic signal line for the real world example; a plastic bottle and a string for the real object; and an interactive worksheet for the virtual object. The latter two materials are the major target of this study. We also took the approach of showing a real world example first, offering the opportunity to handle real objects next, and then transferring into virtual space for the introductory lesson of trigonometric functions, where students could gradually recognize the two aspects of the 3D structure when they saw it from different viewpoints.

As the real world example for electric engineers, we selected a series of pictures of an electronic signal line hanging over the road (Figure 2). It was an amusing quiz for the students to find the helix in these pictures. Finding it in the scenery (Left picture of Figure 2) was rather difficult, but it became easier when it was zoomed in (Center or right pictures of Figure 2). The discovery was unexpected and thus remembered for a certain time period. The signal line looked like a sinusoidal wave on the 2D pictures, but it actually was a 3D object: a helix coiling around the supporting cable. So, it made a good introduction to the next step.

Figure 2. Real word example of a sinusoidal wave

As the real object, we used a transparent plastic bottle and a string with one end stuck to the bottom of the bottle to make a helix (Figure 3). A student could wind the string around the side of the bottle and hold the other end of the string with his finger. The raw materials were easy to collect, and the coiled string just reminded us of the electronic signal line, which we used as the real world example. The student could observe the coiled string from several viewpoints: looking at the string from the top or the bottom of the bottle, the string looked like a unit circle. The same string looked like a coil from most viewpoints, but like a wave if when viewed from a vertical viewpoint along the length of the bottle.

It was important to let the student observe the coiled string from a viewpoint which showed it as a wave or a 2D curve. Then, by rotating the bottle at the neck, students could find the angle where the string appears just as a sine curve. Starting from that angle, students could observe the string shifting gradually into a cosine curve when rotating the bottle a further 90 degrees counter-clockwise. This could be a direct experience of observing the phase shift of a sine wave for the student, which is controlled by the rotating angle of the bottle. It must be an unexpected experience for students and leave a strong impression. The helix was a simple enough structure to reconstruct at any time and easy to reuse in later lessons. It was hoped that this experience would help the students to visualize the phase shift of a sine curve.

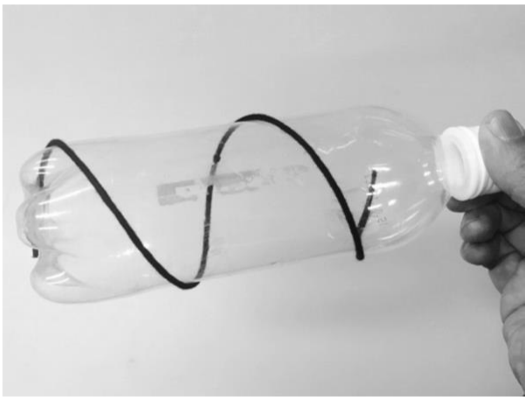

Figure 3. Helix made from a plastic bottle and a string

Although the string around a plastic bottle was handy to use, more precise and flexible visualization was possible with the interactive worksheet (Figure 4). The worksheet had a virtual 3D space and a menu was attached to the topside. In the virtual space, a transparent cylinder, encapsulated in a transparent box, was laid with its bottom on the left and its top on the right. At the bottom edge of the cylinder a wheel was located, which rotated around the central axis of the cylinder. A spoke connected a fixed point on the wheel and the centre of the wheel, so the rotation of the wheel was visible as the movement of the spoke. As the rotation of the wheel was linearly linked with its side shifting along the central axis of the cylinder, the tip of the spoke left a spiral path as the trajectory, which made a helix on the side surface of the cylinder. In the worksheet, the nob on the slider in the menu controlled the side shifting of the wheel, and thus the length of the helix. The cylinder could be rotated in any direction in the virtual space, and viewed from any direction. Three buttons on the menu were the ON/OFF switches. If the “Sin” button was switched on, a sine curve appeared on the vertical wall of the box as the projected shadow of the helix.

The wheel and the spoke in the worksheet became a good visualization, which connected the phase angle in $y(x) = \sin(x + \quad)$ with the direction of the spoke at the bottom of the cylinder (at $x = 0$) when a student viewed it from the side of the cylinder and rotated it around the central axis of the cylinder. As in the case of a real helix, the observed helix looked like a sine curve and it shifted horizontally by the rotation.

We built the interactive worksheets as computable document format (CDF) files coded with a CAS: Mathematica’s programming language (Wolfram, 2015). Mathematica’s built-in functions offer a powerful interface, by which we can manipulate or animate a graphic object and also observe the object from different viewpoints. The students could use the worksheet directly on their local computers or use it by accessing to our Mathematics Learning System website (MLSS, 2015) through the computer network or the Internet. In both cases, the local PC needs the Wolfram’s CDF player, which is freely downloadable from the developer’s website (Wolfram, 2015), so the students can use the interactive worksheet embedded into the webpage.

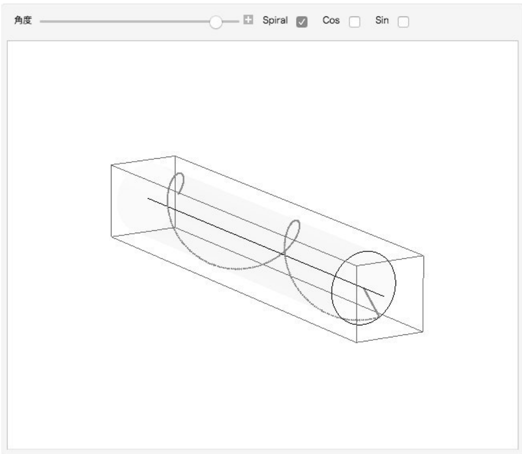

Figure 4. Interactive worksheet showing a graph of a trigonometric function

Subjects and Lessons

The subjects of this study were 42-43 first-year students (aged 15) in a college of technology in Japan from 2011 to 2014 academic years. They had completed nine years of elementary education in primary school and junior high school, and were taking four 90-minute weekly lessons of mathematics: (A) two lesson of algebra; (B) a lesson of geometry; and (C) a lesson of mathematical functions, throughout the first year of the college (Table 1). Lesson time and curricula were the same during the four academic years, but the instructor of lessons (A) and (B) changed in term. The instructor of lesson (C), one of the authors, stayed the same, and the visualization activities of trigonometric functions were conducted in lesson (C).

In the spring semester, the first semester, the students had learned linear, quadratic, and rational functions in lessons (A) and (C), and in the autumn semester, the second semester, they had learned irrational, exponential, and logarithmic functions in lessons (A) and (C) before the learning of trigonometric functions. Trigonometric ratio was taught earlier in lesson (A), but trigonometric functions were taught only in lesson (C) during the two months of this study, from early November to early January.

Academic year	(A) Algebra Two lessons / week	(B) Geometry One lesson / week	(C) Functions One lesson / week
2011	Instructor A	Instructor B	Instructor F
2012	Instructor C	Instructor D	
2013	Instructor C	Instructor B	
2014	Instructor E	Instructor C	

Table 1. Mathematics lessons and instructors in the first year

When introducing the graph of trigonometric functions, the pictures of electronic signal wire (Figure 2) had been used since the 2011 academic year. It usually took five minutes to show the pictures on the projector in class.

For this study, in the lesson (C) on November 11th, 2014, the traditional 50-minute chalkboard-based lesson was replaced by the following experimental one. In the experimental lesson, the first 20 minutes was set to observe a real object (Figure 3). The instructor held a transparent plastic bottle, wound a black string around it, and fixed it on the bottle with sticky tape. Then he showed it in front of the class, and slowly rotated the bottle in two directions. When he rotated the bottle around an axis perpendicular to the length of the bottle, the students observed the coiled string changing its shape from a circle to a wave and back from a wave to a circle in front of them. When he showed the side of the bottle to the students and rolled it around the length of it, careful students could observe the phase shift of the wave. He also passed the plastic bottle to the students in the class and encouraged them to view it by themselves from different angles.

Finally, the instructor showed the interactive worksheet (Figure 4) projected on the screen in front of the classroom. He demonstrated how to rotate the 3D graphic object in the virtual space, to animate the curves, and to switch on or off the selected curves. A few selected students were invited to manipulate it on the interactive worksheet for a guided demonstration. Using the interactive worksheet in the lesson took 30 minutes. He also encouraged the students to examine the interactive worksheet on the Web after the lesson.

Tests

During the next three to five weeks, symmetry and periodicity of trigonometric functions, Pythagorean identity, sum and difference formulas, and other formulas were taught and discussed in the lessons. At the end of this period, the effect of the experimental lesson was tested by a paper test. The test problem was to draw a graph of $sen\left(x+\frac{\pi}{3}\right)$. The full score for the problem was five points. 3.5 were given to the answers that drew the curve correctly, and additional 1.5 was added if correct coordinates of three cross points of the curve and two axes were written. The test time was so short that the students could not have time to calculate the actual data, make the table, and plot several points to draw the graph. Graphic calculators were not used in the test or in any mathematics lesson for the first year students. Students needed to visualize the graph in their mind to draw the graph correctly on the paper in the limited time.

The test scores were compared with the scores of the students from 2011 to 2013 academic years, who saw the pictures of electric wire (Figure 1) but did not handle the plastic bottle or the interactive worksheet in their introductory lesson nor after the lesson. The test problem in each academic year was different only in the phase angle, which in the case for 2014 was $\frac{\pi}{3}$. The task was fundamental, and it was usual that about a half of the students got full marks for the problem every year. What we wanted to see was if slow learners in the 2014 academic year would score higher than their former slow learners in the paper test; perhaps there could be observable benefits of the new introductory lesson using the real and virtual helix.

RESULTS

The test scores of four groups of students from 2011 to 2014 academic years were shown in Table 2. The time between the introductory lesson and the paper test was from three weeks to five weeks, 2014 being the longest. The 2013 group, which had the second longest interval between the introduction and the test, had the lowest average score among three groups from the 2011 to 2013 academic years, but the difference between 2013 and 2011, or 2013 and 2012 were not significant (p = .06 or .10 respectively). From 8 to 12 students (from 21 to 28 %) scored lower than 3 in each year. It meant that they could not draw the sine curve correctly. Compared to these three groups, the students of 2014 academic year had higher average score (p < .005) and fewer low scorers. There was only one student who could not draw the curve correctly in the 2014 academic year.

Academic year	Contents used in the introductory lesson	Time between the introduction & the test	Average (SD)	N of low scorers*
2011	Pictures of electronic signal line (Figure 2)	3 weeks	3.94 (1.70)	8 /38 (21%)
2012		3 weeks	3.80 (1.56)	11/ 43 (26%)
2013		4 weeks	3.35 (1.68)	12 / 43 (28%)
2014	Pictures of electronic signal line (Figure 2), coiled string on a plastic bottle (Figure 3), and the interactive worksheet (Figure 4)	5 weeks	4.76 (0.45)	1 / 41 (2%)

* low score was < 3, which meant the curve was incorrect or not drawn

Table 2. Test scores of drawing graph of a trigonometric function

DISCUSSIONS

The result suggests the possibility that handling the real and virtual helix made an effective introduction to the graph of trigonometric functions, and helped students to understand how to draw the graph of sine wave with a phase shift. In this study, after recognizing an electronic signal wire as a helix, students observed real and virtual helixes closely, and some of them rotated the helixes around the central axes by themselves. After five weeks, most of them succeeded to draw a graph of $v(x) = sen(x + \alpha)$, scoring higher points in the paper test. Even the slowest learners in 2014 must be helped to visualize the structure causing the phase shift of the graph by α. We would like to argue that students' experience of rotating the helix by their own hands and observing the horizontal shift of sine curve simultaneously was a quite effective introduction to trigonometric functions.

However, the test result was not enough evidence to confirm the effect of the real and virtual helixes, because the time between the introductory lesson and the paper test was the longest in 2014, and the influence of how the other instructors of lessons (A) and (B) had taught was not controlled in this study. It is necessary to measure the effect on plural academic years such as in 2015 and 2016.

CONCLUSIONS

Observing real and virtual helixes from different viewpoints and handling them may be helpful introductory activities for slow learners to recognize the 3D structure that could be seen as a sine curve, a cosine curve, or a rotating vector on a unit circle depending on the viewpoint. A group of students could draw a graph of sine function with a phase shift better after the experience of handling the helixes. Further studies were needed to confirm the effectiveness.

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PARSING OF FUNCTIONS USING *MATHEMATICA* AND ITS APPLICATION FOR TEACHING OF DIFFERENTIATION

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Computer algebra systems (CAS) are an important component of Information and Communication Technologies (ICT), which turned into a constituent part of education. In this paper, the application of Mathematica and its Internet version webMathematica for teaching students the art of differentiation is considered. Mathematica ideally suits to the developing skills of differentiation insofar it can reveal the syntactic structure of formulas that determine functions. The latter greatly facilitates the application of the proper differentiation rules. There is a special command in Mathematica that identifies the syntactic structure of expressions and presents it in a form of a tree. Mathematica and webMathematica enable significant enrichment of calculus didactics at least concerning differentiation.

Keywords: CAS, Mathematica, differentiation, syntax of elementary functions

INTRODUCTION

Computer algebra systems (CAS) emerged as a tool for the automation of algebraic calculations. It soon became clear that those of them called integrated mathematical systems (Maple, Mathematica, Mathcad, Matlab, etc.) are extremely useful for teaching students because they allowed performing not only algebraic but also symbolic, graphical, and numeric, calculations. These calculations can be done during lecture hours as well as in practical classes. Due to the rapidly increasing number of computers in educational institutions, a growing number of enthusiastic teachers started using CAS to teach mathematical disciplines.

However, many teachers of mathematics do not share this enthusiasm. They argue that the application of CAS in teaching of mathematical disciplines raises many problems. The simplest of these problems is the need to spend at least a minimum amount of time for the development of computer skills and learning the commands of CAS. If the application of CAS is not limited to 'a simple transfer of the traditional curriculum from print to the computer screen' [1], then using of CAS can change the emphasis in teaching and completely transform the look and the content of a discipline. New tools change cognition.

The above-mentioned objections are gradually losing their persuasive force. First, with the spread of the Internet, computers have become an integral part of the households of millions of people. Second, web interfaces for CAS allow eliminating the user's interaction with their computing engines. The user only needs to know the input syntax of mathematical formulas adopted in the CAS. This significantly simplifies the user's life. We hope that a new generation of browsers will make special input syntax unnecessary. The user will enter mathematical expressions in the usual format using traditional mathematical symbols. Third, both practical and theoretical issues of ICT have been the subject of intense and interesting discussions at numerous venues held under the auspices of SEFI, ICIAM, ICMI, and other authoritative organizations.

For example, the following topics were discussed at the recent 17th ICMI Study: design of learning environments and curricula; learning and assessing mathematics with and through digital technologies; teachers and technology; technology for mathematics education: equity, access and agency, and many others. The editors of [2] conclude that 'appreciating the interdependencies of tools, activities, pedagogies and learning outcomes and designing accordingly is a challenge that mathematics educators will continue to face as digital technologies evolve and extend their reach'.

The authors of the current paper act as practitioners who suppose that the developing of didactics and designing of tutorials based on the CAS usage is a matter of paramount importance although the development of various theoretical approaches is also necessary and desirable. Only tested methods and examples of the successful use of CAS can convince the educational community of practical utility of CAS technologies. In this paper, the didactics and application of *Mathematica* [3] and its Internet version *webMathematica* for teaching students the art of differentiation is considered. We limit ourselves with differentiation of elementary functions, i.e., the functions defined by explicit formulas. We will show that *Mathematica* is ideally suited for the development of differentiation skills.

ELEMENTARY FUNCTIONS AND THEIR SYNTAX

Elementary functions form the most important class of functions students learn in Calculus. Informally, these functions are determined by formulae. Power x^n, exponential a^x , trigonometric functions, and functions that are inverse to them, are called basic elementary functions. The other elementary functions are obtainable from the basic ones by real number multiplication, arithmetic operations and substitutions repeated in a finite number. We can describe the class of elementary functions as algebra over the ring of real numbers with arithmetic operations and substitution as algebraic operations.

In the early stages of learning calculus, students study to examine the properties of continuity of elementary functions and learn to differentiate these functions. For each of these tasks, the syntactic structure of elementary functions could be very useful. In fact, the differentiation is not a mechanical operation, and the art of differentiation is the ability to choose and correctly apply the rules of differentiation. The first is impossible to do without understanding the structure of the formula that determines the function. Let us illustrate this assertion with an example.

Example. Consider the function $f(x) = x^2 + \cos(x + 3^x)$. We see that this function is a sum of the power function x^2 and the composition of cosine and the function $x + 3^x$, which is, in its turn, the sum of two basic elementary functions. The syntactic structure of the formula that determines the function $f(x)$ can be represented by an expression $Plus[Power[x, 2], Cos[Plus[Power[3, x], x]]]$, where *Plus* is a name of an arithmetic operation, *Cos* is a name of a trigonometric function, $Power[x, 2]$ represents the power function, and $Power[3, x]$ represents the exponential function.

Figure 1 represents the syntactic structure of the formula in a form of a tree.

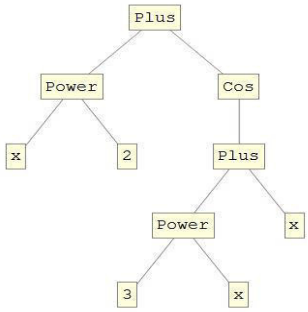

Figure 1. Tree representation of the function determined by the formula

$$\boldsymbol{x^2 + \cos(x + 3^x)}$$

We see that a mathematical formula arises from the bottom to the top, from symbols and numbers to the last used operation.

The user of *Mathematica* can obtain Figure 1 by using a special command **TreeForm**. This is possible because formulas of elementary functions are a special case of *expressions* of *Mathematica*. An expression is one of two basic concepts of *Mathematica*; the other is evaluation. The syntax of *Mathematica* provides a detailed description of the expression and, therefore, gives a more accurate and detailed definition of a formula than in traditional calculus textbooks.

RULES OF DIFFERENTIATION

The following rules are a basis for differentiation of elementary functions.

- Differentiation of a sum: $\frac{d(f(x)+g(x))}{d\,x} = \frac{d\,f(x)}{d\,x} + \frac{d\,g(x)}{d\,x}$;
- The Leibnitz rule: $\frac{d\,f(x)\,g(x)}{d\,x} = f(x)\frac{d\,g(x)}{d\,x} + g(x)\frac{d\,f(x)}{d\,x}$;
- Differentiation of a quotient: $\frac{d\,f(x)/g(x)}{d\,x} = \frac{1}{g(x)}\frac{d\,f(x)}{d\,x} - \frac{f(x)}{g(x)^2}\frac{d\,g(x)}{d\,x}$;
- Chain rule: $\frac{d\,f(\,g(x)\,)}{d\,x} = f'(g(x))g'(x)$;
- Differentiation of a constant: $\frac{d\,c}{d\,x} = 0$, *provided c does not depend of x*;
- Differentiation of the power function: $\frac{d\,x^a}{d\,x} = a\,x^{a-1}$, *provided a does not depend of x*;
- Differentiation of the exponential function: $\frac{d\,a^x}{d\,x} = a^x \ln(a)$, *provided a does not depend of x*;
- Differentiation of the logarithmic function: $\frac{d\,log_a(x)}{d\,x} = \frac{1}{x\ln(a)}$, *provided a does not depend of x*;
- Differentiation of trigonometric functions: $\frac{d\sin(x)}{d\,x} = \cos(x)$, $\frac{d\cos(x)}{d\,x} = -\sin(x)$, $\frac{d\sin(x)}{d\,x} = \cos(x)$; $\frac{d\tan(x)}{d\,x} = \frac{1}{\cos(x)^2}$, $\frac{d\cot(x)}{d\,x} = -\frac{1}{\sin(x)^2}$;
- Differentiation of inverse trigonometric functions: $\frac{d\arcsin(x)}{d\,x} = \frac{1}{\sqrt{1-x^2}}$, $\frac{d\arccos(x)}{d\,x} = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d\arctan(x)}{d\,x} = \frac{1}{1+x^2}$, $\frac{d\,\mathrm{arccot}(x)}{d\,x} = -\frac{1}{1+x^2}$.

In order to differentiate the elementary function, a student needs to apply step-by-step the rules of differentiation going in the direction from the top of the formula tree to its bottom and passing through all its vertices. In our experience, the most difficult for students to begin differentiation. At this stage, the representation of a mathematical formula in the tree form essentially facilitates the successful differentiation.

TEACHING OF DIFFERENTIATON WITH *MATHEMATICA*

Teaching of differentiation is carried out with the help of an electronic tutorial **Steps of Differentiation** developed by the authors and sited on *http://webmath.mesi.ru*. Figure 2 presents the *webMathematica* version of the main page of the tutorial. We choose online version of the tutorial in order that all readers can obtain a clear and comprehensive overview of it.

Figure 2 demonstrates that the tutorial consists of three sections. The first two are of theoretical nature. They contain the basics of differentiation and discuss the rules of differentiation and syntax structure of formulas. There are a list of problems and a pattern of a problem solution at the most interesting third section **Problem solution**. Let us consider the subsection *Pattern of a problem solution* in more detail. We choose the following problem as a pattern.

Problem. Calculate the first derivative of the function $f(x)$ determined by the formula $f(x) = x^2 + cos[x^3]/(x+1)$.

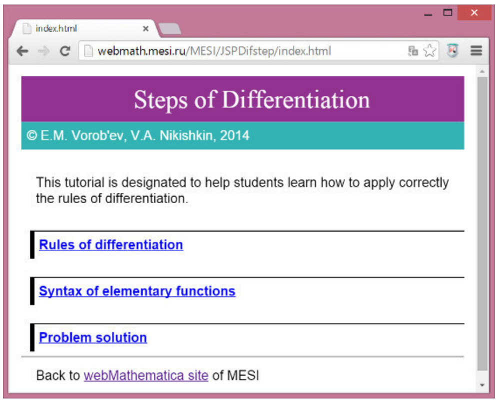

Figure 2. Main page of the tutorial Steps of Differentiation

Solution. Let us calculate the syntax structure of the formula. To this aim, we resort to the HTML form **Tree Form** (see Figure 3).

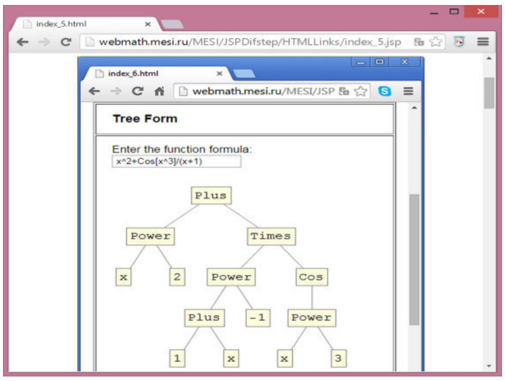

Figure 3. Tree form of the differentiated function

We see that the formula is a sum of two terms: x^2 and $\mathrm{Cos}[x^3]/(x+1)$. Therefore, first we apply the rule of differentiation of sums. We can do the differentiation using the HTML form **Step of Differentiation** (Figure 4).

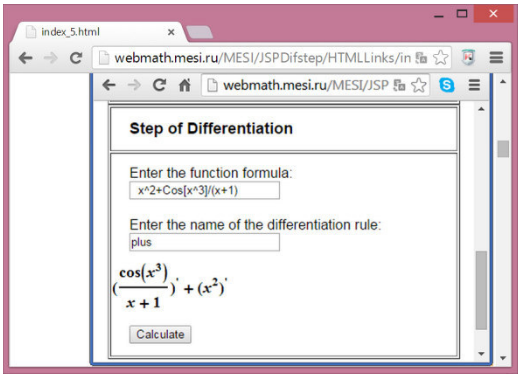

Figure 4. The first step of differentiation

The derivative of the sum is equal to the sum of the derivatives of terms, so we calculate the derivatives of the terms. The second term is evidently the power function. The student can either directly apply the rule for differentiation of the power function, or resort again to the HTML form **Step of Differentiation.**

Mathematica represented the first term $\cos(x^3)\,/(x+1)$ as a product. Therefore, when applying the form **Step of Differentiation** to differentiate the first term, a student can type “times” into the second input field of the form. Continuing in this way, a student comes to the complete solution of the problem. By the way, if a student enters an incorrect rule of differentiation, then, instead of the result of differentiation, he/she receives a message ‘*Improper rule*’.

CONCLUSIONS

Our teaching experience demonstrated the positive effect of the tutorial considered in the paper. It deepens the background and improves drilling skills. Students who used it much more successfully cope with their jobs than their counterparts.

As for *webMathematica*, one of the advantages of its use is that there is no need to install special software on each of the students` computers. Students only need to have an access to *webMathematica* site and a web browser on their computers. The latter is not problem.

We are at the beginning of a long journey towards understanding how to use CAS most effectively in teaching, and we hope that our paper has made a modest contribution to the achieving this goal. Thanks to the combination of visualization and calculation capabilities, *Mathematica* and *webMathematica* enable significant enrichment of calculus didactics. We demonstrated that the didactics of differentiation teaching should be changed in order to reveal the full computational potential of *Mathematica*. We are convinced that such a change of didactics should be done with all topics of calculus.

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EXPLORING THE HISTORICAL DEVELOPMENT OF COMPUTER GAMES RESEARCH IN MATHEMATICS EDUCATION

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About 40 years have passed since the initial efforts to capitalize on the potential of computer games for supporting the teaching and learning of mathematics. A historical review is presented of the empirical research on computer games for mathematics education. First, it is observed that research remained virtually absent while the videogame industry achieved important developments. It has grown steadily during the last decade, reflecting the widespread availability of technologies for producing and playing games. Second, an analysis of the populations and mathematical domains in which computer games have been researched revealed a concentration on arithmetic for elementary school students. Other populations and domains remain largely unexplored. To conclude, we outline alternatives for a further development of computer game research for mathematics education.

Keywords: Computer games, serious games, mathematics education

INTRODUCTION

The interest for using computer games [1] in mathematics education can be traced back to the 70's and 80's. Right from the beginning scholars recognised that the motivating nature of computer games might be used to precipitate useful cognitive activity in the mathematics classroom (Ahl, 1981; Bloomer, 1974; Bright, Harvey, & Wheeler, 1985). Contemporary researchers such as Devlin (2011) argue that the technological sophistication of modern computer games offers opportunities for learning mathematics through immersion in realistic simulations and the mastering of complex rules and mechanics. These characteristics open possibilities for using computer games beyond the drill and practice of basic skills and formulas, allowing ways to support more comprehensive forms of mathematical thinking. After 40 years of efforts, we believe that it is time to review the development of computer game research in mathematics education. This issue has received little attention in recent years. The latest reviews of educational computer games in the general sense offer limited information about the mathematics domain. For instance, Young et al. (2012) reviewed only eight articles related to mathematics and focused on the effects of games on learners' achievement. Similarly, Connolly, Boyle, MacArthur, Hainey, & Boyle (2012), reported only a small number of articles addressing mathematics (4 out of 129), presumably because they constrained their review to articles involving learners over the age of 14 and published until 2009.

Here we present results of a review that focuses on mathematics, including a large number of articles involving both teachers and learners of all academic levels and published until 2014. The current work focuses on the context in which computer games research for mathematics education has developed, and the interests that have driven it. This is part of a larger project aimed to analyse a wider set of factors. In this paper, two research questions are addressed:

How does the development of computer games research in mathematics education relates to the development of videogame culture?

What populations and mathematical domains have been addressed by computer game research in mathematics education?

METHOD

Data collection

The information was gathered from databases relevant to education, social sciences, and information technology, including: ACM (Association for Computing Machinery), Cambridge Journals Online, EBSCO (Psychology and Behavioural Science, SocINDEX, Information Science and Technology Abstracts, CINAHL, ERIC, Academic search complete, Fuente academica, Education source, and SocINDEX), Emerald Group Publishing Ltd, IEEE Explore, JSTOR, Oxford University Press (Journals), Proquest, Science Direct, SCOPUS, PsycINFO, Royal Society Publishing, and Web of Science (Web of Science Core Collection, KCI-Korean Journal Database, Inspec, SciELO, SSCI, SCI-EXPANDED). The search terms combined keywords for computer games with terms related to mathematics education. We employed the same computer game terms as Connolly et al. (2012), including: “computer game”, “video game”, “serious game”, “simulation games”, “game based learning”, “online games”, “MMOG”, “MMORPG”, and “MUD”. The mathematics education terms included: “mathematics learning”, “mathematics teaching”, and “mathematics education”.

Further selection criteria for the articles to be reviewed included: (a) being published in a peer reviewed journal, excluding for instance contributions to conference proceedings and book chapters, (b) include an abstract in English, (c) report an empirical study, (d) explicitly address a mathematical domain, excluding studies addressing spatial abilities or problem solving skills without making a direct connection to mathematics, (e) report technologies designed to be games, excluding for example paper-based games, or playing activities with robots or calculators, and (f) published up to 2014, excluding articles in press. The search was made during January 2015. 116 articles met these criteria and were included in the analyses. Due to space limitations, the full list of reviewed articles is not included here, but it is available online (http://wp.me/p5KVPm-3).

Data analysis

The articles were categorized along a number of dimensions, including*, platform/delivery, game genre, research methodology, game design framework, year published, mathematical domain,* and *population.* Here we focus on the findings resulting from the later three dimensions. The analysis was exploratory and based in descriptive statistics.

RESULTS AND DISCUSSION

How does the development of computer games research in mathematics education relates to the development of videogame culture?

In order to have a deeper understanding of how the research involving educational games for teaching or learning of mathematics, we positioned our review within the context of the social, cultural, technological and economical factors of videogame culture. Educational computer games research has grown significantly. This suggests the overcoming of some of the barriers (Klopfer, Osterweil & Salen 2009) that have slowed or stopped the adoption and development of video games in the field of education. For example, during the 80’s the development of a video game was costly and involved much difficulty. It was not easy group for educational researchers to work with the emerging technology of video games, which required expensive hardware and highly specialized technical knowledge. All this changed as the costs of computational processing and storage became lower. Nowadays powerful desktops or laptop computers are much more affordable, so many more people have access to the hardware required to develop a video game.

Another aspect is the democratization of video games software. For example, when the first video games were developed in the 50's, it was not possible to create a game without advanced programming skills. However, today there are free and commercial "game engines" that allow the development of commercial quality games with little or no formal programming skills. Another situation is the academic training. For example, in Mexico and presumably in other countries, during the late 80's and early 90's, if you wanted a job creating video games, you had to study an engineering degree related to electronics or computer science. Although you did not get to learn how to design and program video games, you were learning a set of knowledge required to create them. Moreover, after graduation, it was very difficult to find a position, either commercial or academic; programming video games, so there was no opportunity for creation.

The aspects outlined above picture the general context of the videogame culture. No we turn to locate the production of research about mathematics computer games in the wider context of the videogame culture. Figure 1 shows the number of articles by year, plotted against key events in the videogame industry. Some parallelisms are worth highlighting.

First we show that in 1981-1995, only one article was reported per year, while in the field of video games happened the following events, as described in http://www.museumofplay.org/icheg-game-history/timeline/):

- 1980. Namco's Toru Iwatani created Pac-Man, and released it in July 1980, becoming the best-selling arcade game at the time.
- 1985. The Nintendo Entertainment System (NES) revives an ailing United States video game industry.
- 1986. The emerging educational software market leaps ahead. The educational computer business flourished with the introduction of CD-ROMs in the 1990s, but crashes with the rise of the Internet.
- 1989. Nintendo's Game Boy popularizes handheld gaming.
- 1990. Microsoft bundles a video game version of the classic card game solitaire with Windows 3.0. It provides a gaming model for quick, easy-to-play, casual games like Bejeweled.
- 1995. Sony releases PlayStation in the United States.

From 1999-2006, the production of articles is still low, ranging from 1 to 6 papers by year.

- 1999. Sony Online Entertainment's Everquest leads hundreds of thousands of users to join in the multiplayer online world of Norrath.
- 2000. Emerging the social simulators games with The Sims, Utopia (1982), Populous (1989) and Civilization (1991).
- 2001. Microsoft enters the video game market with Xbox.

- 2005. Microsoft release Xbox 360.
- 2006. Nintendo release Wii. With its innovative gameplay based in sensors.

From 2007 to 2010, articles production remains modest, but increases from 7 to 14.

- 2008. More than 10 million worldwide subscribers make World of Warcraft the most popular massively multiplayer online (MMOG) game.
- 2009. Social games like Farmville and mobile games like Angry Birds make their appearance in Facebook and the iPhone.

From 2010 to 2013, the production of articles reaches its historical top, increasing from 14 to 20, with a slight reduction towards 2014.

- 2010. The indie game movement becomes prominent with Minecraft
- 2011. Skylanders: Spyro's Adventure becomes the first augmented-reality. Two years later Disney Infinity joins the ranks of toy-video game hybrids.
- 2013. Gone Home, The Last of Us, and Papers, release a mature video game stories that confront players with tough emotional choices in ethically-complex worlds.
- 2014. "Free-to-play" becomes a dominant business model.

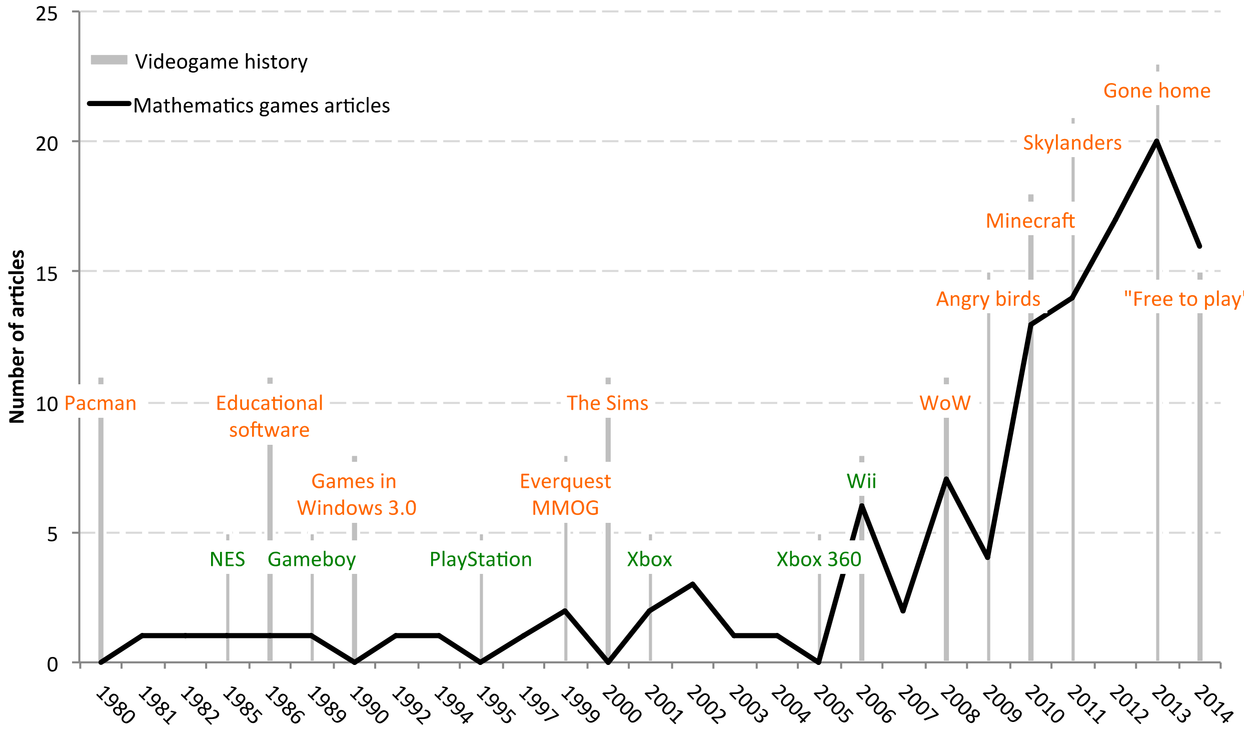

Figure 1 Number of articles published by year involving computer games in mathematics education, and key hardware (green) and software (orange) developments in the videogame industry. Source of videogame information: http://www.museumofplay.org/icheg-game-history/timeline/

What populations and mathematical domains have been addressed by computer game research in mathematics education?

Table 1 shows the distribution of the reviewed articles in a cross-classification of population by mathematical domains. 10 articles were excluded because their abstracts did not clarify either the mathematical domain covered or the population involved.

Populations. Articles have concentrated in elementary school. More than 60 % of the research addresses this population, whereas other populations have received much less attention. Only about 10% of the articles involve Middle school students. Less than 10% of the articles attend students with special needs, such as those diagnosed with dyscalculia or Attention Deficit Hyperactivity Disorder (ADHD). Similar percentages of articles have involved students in high school, higher education, or student teachers. It is striking that only three articles have studied the possibilities of computer games for teaching mathematics in preschool. Only one study addressed the potential of computer games for supporting the learning of mathematics in the general population.

Mathematical domains. More than 25% of the articles involved arithmetical knowledge and skills, including operations such as addition, subtraction, multiplication and division, with natural or rational numbers. About 15% of the reported studies described the application of computer games for supporting mathematics in general, e.g., games that integrate content addressing arithmetic, algebra, and problem solving. A similar number of articles reported studies in which games were used to support the acquisition of numerical skills or knowledge, such as number sense, number line, or decimal numbers. A little less than 15% of the articles involved the development of general cognitive abilities that are essential for succeeding in mathematics, including problem solving, modelling, logical reasoning, or concentration. Efforts to apply computer games technology in the teaching and learning of algebra are reported by about 10% by articles. Marginally smaller percentages of articles covered data handling, including probability, statistics, or measurement; and geometry, including areas, transformations, or spatial skills. Only 2 articles addressed topics such as the history of mathematics or the design of mathematics games. Strikingly, only one article addressed advanced mathematics, i.e., calculus.

Mathematical domain by population. Researchers have investigated different mathematical domains across populations. Articles involving elementary school learners tend to concentrate in arithmetic. Other domains have received less attention at this level, including overall mathematics, numerical skills, cognitive abilities, algebra and geometry. Noticeably, the data domain has been largely neglected. In middle school populations, the modest number of articles is, in general, evenly distributed across mathematical domains, although there is a total absence of investigations in the data domain. Similarly, there are no obvious preferences for a particular domain in the few investigations devoted to higher education. Research involving special needs learners have concentrated on the development of their arithmetic and numerical skills. The research conducted with student teachers has attended mathematical domains rarely addressed with other populations, such as data and what we labelled “other”, which includes computer games design and history of mathematics. The very small number of articles devoted to preschool children address numerical skills, mostly number sense. The only one article involving general populations covers arithmetical topics.

Population	Mathematical domain								
	Arithmetic	Overall	Numerical skills	Cognitive abilities	Algebra	Data	Geometry	Other	Calculus	Total
Elementary school	23	17	11	10	7	2	8			78
Middle school	2	2	1	3	3		1			12
Special needs	3		4	1						8
High school	2	1		1	1	1	1			7
Higher education	1		1	1	1	2			1	7
Student teachers					1	4		2		7
Preschool			2			1				3
General	1									1
Total	32	20	19	16	13	10	10	2	1	123

Note: The total is larger than the number of articles reviewed (106) because one article could involve more than one population and/or cover more than one mathematical domain

Table 1. Number of articles by population and mathematical domain

Computer game research has concentrated on the mathematics education needs of elementary school learners, most frequently in the arithmetic domain. One might say that this is only natural considering that elementary school curriculums have traditionally concentrated on this domain. Only until recent years there has been inclusion of other domains in the elementary curriculum, for example algebra in the U.S. (www.nctm.org/examples/). This tendency to move from concentrating on arithmetic is yet to be reflected in computer game research. Nevertheless, the concentration of research on elementary school suggest that researchers held the view that games are mostly played by children (e.g., Kirriemuir & McFarlane, 2004). This is not necessarily true. Children represent a relatively minor proportion of the gamers population, since the demography of the video game industry has expanded dramatically to reach older segments of the population (e. g., see ESA, 2014 for U. S. data).

Unfortunately, mathematics education research is not capitalizing on the fact that learners in higher levels of education are probably more likely to be players of computer games than children in elementary schools. One of the challenges to do so might be that at higher levels the mathematical concepts are much more complex. In order to be successful, an educational computer game needs to integrate the conceptual or procedural knowledge to be learnt with the mechanics and fantasy elements of the game (Habgood & Ainsworth, 2011). Perhaps is easier to do this with arithmetical operations than with more sophisticated concepts- but see for example *Dimenxian* for an attempt with algebra (http://www.dimensionu.com/dimu/home/games_Dimenxian.aspx).

More studies with preeschoolers are also required. The very small number of articles involving young children might reflect the difficulty of designing software for young children. Educational technology researchers soon discovered that children struggled to control a mouse or a joystick (Revelle & Strommen, 1990). However, the development of touchscreen and motion sensing input devices, e.g., Kinect, is already facilitating the development of mathematics software for preschool learners, e.g., Sinclair & Heyd-Metzuyanim, 2014. Thus, technological advances in user interface design should be reflected in more mathematics computer games for preschool education.

CONCLUDING REMARKS

We have seen that computer games research in mathematics education remained dormant during various decades while the videogame industry achieved important progresses in hardware and software. It is until the last years or so that the body of research has started to increase steadily, perhaps reflecting the increasing availability of videogame hardware and software. This is certainly positive. However, research remains to develop in breadth and depth. Interesting and necessary research lines need to be opened that take advantage of the sophisticated technologies that are nowadays available, to address populations and themes other than arithmetic with children in elementary school. This is certainly an important segment of learners, but children in preschool, adolescents in high school, and young adults in universities also play computer games. In future, we should expect to see innovative games addressing a range of mathematical topics for these and other populations, e.g., adult learners or learners with special needs.

NOTES

1. The term computer game and videogames are used interchangeably

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Theme: Teachers

A LENS TO INVESTIGATE TEACHERS' USES OF TECHNOLOGY IN SECONDARY MATHEMATICS CLASSES

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This paper focuses on how ordinary teachers develop practices in technological environments. It examines also the factors that determine these practices. Broadly based on the frame of the activity theory, several theoretical tools and constructs were combined, adapted and used to identify and analyse relevant elements shedding light onto the complexity of teaching mathematics in such environments. In order to exemplify this theoretical development, a case study is presented about the use of Geogebra by a secondary mathematics teacher. The research lens was both focused onto the tensions within the teacher's classroom activity, and onto the factors that determine this situated activity. The study provided deep insight into the teacher's technology-mediated activity and brought forth the contribution of this new theoretical development to a better understanding of the complexity of integration technology into day-to-day mathematics teaching.

Keywords: Teachers, Activity, Determinants, Tensions, Technology uses.

INTRODUCTION

Nowadays there is an increasing emphasis on how (and certainly why) do teachers use digital technologies (DT) in their classes. This trend is relatively recent in mathematics educational research (Artigue, 2010) given the emphasis of research outcomes about mathematical potential of DT and its effects on students learning (Hoyles & Lagrange, 2010). This is currently noticeable in many studies published lately, such as the set of research articles gathered in "The mathematics teacher in the digital era" (Clark-Wilson, Robutti & Sinclair, 2014). In the penultimate chapter of this book, Ruthven (p. 373) examines and illustrates three contemporary frameworks for analysing the teacher's expertise while integrating DT into everyday teaching practice. He concludes that an important contribution researchers can make in this field is to identify such expertise and provide means of representing and analysing it (p. 390).

In my earlier research studies, I focused on the ordinary practice of mathematics teaching when DT are in use and tried to describe and understand teachers' practices by searching at a local and a global level. In this current research, I continue this path but rather at a micro level. My main research questions are: What *tensions* do an ordinary teacher encounters when preparing a lesson and managing it and how he/she deals with? What are the factors that *determine* the teacher's activity and make it viable?

In order to do so, I attempt to connect and adapt frameworks that are primarily derived from the activity theory. The first section provides an overview on these frameworks and on the different tools borrowed or constructed enabling a deeper analysis of the teacher's activity at a micro level while considering at the same time the complexity of practices at a global level. The second section illustrates such an analysis via a case study. The last section is about more research questions and perspectives.

THEORISING ABOUT MATHEMATICS TEACHERS USES OF TECHNOLOGY

Far from research projects and experimental projects, technological evolutions, schools material equipment, institutional and social pressures gradually led ordinary teachers to develop uses of technologies that become less anecdotic, more integrated into the day-to-day practice and have reached a certain degree of stability. My concern in studying these uses took shape throughout a body of work addressing complementary aspects of DT integration and the associated professional learning and development of teachers (Abboud-Blanchard, 2013). To do so, I exploit in my work resources from the double didactic and ergonomic approach (Robert & Rogalski, 2005) alongside with those from the instrumental approach (Rabardel, 2002), coordinated within the overarching frame of the activity theory (Cole & Engström, 1993) to which both can be related. This theoretical work gave birth to a conceptual framework, shaped as analytical structure, that enables to understand how ordinary teachers face professional challenges and constraints and what changes could be observed in their technology-based activity at the classroom level (for more details, see Abboud-Blanchard, 2014). This structure considers three main axes to analyse practices and three factors that determine them.

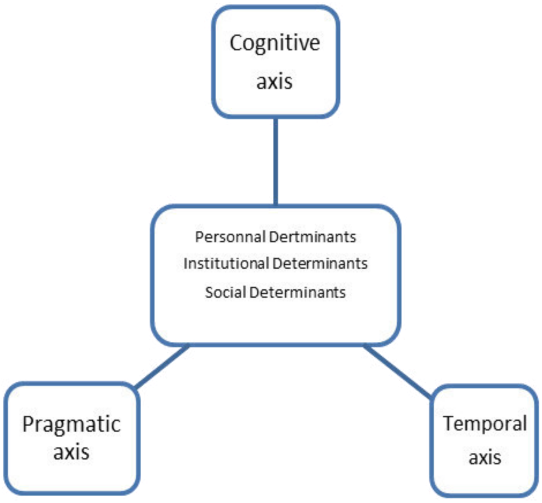

Figure 1: CPT Structure

The *Cognitive* axis relates to the mathematical intentions and goals of the teacher. The analysis with respect to this axis focuses mainly on the scenario the teacher sets for students in terms of mathematical tasks. The *Pragmatic* axis is first based on the effective observation of teacher classroom activity, i.e. what really happened and not what might have. The analysis with respect to this axis incorporates teacher-students interactions, teacher discourse and also transversal aspects in the lesson management related to the presence of DT and to material aspects of the environment. The *Temporal* axis focuses on analysis relatively to the complexity of teaching in technology environments with respect to time. This complexity concerns several aspects: the time needed for the organisation of teacher's work (preparing lessons, planning lessons, evaluating the outcomes of lessons), the dynamic time of the class and the didactical time of learning.

Personal determinants deal with the teacher's conception of mathematical knowledge, of teaching processes and of his/her familiarity with technology or beliefs related to the impact of technology on mathematics learning. *Social* determinants are about how teacher adapts to the conditions of the work environment in a given school, to the habits of the class, to the colleagues as individuals and also as a community. *Institutional* determinants mainly concern the influence of institution, for example, via the curriculum, textbooks, hierarchy requirements, and so on. These factors are often considered by the teacher as constraints to deal with while practicing the teaching profession.

An analysis using this CPT structure aims to locate the characteristics with respect to each axis and then to attempt recombination and interpretation inferred by the determinants in order to provides access to the wholeness of the activity of the teacher in situ.

In the present *ongoing research*, I go back to instrumental approach and specifically to the model of "Instrument-mediated Activity Situations" elaborated by Rabardel as a tool for the analysis of tasks and activities. Rabardel (2002, p.42-43) explains this model by stating:

> Beyond direct subject-object interactions (dS-O), many other interactions must be considered: interactions between the subject and the instrument (S-I), interactions between the instrument and the object on which it allows one to act (I-O), and finally subject-object interactions mediated by an instrument (S-Om). Furthermore, this whole is thrown into an environment made up of all the conditions the subject must take into consideration in his/her finalized activity.

By adopting this model, I add to it the notion of *tensions* adapted from the activity theory. Indeed, activity systems are characterised by internal tensions that can promote and trigger innovation and changes and can be a source of development (Cole & Engeström, 1993). It is worth to note that tensions is not necessarily understood as problems, obstacles or conflicts and that I extended it to better understand the complex context of teaching with DT. In fact, teachers have often to deal with tensions due to the presence of the tool and its role in the student's activity but also its interaction with the mathematical knowledge at stake. Some of these tensions might be predicted by the teacher and so he/she plans how to manage them. Others are unexpected and make the teacher take decisions, in situ, that direct his/her actual activity. More the teacher has a low level of instrumental genesis, more the number of unexpected tensions is higher and more the management of these tensions leads to disturbances in the predicted activity (of both teacher and students).

Based on these theoretical considerations, I illustrate in the following the usefulness of this conceptual framework for the analysis at a detail level, of an ordinary teacher's activity when using digital technologies.

A CASE STUDY

Methodology issues and context of the Study

Data gathered for this research include semi-structured interviews with the teachers and video-recording of technology-based sessions carried out with their classes. The teacher chooses the session to be observed and video-recorded it herself (with a fixed camera at the back of the classroom).

In this case study, the teacher investigated is Dave, an experienced (10 years career) secondary teacher. He was chosen because, on the one hand he is not involved in any experimental project and is not a technology-expert, and on the other hand he supports the use of technology in mathematics education. Within Dave's interview the focus is on his teaching experience, the professional context in which he is working, his use of institutional resources (curriculum, textbooks, academic websites, etc.), and on how and under what conditions he integrates DT into his practices. Dave chose a geometry session where he uses Geogebra. In addition of the video-recording, he provided a document explaining choices made and rationales for the students' task in this session.

The analyses combine the two sources of data in order to capture features and determinants of Dave's activity by means of the conceptual tools presented above.

Summarising the session

The session took place in an 8th grade class, in computer room, with a data projector screen on which the teacher's computer is displayed. The students are asked to download a file previously prepared by the teacher. When opening the Geogebra file, students discover the following screen:

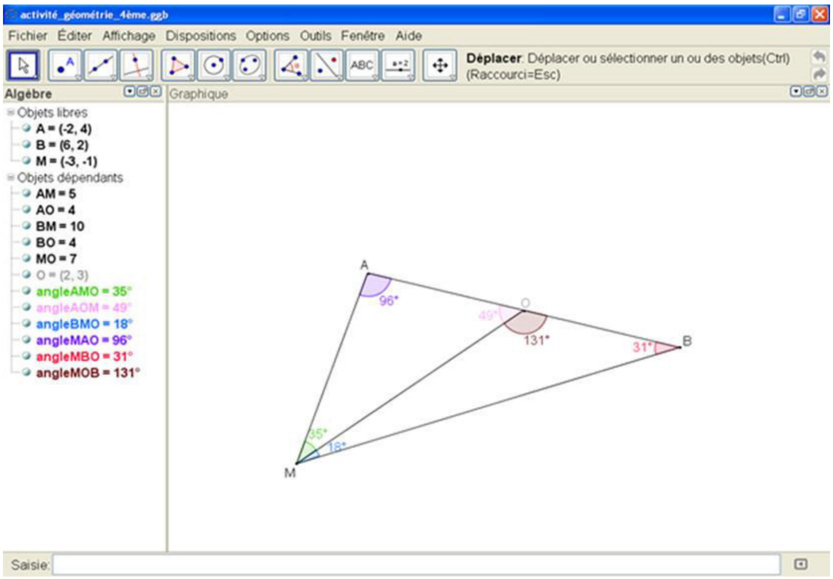

Figure 2: students' computer screen

The teacher gives then a preliminary remark: "*please recall that every representation (on paper or computer) of a geometrical figure is inaccurate; measures given by Geogebra are approximated values*"

Students are first asked to move point M in order to have both triangles AOM and BOM become isosceles at O. Second, they have to find other positions of M satisfying this condition, to observe the AMB triangle and to make a conjecture about the M angle. Last, they have to prove this conjecture.

Approximatively half-hour after the starting, the students were still trying (or succeeding for some of them) to have angles A, B and M equal to 45°. The teacher makes several individual and collective interventions: "*You charge yourself with supplementary constraints, so it is difficult to find several positions*". Finally after a "right" example was proposed by a student on the projected screen, the teacher shows several cases where the two triangles are isosceles without having OM perpendicular to AB. Within the ten minutes that follow, the teacher moves from the demand of finding more than one (general) configuration to finding "a maximum" of positions for M, and then he asks for "all possible positions". Let us notice here that from this moment on the teacher's initial goal is changed. He does not really discuss students' proposals, but insists on his present goal i.e. finding that the set of all possible positions is a circle. When a student proposes this idea, he immediately approves and draws the circle and places M on it. It is only after this episode that he gets back to the conjecture and (re)formulates a student's proposal: "*it is always a right angle, yes, that is the triangle seems to always be a rectangular triangle at M*". He decides then to dictate the present state of shared or – supposedly shared – knowledge that is to say the set of all possible positions of M is a circle. He postpones the proof of the conjecture because it is already the end of this session.

Analysing the teacher's choices according to the cognitive and the temporal axes

What is at stake in this session is the theorem related to the circumscribed circle to a rectangular triangle. It is an unusual task concerning this theorem in the French curriculum. The presence here of a dynamic geometry software allows a process of investigation that is difficult to achieve in paper-and-pencil environment. According to Dave, this theorem was not yet seen by his students, when engaged in the task. This is meant to be an introductory task, and not a task requiring a functional use of this theorem in problem solving. Let us analyse the two main choices made by the teacher when preparing the task. The first choice is to construct the Geogebra figure himself and to let students only download the corresponding file. Such a choice limits the instrumented students' activity as concerning the construction of figures through Geogebra. The second choice is related to what is made visible to students on the screen. In the algebraic window, Dave decided to give usual geometrical names to the sides and angles; in the graphical window, he indicates the angles measures. Another choice in the Geogebra options is to round measures up to units.

Dave explains these choices by the fact that his aim was to bring students directly to the mathematics exploration of the figure and not to spend time doing it. This was thus meant to restrain the students' instrumented action (limited to handling skills) and to focus their attention on the geometrical exploration and to devote more time to the process of conjecture validation involved in the last question.

Unfortunately (for Dave), moving M in such a way that the angles become equal is not simple at all. Firstly, the coordination between looking at measures and moving the point is somehow complex. Secondly, the teacher wanted students to focus on angles measures when considering the side lengths would make handling easier. Moreover, it is quite possible that when preparing the task, Dave was testing the exploration task on Geogebra, guided by all his mathematical knowledge, particularly the locus of M (the circumscribed circle itself).

Identifying tensions and analysing their impact according to the pragmatic and temporal axes

Tension between what does the teacher expected from the use of Geogebra and how students actually use it. A part of this tension was indeed predicted by Dave. That explains the choices he made when preparing the task (see above) in order to minimise the impact of this tension. But other parts were not predicted, which called for the teacher's specific interventions such as the following.

First, students tried to move points A & B where Dave did not want them to do so. But Geogebra enables them this manipulation, and they thought that searching for isosceles triangles might be easier if they move not only M but also A & B. The teacher intervened throughout the session to forbid explicitly many students to move A & B. When a student was still moving the two points half an hour after the beginning of the session, Dave took the control of the computer himself, reset to the initial state and re-explained the task to the student. This difference in the rhythm of task performing among students in computer room is often noticed in technology-based lesson and could be highlighted as a characteristic of such a context (Abboud-Blanchard, 2014). It often magnifies the tension related to the time (see below).

Second, the algebraic window of Geogebra allows students to observe not only the equality of angle measures but also that of side lengths. Dave perfectly knew this possibility that is why he indicates in the graphic window only the measures of angles in order to direct the attention on them. However, some students still observed all of what was available in the graphic window and the teacher was not aware of this fact during the session. For example, a student encountered the following phenomenon: in the OBM triangle, angle B and M were not equal (45°; 46°), whereas the sides OM and OB were equal (2; 2). The teacher, focusing on the angles (not seeing the sides values) reacted by saying that the equality must be more precise. The student mumbled after the teacher moved away: "*I don't understand… it is precise!*"

*Tension between how each of students and the teacher consider the available mathematical knowledge within Geogebra***.** Dave considers that Geogebra provides approximated mathematical information (see determinants below) whereas students gather from Geogebra information that they consider as reliable. The latter is strengthened by the fact that Dave rounded all measures to units. This tension provoked throughout the session several interventions (collective or individual): the teacher reminding students that they must not forget the "approximately character" of what they see on the computer screen whereas at the same time he asks them to use what is seen on the screen to make conjectures.

By rounding to units, the number of M possible positions, where there is angles equality, becomes a discreet one. When Dave changed the initial task by adding another sub-task aiming to find the "set" of all possible positions, he had to state that even if Geogebra gives a limited number of such positions, there are actually an unlimited number. He hastened to bring an end to this contradiction by immediately drawing the circle.

These kinds of tensions are without any doubt related to the weak instrumental genesis of Dave himself regarding Geogebra. This fact generates *tensions between the limited role he ascribed to Geogebra* in the "geometry to be learnt", i.e. a means of simple exploration, and *the actual role that Geogebra is playing during the session.* Students had computer handling difficulties and were focusing on secondary visual information or adding constraints facilitated by the visualisation potential of Geogebra. Dave was not aware of all these difficulties when he designed the task and prepared the session; he did not know how to deal with this tension and with the disturbances it generated.

Tension between the planned time for instrumental task (an average of 1/3 of the total duration of the session) *and the actual time this task is taken in the lesson in progress***.** After 2/3 of the time session, students were still trying to find several positions of M enabling them to make a conjecture about the angle AMB. Being aware of the slow progress of the students' activity, Dave decided to interrupt them and called for a "first assessing" where he gave the right answers and dictated the conjecture, ending thereby the instrumental task.

Inferring determinants of the teacher's activity

Dave's activity is determined by different combined factors. The analysis of the lesson in progress, allows inferring impact and articulation of these factors. The analysis of his interview puts forward particularly personal and institutional determinants.

Indeed managing conjectures in an investigation process is promoted by mathematics curriculum for the French lower secondary school (6^{th} to 9^{th} grade). The curriculum also promotes the use of dynamic geometry software for constructing figures and investigating these figures. Geogebra is often used in this line, as designing several figures through the software is (generally) easier than with paper-and-pencil and allows students to look for invariants, and propose conjectures. Dave explains his choice of this particular task by referring to these *institutional determinants*. A plausible inference is that he was expecting (and hoping) that students would enter in a real investigation work relatively to a new geometrical content.

*Interactions between personal and social/institutional determinants:*Dave chose to present (on the Geogebra figures) information about the measures of the angles of the "to-be" isosceles triangles and not about the lengths of their sides; this is a bit unusual. In fact, past curricula (6^{th} and 7^{th} grades) present equality of angles as a property, not as a definition of isosceles triangles (even if this property is a characteristic of isosceles triangles). However, starting from the measures of angles allows to validate the conjecture that the angle AMB can be computed and shown equal to 90°, using the theorem of the sum of the angles of a triangle, previously known, and the fact – implied by the design of AMB – that angle M is composed of two angles, equal to the other angles of AMB.

We can consider here that the use of Geogebra interacts with Dave's will to modify his teaching practices (as he declared). That is to say he sees an opportunity to introduce a new way for teaching the geometrical chapter devoted to the circumscribed circle and the rectangular triangle where the current teaching habits (as reflected in textbooks) presenting the other way round.

The personal determinant of "being rigourous": The teacher's choice (about the angles) opens the possibility for a real mathematical proof of the central property about the rectangular triangle and the circumscribed circle - using wide scope knowledge, instead of referring to "figural properties" of the rectangle (draw on AMB by a central symmetry).

In fact, during the session, Dave frequently employs logical connectors (so, because, as, then…) in his discourse. An interpretation of this observation, along with considering his will to let students spend more time on the proof process, is that Dave is strongly oriented toward students developing a logical treatment of mathematical tasks. Such an orientation seems to be a personal determinant of his choice in this particular task.

Interactions between personal and social determinants: The wordings of the task (e.g. the preliminary remark) also give some evidence both of a collective determinant (shared by secondary mathematics teachers) and of a personal determinant (being rigorous). This might be related to the common preoccupation of secondary mathematics teachers to make students move from the geometry of "material traces" of primary school to a *per se* geometry of abstract figures. The personal choice appears when we observe Dave recalling this same remark throughout the session. It can also be deduced from several interventions of Dave such as: "*is it really precise? – of course, when I say really it is roughly, as it is on the screen", or:* "*with regard to what I see on the screen,* *it must be more precise*". As the task requires finding equal angles by using Geogebra, this preoccupation of precision became a critical issue that the teacher had serious difficulty to manage.

CONCLUDING REMARKS AND PERSPECTIVES

Through this case study, I tried to show that using the theoretical lens I am developing reveals on the one hand how complex the teacher's activity in technology-based-lessons can be, and on the other hand provides in-depth insight into tensions that regulate or disturb this activity. It also stresses the necessity to connect understanding teacher's activity and carful reflection on its determinants. At the same time, it raises the question of the possibility to a teacher to maintain the students on the "cognitive trajectory" he planned for them in DT environment, giving all the tensions that can be identified.

This fine-grain analysis is nowadays needed to improve our understanding, as researchers, of the increasing complexity of ordinary teachers' practices related to DT. It is also needed to enhance our action as teacher educators in order to help teachers adapt to new challenges they are facing with artefacts in perpetual growth.

I would like in this closing section to give thoughts on what is planned about how to go further and more deeply in this ongoing investigation. Two directions of gathering data are planned.

The first is to observe the same teacher over several years relatively to the same type of use of DT. For example, Dave proposed recently to provide a video-recording of a similar session that he carried out in his class this year. He states that he made several changes and different choices following what he experienced last year. The aim of the analysis is particularly here to have insights on how the teacher's activity develops and what elements of the above analyses have a direct impact on its development.

The second is to investigate teachers having different personal profiles and/or working in different social/institutional context. For example, I am currently analysing the session of an experienced teacher, also in dynamic geometry environment, but who is less enthusiastic than Dave to use DT with her "weak" students. The aim of the analysis is particularly here to explore more deeply the impact of these determinants on the technology-based activity and on its development.

The research presented in this paper is an ongoing one; more data and more analyses would allow testing the robustness of our new theoretical lens. Indeed, frames we use help us to address what we aim at and new approaches we choose are dependent on what we are looking for. We need also to be aware of their limitations!

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TEACHERS' SUPPORT OF STUDENTS' INSTRUMENTATION IN A COLLABORATIVE, DYNAMIC GEOMETRY ENVIRONMENT [1]

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We report on a case study that seeks to understand how teachers' pedagogical interventions influence students' instrumentation and mathematical reasoning in a collaborative, dynamic geometry environment. A high school teacher engaged a class of students in the Virtual Math Teams with GeoGebra environment (VMTwG) to solve geometrical tasks. The VMTwG allows users to share both GeoGebra and chat windows to engage in joint problem solving. Our analysis of the teacher's implementation and students' interactions in VMTwG shows that his instrumental orchestration (Trouche, 2004, 2005) supported students' instrumentation (Rabardel & Beguin, 2005) and shaped their movement between empirical explorations and deductive justifications. This study contributes to understanding the interplay between a teacher's instrumental orchestration and students' instrumentation and movement towards more deductive justifications.

Keywords: dynamic geometry, teacher practice, student reasoning, mathematical discourse, collaboration

INTRODUCTION

Dynamic geometry environments (DGEs) afford learners the ability to construct, visualize, and manipulate geometric objects, relations, and dependencies. These affordances support empirical explorations and theoretical justifications or proofs (Christou, Mousoulides, & Pittalis, 2004). In DEGs, empirical explorations are experienced immediately while the need to formulate proofs is latent and requires either a learner's disposition towards justification or pedagogical intervention. Pedagogically motivated transitions from empirical explorations to theoretical justifications depend on carefully designed tasks, teacher guidance, and classroom climates that support conjecturing and deductive justifications (Arzarello, Olivero, Paola, & Robutti, 2002; Hölzl, 2001; Öner, 2008).

Technologically-enabled collaboration, which for mathematics teaching and learning supports social conjecturing and justification, can occur in computer-supported, collaborative-learning (CSCL) environments (Öner, 2008; Silverman, 2011). In such CSCL environments, mathematics education researchers can investigate how pedagogical interventions support learners' appropriation of the environment and promote their movement between exploration and deductive justification. Knowing how to support this appropriation and promote this movement will enable mathematics education researchers and educators to realize the potential of DGEs to improve geometry learning and of CSCL environments to engage learners in developing mathematical ideas through online collaboration that parallel the real-world online, collaborative work of mathematicians, including Fields Medal recipients (Alagic & Alagic, 2013).

From a larger design-experiment project, this paper reports on a case study that aims to understand and describe a teacher's pedagogical interventions–using the framework of instrumental

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orchestration (Trouche, 2004, 2005)–that support learners' instrumentation (Rabardel & Beguin, 2005) and shape their movement between exploration and deductive justification as they work on geometric tasks in a CSCL environment. We understand pedagogical interventions to be instructional actions initiated by teachers that precede, invite, sustain, monitor, or reflect on students' activity. These actions are organized according to instrumental orchestration to understand the teacher's support of students' instrumentation of the Virtual Math Teams with GeoGebra environment (VMTwG). By instrumentation and "movement between exploration and deductive justification," we mean discursive, recursive actions in VMTwG through which learners are motivated to notice relations while manipulating objects to develop and communicate convincing arguments that satisfy their peers about the relations. A guiding research question frames our analyses: What pedagogical interventions promote learners' instrumentation and their movement between exploration and deductive justification? To understand students' instrumentation and their movement between exploration and deductive justification, we analyze a teacher's pedagogical interventions and his students' consequent actions.

RELATED LITERATURE AND CONCEPTUAL FRAMEWORK

When integrating technology in the learning, the instrumental approach provides insight into understanding how learners interact with technological tools. Some studies investigate how learners appropriate technological tools while learning mathematics (Guin & Trouche, 1998; Kieran & Drijvers, 2006). To understand this appropriation, some researchers examine the role that mathematics teachers play when integrating technology (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Sutherland, Olivero, & Weeden, 2004). Using Trouche's (2004, 2005) notion of instrumental orchestration, investigators study how mathematics teachers support students' appropriation of technological tools. Sutherland et al. (2004) studied how a secondary school mathematics teacher taught proofs with a DGE. They found that the teacher's orchestration focused on relationships between geometric construction of objects and their properties, where students used the "drag test" to identify mathematical properties. Drijvers et al. (2010) investigated how three mathematics teachers orchestrated the teaching of functions using an applet that allows users to create functions and visualize their graphs. They identified six different orchestration types that the teachers used to support students' appropriation of the tool. In these two studies, the teachers were the focal point of classroom interaction as they orchestrated students' learning with technology, which corresponds to the theory of instrumental orchestration (Trouche & Drijvers, 2014). In contrast, research is needed to understand teachers' instrumental orchestration when they are not the focal point of classroom interaction, for example, when students work in a collaborative environment. In such environment, students are the focal point of classroom interactions and the teacher's role is significant but peripheral.

Instrumental orchestration aims explain teachers' role in supporting learners' appropriation of technological tools. To understand this appropriation, we draw on a Vygotskian perspective about goal-directed, instrument-mediated action and activity. Instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) theorizes how learners interact with tools that mediate their activity on a task. To appropriate a tool, learners (teachers or students) develop their own knowledge of how to use it, which turns the tool into an instrument that mediates an activity between learners and a task. Learners engage in an activity in which actions are performed upon an object (matter, reality, object of work…) in order to achieve a goal using a tool (technical or material component). Rabardel and

Beguin (2005) emphasize that the instrument is not just the tool or the artifact, the material device or semiotic construct, it "is a composite entity made up of an artifact component and a scheme component." (p. 442). Learners appropriate artifacts by developing their own utilization schemes. The transformation of an artifact into an instrument, or instrumental genesis, occurs through two important dialectical processes that account for potential changes in the instrument and in leaners, instrumentalization and instrumentation. Instrumentalization is "the process in which the learner enriches the artifact properties" (Rabardel & Beguin, 2005, p. 444). Instrumentation is about the development of the learner side of the instrument; the learner assimilates an artifact to a scheme or adapts utilization schemes. Instrumentation plays an important role in understanding the relationship between technology and mathematics but it can be a complex process (Artigue, 2002).

Part of the complexity of instrumentation lies in its multidimensionality. It has an individual as well as social dimensions (Trouche, 2005). Instrumental genesis mainly considers the individual aspect of instrumentation. To account for the social aspect, Trouche (2004, 2005) introduces instrumental orchestration to describe how teachers manage mathematics classroom when integrating technological tools. It is defined by the arrangements of artifacts in the environment, didactical configurations, and teacher and student moves within these configurations, exploitation modes (Trouche, 2004, 2005). Teachers use different combinations of didactical configurations and exploitation modes to support their students' instrumentation. Their orchestration acts on three levels: artifacts, instruments, and students' relationship with the instruments. In each level, teachers choose certain configurations and exploitation modes to support students' instrumentation. Ruthven (2014) provides a summary of Trouche's example of instrumental orchestration with using "Customised calculator" to teach about limits in which he describes the specific didactical configurations and exploitation modes. For example, the artifact level has two configuration modes: a) "classroom calculators are 'fitted out' with a guide affording three levels of study of the limit concept" and b) "these are designed to support the shift from kinetic concept of limit to an approximative concept". The exploitation modes for this level are a) "guide can be available always or only during a specific teaching phase", b) "students can use guide freely when available, or be constrained to follow the order of the levels", c) "components can be fixed, or updated in response to classroom lessons", and d) "recording of steps of instrumented work, can be required, or not" (p. 382). For Trouche (2005), instrumental orchestration tries to answer questions about what technological artifacts mathematics educators should introduce to learners and what guidance they should provide so learners can appropriate and use artifacts as instruments to mediate their activity.

METHODS

The research setting is a professional development project that involves middle and high school teachers in two semester-long, technology-focused online courses. The first course engages teachers in interactive, discursive learning of dynamic geometry through collaborating on construction and problem-solving tasks in VMTwG. The teachers reflect in writing on the mathematics, collaboration, and technological components of the course and collaboratively plan how to implement course modules with their students. In the second course, a reflective practicum, the teachers engage their students in at least 10 hours of class sessions to learn dynamic geometry through use of VMTwG to work on construction and problem-solving tasks.

The online environment, VMTwG, is an interactional, synchronous space. It contains support for chat rooms with collaborative tools for mathematical explorations, including a multi-user version of GeoGebra, where team members can construct dynamic objects and drag elements to visually explore relationships (see Figure 1). VMTwG records users' chat postings and GeoGebra actions, which participants can review and even replay at various speeds. The research team designed dynamic-geometry tasks to encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice relations and dependencies among the objects, make conjectures, and build justifications.

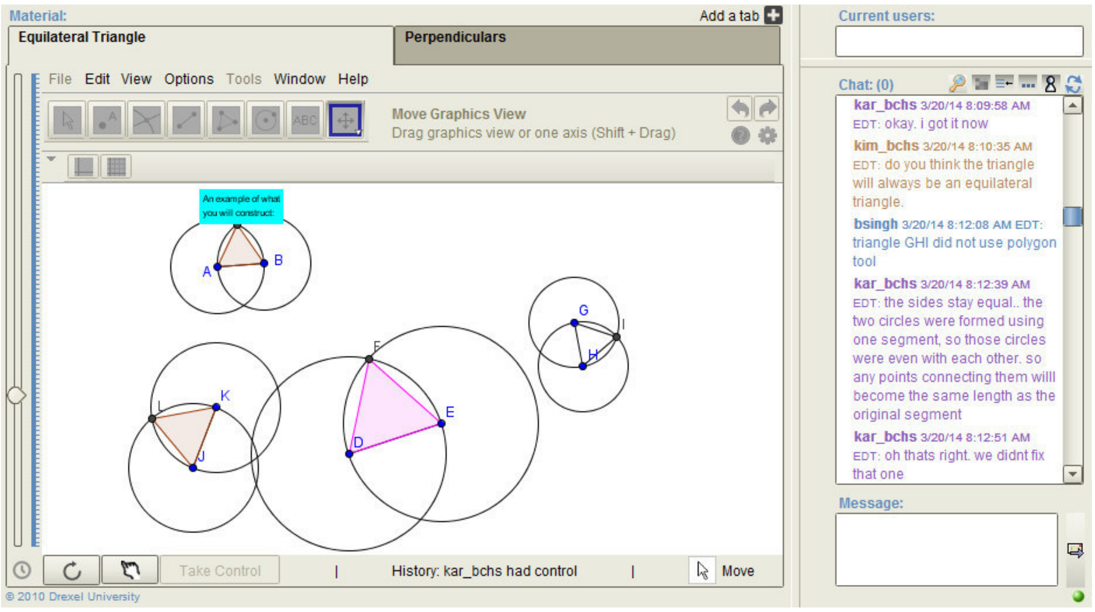

Figure 1: Screenshot of VMTwG environment with the work of Mr. S.'s students

The data for this case study come from the second course and concerns the work of a high school mathematics teacher, Mr. S. He engaged his class in VMTwG in small teams of three to four students each. The class worked in a computer lab, and Mr. S. encouraged students to communicate only through VMTwG. To understand his instrumental orchestration and how his pedagogical interventions promote students' movement between explorations and justifications, we analyze qualitatively four sources of data: (1) the tasks he used with his students, (2) the modifications he made to the tasks after reviewing teams' work, (3) the logged VMTwG interactions of two teams of his students on the tasks, and (4) his reflections on their work, which he wrote after each class session. We chose to analyze two teams, Team 1 and Team 6, since Mr. S. considered those teams to be most collaborative.

On each of the four data sources, we performed conventional and directed content analysis (Hsieh & Shannon, 2005). We were particularly interested in coding and categorizing both Mr. S.'s pedagogical interventions and the deductive justifications of two teams of his students. The data drives our analysis, and we interpret them using the theories of instrumental genesis and orchestration whenever there are links between the data and the theories.

FINDINGS

Based on our analyses of Mr. S.'s implementation of the project design, his interventions were directed at supporting students' actions that can be grouped in three categories: collaboration, mathematical reasoning, and the use of technology. In addition, the analysis reveals that Mr. S. followed a trajectory of pedagogical interventions focused on his students' discursive interactions and their emerging knowledge of dynamic geometry. In his reflections on his students' work, Mr. S. expresses an overall goal that, within their teams, students manipulate and construct dynamic geometric objects and notice and discuss relations among them, particularly relations of dependency. To achieve this goal, Mr. S.'s pedagogical interventions focused on how the teams of students collaborate. Having given his students a task designed to promote collaboration, Mr. S. expressed concern in his weekly reflection that the teams did not collaborate successfully. He reported that to ensure successful collaboration sessions he subsequently discussed with his class features of successful collaborative sessions and presented examples of what he considered good collaboration moves. To underscore his advice, he distributed a list of behaviors that can help to ensure successful collaboration and called it "The Pledge". It contained behaviors such as "Include everyone's ideas" and "Ask what my team members think and what their reasons are".

These pedagogical interventions and ones that we will present below focused on collaboration reveal that Mr. S. is choosing exploitation modes (instructional decisions) that encourages students to be reflective of their work within their groups. His pedagogical interventions are mostly focused on the second and third level of his instrumental orchestration. Those levels are concerned, respectively, with the instrument and the students' relations with the instrument. Mr. S. used collaboration as a vehicle to orchestrate his students' appropriation of VMTwG artifacts and movement towards deductive justifications. In his weekly reflections, he assessed his students' reasoning by tracking their collaboration and their use of mathematical language.

Closely following Mr. S.'s interventions concerning collaboration, he focused on aspects of students' use of the technology. This focus is at the first level–artifact level–of his instrumental orchestration. In his weekly reflections, he reported that during his students' engagement in VMTwG, he "monitored progress and resolved some tech issues." He helped students gain insights into the use of particular GeoGebra commands by modifying tasks and by directing his students to view specific YouTube GeoGebra clips.

As Mr. S.'s teams of students increased their effective collaborative interactions, he shifted his pedagogical interventions more explicitly towards supporting their mathematical reasoning. He discussed with his class the concept of dependency in dynamic geometry to contrast it with dependency in other mathematical domains and modified the tasks to explicate particular mathematical ideas. He posed detailed questions to foreground mathematical discourse. For example, he found that the tasks' original questions were not specific enough to elicit mathematical reasoning, so he included the following questions in one of the tasks, "Constructing an Equilateral Triangle":

1. What kinds of triangles can you find here?
2. Drag the points. Do any of the triangles change kind? Discuss this in the chat.
3. Are there some kinds [of triangles] you are not sure about?
4. Why are you sure about some relationships?
5. Does everyone in the team agree?

These questions prompted students to attend to particular objects and relations in the construction and to discuss their behavior.

Mr. S.'s instrumental orchestration and his other pedagogical interventions contributed to his teams of students' instrumentation and movement towards greater collaboration and deductive justifications. For example, according to Mr. S.'s and our analyses, a team of three students (Team 6) improved their collaboration, explorations, and mathematical reasoning. In their third session, the task asked them to construct an equilateral triangle, find the relationships among objects in their construction, and justify their claims. The students first dragged a pre-constructed figure of an equilateral triangle (see triangle ABC in Figure 1) to explore elements of the construction and their behavior. Afterward, they each constructed a similar figure (see Figure 1) and dragged their construction vigorously to validate and justify their construction. Below, an excerpt of their discussion shows how they articulated a valid justification of why their constructions were of equilateral triangles.

18	kar_bchs:	looks like we both got it [both successfully construct and drag the figures vigorously]
19	kim_bchs:	yay, it seem like for a second one of the circles appeared much larger. but that was my imagination.
20	kar_bchs:	oh. lol. why is the third point dependent on the distance between the first two points? (number 7)
21	kar_bchs:	it just connects the points and the circles. making them all one piece
22	kim_bchs:	as the segments change sides so does the radius of the circle. However, the triangle remains an equilateral traiangle
23	bsingh:	[the teacher] be sure to read directions, ALL, and make the pledge
24	kim_bchs:	triangle
25	kar_bchs:	yea. even though the sizes of the sides change, the fact that it is an equilateral triangle doesn't
26	kar_bchs:	each side has the same distance in between it. even when you move the points
27	kim_bchs:	i notice that point d and e are on the circumference of one circle. while point f is an intercetion of both circle. making it dependent on both points.
28	kar_bchs:	if you try and move the intersected point (F and I), it wont move. but yea you're right, the intersecting point depends on the segment that was made
29	kim_bchs:	*point f is an intersect of both circle
30	bsingh:	[the teacher] there is something missing, are you reading the directions
31	bsingh:	[the teacher] we are only doing tab 1 today
32	kar_bchs:	i didnt use the polygon tool.. thats missing in mine
33	kim_bchs:	i just notice that.
34	kar_bchs:	can i try?
35	kar_bchs:	okay. i got it now
36	kim_bchs:	do you think the triangle will always be an equilateral triangle.

37	kar_bchs:	the sides stay equal.. the two circles were formed using one segment, so those circles were even with each other. so any points connecting them willl become the same length as the original segment
.	.	.
.	.	.
.	.	.
50	kim_bchs:	the radius of a circle is the same distance. segment AB is Sure. the radii of both circles and Segment AC and BC are also radii of both cicles. hense, the triangle should be euilateral.
51	kar_bchs:	the circles are equal. Making the circumference of each, equal to one another

The students noticed that the equilateral triangle depended on the relationship between the two circles that they created. They discussed their constructions and the relationships they noticed (lines: 18 - 29). Both students noticed that the construction maintains equilateral triangle with dragging (lines 22 and 25). They tried to explain how the intersection points of the circles are dependent on the centers of the circles (lines 27 - 29). In line 36, kim_bchs asks whether the triangle is always equilateral. In response, kar_bchs states that the sides of the triangle are equal and mentions that the two circles are "even" or congruent. In line 50, it seems that kim_bchs builds on kar_bchs's observation and notes that the radii of both circles are equal and that implies that the triangle is equilateral and, in line 51, that the circumferences of the two circles are equal. The students successfully build on each other's ideas and justify why their constructions yield equilateral triangles and justify other equivalences that they notice. They also note that the congruence of their circles depends on the segment that they share (line 37: "the two circles were formed using one segment, so those circles were even with each other") and that two sides of the given triangle are dependent on segment AB (line 50: "the radius of a circle is the same distance. Segment AB is Sure. The radii of both circles and Segment AC and BC are also radii of both cycles. Hence, the triangle should be equilateral."). This provides further evidence that these students are justifying mathematical relations, which moves them in the direction of deductive justification. This also indicates that the students transformed artifacts of the environment into instruments such as chat, dragging, and tools involved in constructing equilateral triangles.

CONCLUSIONS AND IMPLICATIONS

While integrating technology in mathematics, examining a teacher's pedagogical interventions provides insight into that part concerned with his instrumental orchestration and with fostering deductive reasoning. To promote learners' movement between exploration and deductive justification, our study indicates that the teacher's pedagogical interventions addressed different aspects of his geometry lessons—collaboration, mathematical content and practices, task design and instructions, and tool use—and were coded to be acting in the three different levels of instrumental orchestration. His orchestration followed a trajectory of pedagogical interventions that began with a focus on supporting teams of his students with different didactical configurations and exploitation modes to have effective collaborative interactions. Once he was satisfied those students within teams were listening to each other and building on each other's ideas, he shifted to focus his instructional interventions on mathematical reasoning and justifications. Our analysis of his weekly reflections and of his students' work show that, in parallel with his trajectory, his students progressed toward more pointed justifications of geometric relations they noticed, including relations of dependencies. Such relations are mathematically significant (Stahl, 2013; Talmon & Yerushalmy, 2004).

Finally, further research is needed to determine in general whether students' instrumentation and movement between exploration and deductive justification in a CSCL environment can be promoted effectively by a trajectory of pedagogical interventions that first focuses on their discursive interactions in collaboration and then attends to mathematical reasoning and justifications. Research would also need to account for effects stemming from the task design and the VMTwG environment. Such research would inform mathematics teacher educators about the instrumental orchestrations and pedagogical interventions teachers could employ to support students' collaborative instrumentation and learning of deductive justifications in dynamic geometry.

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COMPLEX FUNCTIONS WITH GEOGEBRA

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Complex functions, generally feature some interesting peculiarities, seen as extensions real functions, complementing the study of real analysis. However, the visualization of some complex functions properties requires the simultaneous visualization of two-dimensional spaces. The multiple Windows of GeoGebra, combined with its ability of algebraic computation with complex numbers, allow the study of the functions defined from $\mathbb{C}$ to $\mathbb{C}$ through traditional techniques and by the use of Domain Colouring. Here, we will show how we can use GeoGebra for the study of complex functions, using several representations and creating tools which complement the tools already provided by the software. Our proposals designed for students of the first year of engineering and science courses can and should be used as an educational tool in collaborative learning environments. The main advantage in its use in individual terms is the promotion of the deductive reasoning (conjecture / proof). In performed the literature review few references were found involving this educational topic and by the use of a single software.

Keywords: Complex Functions, Domain Colouring, Visualization, GeoGebra

INTRODUCTION

In *Visual complex analysis* Tristan Nendhan tell us how in the last one hundred years the bourbakista view of mathematics led to the "divorce from one's sensory experience of the world, despite the fact that the very phenomena one is studying were often discovered by appealing to geometric (and perhaps physical) intuition." (Nenglam, 1997, p.vii).

Nenglam (1997) is one of the mathematicians who tries to reverse this situation, making even appeal to the use of computers to visualize mathematical objects. For Nenglam mathematicians must think computers "as a physicist would his laboratory – it may be used to check existing ideas about the construction of the world, or as a tool for discovering new phenomena which then demand new ideas for their explanation (Nenglam, 1997, p.viii)

In an attitude of perfect agreement with Nendham ideas is Vieira Alves, who explains his attempt to illustrate the Fundamental Theorem of Algebra (Vieira Alves, 2013). He uses the features of GeoGebra 2D and the Maple's CAS and 3D capabilities. Of course, the use of single software for the features mentioned is an added value. The great advantage of using 3D GeoGebra is just this.

Although the need to carry out further literature review, for this type of mathematical content and having in mind educational purposes and their application with groups of students, we found references to MatLab, used in the context of simulation problems (Meng, Jiang, Shi, & Liang, 2014). Another study was conducted by Yong-Ming and Dan mentions the use of computational tools to calculate integrals of complex functions and they still refer mathematical experiences with students from collaborative nature. It should be pointed out that these studies are focus in mathematical calculations and about visualization issues and don't use GeoGebra. However, the GeoGebra was used in a research with a control group as a tool in collaborative and cooperative learning environments, in spite of being embedded in a real analysis context (Takači, Stankov, & Milanovic, 2015).

Starting from several examples of complex functions, in the first section of this article, we illustrate various forms to represent such functions, contributing to the understanding of their geometric properties and how they can be viewed as extensions of real functions. One of the strategies used to analyse some properties of complex functions is the exploration/visualization of the behaviour of the image set by a complex function of affixes of complex points on subset of points in the domain. Within this purpose, it were developed for GeoGebra tools creating some particular type of grids in a *Graphics* window of GeoGebra, representing the function domain, at the same time, the image of the grid by the function is visualized in a *Graphics 2* window. In fact, the graphical windows of GeoGebra correspond to the Cartesian representations of $\mathbb{R}^2$, isomorphic to $\mathbb{C}$, being models of excellence for the representation of the domain and codomain of the function that we want to analyze. Being the GeoGebra software that allows the user interaction with geometric and algebraic properties simultaneously, the possibility of moving points on particular subsets of the domain, allows us to stimulate the students view by their assessment to instantaneous images of the image set of these points, allowing conjectures about properties of complex functions of complex variable.

In the second section of this article, we will use the 3D capabilities of GeoGebra to analyze the real part and the imaginary part of a complex function, as a strategy to deepen the analysis of complex functions of complex variable. GeoGebra has a 3D graphic window which leads to the possibility to obtain an immediate graphical representation of the components of the complex function (Breda, Trocado, & Santos, 2013). There is an increased interest in the use of GeoGebra to the teaching and learning of complex analysis considering representations of the domain and the codomain of complex functions, the graphic representation of their components and the interaction between distinct representations.

Another way to represent functions from $\mathbb{C}$ to $\mathbb{C}$, or from $\mathbb{R}^2$ to $\mathbb{R}^2$, is the use of the so-called colouring domain technique. This will be focussed in the third section of this paper.

The colouring domain is a procedure popularized by Frank Farris (1997), previously used by Larry Crone and Hans Lundmark, is based on the use of a spectrum of colours that act as elements of replacement of the not accessible dimension and it was used to represent complex functions of complex variable. Thus, applying this technique using GeoGebra (Breda et al., 2013) we will obtain graphic representations of complex functions, weaving some considerations about the information obtained by the analysis of these graphics.

Finally, it will be presented some considerations about the capabilities of GeoGebra for the analysis of complex functions. We will also presented some clues and concerns for the development of future work, in particular, on the implementation of this technological device to the study of other properties of complex functions, as well as its potential in the teaching and learning processes of elementary complex analysis.

GRAPHIC WINDOWS OF GEOGEBRA AND GEOMETRIC REPRESENTATION OF COMPLEX FUNCTIONS

One of the strategies to analyse properties of a complex function, f, is given by considering a subset of the domain of the function, $G \subset Df \subseteq \mathbb{C}$, and analysing the properties of the image set, $f(G) \subseteq \mathbb{C}$. In general, the subsets are chosen to be Cartesian grids, however with GeoGebra is possible to use a wider set of domain subsets. In a first approach several tools were built allowing, through some parameters, to obtain some particular type of grids (table 1).

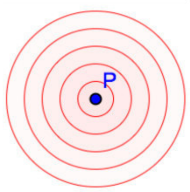
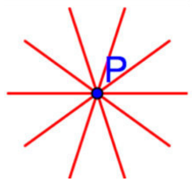
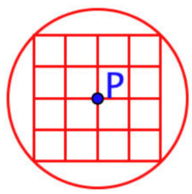
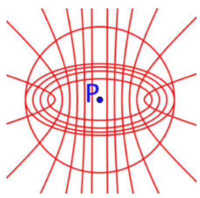
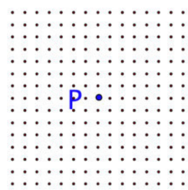
Tool name:	**CircleConcentricGrid**	**RayConcentricGrid**	**QuadrangularConcentricGrid**	**EccentricityConcentricGrid**	**PointConcentricGrid**
Command name:	**CCgrid**	**RCgrid**	**QCgrid**	**ECgrid**	**PCgrid**
Parameters:	**n**=Slider[1, 10, 1, 1,100, false, true, false, false] **P**=1+i **d**=Slider[0, 4, 0.1, 1,100, false, true, false, false]	**n**=Slider[1, 10, 1, 1,100, false, true, false, false] **P**=1+i	**n**=Slider[1, 10, 1, 1,100, false, true, false, false] **P**=1+i **d**=Slider[0, 4, 0.1, 1,100, false, true, false, false]	**n**=Slider[1, 10, 1, 1,100, false, true, false, false] **P**=1+i **d**=Slider[0, 4, 0.1, 1,100, false, true, false, false]	**n**=Slider[1, 10, 1, 1,100, false, true, false, false] **P**=1+i **d**=Slider[0, 4, 0.1, 1,100, false, true, false, false]
Code:	CC=Sequence[Circle[(real(P),imaginary(P)) , k d / n], k, 1, n, 1]	RC=Sequence[Rotate[Segment[(real(P) , imaginary(P)), (real(P)) + d, imaginary(P))], i 2 π / n, (real(P), imaginary(P))], i, 0, n, 1]	QC= {Circle[P, d], Sequence[Translate[Segment[P - d sqrt(2) / 2 (1, 1), P - d sqrt(2) / 2 (-1, 1)], (0, i d sqrt(2) / n)], i, 0, n, 1], Sequence[Translate[Segment[P - d sqrt(2) / 2 (1, 1), P + d sqrt(2) / 2 (-1, 1)], (d i sqrt(2) / n, 0)], i, 0, n, 1]}	EC={Circle[P,d],Sequence[(x - real(P))² / (i d / n)² + (y - imaginary(P))² / (1 - 1 / (i d / n)²) = 1, i, 0, n, 1]}	Sequence[Sequence[P + sqrt(2) d k / 2 + sqrt(2) í d j / 2, k, -1, 1, sqrt(2) d / n], j, -1, 1, sqrt(2) d / n]
Icon:					
Purpose:	Given P (in Argand plane) d, and n. The tool draw a family of circles with centre in P and radius multiple of d/n.	Given P (in Argand plane) d, and n. The tool draw a family of Rays cantered at P, with length d and making 2 π / n as a minimum angle.	Given P (in Argand plane) d, and n. The tool draw an orthogonal gird with 2n segments centred at P and in-script in a circle with centre in P and radius d.	Given P (in Argand plane) d, and n. The tool draw a family of conics centred at P with eccentricity d/n and making 2 π / n as a minimum angle.	Given P (in Argand plane) d, and n. The tool draw an squared orthogonal gird with points centre in P and in-script in a circle centre in P and radius d.

Table 1. Tools that create subsets of points in the Argand plane

Let us assume that we wanted to study the function f: ℂ →ℂ: f (z) = z^2. The classical geometric analysis uses the images by *f* of a subset of points in the complex plane. Using the tools presented in table 1, one we obtain the images illustrated in Figure 1. Handling the mobile point on the girds the user obtain relevant information about the behaviour of the complex function f (z)=z^2 [2]. Looking at the first image of Figure 1 we can see a quadrangular concentric grid centred at $0+0i$ and a circle. The handling process leads, to the observation that circles of radius r, $\rho = r$, are sent to circles of radius r^2, $\rho = r^2$, and straight lines strictly parallel to the real and imaginary axes are sent to parables. Besides, the real axis is sent to the positive real axis and the imaginary axis is sent to the real negative axis.

The second image, in Figure 1, allows you to strengthen the conjecture about the circles centred at the origin and displays that the family of six segments that make minimum angles of a sixth of 180 are sent in three segments which form an angle of at least one-third of 180, it seems so to say that the ray $\theta=k$ are sent by f in the ray of equation $\theta=2k$. In the third image we can conjecture that ellipses are sent to ellipses and hyperbolas in parables.

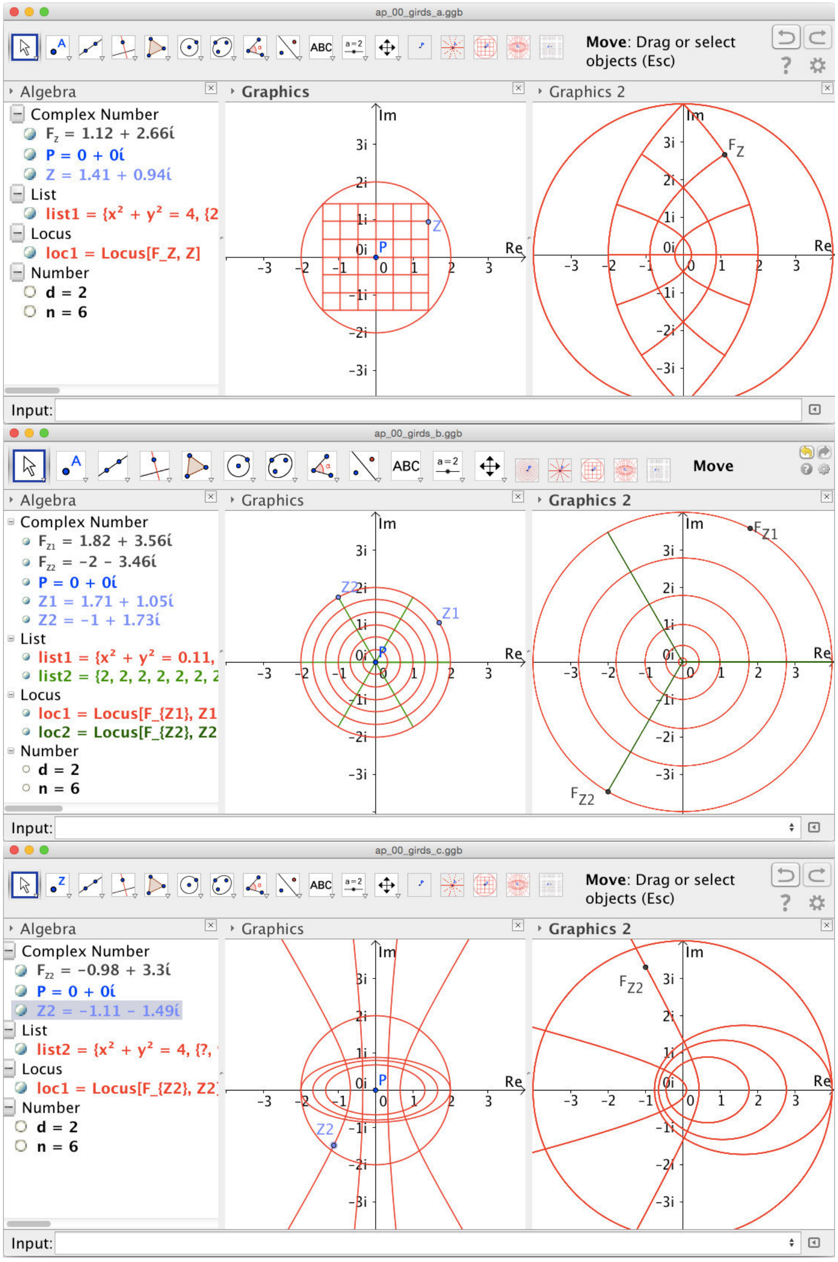


Figure 1. Tools that create subsets of points in the Argand plane and their images by *f*: ℂ→ ℂ, $f(z)=z^2$.

The visualization and handling of the applications built with the GeoGebra display properties of this function, providing moments for conjectures and acting as facilitators to the mathematical proof. Note that here we have chosen a function allowing the realization of conjectures with a very simple evidence. Using other functions we may provide circumstances where the first strategy points toward a conjecture that is refuted by the analysis of their component functions (Breda et al., 2013: 80).

3D GRAPHIC WINDOWS OF GEOGEBRA AND REPRESENTATION OF THE COMPONENTS FUNCTIONS OF A COMPLEX FUNCTION

Considering the function used in the previous section we can easily obtain their components graphs. So considering $z=x+yi$, with $x\in\mathbb{R}$ and $y\in\mathbb{R}$, the components of f are the functions: $If(z)=2xy$ and $Rf(z)=x^2-y^2$. In GeoGebra we can represent the graph of functions of two variables, we can write the real component as $f1(x,y)=real((x+yi)^2)$ and the imaginary component as $f2(x,y)=imaginary((x+yi)^2)$. In alternative we can also write $f1(x,y)=x^2-y^2$ and $f2(x,y)=2x$, in any case their graphic representations correspond to hyperbolic paraboloids (see fig. 2).

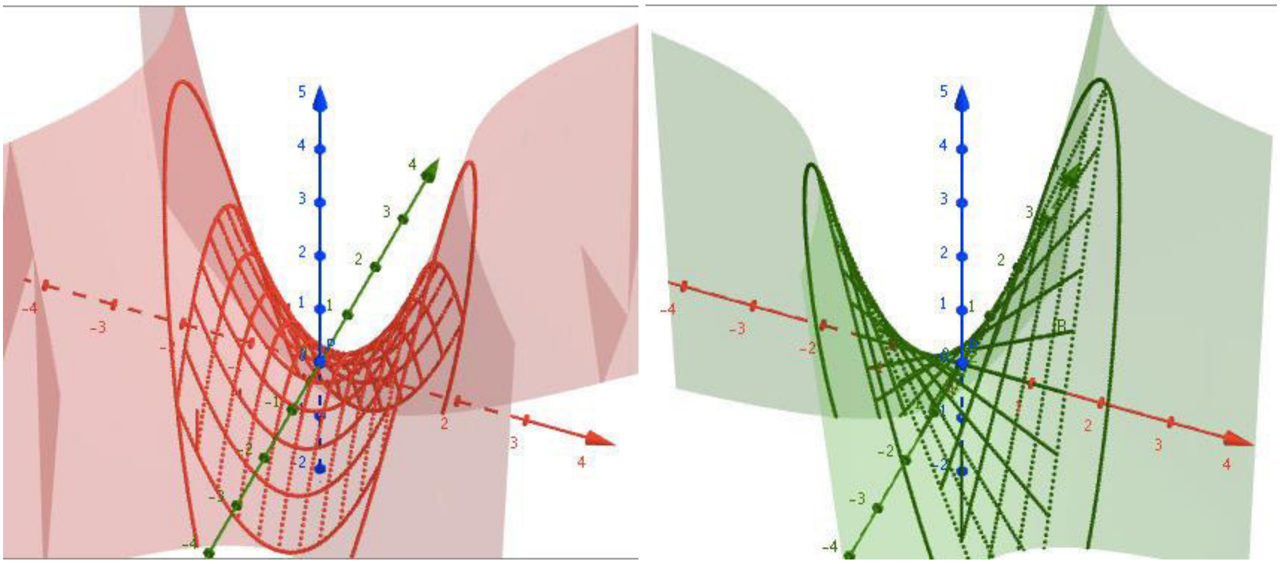


Figure 2. Graphical representation of components functions to f: $\mathbb{C} \rightarrow \mathbb{C}$, $f(z)=z^2$, on the left the function component corresponding to real part and, on the right, the corresponding to imaginary part.

To analyse the image of a Cartesian grid we note that on the hyperbolic paraboloid, the image of the real part of the straight lines correspond to parables, i.e., the intersection of the surface defined by $z=x^2-y^2$ with planes with equation $x=k$ or $y=k$ are parables. The intersection of the planes of equation $x=k$ or $y=k$ with the hyperbolic paraboloid is not easily visualized, it seems to be straight lines, but observing carefully the algebraic expression of the imaginary part of the function, we may easily conclude that these are straight lines satisfying the equations $x=k \wedge z=2ky$ or $y=k \wedge z=2kx$.

GEOGEBRA AND COLOURING DOMAINS IN REPRESENTATION OF THE GRAPH OF A COMPLEX FUNCTION

The technique of the colouring domain was implemented in Geiger by the authors of this paper in 2013, based on a preliminary work of Rafael Losada. In this technique each point has an associated and the colouring is done according to certain rules (Breda et al., 2013, p. 78). In the production of a colouring domain in GeoGebra a geometric scanning supported by values calculated in the spreadsheet is used, the colour of a point is given according to a given criterion, and the representation obtained can be contained in the Graphic windows (Breda et al., 2013) or Graphics 3D Windows (Breda & Santos, 2015). In Figure 3 is shown the application that generates the colouring domain of the identity function, the functions g and h which are crucial in the criterion for colouring domain, are visible in the algebraic window.

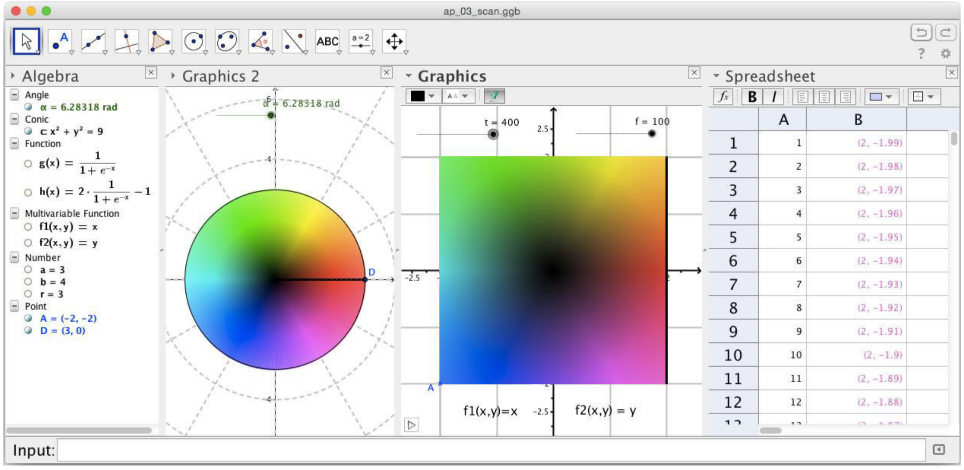
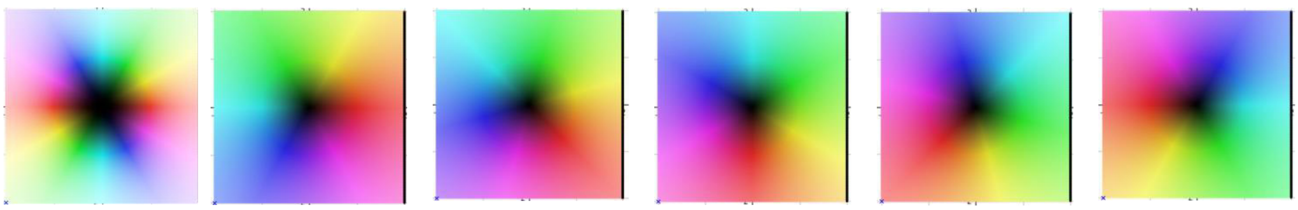


Figure 3. Application creating the scan of a colourful two-dimensional domain in GeoGebra.

The interest in the colouring domains is the representation of four dimension objects in a bi-dimensional colourful world, where the colour has a very important meaning. The colouring of the identity function gives us the starting point to gather information about the action of the function in study. Dark areas represent an approximation to zero and areas with a lot of brightness indicate the proximity of the pole of the Riemann sphere, i.e. values approaching infinity. In Figure 4 we can see the coloured areas associated with the complex functions $j(z)=ze^{i\theta}$. Looking at the images we may induce that these functions represent rotations of the complex plane.



$f(z) = z^2$ $I(z) = z$ $j(z) = z\, e^{i\,\pi/4}$ $j(z) = z\, e^{i\,\pi/2}$ $j(z) = z\, e^{i\,3\pi/4}$ $j(z) = z\, e^{i\,\pi}$

Figure 4. Colouring Domain $f(z) = z^2$, of identity and $j_\theta(z) = ze^{i\theta}$, $\theta \in \{\pi/4, \pi/2, 3\pi/4, \pi\}$.

Let us, now focus our attention on the first two images shown in Figure 4. On the left, comparing the colouring domain of the function $f(z)=z^2$ with the colouring of the identity function, you will see the modification of the above mentioned angle duplication, the colour spectrum is repeated. Besides, an increase of the module by the function f, is also visible, the luminosity increases as we move away from the central point, at the same time, the dark zone close to zero also increases.

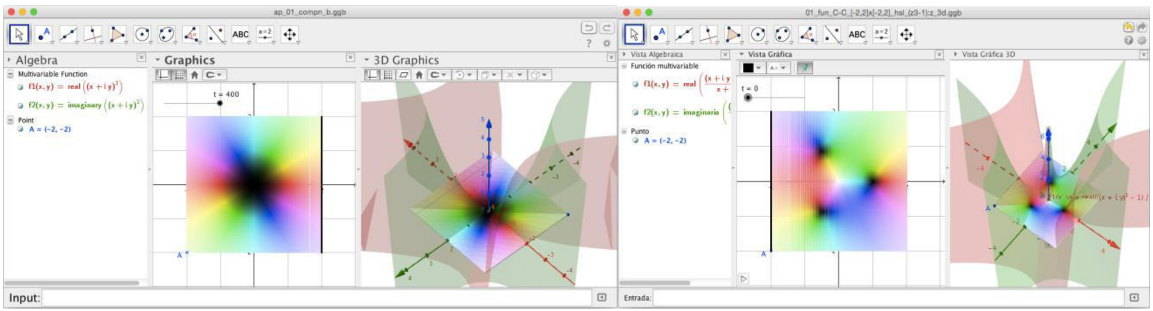


Figure 6. Several representation of $f(z)=z^2$ and $l(z)=(z^3-1)/z$, $z \in \mathbb{C}$.

In fact, the search of zeros of a complex function is one of the situations where the use of the colouring domain is particularly suitable.

In Figure 6, looking at the colouring of the complex functions $f(z)=z^2$ and $l(z)=(z^3-1)/z$, $z\in\mathbb{C}$ provided by GeoGebra, the perception of the zeros of these functions is very clear. In the case of f it corresponds to $z=0+0i$, in the case of the function l we may identify its zeros which correspond to the three roots of the unity in $\mathbb{C}$.

CONCLUSIONS

In the last few pages, we summarize the work done that so far with the purpose of studying complex functions using colouring domains developed in GeoGebra. We can say that this software allow the junction of several available mathematical resources with the ability to produce various graphical representations. The GeoGebra graphic capacities still need to be improved. With the GeoGebra current versions are still slow, especially in representations of 3D colouring domains. On the other hand the exploitation and dissemination of these applications are essential so that, this powerful tool can get to where it is most needed – to "classrooms". The knowledge about the role of the visualization in teaching and learning complex elementary analysis is still very incipient. The use of GeoGebra in this topic is still very fresh, however we have exposed that it is possible to make use of several representations to support the study, from multiple views, of some properties of complex functions.

NOTES

1. The GeoGebra version used in the applications referenced in this paper is 5.0.64.0-3D, February 13, 2015. However, these applications were created with version 4.9.183, December 21, 2013.
2. Note that $z = \rho cis(\theta)$ and so $z^2 = \rho^2 cis(2\theta)$, where $\rho=|z|$.

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ABOUT THE AWKWARD PROCESS OF INTEGRATING TECHNOLOGY INTO MATH CLASS

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This paper briefly reports and discusses the findings of some studies (carried out over the past years within our research group) on the use of technology in mathematics teaching and learning, thus taking the shape of an overall a posteriori reflection with the aim of promoting further development. The first study concerned teachers' perceptions of technology in math class. The second study aimed at investigating how teachers orchestrate activities in a technology-rich class. The aim of the third study was to analyse the relationship between work with manipulatives and technologically instrumented work within a laboratory approach. The important role of the teacher is highlighted, seeking to individuate the crucial factors influencing the awkward process of integrating technology into math class.

Keywords: Integrating technology, teacher's perceptions, teacher's role, teachers' training

INTRODUCTION

According to findings in the research field on technology in mathematics education, it can be assumed that technological instruments can play a crucial role in the teaching and learning process (Arzarello et al., 2006). Calculators, Interactive Whiteboards (IWB), and other technological tools, such as Dynamic Geometry Software (DGS), Computer Algebra Systems (CAS), applets and spreadsheets, can be considered as vital components of high-quality mathematics education in the 21st century. Scholars agree in believing that, thanks to the possibilities provided by the use of technology, a shift from processing algorithms and calculations to constructing models, reflecting, or evaluating results is possible during teaching and learning math process. For instance, it is possible to shift from using static representations to experimenting with dynamic and interactive modes of visualization and exploration (e.g. Holyles & Lagrange, 2010).

However, it is extremely important and urgent to understand and let teachers become aware of how, when and why technologies can influence, support and mould the way that students learn mathematics. This urgency is also justified, at least in Italy (as in many other countries), by the national policies aimed at fostering the integration of technologies (revising school curricula and lately, increasing the number of IWBs in all kinds of Italian schools). Although the use of technology in Italian schools is on the cutting-edge of the national policies, we contend that it must still be considered at the edge of a meaningful effectiveness in the teaching and learning of mathematics.

It could be said that an "adequate" integration of technology within classroom activities should bridge the gap between the planned curriculum and the ones implemented, enhancing mathematics teaching and learning (NCTM, 2000). This paper will focus on the meaning of the term "adequate" when referred to the integration of technology in math class.

RATIONALE AND RESEARCH QUESTION

In the mid-1980s, the ICMI Executive Committee launched a set of activities, named ICMI Studies, with the aim of contributing to a better understanding and resolution of the challenges that face multidisciplinary and culturally diverse research and development in mathematics education. For the very first ICMI Study the chosen topic was: "The Influence of Computers and Informatics on Mathematics and its Teaching". As Jean Pierre Kahane (President of the ICMI in 1985) recently explained, at that time it seemed evident that informatics was likely to have an important influence on mathematics education but many professional mathematicians were not yet convinced that it would have a substantial influence on their mathematical practices. Twenty years later, the scenery has radically changed:

> no one would deny the influence of informatics and digital technologies on the professional practices and life of mathematicians and on the mathematical sciences themselves, but regarding the influence on mathematics education, the situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study have been fulfilled (Artigue, 2010, p.464)

so that the ICMI decided to return to the theme, launching an ICMI Study, the 17th, to be called "Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain" (Hoyles & Lagrange, 2010).

As some scholars have underlined, various teaching and instrumental factors that foster digital technology integration can be identified. Assude and colleagues (2010), for instance, pointed out that among these factors, one concerns the didactical transposition, while another the problem of management in the classroom. In addition, the relationship between technical and conceptual mathematics must be taken into account.

In this paper we focus on the decision to use technology for teaching and learning maths that is an important responsibility for teachers. Their duty, indeed, is to find learning environments, activities, ways and tools that allow students to benefit from fields of experience (that are important and useful), and also promote socialization, a process in which students are encouraged to learn maths.

Herein, we attempt to answer the following research question: what are the crucial factors influencing the awkward process of integrating technology into math class?

THEORETICAL FRAMEWORK AND RELATED LITERATURE

In agreement with general results in this field, we believe that a strategic use of technology, by offering opportunities for change in pedagogical practice, could contribute to mathematical reflection, problem identification and decision making. However, technology cannot replace conceptual understanding, computational fluency or problem-solving skills, nor can it be taken for granted that technology alone can change essential aspects of teaching and learning (Mously et al., 2003). It is a well-known fact that the use of any kind of tools in a classroom, although they can help some students to find an explanation, is not enough to guarantee a permanent understanding, still less to promote conscious and thoughtful learning. If technology was used only as an auxiliary tool to generate and show images, expand human memory or increase the turnaround in feedback, it would be unable to foster the progressive construction of a personal heritage of meaningful mathematical knowledge, skills and attitudes.

For these reasons, it is extremely important that teachers understand and become aware of the affordances, constraints, and general pedagogical nature of technology as a new resource in relation to the specific mathematical topics addressed in school (Ruthven & Hennessy, 2002).

As part of the instrumental approach developed from the studies of Vérillon and Rabardel (1995), the “instrumental genesis” expression has been coined to reflect the long and complex process (at the same time social and individual, connected to the limits and potential of the artefact and to the student’s qualities) during which a student turns an artefact into a tool, developing techniques and mental patterns that allow him to use the artefact for a well-defined purpose.

As Pierce and Ball (2009) underlined, the knowledge of, the experience with and the views on, mathematics education and the role of technology within class activities guide the process of a teacher developing instrumental orchestrations. In particular, these often follow implicit guidelines, such as the teachers’ own knowledge and skills concerning the integration of technology and their concerns about time constraints and behavioural control. These drive the teacher’s choices and result in types of invariant teacher behaviour, which are instrumented by the available tools (Gueudet & Trouche, 2009). By promoting the creation of meanings through an orchestration process, the teacher can strongly stimulate the student’s educational process (Lagrange et al., 2003).

In this paper, we use the term “orchestration” to refer to Trouche’s idea of instrumental orchestration that, within the framework of the instrumental approach, points out the necessity for a given teacher to rely on external steering of the students’ instrumental genesis. An instrumental orchestration is defined (Trouche, 2004) as the teacher’s intentional and systematic organization and use of the various artefacts available in a learning environment in a given mathematical task situation, in order to guide the students’ instrumental genesis. It is based on the combined action of three elements: “didactic configuration”: arranging artefacts according to the teaching purposes fixed in advance; “exploitation mode”: deciding on the roles that artefacts, teachers and students should play and choosing the technologies and procedures to develop as regards the didactic configuration; “didactic performance”: assessing all the choices that a teacher should make during their implementation and envisaging possible inputs from students and any consequent choices to adopt.

METHODOLOGY

In order to answer our research question, we report and discuss results from three different studies all of which attempt to throw some light on the use of technology in math class activities. The first study (Study A) concerned teachers’ perceptions of technology in math class. The second study (Study B) aimed at investigating how teachers orchestrate activities in a technology-rich class. The aim of the third study (Study C) was to analyse the relationship between work with manipulatives and technologically instrumented work within a laboratory approach.

In particular, it is important to underline that as far as the Study C concerns, we apparently moved toward a different level of investigation even though our interest from the point of view of this reflection is still related to the role of teachers and their professional development. As a matter of fact, we were also interested in figure out the crucial factors influencing teachers’ choices (how, when and why) and how teachers can be helped to design, realise and evaluate the effectiveness of technology-rich activities.

FINDINGS FROM PREVIOUS RESEARCHES

As declared above, this paper reports and discusses results from three different studies, all of which attempt to throw some light on the use of technology in math class activities.

Study A, on teachers' perceptions of technology in math class

The aim of Study A was to understand how deeply math teachers do perceive the opportunities technologies can bring about for change in pedagogical practice, in order to effectively use them for the students' construction of mathematical meanings (Faggiano, 2009). Moreover, it aimed at verifying whether or not teachers realise that, in order to successfully deal with perturbation introduced by technologies, they have to keep continuously up-to-date and to acquire not only a specific knowledge about powerful tools, but also a new didactical and professional knowledge emerging from the deep changes in teaching, learning and epistemological phenomena. An anonymous questionnaire was submitted to 16 in-service high school teachers and 113 pre-service teachers. Key questions in the questionnaire included the following:

1. Do you think ICT could be useful for your teaching activities? Why?
2. Do you think that the use of ICT can somehow change the learning environment? And the way to teach? And the dynamics among actors in the teaching/learning situations?
3. Which difficulties do you think can be encountered when designing and developing a math lessons using somehow ICT?
4. As a teacher, do you think you need to have some didactical competences in order to properly use ICT? Eventually, which ones? And anyway, why?

Findings from the questionnaire revealed that in-service teachers perceived that technology can bring support to their teaching, but only inasmuch as it is a motivating tool enabling students to gain understanding per se. Answers given by the pre-service teachers were, instead, a little more didactically oriented: some of them recognise that, if nothing else, a knowledge of the instrument's functions is probably not enough for a teacher to use it in an effective way in terms of promoting construction of meanings by the students. An awareness of the opportunity to create a new "milieu" and change the "economy" of the solving process was totally lacking in the perception of the use of technology in mathematics teaching/learning activities, both of in-service and of pre-service teachers. As to the difficulties they think can be encountered when designing and developing a math lesson using technology somehow, they mostly ascribed possible difficulties to the lack of an adequate number of PCs and the technical problems that might occur, but also to the natural students' tendency to get distracted and to take a mental break, especially when facing a PC. As a consequence they did not feel the need to gain further technology skills for their teaching and did not usually consider that their lack of skills might present them with any difficulties. And, although 75% of the student teachers recognized the need to possess some didactical competences in order to use new technology, what they asked about was, in most cases, just software functionalities (not potential, nor constraints): only some of the pre-service teachers also asked to know how to effectively integrate their use in the teaching practice.

The process of a teacher developing instrumental orchestrations, therefore, has to be guided and, as highlighted by Pierce and Ball, a knowledge of, experience with and clear views on mathematics education and the role of technology within the class activities, are crucial needs.

Study B, on teachers training "in action": the case of Enza

The aim of Study B was to investigate how teachers orchestrate activities in a technology-rich class

(Faggiano et al., submitted). Some teachers with different experience and background have been videotaped and interviewed. Herein we focus on the case of Enza.

Enza is a secondary-school math teacher with a good knowledge of pedagogical content and with good skills in the use of technology. She decided to introduce, in her 9th grade class, the concept and properties of the circumference (in the plane), involving students in activities based on the use of the dynamic software GeoGebra. The topic, that can well be addressed with the chosen technology, offered the opportunity to verify the effectiveness of the experience, when bringing the matter up again after the school summer holidays.

In the overall experience, three different types of orchestrations could mainly be observed: starting from students' solutions to a task and guided by what happens when dealing with the task on the Interactive Whiteboard (IWB), Enza explained the mathematical content involved. Her orchestration often included a whole class discussion about what happens on the IWB, aiming to enhance the collective instrumental genesis; sometimes, especially in the last part of the experience, she brought to the fore an interesting answer among student's solutions and discussed it with the whole class. The "didactic configuration" allowed an obvious choice of the "exploitation mode" in which technology played a very important role. The roles played by each component of this "instrumental orchestration" clearly changed as the teaching activity developed, thus allowing the "observer" teacher to assess all the choices made during the implementation phase and to compare the students' input and their consequent choices to those established beforehand, in the "didactic performance" perspective.

It is noteworthy that although she observed that activities succeeded in bringing the lesson alive, when reflecting on the whole experience she concluded that the transfer of what students understood through the use of GeoGebra ("yes, I can see. It is clear") was problematic for conventional paper and pencil mathematics. Indeed, at the end of the experience students demonstrated that they had understood the concept and properties of the circumference, but, after the school summer holidays, they proved they had not fully dealt with the subject in the least. For this reason, Enza now says that she will repeat the experience in a new class, paying particular attention to this aspect:

> the introduction of the circumference and its properties can be done with the use of GeoGebra, discussing with the students their solution to some tasks, but it is extremely important to take care of the students' meaningful understanding. Technology is a means to achieve the discovery of a property, and to implement an interactive process in which students can have a voice, but knowledge thus obtained also needs to be transferred to the paper and pencil environment.

In conclusion, the described teacher training "in action" experience helped Enza to verify potentialities and constraints of her orchestrations and to focus on the opportunity to use technology within an integrated learning environment, in which a central aspect is the continuous alternation between technology and paper and pencil.

Study C, on technology-rich activities within a laboratory approach

The aim of Study C was to analyse the relationship between the work with manipulatives and the technologically instrumented work at primary school level (Faggiano, 2012). The research aimed at verifying if by manipulating both physical and technological instruments, within a laboratory approach, pupils can really achieve the building of new geometric concepts and a firm understanding of geometrical relationships. Herein we report a teaching experiment that has been conducted with 5th grade pupils: it integrates the technological opportunities offered by the Interactive Collection 123…Cabri (http://www.cabri.com/special-pages/bett2010/) within authentic learning situations, in which important steps have been based on the manipulation of physical objects such as paper, scissors, strings and straws.

Students worked mainly in pairs or small groups. Collective discussions aimed at exchanging ideas and find common solutions, as well as at comparing different strategies. Moments of individual reflection were finally dedicated to reflect on what was happening and culminated in the production of reports. The data collected included classroom observations notes, audio-recordings of lessons (transcribed), and other field notes. During each lesson, photographs were taken to provide information which could not be recorded by audio-recorder or field notes (for example, recording work presented on the blackboard).

During 3-hour afternoon meetings for a total of 30 hours within three months (from March to May), students were engaged in different kind of activities.

The first three meetings were devoted to creating a "classroom climate", breaking away from the usual "didactical contract" (in Brousseau's sense, 1997). The children were posed a series of challenges suited to their age. For instance, using the extremely attractive Tangram interactive activity of the collection "123...Cabri", diagrams were shown to be reproduced using the classical tangram seven pieces. From page to page the indicators to the solutions diminished from a multi coloured a black diagram without any indication to solving the problem. Students were then involved in reproducing themselves the tangram square, exploring the different pieces available and trying to use them for various creative productions. This crucial intuitive exploration and manipulation of the material contributed to highlighting in particular those properties of figures that relate to the dynamic nature of the position figures take up in the plane or to the different configurations of the borderline between tangram shapes in the composed figure, thus leading to the enrichment of the geometric vocabulary.

Further meetings concerned the characteristics and classification of triangles and quadrilaterals. Pupils were firstly asked to construct triangles and quadrilaterals using pieces of straws (with different length) and a string. They realised that it is not always possible to obtain a triangle while with four pieces of straw it is possible to construct "more than one" quadrilateral. Then, using some paperweight, students focused their attention to the diagonals of the quadrilateral. Discussion leaded to the use of ruler, paper and scissors in order to make lots of different quadrilaterals to be drawn on their exercise book. Geometrical intuition had at this time a crucial role. With the aim to overcame the difficulties learners have with coming to an understanding of the hierarchical relationship between quadrilaterals further activities were carried out. In particular, pupils interacted with some of the activities of the collection "123...Cabri" which have been designed with the aim to focus on the properties of the figure by means of an adequate selection of tools and of the "drag-mode".

Analysis of the results revealed that while working on learning activities in an integrated laboratory approach the students gained some insight into the structure of plane geometry, thus fostering not only the building of new geometric concepts but also a firm understanding of geometrical relationships. Moving from the physical object that can be manipulated to the geometrical drawing (by identifying features) and from the drawing to the geometrical object (by means of dynamic environment activities and of interpretations of the feedbacks thus obtained) can show how the use of manipulatives, graphic activities and gradual refinements are all consequences and sources of learning.

According to the research results, it can be claimed that within an integrated laboratory approach, making hypotheses about the relationships between geometrical objects, manipulating and constructing the objects for themselves, and verifying the truth of their conjectures in various ways, students may develop not only a change in their geometrical work but also, in a spiral and iterative fashion, a feeling for the need of proof of any explanation which will be very important in their further math education. From this point of view, well designed authentic learning activities (using both physical and technological instruments) can offer opportunities for a progressive geometric mathematization.

CONCLUSIONS AND FUTURE DEVELOPMENTS

In conclusion, reconsidering findings arisen from the three studies, we deem that the crucial factors influencing the awkward process of integrating technology in math class are: the teacher's knowledge of pedagogical content related to the use of technology; the decisions the teacher takes in determining when integrating technology in everyday teaching practice and how to structure the learning environment; the choices s/he makes when facing the problems that new environments require a new set of mathematical problems, that both the constraints and potentialities of the artefacts need to be understood and that the instrumentation process and its variability needs to be managed.

In particular, the studies we described above have strengthened our conviction that an awareness of the potential and limitations of technology in the teaching field, as well as a knowledge of the related pedagogical content, are essential conditions to ensure an adequate orchestration of the teaching activities. It is highly important, therefore, for the teacher to be able to review the teaching activities s/he has conducted and reflect on what happened in class. To do this, proper tools emerging both from teaching practice and from research findings are a fundamental requirement. These will allow the teacher to refine her/his pedagogical content knowledge and improve subsequent steps in the process. Clearly, for the teacher this is a long, complex undertaking that is both demanding and time-consuming. For this reason, it will probably take quite a few years to bridge the gap between the planned curriculum and those implemented.

Our plan here is to start from the positive results obtained in previous studies, using them as a springboard for developing a long term teacher training "in action" project with some in-service teachers, in order to promote the integration of technology in their teaching practice, and to assess the impact and results. In parallel, we aim to continue to work with pre-service teachers, giving them the chance to be the subject of a "mise en situation". That is, we aim to allow teachers to be an active part of a learning situation, engaging them to solve unusual problems which require non-standard strategies. In this way, teachers can experience for themselves the difficulties students may encounter, the cognitive processes that they can apply, and the attainments they can achieve. They will also have the opportunity to understand and manage students' instrumental genesis and to become more skilful and self-confident when deciding to exploit the potential of software in mathematics education. In particular, we aim to verify whether the "mise en situation" experiences could allow teachers to become more aware of the important relationship between the specific knowledge to be acquired by the students and the knowledge possessed by the teacher, as well as between the specific knowledge to be acquired by the students and their prior knowledge.

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HOW CAN TECHNOLOGY SUPPORT EFFECTIVELY FORMATIVE ASSESSMENT PRACTICES? A PRELIMINARY STUDY

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Formative assessment is a process that can inform both teacher and students of their understanding of knowledge at stake. Technology allows to get data, to arrange them and to share them. The FaSMEd project aims to study the effective role of technology within a formative assessment process. This paper presents a preliminary case study allowing to better understand how the teacher processes data from students using technology (i.e., tablets, student response system, IWB) and how he uses them to inform his teaching.

Keywords: formative assessment, low achieving, technology.

INTRODUCTION

This paper reports on an ongoing research we are carrying out within a wider European project, titled FaSMEd (Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education). The preliminary study that is presented here is just a picture of a particular moment at the beginning of the project. Nevertheless, starting from what we observed and analysed, some more information about the influence of this study on the following phases of our work will be provided and justified. The research aim of the FaSMEd project is to investigate the role of technologically enhanced formative assessment methods in raising the attainment levels of low-achieving students. As specified in the shared glossary, formative assessment is a method of teaching where

> "[…] evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited." (Black & Wiliam, 2009, p. 7).

A digital environment which enhances connectivity and feedback can assist teachers in making more timely formative interpretations. The project relies on the hypothesis that creating such a digital environment has the potential of amplifying the quality of the evidence elicited about student achievement (Hattie, 2009). Moreover, both teachers and students have access to detected data for real-time interpretation and further use.

In particular, in this preliminary phase of the project we followed the "school life" of a connected tablet classroom, both directly observing some lessons and indirectly reading reports written by the teachers. We have experienced the deep change of a classroom's reality when tablets enter it, being witness of Walling's words:

> "In the emerging world of a tablet classroom the teacher is likely to be a principal learning designer. […] In an ideal educational environment, of course, teachers would have adequate training prior to being thrown into a tablet classroom. Most often this ideal is not realized, and training is sketchy at best. Consequently, effective teachers draw on both art and science to craft teaching and learning for their students, whether collectively or individually. If we compare effective teachers to jazz musicians, they must be exemplary players, more than merely

technically competent. They know when to follow the score (the curriculum) and when to improvise." (Walling, 2014, pp. 26-27).

Tablets support, accompany and sometimes replace students' notebooks and the paper and pencil environment. On the technical side, several competences are needed by the teacher to make the lesson develop in a natural way for students. On the didactic side, the usual activities have to be adapted and new activities can be designed and proposed. Moreover, the way of exploiting them in the classroom can change thanks to the possibilities offered by connected classrooms technologies. Nonetheless, the challenges are great. Different studies have highlighted that connected classrooms technologies have increased the complexity of the teacher's role with respect to orchestrating the lesson (Clark-Wilson, 2010, Roschelle & Pea, 2002). As a "conductor-of-performances", in fact, she has responsibility for choosing and sequencing the material to be performed, interpreting the performance, and guiding it toward its desired forms (Roschelle & Pea, 2002).

In our research, we leave to teachers the responsibility for designing their lessons, being at their disposal for discussion and advice if they wish. Then, we observe and analyse some lessons in order to come back and exchange on them with the involved teachers. This process generates successive cycles of design, observation, analysis and redesign of classroom sequences (Swan, 2014). The resources for the classroom, designed and redesigned through this process, will inform the production of a "toolkit", that is a set of curriculum materials and methods for teachers to support the development of practice.

In line with the project purposes, we carried out this preliminary study in order to understand which possible formative assessment practices involving technology could be efficiently proposed in classroom. More precisely, we analyse a mathematics lesson in a grade 9 tablet classroom, in which the teacher is testing the student response system provided by the classroom network NetSupport School. We will try to answer the following questions: how does the teacher process data from students using technology (i.e., tablets, NetSupport School, student response system, IWB) and how does he use them to inform his teaching?

THE CONTEXT OF THE STUDY

In the preliminary study, as well as in the course of the whole project, we intend to maintain those tools teachers have already tested, combined with other supporting tools if necessary. This choice allows teachers to collect and use feedback from students in a way that is, insofar as possible, independent from the teachers' and the students' unfamiliarity with the tools.

In particular, in the observed grade 9 tablet classroom, each student has been equipped and is responsible for a specific tablet, using them for all the subjects. The choice of a personal use of tablets in the classroom encourages the students to appropriate them and allows the teacher to follow more directly the progress of each student.

The leading idea of this study is to get a first insight into the way a teacher can adapt the technological tools available in the classroom for formative assessment, with a particular attention to low achievers. The choice of the observed teacher consists of integrating the use of a student response system offered by NetSupport School: the tablets connecting network he is already exploiting in the classroom.

THEORETICAL FRAMEWORK AND METHODOLOGY

When technology intervenes in the classroom as a learning tool, we can describe the occurring situation referring to the Theory of Didactical Situations (Brousseau, 1997). The teacher creates a *milieu* the student has to cope with, and she modifies it depending on the student-*milieu* interaction. According to Brousseau, "Within a situation of action, everything that acts on the student or that she acts on is called the '*milieu*'" (Brousseau, 1997, p. 9). In our study, we consider the employed technology as a part of the *milieu* that plays a fundamental role in informing the students. Brousseau further specifies the teacher's role, by stating that "Teaching is the devolution to the student of an adidactical, appropriate situation; learning is the student's adaptation to this situation" (Brousseau, 1997, p. 56). In a complementary way, the institutionalization corresponds to the phase in which the teacher "defines the relationships that can be allowed between the student's 'free' behaviour or production and the cultural or scientific knowledge and the didactical project; she provides a way of 'reading' these activities and gives them a status" (Brousseau, 1997, p. 56).

In presence of technology, the role of the teacher evolves as she manages the essential task of orchestrating its use in the classroom. Indeed, each student working with a particular technology develops her own schemes of use with respect to it, through a process that is called instrumental genesis (Rabardel, 1995). At the same time, each student shapes the technology in the so-called instrumentation. For each student, the technology from a simple artefact becomes an instrument through this double movement from the artefact to the user and from the user to the artefact, but the time for the instrumental genesis can be very different from student to student. In the context of a tablet classroom, where each student is appropriating her own tablet, the orchestration of all the different schemes of use developed and of all the instrumental genesis occurred at different levels is a crucial task for the teacher. With this concern, Trouche (2004) speaks about "instrumental orchestration" to indicate didactic configurations and exploitation modes of these configurations. In the observed tablet classroom, this framework is particularly suitable to describe the arrangement of the technological environment and the teacher's exploitation of it. The teacher networks all the tablets in the classroom, so that each tablet can communicate with the central system. He acts directly on NetSupport School to communicate with all the students. He uses the IWB as a common screen to collected all the data sent by the students. In this particular environment, he exploits a NetSupport School functionality that works as a student response system. So he sends to each student a question, taping and hiding the correct answer; then he gets an elaboration of the set of answers taped by each student on her own tablet, compared with the correct one (so they appear in red or green). The particular orchestration chosen by the teacher provides him with data that can potentially inform his teaching and produce other modes of exploiting the arranged didactic configuration, perhaps decided on the spot, during the lesson. It is interesting also to notice that the teacher in this technological environment wants to maintain a written mark of the work done during the lesson. Instrumental orchestration is then combined with the use of paper and pencil as illustrated in Figure 1.

Figure 1. Instrumental orchestration combined with the use of paper and pencil.

The feedback coming from technology is useful for both the student and the teacher. The student can use it to improve her performance in front of questions or to change her strategy in the resolution of a problem. Nonetheless, the feedback has to be problematic, sometimes negative or doubting. This surely entails a moment of difficulty for the student, but the way in which she manages it can actually inform herself and the teacher about her understanding of the involved piece of knowledge. The teacher, in turn, can use this feedback to have a class overview, to identify what are the problematic notions and which students have more difficulties with a particular concept, and then to adapt his didactic strategy. It is in conditions like this that assessment becomes "formative" and can efficiently contribute to the students' learning. Observing a lesson, thus, we are interested in those moments in which the teacher collects data and draws on them for deciding his didactic technique. Starting from what he plans before the lesson, all these local variations contribute to shape his actual practice in the classroom.

For the case presented in this paper, we have attended a one-hour lesson as observers in the classroom, without participating to the lesson design and implementation. The collected data at our disposal encompass the audio recording of the whole sequence, some short videos and pictures.

In the next paragraph, part of these data are analysed, according to the theoretical framework that is described above. More precisely, we have selected two specific moments in which teacher and students communicate via tablet. The teacher poses a question through NetSupport School and asks the students to send him their answers. The first question deals with the result of a given problem. The second one is related to an argument proposed by a student, who is in difficulty during this lesson.

DATA ANALYSIS

The teacher proposes two geometrical problems that require to determine the length of a chord, given the radius of the circle and the angle subtended at the centre by the chord. In the first case, the radius is 3 cm long and the angle is 60° wide. Each student works on her own tablet, but she can discuss with her schoolmates. The teacher recalls all the possible supports students can draw upon to solve the problem:

1 Teacher: You have many possibilities: you can draw the figures by hand in real dimensions, you can do some calculation […] you can also draw the figure in real dimensions with GeoGebra if you want. Do whatever you want. I give you, it's 27, at 32 I want that there are some answers […] that everyone

has an answer to propose, right or wrong it doesn't matter, but by 5 minutes I want everyone to have an answer to propose with a written argumentation.

With his words, the teacher devolves the problem to the students. He makes them cope with a *milieu* that encompasses their geometrical knowledge, the given geometrical problem and the tools they dispose of. He leaves them complete freedom in choosing their resolution strategy ("Do whatever you want"), but he specifies that a justification is needed ("with a written argumentation") and not simply the answer. We find really interesting his clarification about the allowed answers: "right or wrong, it doesn't matter". He encourages the students to make their proposal and to defend it. Then, the students work alone or in pairs on the task. There can be interaction with the tablet if they draw and explore the figure with GeoGebra. So, the teacher a priori permits the work on tablet and in paper and pencil as well, as the students prefer. This is another important element of orchestration that is explicitly declared in classroom. However, actually, few students open GeoGebra. They generally prefer to work on their notebooks (Fig. 2).

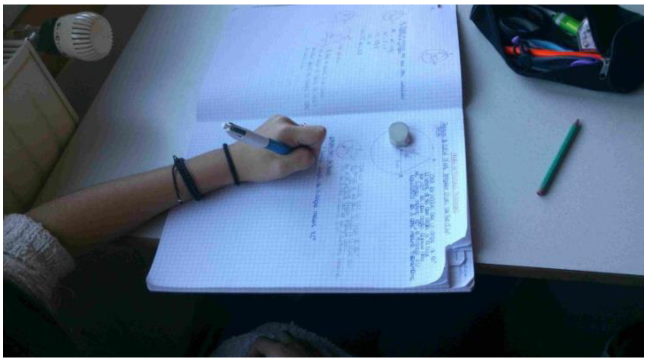
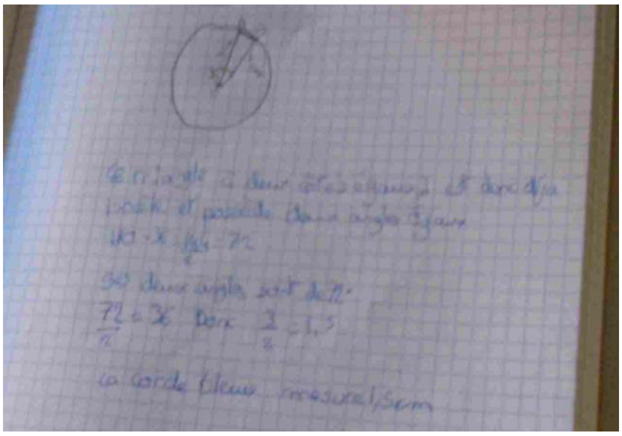


Figure 2. A student's research on her notebook.

After a while, the teacher asks to each student to submit the obtained result. The answers are compared by the system with the correct answer, taped and hidden by the teacher. In this moment, all the students interact with the tablet, and the classroom attention focuses on the common grid that collects all the answers available at the IWB. We clearly see that the students use the tablet mainly as a mean of communication. One of the answers appears in red, even though the student swears to have taped 3, that is the correct answer. The teacher reviews the exercise on the IWB, asking the students' participation for giving reason of all the steps. This first problem has been useful for the teacher to introduce the task to the students and also to test the student response system.

Afterwards, he invites the students to focus on the second problem. The *milieu* changes with the adding of the first problem now solved and of the second given problem. The geometrical situation is similar to the first problem, the radius of the circle is 3 cm long, but this time the angle at the centre is 36° wide. The task is the same: finding the length of the chord. The teacher's suggestion is to exploit what they know about the right-angled triangle (i.e., the relation between leg and hypotenuse, in terms of sine and cosine) and to try and obtain a right-angled triangle in the figure. Following another reasoning a student, we will call Student1) finishes very fast and proposes his resolution to the teacher (see Fig. 3). It is a wrong argument, but the teacher shows it to the classroom in order to discuss about it.



The triangle has two equal sides then it is isosceles and has two equal angles.

180°-36=144/2=72

Its two angles are 72° wide

72/2=36 then 3/2=1.5

The blue cord is 1.5 cm long. [1]

Figure 3. Student1's solution (translation on the right).

2	Teacher:	I highlight Student1's remark, on which we stop two minutes to discuss. Let's simply note it, we are not going to discuss about it, let's simply note it without reasoning and we are going to further answer it. So, note Student1's proposal. He proposed something that would be so practical! […] So, he said this [angle at the centre] is 36, so those two [angles at the base] are 72. 36 is the half of 72, so AB is the half of 3.
3	Students:	Uh!
4	Teacher:	[…] Student2, is your conclusion different from that of Student1 or did you get to the same result? Did you conclude, didn't you?
5	Student2:	Yes.
6	Teacher:	So, I'm going to take Student1, then you will tell me the way you concluded.
7	Student2:	It isn't right.
8	Teacher:	I don't know. If it isn't right you will tell me why it isn't right. The other two angles of the triangle are 72 wide, 36 is the half of 72, then AB is the half of OB.
9	Student3:	But how do you know it is the half? You don't see it there!
10	Teacher:	I agree with you, this is a question we should ask to Student1. Nonetheless, what I ask you with respect to this is what is the mathematical notion that […] Student1 presupposes, when he does so. What mathematical property is he using? What is he using in mathematics? […] I'm going to ask you the question on the tablet, you will answer on the tablet.

The teacher chooses the production of a student and rewrites it on the IWB, in order to share it with the whole classroom. He specifies that his intention is not to judge Student1's proposal ("Let's simply note it without reasoning", line 2) but to discuss about it. His choice is particularly interesting because Student1's proposal and solution are wrong, even if the reasoning begins correctly with the calculation of the angles width. Discussing about a classmate's proposal can be an effective technique to foster formative assessment in the classroom. Every student has the possibility to compare his production with the presented one and the teacher can be informed by the other students' reaction. In our case, the fact that several students react with an exclamation of surprise (line 3) to Student1's wrong solution informs the teacher about the direction he must give to his intervention. Orchestrating the works of different students is his explicit intention (line 4-8). Even though a student, probably a high achiever, notices that there is something wrong with the proposed solution (line 9), the teacher quickly admits to agree with him ("I agree with you, this is a question we should ask to Student1", line 10), but he goes on discussing Student1's argument. In particular, he wants to focus students' attention on the mathematical property that is behind Student1's proposal. His aim indeed is not simply to lead the students to reject the proposal. He wants to be sure that the students get to understand the mathematical reason why it has to be rejected. Then, he poses the question via tablet.

11	Teacher:	What I am interested in is knowing, to be able to compare then with the computer, orally is there anybody that can tell me what is the name of the property 36 is the half of 72, then AB is the half of OB? What is it?
12	Student4:	Thales[2].
13	Teacher:	It recalls Thales. What is behind Thales? Student2?
14	Student2:	1 over 2.
15	Teacher:	Yes, it is the equal ratios. And these questions of equal ratios, when we make this kind of work...
16	Student5:	Cosine.
17	Teacher:	No, it is not the cosine.
18	Student6:	Proportionality.
19	Teacher:	It is the proportionality. Raise the hand up those who have answered, honestly, to have a feedback, those who have found that it was the proportionality. Ok, did you tape it or not?
20	Student7:	Yes.
21	Teacher:	And you don't appear...
22	Student8:	The same for me.
23	Teacher:	You wrote proportionality (in French *proportionnalité*). How many "n" did you tape? It can be linked to this.

While the students are sending their answers and the system is elaborating them, the teacher asks the students to share their ideas with him. This will allow him to make a comparison with the collected data (line 11). Such a comparison between oral answers and taped answers reveals essential for the teacher when, in the end, the right name of the property comes out (line 18). The correct answers in the common grid appear to be less than expected (line 19). The quick oral survey the teacher has carried out helps him in interpreting the data and in understanding what problem could have occurred (line 23). In particular, he realises that part of the wrong answers he sees in the common grid are not due to a mathematical misunderstanding or a conceptual error. They are probably due to an error in taping the good answer.

DISCUSSION

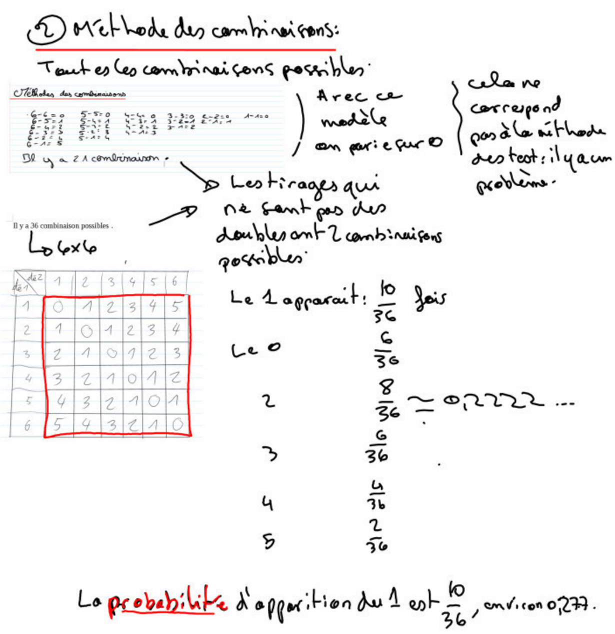
The preliminary study we conducted aims to identify elements for an efficient formative assessment practice with the support of technology. Observing the teachers' usual employ of technology in the classroom can enable us to support them more effectively in developing and adapting their formative assessment practices and to interpret possible changes in their usual techniques.

The case discussed above is an example of a way of using a student response system to support the teacher's formative assessment practice in classroom. For the observed teacher, it is a first test of the possibilities offered by the connected classroom technology. From a global perspective, we can observe that the teacher tries to integrate technology in his usual practices, but not in an exclusive way. We can also detect some moments in which technology could help but it is not exploited. For example, if every student had worked on her tablet, Student1 could have shared his screen with the schoolmates in order to explain his reasoning. Thus, the teacher could have extended the *milieu* including the students' tablet productions. The analysis and the *a posteriori* discussion with the teacher concerning this lesson allow us to highlight the possible modifications of the *milieu*, in order to improve what has been done and to try what has not been done yet.

We visited again the tablet classroom two months later and we could observe a remarkable evolution in the teacher's appropriation of the connected classrooms technologies and in his didactic practices with them, especially with respect to formative assessment. We attended one lesson about the introduction of probability and the teacher reported on the following two lessons we could not attend. In a first a-didactic phase, the teacher proposes to the students to play a game: betting on the difference of two dices with 6 faces. The students work in small groups for exploring the problem. Each component of the group works on her tablet, aiming to get a shared conclusion: on which result they would bet and why. In a second phase, the teacher collects one production for each group by making tablet screen shots. He shows the different proposals at the IWB, discussing and commenting them with the classroom. The tablets are blocked during this central phase of the lesson, since the teacher wants to have the complete attention of the students (see Figure 4).

Figure 4. Tablets are blocked during discussion to catch students' attention.

In the last phase, the teacher gets to the institutionalization of the definition of probability, starting from students' productions available at the IWB (see Figure 5, as an example) : this allows him to validate students' work, in a perspective of formative assessment within the learning process. Connected classrooms technologies play a relevant role in the way the teacher orchestrates the classroom and guides the lesson. NetSupport School permits him to collect in real-time the students' work, to foster discussion and debate in the classroom and to use such data for constructing the lesson notes at the whiteboard.



2) <u>Combinations method</u>:

All the possible combinations

S1's proposal: There are 21 combinations (list).

With this model we bet on 0; this does not correspond with the tests method [before, throwing the dices 100 times students have found that 1 is the most frequent result]; there is a problem.

The throws that are not double have two possible combinations.

S2's proposal: There are 36 combinations (table).

1 appears : 10/36 times …

The <u>probability</u> of appearance of 1 is 10/36, about 0,277.

Figure 5. Lesson notes at the IWB using the students' productions as a base (translation on the right).

Technology allows the teacher to enrich the students' *milieu* by sharing the different proposals and ideas produced by the students in the a-didactic phase.

FINAL REMARKS

The analysis of these examples gives us some precious indications about how teachers can process data from students using technology and consequently how they can use them to inform their teaching. Comparing the probability learning sequence with the geometry one already shows a modification in the teacher's practices and the essential support of technology. The new orchestration skills of the teacher allow him to modify the students' *milieu* and to enrich it through data collection, discussion and lesson notes constructed on the students' proposals.

The results of our preliminary analysis have been useful to develop case studies and to analyse them more deeply in the frame of the FaSMEd project. As a result, we conceive formative assessment as a process that requires time, but in the same time the way teachers decide to exploit data for modifying their teaching evolves over time. So, in our case studies, we are planning to visit and observe classrooms in different moments of the school year.

We will build our presentation for the conference on both these preliminary experiments and the first results of different case studies that will be at our disposal.

NOTES

1. Our translation of:

Le triangle a deux côtés égaux, et donc déjà isocèle et possède deux angles égaux. / 180-36=144/2=72.
Ses deux angles mesurent 72°. / 72/2=36 donc 3/2=1,5. / La corde bleue mesure 1,5 cm.

2. The students refer to the Intercept Theorem (also known as Thales' Theorem) concerning the ratios of the line segments that are created if two intersecting lines are intercepted by a pair of parallels (and, by extension, the ratios of the sides of similar triangles).

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THE PREDICTIVE NATURE OF PERCEIVED LEARNING FIT ON TEACHERS' INTENTION TO USE DGS IN GEOMETRY TEACHING

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The purpose of this study was twofold. Firstly, to extend Technology Acceptance Model to assess secondary school teachers' intention to use Dynamic Geometry Software (DGS) in geometry teaching and, second, to examine the relations among the parameters of TAM and the role of external factors. We enriched TAM by integrating in the model the notion of "perceived pedagogical-learning fit", which refers to evaluating the pedagogical-learning appropriateness of teaching geometry with DGS based on a cognitive-learning model. The results of the study showed that perceived pedagogical-learning fit is the strongest predictive factor of intention to use, while the external factors affect only the "perceived of ease to use". "Perceived usefulness" and "perceived ease of use" had weak indirect effects on intention to use through attitude.

Keywords: TAM, perceived learning fit, technology acceptance model, task technology fit

INTRODUCTION

Dynamic geometry software (DGS) has been considered as an effective tool in the teaching and learning of geometry and proved to have the potential to regenerate geometry in schools (Hollebrands, Laborde, & Straesser, 2008). Numerous research studies support that DGS can be a powerful teaching and learning medium and it can contribute to enhance mathematics teaching, enrich visualisation of geometry and improve visualization skills, provide a foundation for analysis and deductive proof, study the interrelationships of the structural elements of geometric shapes and create opportunities for creative thinking. However, research studies have shown that DGS classroom use has remained limited (Ofsted, 2004). Stols and Kriek set the research question why don't all mathematics teachers use DGS in their classrooms since DGS has been proved to be such a powerful teaching and learning tool (2011). Cuban supported that problems may emerge when teachers' beliefs are ignored, because "beliefs and values that teachers hold drive many of the choices they make in the classroom" (2001). Thus, there is a need of a deeper understanding of teachers' beliefs that may influence their behaviour and the way in which these beliefs are manifested (Stols & Kriek, 2011).

The purpose of this study is to empirically examine the way in which teachers' behaviour intention to use DGS in their classrooms is influenced by their behavioural, normative and control beliefs. To this end, we adopted and extended the Technology Acceptance Model (TAM), which is well known and widely accepted in the study of specific behaviours to understand the way in which users' beliefs and attitudes affect their technology usage behaviour (Chung, Park, Wand, Fulk, & McLaughlin, 2010; Venkatesh & Davis, 2000). TAM is considered as a simple, precise and robust model for understanding IT usage with few but salient constructs. Dishaw and Strong (1999) supported that a weakness of TAM is its lack of task focus and extended the model to integrate task-technology fit (TTF), which refers to matching the capabilities of the technology to the demands of the task. In the present study, we integrate TAM and TTF theoretical considerations to extend and propose a technology acceptance model that assesses the intention of pre and in-service secondary school teachers to use DGS in geometry teaching. In particular, we modified TAM by adding in the model personal traits and environmental factors and teachers' perceived learning fit parameter on a task-technology fit theoretical assumption.

THEORETICAL BACKGROUND

Technology Acceptance Model

A number of research studies examined factors that influence teachers' and students' technology acceptance. More specifically, these included personal factors such as attitudes towards computers (Teo & Noyes, 2011), computer self-efficacy, technical factors such as technological complexity, and environmental factors such as facilitating conditions (Ngai, Poon, & Chan, 2007). The most dominant, parsimonious and widely accepted model is the Technology Acceptance Model (TAM), which is based on the Theory of Reasoned Action (Ajzen & Fishbein, 1980; Ajzen, 1991) and the theory of planned behavior and explains how users' beliefs and attitudes affect their intention to use a specific technological device. The TAM includes the very important assumption that the behavior is volitional. TAM explains the interactions among attitudes, beliefs and intention to use technology. The two belief variables refer to perceived usefulness and perceived ease of use. Perceived usefulness refers to the subjective belief that the use of the new technology will improve job performance and productivity. Perceived ease of use refers to the subjective belief that the use of the new technology does not demand considerable time and effort. Recent studies have shown that the above variables affect users' intention to use and their attitude towards technology use (Cheung & Huang, 2002; Raaij & Schepers, 2008). A meta-analysis has shown the robustness of paths from perceived ease of use to perceived usefulness and from perceived usefulness to behavioral intention (Sun & Zhang, 2006).

Having in mind the business and commercial origins of TAM, not surprisingly, it has had limited applications in education. Recent research studies of TAM in education have explored students' or teachers' acceptance towards new technologies such as online learning, and technology in education (Lee, Yoon, & Lee, 2009; Raaij & Schepers, 2008; Stols, 2007). Results showed that perceived usefulness and perceived ease of use proved to be critical parameters of the acceptance and usage of the innovation as an effective and efficient learning technology. Arguments against the theoretical contributions of the model claimed that TAM's emphasis on two key variables prevailed researchers from investigating other essential determinants of technology adoption decisions (Bagozzi, 2007). A major concern for the use of TAM is whether it actually predicts actual usage. A metanalysis by Turner, Kitchenham, Brereton, Charters and Budgen (2010) showed that the TAM variables are less likely to be correlated with actual usage. Although TAM'S perceived usefulness concept implicitly includes task, the model has also been criticized for the lack of task focus and its application revealed mixed results in information technology evaluations (Dishaw & Strong, 1999).

Task-Technology Fit Model

In contrast to TAM, the theoretical foundation of the task-technology fit model (TTF) lies on the assumption that technology will be used if, and only if, the functions available to the user fit the activities and needs of the user and the demands of the task. Thus, TTF explicitly includes task characteristics and tests for direct effects of task and technology characteristics on utilisation. Dishaw and Strong emphasized that information technology that does not offer sufficient advantage will not be used (1999). According to the TTF theory, understanding software use decisions prerequisites understanding the way in which the functions provided by the software fit the perceived needs of the user. The structure of the TTF model hypothesizes that the variables task requirements and tool functionality have direct effects on task-technology fit and task-technology fit has a direct effect on actual tool use. Dishaw and Strong argued that the general argument for combining TAM and TTF is that they capture two different aspects of users' choices to utilize the examined technology (1999). Thus, integrating the two models is likely to provide a better explanation of the technology usage than either an attitude or a fit model could provide separately.

The role of control variables

The effect of computer anxiety on teachers' disposition to use technology in teachings has been studied extensively (Raaij & Schepers, 2008). Computer anxiety is defined as the emotional distress or the tendency of an individual to be uneasy, apprehensive and/or phobic towards the use of computers. It involves the anxiety regarding the loss of data, damaging computer equipment or making other mistakes. Computer anxiety may result to negative perceptions towards technology and information systems, reduced usage, and lower end user satisfaction. Moreover, personal innovativeness was regarded as a form of openness to change and was used as an external variable in the TAM model (Raaij, & Schepers, 2008). In addition, facilitating conditions, such as administrative support, were rated as significant environmental factor in affecting teachers' willingness to integrate technology in teaching (Groves & Zemel, 2000).

THE PRESENT STUDY

There is a research need to establish an empirical link between TAM and specific mathematics geometry software. Thus, the main purpose of the study is to extend TAM and propose a structural and measurement technology acceptance model that could be used to evaluate the intention of teachers to use a DGS in geometry teaching. The present study adds to the research literature on TAM and DGS in a number of ways. It integrates TAM and TTF theoretical considerations by proposing a model that evaluates the task-technology fit of DGS based on teachers' perceived pedagogical-learning fit of the software. By the term "perceived pedagogical-learning fit" we refer to teachers' perception about the quality of teaching and learning of geometry with DGS and whether the specific software could meet the learning needs of students in geometry. The proposed theoretical notion was first suggested and validated by Pittalis and Christou (2011). In addition, ideas arising from the Research Project KeyCoMath regarding digital competence were utilized.

The purpose of the present study is to propose a model that extends TAM to assess teachers' intention to use DGS in geometry teaching based on teachers' perceived pedagogical fit of the software. Specifically, the aims of the study were to (a) to validate the measurement model that describes teachers' perceived pedagogical fit of DGS based on Duval's geometry reasoning model and (b) to extend and modify TAM so it could potentially be used to assess, on a task-technology fit basis, the intention to use DGS by integrating in the model, as a task-technology fit parameter, the effect of teachers' perceived pedagogical fit and control variables, such as computer anxiety, facilitating conditions, personal innovativeness and age. In this paper, we hypothesized that an additional parameter, "perceived pedagogical-learning fit", influences teachers' intention to use DGS in geometry teaching. Specifically, based on the literature we assumed that the theoretical construct "perceived pedagogical-learning fit" describes teachers' perceived pedagogical and learning appropriateness of geometry teaching with DGS to develop students' visualisation, reasoning and construction processes. Based on Duval's model (1998), geometrical reasoning involves three kinds of cognitive processes which fulfil specific cognitive processes; (a) visualization processes that refer to the visual representation of a geometrical concept, (b) construction processes that can be developed in DGS by appropriate tools and (c) reasoning processes that are necessary for the extension of knowledge, for explanation and proof.

Subjects

The sample of this study consisted of 97 pre and in-service secondary school mathematics teachers. Forty teachers were males and 57 females. All the subjects attended a 12-hours module regarding DGS and its pedagogical applications in a master course in the University of Cyprus during the academic years 2013 to 2014. The questionnaire was administered after the completion of the master course.

Instrument construction

A questionnaire was developed for this study. TAM scale items were adopted from previous studies (Dishaw & Strong, 1999) and were modified to meet the needs of the present study. Our research TAM model consists of 12 items that measured "perceived ease of use" (3 items), "perceived usefulness" (3 items), "attitude towards use of Cabri" (3 items) and "use intention" (3 items). In addition, based on the existing literature on geometry reasoning discussed in the previous sections, we developed 10 items that measured teachers' perceived learning fit. For example (see Table 1), the item "Teaching geometry with DGS helps in visualizing geometrical concepts" was used to measure "visualization processes" fit, the item "DGS measurement and dragging tools help students making generalisations" was used to examine the "reasoning processes" fit and the item "DGS tools make easy the construction of complex geometrical constructions, such as locus" was developed to examine the "construction processes" fit. The difference of the perceived pedagogical learning fit items compared to the general perceived-usefulness items is the fact that they are more specific and refer to research-based aspects of learning in geometry. In addition, we used three items to measure students' computer anxiety, such as "Using the computer is so complicated that it is difficult to know what is going on". Three items were used to measure facilitating conditions, such as "When I need help to use computers, specialized instruction is available to help me". Finally, three items were used as measure indicators of personal innovativeness. For instance, "If I heard about a new information technology, I would look for ways to experiment with it". We developed multi-item Likert scales which have been widely used in the questionnaire-based perception studies, using the seven-point Likert scale, with 7 being "Totally Agree" and 1 being "Totally Disagree".

Table 1: Perceived Pedagogical-Learning Fit items.

Factor	Items
Visualization processes	Q1. Teaching geometry with DGS helps in visualizing geometrical concepts. Q2. DGS facilitates the dynamic visualization and understanding of geometric theorems. Q3. DGS functions (i.e. dragging) help students to "see" the properties and characteristics of geometric shapes. Q4. DGS offers dynamic images that promote dynamic visualisation of geometrical concepts.
Reasoning processes	Q5. Teaching geometry with DGS helps in developing students' reasoning and conjecturing thinking. Q6. Manipulating shapes in DGS contributes in understanding geometric shapes' relations. Q7. DGS measurement and dragging tools help students making generalisations.

Construction processes	Q8. DGS tools make possible the construction of geometric shapes based on their properties. Q9. DGS tools make easy the construction of complex geometrical constructions. Q10. Constructing geometric shapes in DGS is not a mechanical process, but it develops students' construction abilities.

Data Analysis

The goal of the analysis was to estimate the relative strength of the proposed models. Because we proposed a theoretically driven model about the components of "perceived pedagogical-learning fit", our first interest was in the assessment of fit of the hypothesized a priori measurement model to the data. Then, we examined the validity of the hypothesized structural model. One of the most widely used structural equation modelling computer programs, MPLUS, was used to test for model fitting (Muthen & Muthen, 2007) and three fit indices were computed: The chi-square to its degrees of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit.

RESULTS

To examine the first aim of the study, we conducted a confirmatory factor analysis (CFA) to validate a measurement which should have been able to model teachers' perceived pedagogical-learning fit of DGS. The descriptive-fit measures indicated support for the hypothesized measurement model (CFI=.97, χ^2/df=1.37, p>0.05, RMSEA=.06). The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large. Moreover, the factor loadings of the first-order factors (visualization, reasoning, and construction processes learning fit) that corresponded to teachers' perceived pedagogical-learning fit were extremely high (.98, .99 and .99 respectively), claiming that a general type of belief that refers to teachers' perceived pedagogical-learning fit could explain very accurately teachers' variances in evaluating DGS.

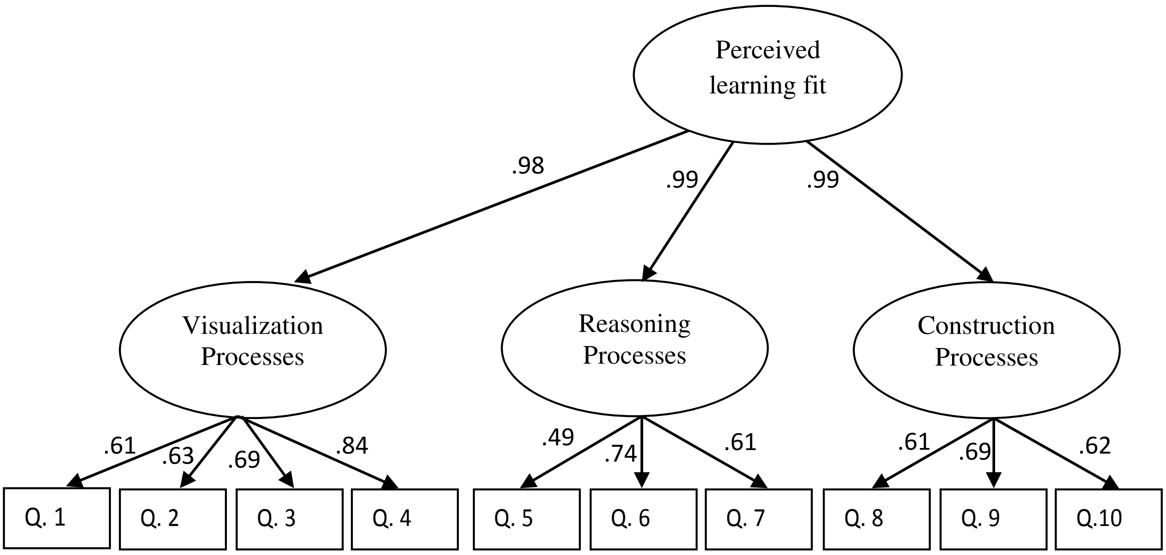

Fig 1: Perceived Pedagogical-Learning Fit structure.

To examine the second aim of the study, we tested the validity of alternative structural models. The first hypothesized structural model claimed that the intention to use DGS is influenced by the factors "perceived usefulness", "attitude towards use of DGS" and "perceived learning fit" and the external factors. However, the descriptive-fit measures did not support the hypothesized structural model. Thus, we examined the validity of alternative structural models to trace the relations between the factors of the model. We examined successively the validity of structural models hypothesizing that the external factors affect "perceived usefulness", "perceived ease of use", "attitude" and "perceived learning-fit". In addition, we examined whether "perceived usefulness" and "perceived ease of use" affect directly the intention to use DGS or indirectly through attitude. The analysis showed a complex structure of direct and indirect relations among the involved factors. Figure 2 presents the modified complex model that best fitted the empirical data (CFI=.96, χ^2/df=143, $p<0.05$, RMSEA=.07). As it is highlighted in Figure 2, the results of the study revealed that the factor "perceived pedagogical-learning fit" is the strongest predictive factor of teachers' intention to use DGS (r=.77, z=13.71, $p<0.05$). Teachers' intention to use DGS is also directly influenced by teachers' attitude (r=.26, z=4.06, $p<0.05$). Teachers' age proved to be a weak negative predictive factor of teachers' intention to use DGE (r=-.17, z=-2.65, $p<0.05$). The structure of the modified model showed that attitude towards the use of DGE is predicted by teachers' perceived usefulness (r=.54, z=7.52, $p<0.05$) and teachers' perceived ease of use (r=.22, z=2.63, $p<0.05$). Further, the solution of the modified model showed only indirect effect of "perceived usefulness" and "perceived ease of use" on intention to use through teachers' attitude to use DGS. The indirect predictive validity of "perceived usefulness was 0.14 (0.54*0.26), while the predictive validity of "perceived ease of use" was weaker (0.22*0.26=0.06).

Moreover, the analysis showed that the external factors personal innovativeness, computer anxiety and facilitating conditions had direct effects on "perceived ease of use" and "perceived learning-fit". Specifically, computer anxiety proved to be a negative direct predictive factor on "perceived ease of use" (r=-.18, z=-3.56, $p<0.05$) and "perceived learning-fit" (r=-.17, z=-2.90, $p<0.05$). Facilitating conditions had positive weak direct effects on "perceived ease of use" (r=.20, z=3.91, $p<0.05$) and "perceived learning-fit". Finally, the external factor personal innovativeness had positive direct effect only on "perceived ease of use" (r=.17, z=3.75, $p<0.05$). Furthermore, the analysis showed that "perceived learning-fit" had a strong direct effect on "perceived usefulness" (r=.83, z=22.19, $p<0.05$). Thus, "perceived learning-fit" had also a small indirect effect on intention to use through "perceived usefulness" and attitude to use (0.83*0.54*0.26=0.12), so the total effect of "perceived learning-fit" on intention to use DGS was .89 (.77 direct and .12 indirect). Summing up, the strongest predictive factor of intention to use was "perceived learning fit", while "perceived ease of use" and "perceived usefulness" had only weak indirect effects on intention through attitude to use. Moreover, the external factors had direct effects on "perceived ease of use" and "perceived learning-fit", while "perceived learning-fit" was the only predictive factor of "perceived usefulness".

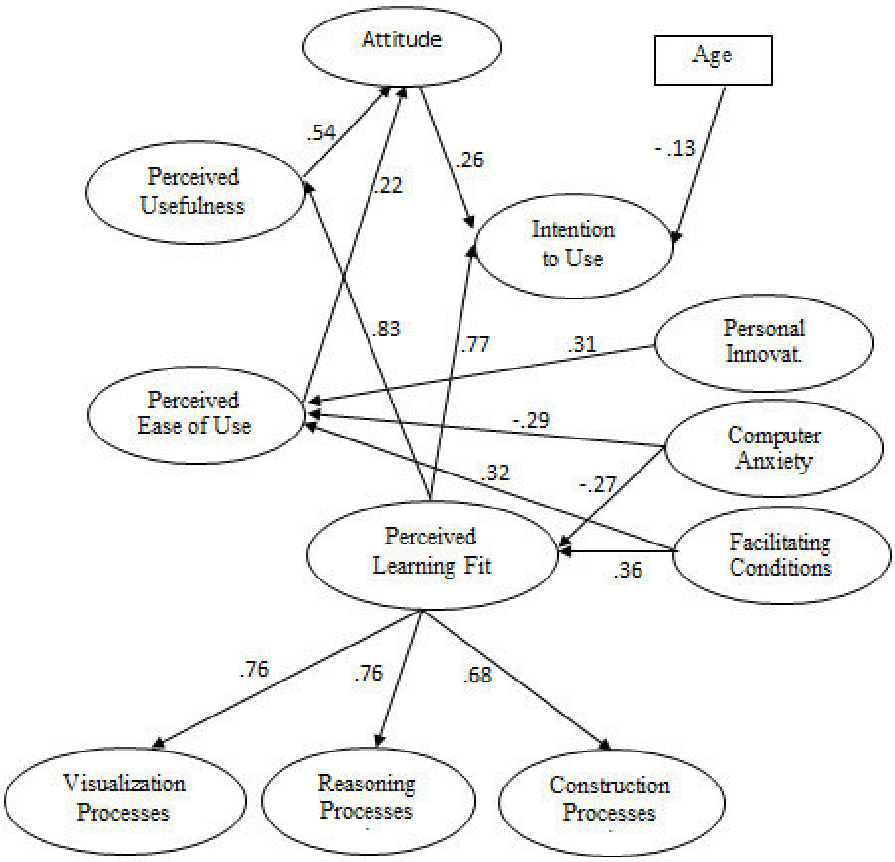

Fig 2: The complex Structural Model.

DISCUSSION

The present study modified and extended the Technology Acceptance Model by introducing the notion of "perceived learning fit". The results of the study validated the structure of "perceived learning-fit" that was proposed by Pittalis and Christou (2011) and involves the assessment of the pedagogical-learning appropriateness of teaching geometry with DGS. In addition, the present study examined extensively the complex relations among the parameters of TAM and external factors. The results showed that the strongest predictive of intention to use was "perceived learning-fit", while "perceived ease of use" and "perceived usefulness" had only weak direct effects through attitude. Another important finding of the study is the fact that the external factors of personal innovativeness, computer anxiety and facilitating conditions had weak direct effects on "perceived ease of use", while "perceived learning-fit" had a strong direct effect on "perceived usefulness". Thus, the present study underlies that personal traits, such as computer anxiety and personal innovativeness, and environmental characteristics, such as the facilitating conditions, may have direct effect only on the foundational pillar of TAM "perceived ease of use". On the contrary, "perceived usefulness" is only affected by "perceived learning fit" and the indirect effect of "perceived ease of use" and of the external factors on the intention to use DGS is too weak. What really affects the intention to use is the "perceived learning fit". In other words, regardless of a teacher's personal or environmental constraints, if the teacher is convinced about the learning capability of a DGS, then it is more likely to use it. Thus, although for the past two decades, numerous studies using the TAM as a research framework have been conducted, there is a need for future research in mathematics education domain to examine extensively and in depth teachers' priorities regarding the learning characteristics of future DGS software.

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THE IMPACT OF TECHNOLOGIES ON THE TEACHER'S USE OF DIFFERENT REPRESENTATIONS

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This study intends to characterize how the teacher uses and integrates the different representations provided by the graphing calculator on the process of teaching and learning functions at the secondary level. Specifically, it intends to understand the balance established between the use of the different representations, and the way these representations are articulated. The conclusions reached point to an active use of the graphic and algebraic representations and to a scarce use of the tabular representation. The conclusions also point to a flexible articulation between the two representations usual used, assuming different forms and frequently an interactive approach, repeatedly switching between representations.

Keywords: different representations, graphing calculators, functions

INTRODUCTION

One of the features of the graphing calculator is to allow access to multiple representations (Heid, 1995; Kaput, 1992). This feature allows the students to establish or strengthen links in a way that would not be possible without the support of technology (Cavanagh & Mitchelmore, 2003), articulating the numerical or tabular, symbolic or algebraic and graphical representations (Goos & Benninson, 2008) and fostering the development of a better understanding of the functions, of the variable concept and the ability to solve problems (Bardini, Pierce, & Stacey, 2004; Burril, 2008). As referred by Kaput (1989), the connection between different representations creates a global vision, which is more than the joining of knowledge on each of the representations. And the technology allows a full exploitation of numeric and graphic approaches in a way that until then was not possible, promoting an integrated approach to different representations and consequently the development of a deeper understanding. Thus, the use of multiple representations has the potential to turn learning into a significant and effective experience (Ford, 2008).

The study presented here is part of a broader research focused on teachers' integration of graphing calculators into their practice. In this paper I focus on the different representations provided by graphing calculators, intending to characterize how the teacher uses and integrates them on the process of teaching and learning functions at the secondary level. Specifically, I intend to understand:

- the balance established between the use of the different representations,
- how different representations are articulated.

DIFFERENT REPRESENTATIONS TO TEACH FUNCTIONS

The three representations provided by the graphing calculator and commonly used in the study of functions have different features and capabilities, as stated by Friedlander and Tabach (2001). According to these authors, the numerical representation allows students to use familiar objects to demonstrate relationships and analyze specific cases. It is, however, a representation that lacks generality. And this can cause that important features of the function are not detected or can result in excessive focus in specific cases. As such, in some cases its utility is very limited.

The graphical representation provides a visual representation, with a wider range of specific cases, allowing a use that transcends the algebraic knowledge of the students, since the graphical approach is more universal than the algebraic, and enabling students to find solutions even when they do not know an analytical approach or even when it does not exist (Friedlander & Tabach, 2001) (for example, finding the zeros of a polynomial function regardless of the degree of the respective polynomial). Being the graphics a more intuitive representation, the solutions obtained in this way may, however, lack accuracy and be influenced by external factors such as the scale used on the interpretation that is made (which may even allow, for example, the existence of a zero to go unnoticed). As in the case of the numerical representation, in the graphical representation only a part of the function's domain is visible (although a somewhat larger part), being so, the usefulness of this representation depends largely on the circumstances.

The algebraic representation is concise, comprehensive and effective in presenting regularities and models, being the algebraic manipulation often the most effective way to make generalizations and achieve results (Friedlander & Tabach, 2001). Still, the exclusive use of this representation may hinder the understanding of mathematical concepts and cause difficulties in the interpretation of the students. This difficulty is, actually, pointed out by Quesada and Dunlap (2008), who suggest the use of numerical and graphical capabilities of the calculators as a way of taking advantage of different representations, and get an introduction of concepts closer to the one as they were developed, which consequently could facilitate its understanding by students. It is also in this perspective that Coulombe and Berenson (2001) report the contribution that fluency with multiple representations can bring to the development of algebraic thinking. Indeed, not only it is common that complex mathematical ideas (as is the case of the concept of function) can not be expressed using a single representation, as it is even more common that it is not easy to understand them in this way (Asp, Dowsey, & Stacey, 1993). The use of different representations allows the student to understand in another way what was not possible to understand at the initial representation or, as Kaput (1992) says, is fundamental for the understanding of the concept. And Ford (2008) points out that this is the importance of working with multiple representations. This is not about using them simply because the technology facilitates access, but to do so because it is necessary to promote the students understanding.

Despite the importance of working with different representations and despite the fact that this work is greatly facilitated by the use of graphing calculators, students find it difficult (Billings & Klanderman, 2000; Kieran, 2007; Ramos & Raposo, 2008) and teachers have not devoted the necessary attention to the flexibility needed to move from one representation to another and to articulate all the information provided by these (Even, 1998). Although there is some concern on the part of the teachers, to articulate and balance the use of different representations, Molenje and Doerr (2006) found that the use of algebraic and graphical representations are dominant in relation to the numerical representation. Furthermore, when the teachers effectively use the three representations, it is possible to identify a pattern in the way they do it. According to the conclusions of their study, some teachers tend to start by an algebraic representation, passing then to a graphic one and, finally, to a numeric representations. Other teachers tend to move from the algebraic representation to a numerical one and only then to a graphic representation. This rigid sequence adopted by the teacher tends to be copied by students (Barling, 1994; Rocha, 2000) who, consequently, cannot develop the desired fluency in moving across representations.

The importance of representational fluency, according to Zbiek et al. (2007), lies in its ability to provide the development of mathematical understanding, which is the reason why this is not limited to the ability to move from one representation to another, carrying the knowledge of one entity to another and articulating it with the new knowledge made available by the new representation. The representational fluency also involves the knowledge of the most appropriate representation, in certain circumstances, to illustrate a particular concept or explain a particular notion and the knowledge of how to link the different representations in relevant ways to support a certain statement. It is therefore a fundamental knowledge for the teacher, but this is also a knowledge with implications for student learning, particularly when the flow between the different representations is subject to some constraints or restrictions. As stated by Almeida and Oliveira (2009), working around the different representations with a strong emphasis on conversion between these is critical to the understanding of the subject by students and to prevent some foreclosure of knowledge. It is therefore important, as recognized by Molenje and Doerr (2006), that the teachers are aware of their options at this level.

METHODOLOGY

The part of the research presented here adopts a qualitative approach involving the implementation of a case study. Teresa was the teacher involved. A teacher who had a long professional experience and also a long experience of using the graphing calculator, having as well a wide training in the use of this technology, aspects that authors like Dunham (2000), Hoben (2002) and Power and Blubaugh (2005) point as particularly influential on the integration of technology.

Data collection involved the monitoring of the teacher's work during the teaching of functions with one of her classes throughout the 10th and the 11th grade. In this sense semi-structured interviews were conducted; 14 classes in the 10th grade and three in 11th grade were observed and several documents were collected. The interviews were of various types, being relevant to the part of the study presented here those carried out before and after each lesson observed. Both the interviews and the observed classes were audio recorded. It was also produced a logbook of the lessons observed. Data analysis was mainly descriptive and interpretive in nature, considering the problem under study. The tasks proposed by the teacher were seen as a structural unit in the light of the elements suggested by the theoretical framework. The process started with the identification of the episodes where different representations have been addressed, and then the questions of the study were used to structure the analysis of the episodes.

TERESA AND THE DIFFERENT REPRESENTATIONS

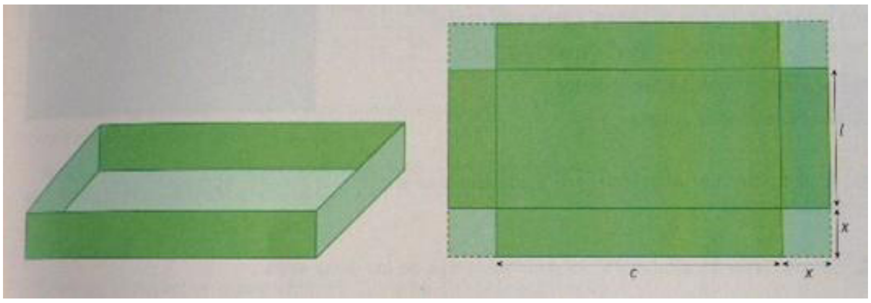
At this section I present some of the tasks proposed by Teresa to her students during the study of functions. Taking into account the issues of the study presented here, I give a particular attention to the description of what the teacher and the students did in each task, in what concerns to the use of the different representations provided by the graphing calculator.

The box

This was one of the several problems that Teresa proposed to her students during the study of functions:

Laura intends to build a box without a lid to store her brother's toys. For this she has a rectangular card with 1.2 m long and 80 cm wide, where she intends to remove four square corners to facilitate folding the sides of the box.

What is the square side length that Laura should cut at every corner of the card to get a box of maximum volume? Display the results in centimeters, rounding it to two decimal places.



Steps to follow in the resolution:

- Show that $V(x) = 4x^3 - 4x^2 + 0{,}96x$, being V the volume of the box.
- Explain the variation of x and use the graphing calculator to get a graphical representation of the function.
- Calculate x so that the volume of the box is maximum.

(adapted from Costa & Rodrigues, 2010, p. 93)

The initial part of this task assumes that the students come to the expression of the volume of the box. This is accomplished without using the graphing calculator and, therefore, without the use of the available representations. The possible values for the variation of x depend on an analysis of the problem and an understanding of how the cut on the card is done. Once again it does not involve the representations provided by the calculator. The calculator is used by the students at the end of the task to, after introducing the expression of the function, get the graph. After that the students use the calculator to calculate the maximum value of the function.

In terms of the calculator's representations, students begin with an algebraic representation, come into a graphic one and then get a numerical value on the graph.

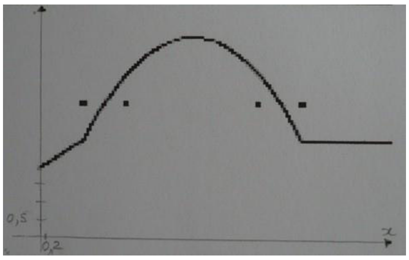
In other situations proposed to the students during the study of functions, the expression that modeled the situation was immediately given, being the students suppose to perform a work similar to the one at the final part of this problem. In some of these cases, the students were requested to find the extremes of the function, their zeros, the values of the function for a given value of x or the value of x corresponding to a certain value of the function. In these cases students used the menu calculate provided by the calculator, using in some cases also a constant function.

Slalom

The task posed by the teacher to the students was as follows:

At a Slalom's competition, the skier must make a route bypassing flags or passing between two flags forming a door.

1. Define the calculator window to $x \in [-1, 8]$ e $y \in [-1, 7]$. Draw the points (1, 4), (2, 4), (5, 4) and (6, 4), that will represent the Slalom flags.
2. Find a course for the skier, defined by a quadratic function and going through both doors without touching the flags.
3. The public attending the competition occupies the area in the plane defined by $y \geq 6$. Find a new quadratic trajectory for the skier so that he is not crossing the area reserved for the public.
4. The following illustration shows the path made by the Swiss champion who won this Slalom. Find the expression of a piecewise function with this graphic representation.



This task was proposed to be solved in pairs while the teacher moves arround the students, supporting their work. The questions require students to find quadratic functions verifying certain conditions. So the students start introducing the expression of a function in the calculator and watching its graph. From this point, they begin a process of successive changes on the expression of the function, followed by the observation of the corresponding graph, until they manage to achieve an expression of a function verifying the conditions imposed by the task. This is an interactive process between the algebraic and graphical representation. However, the students fill that this might not be the process intended by the teacher:

Teacher: Did you manage to find the function?

Student: Yes. But I did it by chance.

Teacher: Are you sure it was just by change?

Student: Actually it was more like trial and error. (…) I knew that the concavity should be facing down, and that the vertice should be… greater than 4.

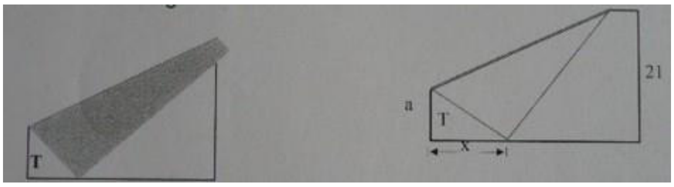
Teacher: Ah! So you knew a few things. So, try to write that down. (…) Write down how you used your mathematical knowledge to find by chance! (class 5)

In this sense there is a work supported by the algebraic and graphic representations that is informed by the mathematics knowledge of students.

Folding the corner of a sheet

By the end of the study of functions (lessons 13 and 14), Teresa proposes to her students the following task:

Fold a sheet of paper so that the upper left corner touch the underside of the sheet as shown in the figure.



What is the triangle (T) of larger area formed in the lower left corner of the sheet by the effect of this fold?

(consider a sheet of paper of 29 cm × 21 cm)

This task begins with a data collection phase, in which students are making successive folds, taking measurements and recording the elements obtained. After getting the data, the students introduce them on the calculator in the form of a table, and draw a graphic of points. Then they start the search for an expression of a function that suits the data. At this part of the task, the students rely on the models provided by the calculator. As the image of the plotted points suggests a parabola, all the students start by considering the quadratic function (actually the function that best fits the set of points obtained in this manner is a cubic function).

This is the first task where the students use the tabular representation. This delay on the use of the tabular representation is a conscious choice of Teresa. She believes it is necessary to introduce the calculator in a progressive way, avoiding the temptation to immediately start using all that it offers. Therefore, she considers it is more appropriate to introduce the use of the table just near the end of the study of the subject, since until then the table is not really needed, but shortly, when the students start the study of statistics, it will become very important:

Teacher: You can not want to do everything at the same time. The machine brings many new things and we have to go managing. They may get the function values with the trace and, therefore, the table is not needed. I only start using the table at the end, because then we will start studying statistics and, then yes, we need the table. (Interview: post-lesson 13)

CONCLUSION

The teacher involved in this study seems to favor the use of algebraic and graphic representations rather than the tabular representation. The way she articulates the different representations can take different forms. In some cases the teacher begins with the graphical representation and moves to an algebraic representation. In other cases she decides to go from an algebraic representation to a graphical representation. In either case the articulation between the different representations tends to assume a somewhat interactive approach, with alternations between the representations and frequent contributions from the two representations. As so, the conclusions reached point to some diversity in how the teacher integrates the different representations, suggesting there is no preference for one particular type of link between the different representations, which contradicts the conclusions reached by Molenje and Doerr (2006). However, the reduced use of the tabular representation provided by the calculator is a reality, one aspect that Molenje and Doerr (2006) also refer. Teresa justifies this option by the need to phase the student's learning of how to operate the calculator and the fact that it is possible to access the numeric values by other processes than the table. And this is an interesting aspect, which turns it inevitable to question to what extent the ease of access to different representations provided by the technology finally results in practice in an action of postponing the tabular representation. In this study it seems to be the case. The algebraic and graphical representations are heavily used, sometimes in ways that would hardly be possible in contexts without technology, and the intensity of their use seems to somehow involve a devaluation of the importance of using the tabular representation. But the question is perhaps a little more complex. While authors such as Goos and Benninson (2008) and Lesser (2001) refer to the numerical representation and tabular representation as two names for the same representation, Cuoco (2001) speaks in numerical, visual, tabular and algebraic representations, suggesting not three but four different representations. To what extent the reading of a graph, based on the trace function of the calculator, has similarities or differences, from the point of view of mathematical understanding of the functions, from the reading of a table, is an issue that the work around the different representations offered by technology seems to arise. And this is an issue that this study suggests as lacking further development.

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MATHEMATICS IN PRE-SERVICE TEACHER EDUCATION AND THE QUALITY OF LEARNING: THE MONTY HALL PROBLEM

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If students acquire a new mathematical notion, according to Gray and Tall (1994), they pass through a proceptual divide. At a higher education institution in Portugal, students from different courses (education and business) came into contact with the Monty Hall problem in Statistics class. As a part of the learning environment the students have at their disposal all the technological apparatus they could use. The correct outcomes of two students (from the two courses) are analysed against the background of a model of analysis based on Tall's theory of the advanced mathematical thinking linked with SOLO taxonomy by Biggs and Collis (1982) and supported by Engeström (2001) third generation model of Activity Theory. In particular, the two different outcomes show that students can attain the same level, but may be operating in different levels: procedural thinking and proceptual thinking.

Keywords: Advanced mathematical thinking, proceptual divide, probability, quality of learning, SOLO taxonomy.

INTRODUCTION

If we look at a mathematical object, we can think of it relating with its mathematical definition, properties, relations, processes and procedures. Gray and Tall (1994) uses the dichotomy between procedure and concept to characterize the proceptual divide. On the one hand procedure relates to routine manipulations, focused on performance and it's somehow inflexible, on the other hand conceptual knowledge calls for relationships and flexible concepts.

> In this sense we can talk about the procept 6. It includes the process of counting 6, and a collection of other representations such as 3+3, 4+2, 2+4, 2 x 3, 8–2, etc. All of these symbols are considered by the child to represent the same object, though obtained through different processes. But it can be decomposed and recomposed in a flexible manner. (Gray & Tall, 1994, p. 6-7)

The above definition reflects the cognitive reality by using the term *procept* to translate the flexibility of the notion starting from an "*elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object." (Gray & Tall, 1994, p.6)

This paper is a report of one episode of an ongoing study of this topics and the underlying question is: does the analysis proposed for student outcomes access the complexity of their thought, and furthermore can it reason about the quality of their mathematical learning? To approach an answer we present an analysis of two correct outcomes, but evaluated in different levels of mathematical thought. Technologies used in this paper are seen as tools to learn mathematics rather than the mathematics used as an excuse to use technological skills.

THEORETICAL FRAMEWORK

Advanced mathematical thinking

In 1988, Tall argued that *advanced mathematical thinking* could be seen in two different ways: (I) Thought related to advanced mathematics, or (ii) advanced ways of mathematical thought (Tall, 1988).

But, what is *advanced mathematical thinking?* Since Ervynck coined the expression in 1985 there is a discussion about it, to some it relates to cognitive changes between secondary and higher education students, others stand for the origin of the cognitive conflicts inherent to mathematical thought. To Tall (1988) advanced mathematical thinking is any part of problem-solving which includes the development of conceptual fields by abstraction.

Gray and Tall (1994) uses the *encapsulation* notion of a process in a mental object, rooted in the works of Piaget to support the cycles of assimilation and accommodation. The use of symbols brings itself and ambiguity between procedure and concept that they can define as a procept. The way students address this ambiguity seems to be the key for the quality of the mathematical learning.

Supported by the use of procepts the characteristics that make a difference between two forms of thinking are: (i) procedural thinking focused on procedure, mathematical objects are concrete entities that can be manipulated based on some rules; (ii) proceptual thinking focused on the flexibility and the ability to use a mathematical object in many ways. "This lack of a developing proceptual structure becomes a major tragedy for the less able which we call the proceptual divide." (Gray & Tall, 1994, p. 18)

SOLO (Structure of the observed learning outcomes) taxonomy

The emphasis on the quality of student outcomes is a key point for the use of this taxonomy in the analytical model proposed. Its focus is not on the correctness of the outcomes, but in their nature, coded in SOLO levels.

To Biggs and Collis (1982) the quality of learning of a student depends on external stimulations, such as the quality of teaching and internal stimulations like the development stage, its previous knowledge about the subject and motivation. But it is hard to identify this quality solely considering a development stage, if we change focus to their outcomes we can identify patterns. These patterns are important components for the terminology used in the taxonomy.

We describe the basic features of the SOLO taxonomy, adapted from Biggs and Collis (1982):

1. *Pre-structural*, the outcomes provide non related information, loose and disorganized with minimal capacity;
2. *Uni-structural*, the outcomes provide simple connections, does not identify its importance, jumps to conclusions on a single aspect;
3. *Multi-structural*, the outcomes provide some connections but without a unifying vision, can isolate relevant data, work with algorithms and perform simple procedures;
4. *Relational*, the outcomes make complex connections, use relevant data and interrelations, explaining the causes;

5. *Extended abstract*, the outcomes goes beyond the topic, make generalizations, use relevant data with no need to give closed responses.

Activity Theory

Initially developed by Vygotsky and Leont'ev centred in the triangle of activity guided by objects (first generation), Engeström (2001) expanded the original centred now in activity (as a process) reflecting the actions and interactions of the subject with the context within learning occurs (second generation). With the notion of activity network a third generation emerged with the centre now in, at least, two activity systems in interaction.

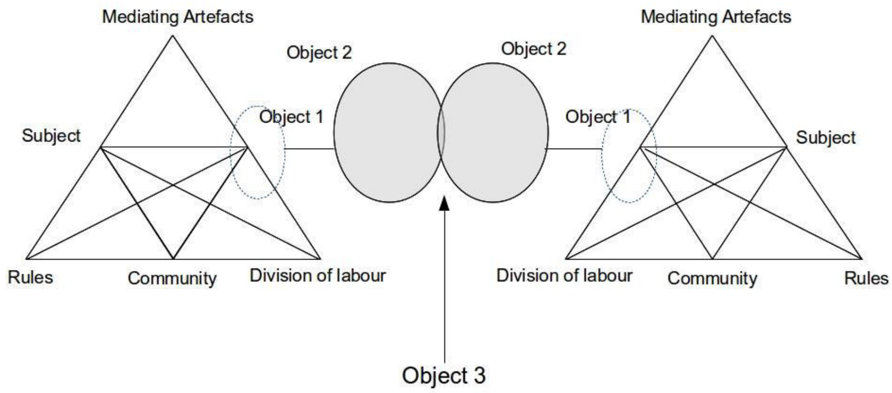

Figure 1. Two activity systems in interaction (Engeström, 2001).

In figure 1, the object goes from an initial outcome (object 1; seen as the answer made by subject and/or the solution by the teacher) to a meaningful outcome constructed by their activity systems (object 2; seen as the expected outcome by each student and teacher) leading the intersection of both expected outcomes to produce a shared object and their evaluation related to the quality of student learning (object 3; seen as a constructed understanding on the outcomes of both activity systems). This is not a static situation; therefore it shows only a picture of a specific outcome.

This third generation of activity theory could be summarized by Engeström (2001) seeing the object of activity as a moving target for an expansive transformation in activity systems supported by the contradictions as a source of development. These contradictions are not conflicts since it evolves a dialectic and multi-directional relation supported by Marx and Hegel in the contradictions of the dialectic relation.

THE MONTY HALL PROBLEM

> In a TV contest, a contestant chooses one of three doors; behind one of the doors there is a prize and behind the other two there is nothing. After the competitor chooses a door, the host opens one of the other and reveals that there is no prize. The host then asks the competitor's choice whether to keep or want to switch. It is advantageous, in statistical terms, to switch or keep?

The solution to this problem caused a great deal of controversy among mathematicians since 1990 answer by Marilyn vos Savant that the contestant should switch. The original problem is based on the TV show *Let's make a deal* starring Monty Hall and it's been discussed since 1975 (at least).

METHODOLOGICAL APPROACH

These episodes are taken from one larger ongoing study, this specific episode was designed based on discussion classes from an education and a business courses in which two students (let's call them Raquel and Mariana) presented different solutions both correct, using all the technological apparatus at their disposal (smartphones, tablets, internet access, and so on). In these episodes one of us acted as a teacher and as a researcher and a two classes are reported, totalling four hours of work, both classes about the Monty Hall problem.

One of the goals of this kind of class (discussion) was to enhance not just the resolution of common exercises in statistics and probability, but to create a kind of problem based learning sustained by a community of enquiry. In these two classes, a variety of mathematical problems are stated and students discussed possible solutions for a 10 or 15 minutes time, then they have 20 minutes to write solutions in order to present to their classmates in the remaining time.

The outcomes were analysed based on SOLO levels and their attributes and deepened by Tall theories covering aspects of procepts and proceptual divide, supported by the third generation of activity theory scheme.

Raquel, a second year student of Business had some interest about the problem and decided to work alone. Grabbed her tablet and searched (in Portuguese) for similar problems switching to English when she found some articles related to Monty Hall.

Her solution is based on the conditional probability, namely *Bayes Theorem* using a decision tree as shown in figure 2 bellow:

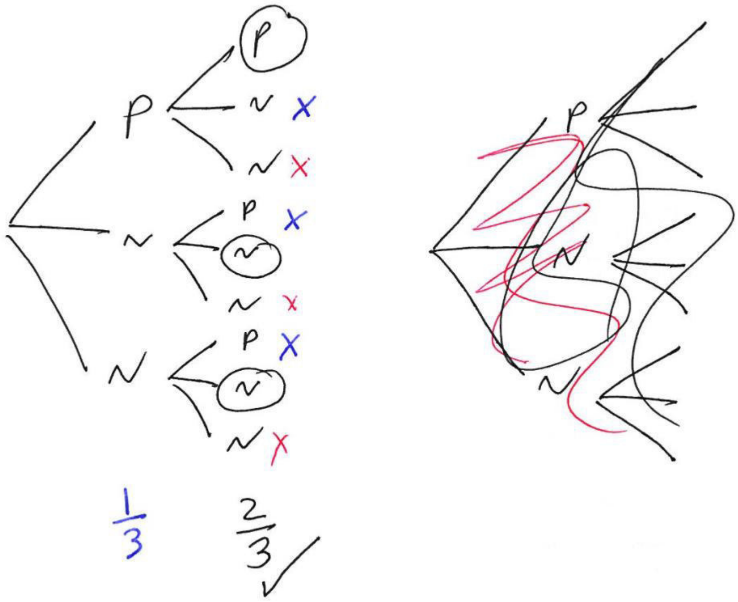

Figure 2. Sketch presented by Raquel to explain her solution where P stands for "Prize" ("Prémio" in Portuguese) and N stands for "Nothing" ("Nada" in Portuguese).

In this decision tree Raquel explains that the red crosses stand for the time that the host reveal one door. Then she continues her explanation (the dialogues were held in Portuguese):

Raquel: The blue crosses are when we stay with the same door leading to Prize-Nothing-Nothing or one third probability.

Teacher: …

Raquel: The black circles are when we change doors leading to Nothing-Prize-Prize or two thirds probability, so we must change to get a better chance to win the prize.

Teacher: Isn't this sketch to confuse...we just understand it when you explain...

Raquel then turns to her tablet and five minutes later comes up with this solution (figure 3):

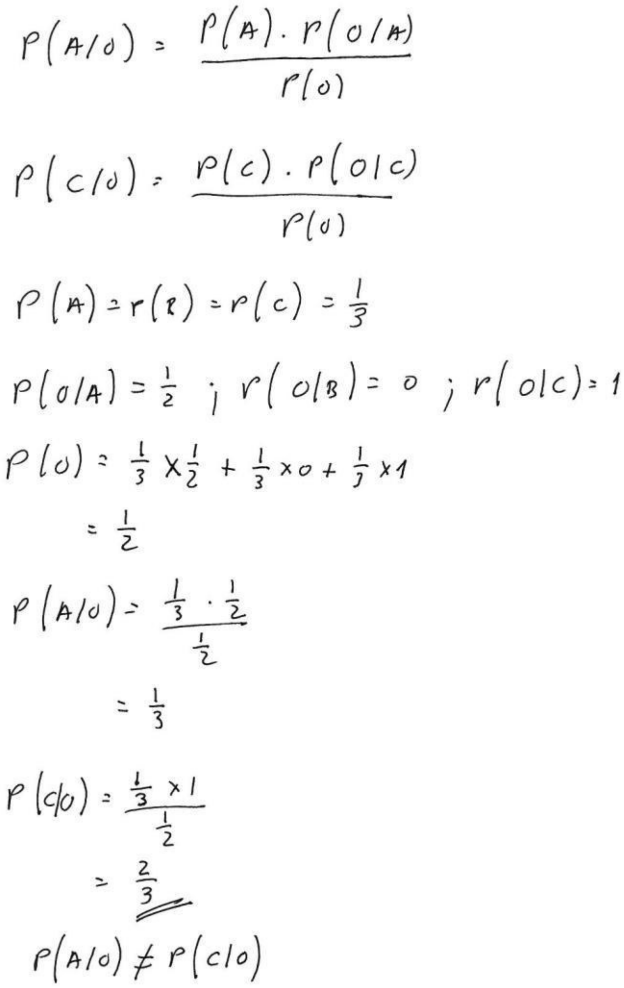

Figure 3. Calculations made by Raquel using Bayes Theorem.

And explained that A, B and C are the events, O is the event that the host opens door number 2 so by calculations made with Bayes Theorem the result is the same of the sketch.

At this time we realize that for her, the problem is solved. When we analyse this episode, Raquel tried, with some success, to use a decision tree to explain the solution of the problem, but the drawing was too confusing, we evaluated this attempt as *relational* in SOLO taxonomy because she makes some complex connections, explains her steps and analyses the solution, but when she was questioned about the confusing design she gave other kind of response, a more *mathematical* solution with the help of the conditional probability.

Her outcome is now classified as *multi-structural* level because she just worked with the algorithms. She just found a webpage with the solution and just copied to the paper. Somehow she felt frustrated that she can't draw a better example and by the use of technology took refuge on the more familiar calculations and algorithms.

The analysis of Raquel outcomes evidenced a procedural thinking, even more when she was asked to explain the first outcome and she goes back to the algorithms evidencing some contradictions in her activity system namely in the *rules* and in the *mediating artefacts* that produces her outcome, as we can see in figure 4.

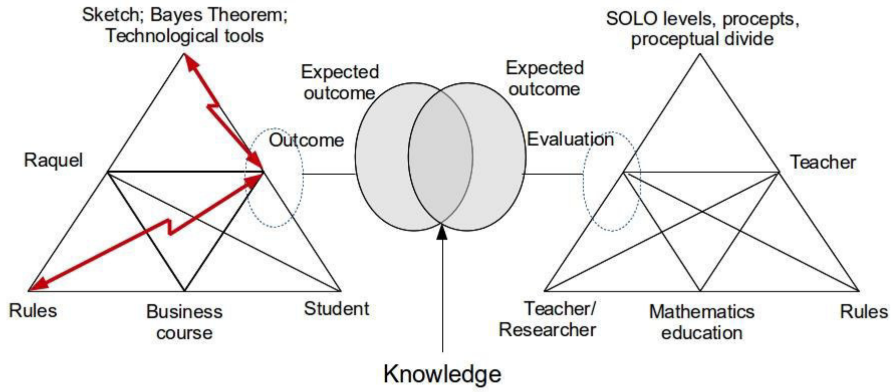

Figure 4. Activity system drawn from Raquel outcomes.

This activity system also evidences a difference on the expected outcome from the two isolated activity systems (student and teacher), the intersection made from this kind of Venn diagram in the middle shows a third object that emerges from the connection of both activity systems.

The next episode with Mariana went up differently. Mariana is a third year student in Education, with no problems using technology, so she chooses the Monty Hall problem and tried to simulate the solution using a spreadsheet. The next section describes all the steps Mariana made to build the simulation:

> On the first cell she wrote =INT(RAND()*3)+1 to generate a whole number from 1 to 3, on the second cell used the same formula to generate a new random number (from 1 to 3) to indicate the door that the contestant could choose, for the third cell the formula was more complicated, but with the help of some spreadsheet cheat sheets she got a conditional formula:
>
> =IF(C4=B4;IF(B4=1;IF(RAND()<0,5;2;3);IF(B4=3;IF(RAND()<0,5;1;2);IF(RAND()< 0,5;1;3)));IF(C4=1;IF(B4=2;3;2);IF(C4=2;IF(B4=1;3;1);IF(B4=2;1;2))))
>
> This formula generates one of three numbers avoiding the numbers of the first two cells; it is the door that the host opens. In the fourth cell she wrote:
>
> =IF(D4=1;IF(C4=2;3;2);IF(D4=2;IF(C4=1;3;1);IF(C4=1;2;1)))
>
> Other conditional formula to prevent the random number to be the one in cell three or in cell one. Now the next formula served to check if the contestant win or lose: number 1 if cell 1 and for match, 0 if it doesn't match:
>
> =IF(E4=B4;1;0).

> To finalize, after she copied the first line 100 times she just made a sum from this hundred counts on the next cell:
>
> =SUM(F4:F1003)
>
> And made a percentage from the value. As the number were randomized she got values around 67, 5% every time arriving to the conclusion that it is advantageous to switch.

This method to find the result is a convincing demonstration and could be found in many web pages around internet, but in this case Mariana didn't just copy the formulas or the demonstration spreadsheets that can be downloaded, she explained to her classmates and replicated the simulation.

Although this simulation isn't a traditional mathematical proof it shows that technological tools could be used to give a new look to mathematics, this outcome was classified as *relational* in SOLO levels close related to the *extended abstract* because, on the one hand Mariana makes complex relations, explain the causes, integrates several areas of knowledge, on the other hand she goes beyond the topic making generalizations to other concepts.

The activity system is different from the one presented in figure 4 although the system contradictions are the same in figure 5.

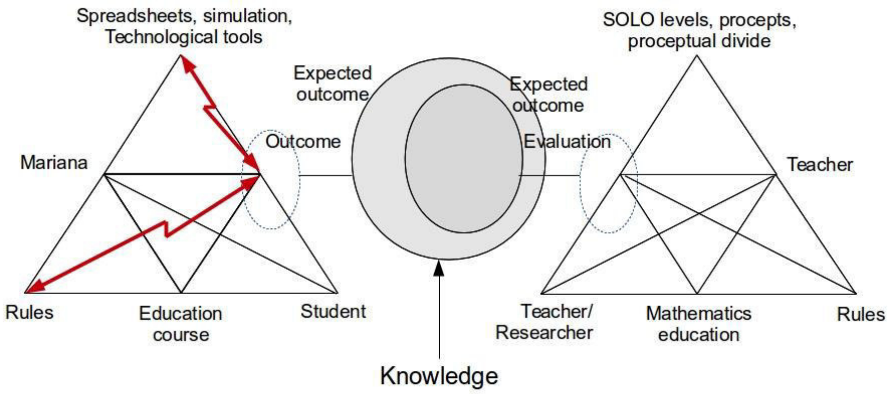

Figure 5. Activity system drawn from Mariana outcome.

This activity system shows the use of more complex procepts clearly a sign of proceptual thinking due to the use of simulation that evidences a proceptual divide. In this case the third object clearly surpasses the expected outcomes of the teacher.

FINAL REMARKS

Both these students worked alone, and both used several technological tools at their disposal, but, possibly due to their different areas and backgrounds their outcomes exposes a proceptual divide. Raquel used a procedural type of thinking, with elementary procepts and started with a *relational* level on the SOLO taxonomy and ended with a *multi-structural* level, we might think as a regression, but one of the characteristics of procedural thinking is the refuge on algorithms and procedures well known without space for new knowledge that she started but was unable to process.

Mariana on the other hand, even with the aid of some simulations found on Internet could reproduce and explain all the processes evolved in her outcome, evidencing a proceptual thinking with some meaningful combination of elementary procepts to form a procept with a flexible combination of derived facts surpassing the barrier of proceptual divide.

Although the ongoing study reported in this paper is not yet closed, it already conjectures interesting results, not only on the model of analysis used to evaluate the outcomes but also on the evidences showing an emergent curriculum for pre-service teacher education (in this institution) based on the outcomes of this study.

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REFLECTIONS OF DEVELOPMENTS IN EDUCATIONAL TECHNIQUES IN THE DESIGN OF A NEW TEXTBOOK ON DESCRIPTIVE GEOMETRY

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The article discusses innovative methods in modernization of teaching descriptive geometry at the Faculty of Mathematics and Physics at Charles University in Prague. Our goal is to increase interest in studying classical and descriptive geometry primarily through 3D computer modelling. I have been seeking to establish a stronger connection between descriptive geometry and its practical application and the extension of descriptive geometry with knowledge of computer graphics and computer geometry. The integration of descriptive geometry with 3D computer modelling appears to follow as a logical step. In order to provide insight into more complex geometric problems and to increase the interest in geometry, I have integrated 3D computer modelling in my descriptive geometry lessons. I plan to use outputs from 3D computer modelling software into my new textbook on descriptive geometry for undergraduate students.

Keywords: descriptive geometry; 3D computer modelling; textbook on descriptive geometry

MOTIVATION AND PRACTICAL APPLICATION

Descriptive geometry (Paré et al., 1996; Pottmann et al., 2007; Robertson, 1966) represents an important area of classical geometry dealing with the representation of three-dimensional objects in two dimensions where 3D computer modelling and interactive software visualization can be applied with potentially significant impacts. Hence, the typical task in descriptive geometry is to represent three-dimensional objects on a two-dimensional display planar surface and to reconstruct 3D objects from the two-dimensional result of the projection. Descriptive geometry deals with those representations which are one-to-one correspondent. In order to gain deep understanding of descriptive geometry it is necessary to have knowledge of the fundamentals of geometry, the properties of geometrical objects in the plane and in the space, and their relations. This means that, in addition to geometrical projection, descriptive geometry should focus on special types of technically important curves and surfaces in engineering practice.

In general, geometry can be conceived as an independent discipline comprising various branches and it also forms the basis for many modern applications. The motivation for studying geometry can be found in building practice, engineering and construction practice, architectural and industrial design, production industries, export of real interiors and exteriors into the virtual worlds of computer games, digitization of real objects by 3D scanning, digital surface reconstruction from point clouds, replication of the shapes of real objects using 3D printing, computer graphics and many more, (Pottmann et al., 2007). The common basis of all these modern applications is the combination of geometric principles and knowledge. Applied methods are often based on elementary geometry.

The role especially of descriptive geometry in practice is irreplaceable in such branches in which correct visualization is crucial. All of the mentioned application fields are dependent on clear illustrations and visualization. Overall, geometry in the plane and in the space, i.e. the properties of geometrical objects and their relations, form a part of many modern and contemporary scientific fields.

Geometry represents one of the highly demanding fields of mathematical science which require logical thinking and which also strongly stimulates spatial imagination, (Hilbert, 1999). The study of geometry, and especially descriptive geometry, represents an ongoing challenge in terms of research and practice.

On the top level, the paper is organized into two parts. The first part explains possible novel methods of teaching descriptive geometry which include 3D computer modelling and interactive software visualization. This part is largely a summary of the existing concepts used in my lessons. The second part contains the main contribution of my work: the description of the upcoming textbook on descriptive geometry for undergraduate students. The conclusion is devoted to my future work and research in geometric fields.

MODERNIZATION OF TEACHING DESCRIPTIVE AND CLASSICAL GEOMETRY WITH 3D COMPUTER MODELLING

There exist professional graphics software applications and environments which provide the required user input tools, and speed up production and are commonly used in the process of designing, design documentation and construction for modelling and drawing, and generally throughout the entire design process (Farin et al., 2002).

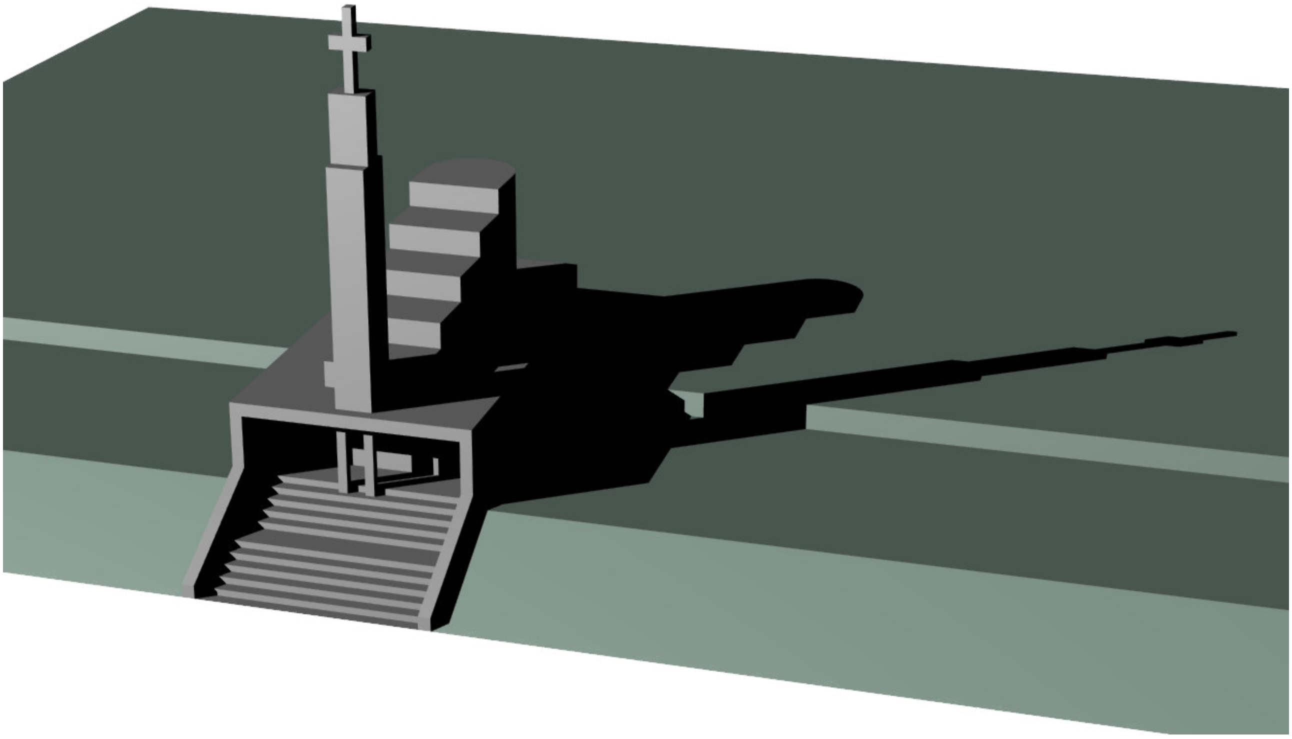

Figure 1: Example of 3D model created in Rhinoceros – the spatial situation of geometry of shadows on real object.

Similar software can be used in teaching traditional geometric subjects, including descriptive geometry. I have integrated 3D computer modelling in my descriptive geometry lessons at the

Faculty of Mathematics and Physics at Charles University in Prague. I work mainly with the *Rhinoceros* (*NURBS Modelling for Windows*) software which is a commercial NURBS-based 3D modelling tool, (McNeel, 1999), commonly used in the process of designing, design documentation and construction. I use Rhinoceros to create 3D models of geometric objects and situations in the space, (Surynková, 2013). It should also be noted that if we work with 3D modelling software, we can change the view of a designed object and see spatial geometric objects from another perspective which provides a clearer idea of the object. Example of spatial 3D model is provided in Figure 1. We also use Rhinoceros to draw up constructions in the plane. Example of computer drawing is shown in Figure 2.

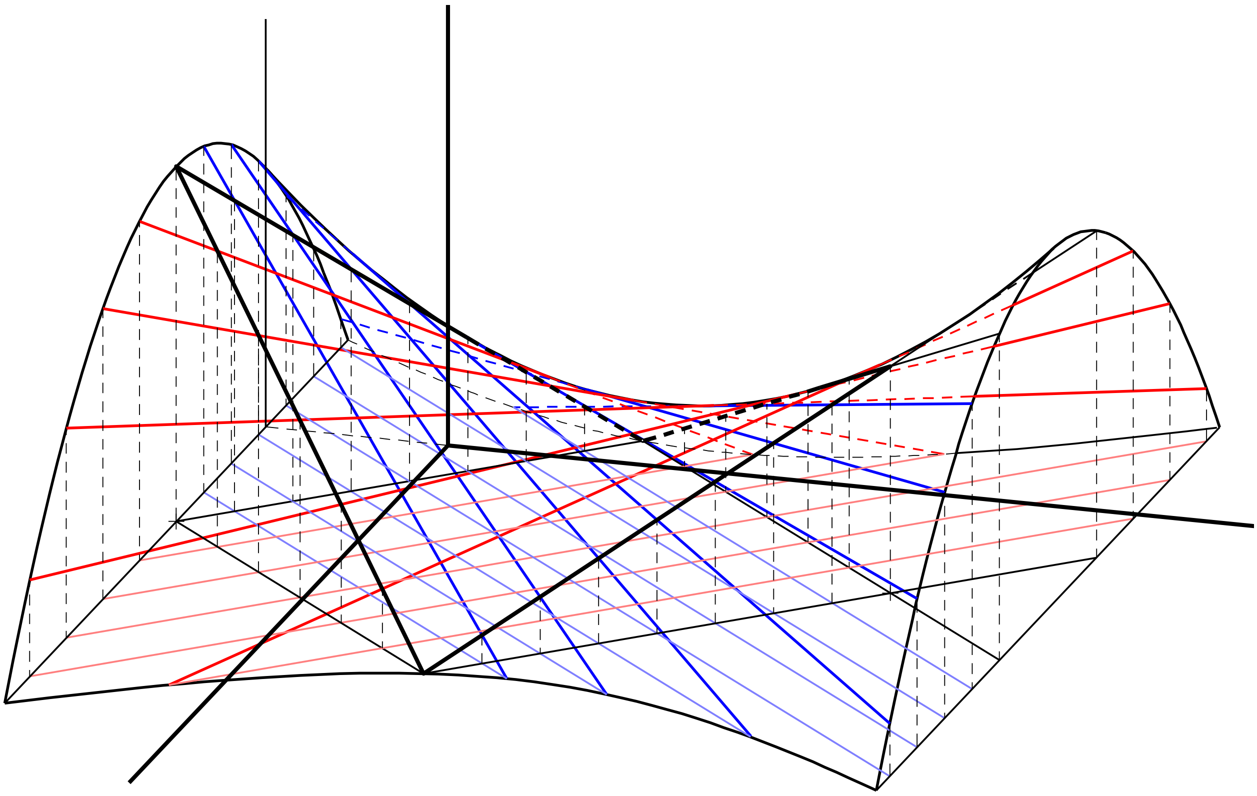

Figure 2: Example of computer drawing created in Rhinoceros – an orthogonal axonometric projection of hyperbolic paraboloid.

The study of descriptive geometry includes, both at secondary schools and colleges, sketching and drawing activities. We do not intend to abandon traditional hand drawing methods because computer drafting is not efficient in developing our skill and thoroughness. Drawing and sketching helps us develop our precision skills and patience and we rely on these tools when developing of our initial ideas and finding solutions to geometrical problems. Computer drafting is a modern auxiliary method which is also capable of yielding more precise results. The examples of hand drawings are provided in Figure 3.

Thus, I combine the both approaches to the teaching of descriptive geometry - the traditional descriptive geometry teaching methods and procedures (sketching and drawing activities) and modern computer-based experiments with digital modelling tools. As has already been pointed out, I use 3D computer modelling to create 3D models of geometric objects and situations in the space which can help my students understand geometrical problems in intuitive and natural way. I use these outputs during my lessons as illustrations of geometrical properties of studied objects. Moreover I show geometrical constructions in the plane and in the space using graphical software tools so that students can discover principles and proofs of geometrical theorems more easily.

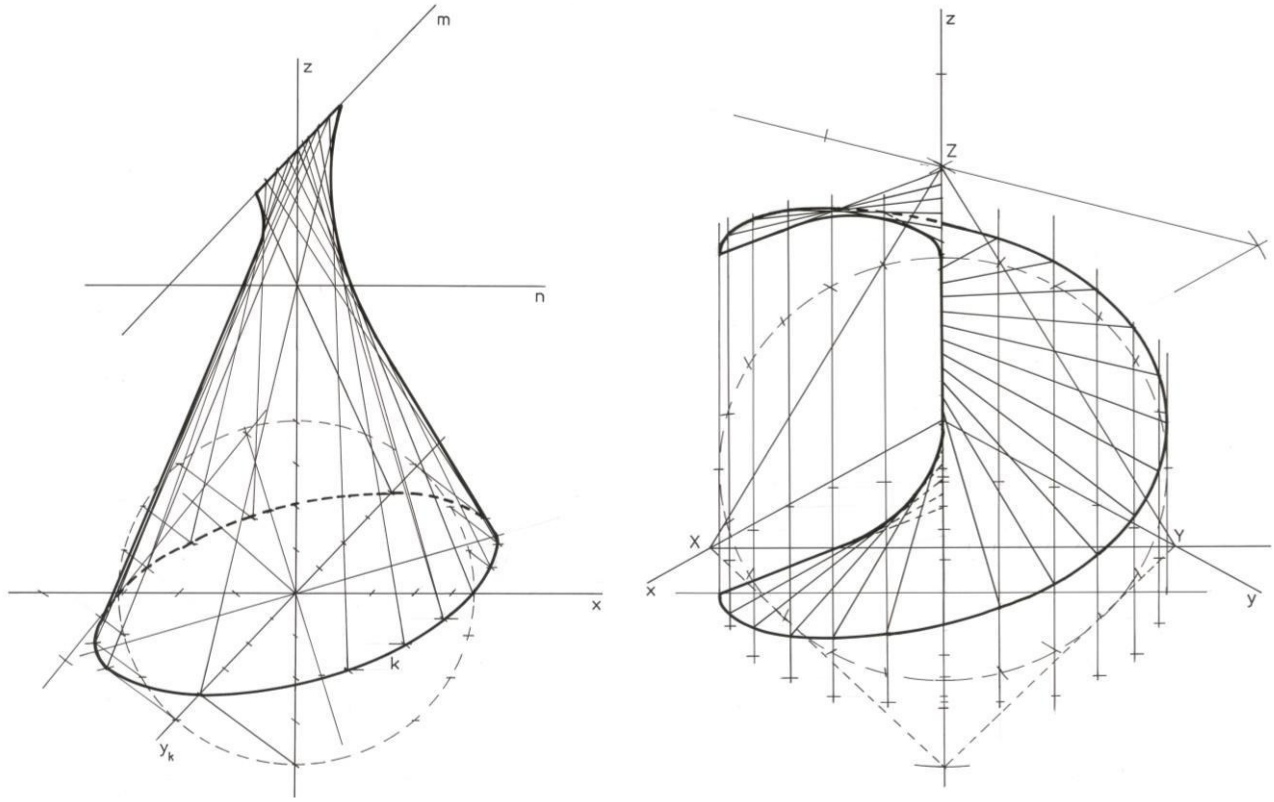

Figure 3: Examples of hand drawings – a parallel projection of ruled surfaces.

Students meet with 3D computer modelling within compulsory lessons of descriptive geometry and also can attend the seminars of applied descriptive geometry where practical applications of descriptive geometry and 3D modelling are mentioned and discussed. Students also use practically 3D modelling software during these lessons and seminars and can create themselves the outputs - 3D computer models and planar constructions.

It is not necessary to work only with Rhinoceros or with expensive CAD applications, which are common commercial 3D modelling tools used for computer aided design (*CAD*). As there is a wide range of inexpensive or free software applications for geometry and mathematics, students and teachers can use them. One of the most widespread free geometrical tools is a mathematics and geometry dynamic software GeoGebra. I use GeoGebra to create planar and spatial constructions and to demonstrate the proofs of geometrical theorems, (Surynková, 2014). Example of planar construction created in GeoGebra is provided in Figure 4. The great advantage of GeoGebra is the possibility to change dynamically the parameters of the designed geometrical objects. My students create their homework or seminar project using GeoGebra for example.

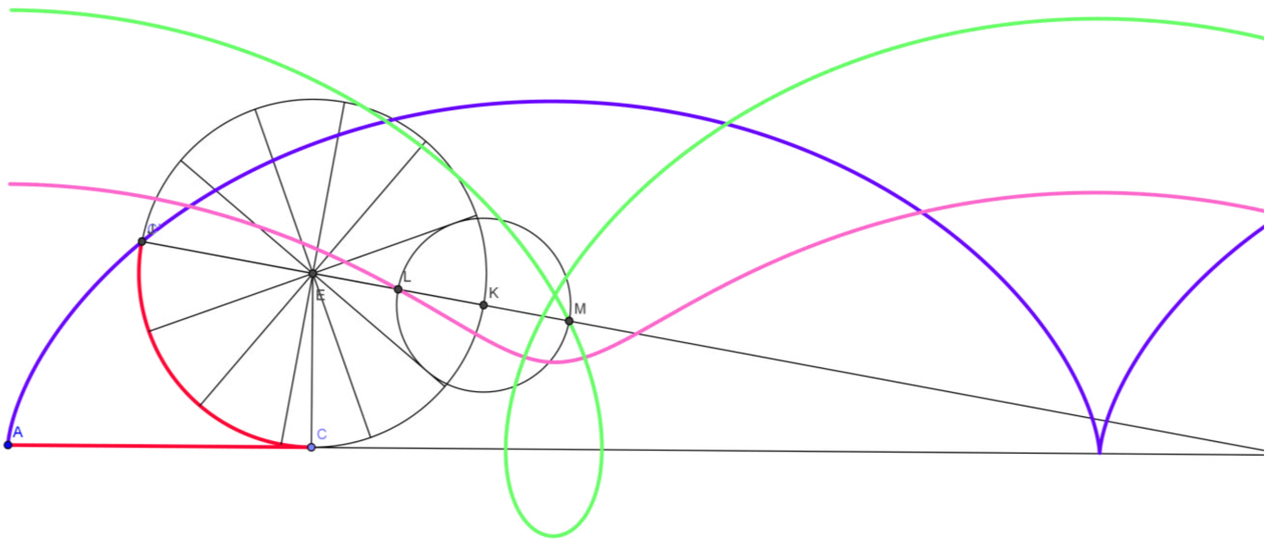

Figure 4: Example of planar construction created in GeoGebra - the paths (cycloids) of points which are obtained by rolling the circle on the straight line.

I have been gathering all of the aforementioned outputs from 3D computer modelling and computer drawings obtained during the preparation of descriptive geometry lessons to create electronic collections of examples as well as for the purposes of new electronic methods of study of materials that relate to various geometric topics. All of these outputs are published on the website http://www.surynkova.info/, (Surynková, 2015). The site is continuously updated and it is intended not only for my students but also for everybody who is interested in geometry (some of the links are in English). New study materials and examples are dedicated to geometric constructions; there are also 3D computer models, examples of students' works and many more.

Visualization of the geometric constructions, 3D models of geometric objects, situations in the space and 3D computer modelling and modern digital tools in general can be used to improve the teaching of geometry. 3D computer modelling strongly stimulates spatial imagination and helps students to understand geometric concepts practically. As it has been shown in my previous teaching experiments, I also find this approach very valuable from the practical point of view as it can demonstrate young students that today's information age practice can be nicely integrated with classroom teaching. Using computer software in classrooms prepares students for their future profession at the same time.

THE UPCOMING TEXTBOOK ON DESCRIPTIVE GEOMETRY DESIGNED WITH 3D COMPUTER MODELS

The use of modelling and graphics software in teaching geometry increases students' interest in the subject and ensures their active involvement in the lessons, which is evident from the reactions of my students and also from their interest in these issues when dealing with their seminar projects or bachelor and master theses. 3D computer modelling is also an efficient aid in innovating the teaching of geometry and achieving better results.

I have been seeking to establish a stronger connection between descriptive geometry and its practical application and the extension of descriptive geometry with knowledge of computer graphics and computer geometry. The integration of descriptive geometry with 3D computer modelling appears to follow as a logical step.

This paper explicitly addresses the content and the design of a new printed textbook on descriptive geometry which I have been working on. This textbook is primarily dedicated to geometric topics such as curves and surfaces, solids, their definitions and properties, their parallel and central projections and the geometry of shadows. The textbook is intended mainly for students of the Faculty of Mathematics and Physics of Charles University in Prague and the first edition is planned to be published in Czech. The textbook will be illustrated using 3D computer modelling and modern software visualizations. The important part of the publication is the collection of examples with solutions and examples for testing purposes.

Case study for textbook chapters

Let us now focus on some parts of the planned chapters in the upcoming book, and describe its expected design. I am currently working on the theoretical aspects of special groups of surfaces used in engineering practice. The book will define each regular geometric surface and introduce its properties. Let us, for instance, look at an example of a part dedicated to helical surfaces; the concept of the chapter is as follows.

First, a theoretical explication regarding the determination of helical surfaces is provided, accompanied with illustrations from the 3D computer modelling software. It is assumed that the source files of most pictures from the textbook are available on the attached removable media to allow practical exercises regarding the properties of the discussed surface or spatial situations directly in the software. The illustration of helical surfaces with a brief description is shown in Figure 5.

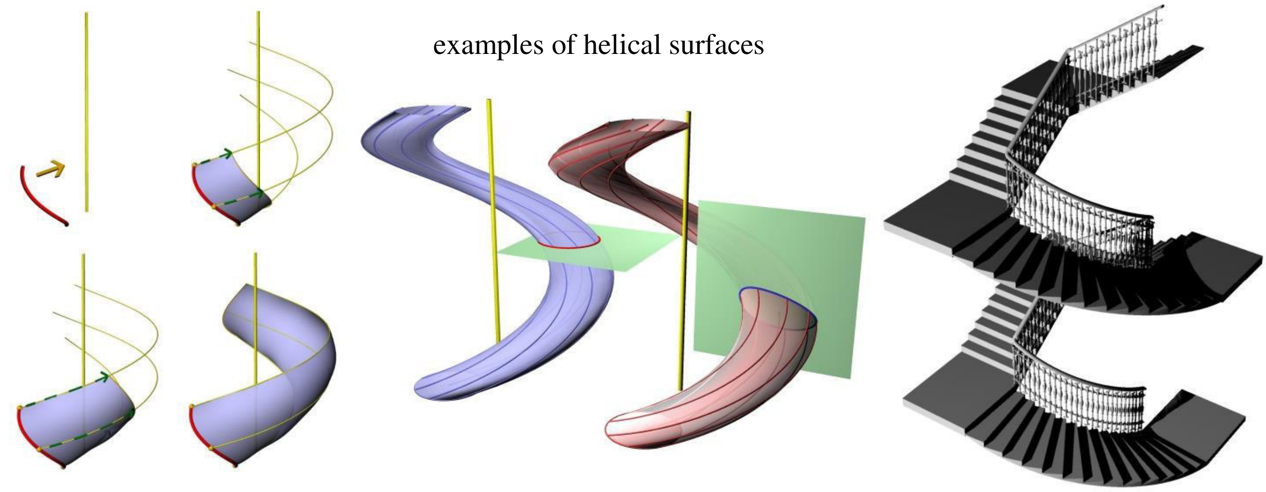

Figure 5: The illustration of helical surfaces.

The second part of each chapter is devoted to parallel and central projections of the studied surfaces, accompanied with a typical example including a detailed step-by-step solution and illustration. The typical task is to construct a parallel or perspective view (a two-dimensional image) of a particular surface. Figure 6 shows an orthogonal axonometric projection and central projection of a helical surface, defined by an axis and screwing segment line. The result of the projection and also the situation in the space are visible. Every illustration is made using 3D computer modelling.

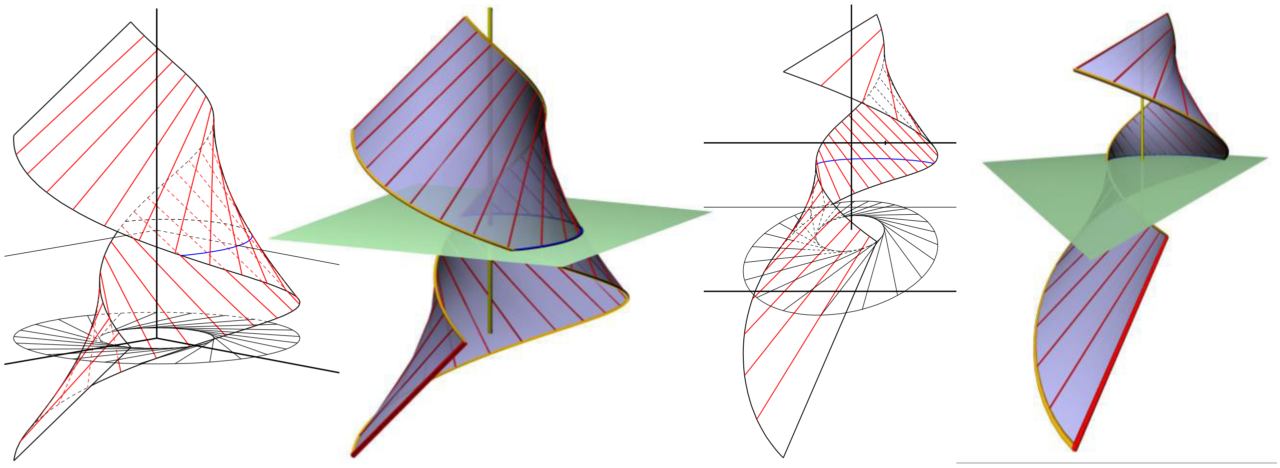

Figure 6: The results of the projections of helical surface and the situations in the space – an orthogonal axonometric projection (left) and a central projection (right) of the same helical surface.

The last part of every chapter comprises a collection of examples for exercising the properties of surfaces in various projections. Students can solve the tasks using 3D modelling or graphics software or they can draw the solutions by hand. When using software, it is necessary to construct the silhouette of the surface; if drawn by hand, the aim is to depict some of the important curves on the surface. In both cases, the result is a planar image.

An interesting additional feature of these examples is the possibility to model the surfaces in 3D modelling computer software in space. The spatial situation and principles of projection can also be demonstrated. The virtual model of the spatial situation and 3D virtual models of surfaces make a significant contribution to the development of spatial imagination. Some examples in the book are added in the form of 3D models on attached removable media and additional 3D computer models can be created in cooperation with my students, for example, as part of their theses.

CONCLUSION AND FUTURE WORK

Two areas are addressed in this paper – the possible *methods of innovation in teaching descriptive geometry* including 3D computer modelling and the creation of study materials and web support for descriptive geometry and *the description of the upcoming textbook on descriptive geometry* for undergraduate students.

The main aim is to improve and innovate the methods of teaching descriptive geometry by using 3D computer modelling and enabling connection with practice. It is planned to integrate the suggested outputs from 3D computer modelling software into my new textbook on descriptive geometry. In the future, it is envisaged to publish the textbook in English translation. Some parts of the textbook are also planned to be published on the Internet.

For the future work I am also considering to improve teaching of descriptive geometry also in other ways. It seems to be promising to intensify the extension of descriptive geometry with knowledge of computer graphics and computer modelling. Each regular geometric surface (and also auxiliary curves) in my new textbook will be described using mathematical equations. Then we can model these geometric objects in mathematical software.

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MATHEMATICS TEACHERS' INSTRUMENTAL GENESIS OF TECHNOLOGICAL MATERIALS

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Using the paradigm of activity theory, the central problem of this paper [1] *is the characterization of the processes through which teachers replicate, adapt, and improvise tasks of textbooks with use of technological resources (CD-ROMs and web portals). In other words, we seek to identify teachers' use of schemas in actions mediated by these technological elements. Two of us accompanied Portuguese secondary mathematics teachers in the assessment of learning tasks involving the use of new technological resources and the analysis of feedback of teaching performance after implementation in the classroom. This feedback was obtained from their peers, trainers and teachers' reflection on the actions in classes and occurred during the sessions of the training activities. The study shows that teachers plan coordinated tasks that integrate technology resources and apply them in classes adjusting them to the technological environment of their schools. However, some difficulties in interpreting feedback are revealed.*

Keywords: Technological resources, textbooks, teachers, instrumental genesis, documental genesis

INTRODUCTION

Since the 2000s graphing calculators became mandatory in Portuguese grades 10th-12th and gradually textbooks for all grades started to include several technological resources (CD-ROMs; web portals; applets built in dynamic geometry programs; programs in flash). To investigate what teachers do with these new curricular resources matters to the understanding of their professional development. This paper analyses how these curricular materials are incorporated into teachers' didactical work.

FRAMEWORK

The object of study is support by two dimensions: the *instrumental* related to the technological format of the resources and the *documentational* related with the work of teachers in the selection of task from the resources, textbooks and technological support.

In the instrumental dimensions, we discuss the concept of human activity us mediated by cultural artefacts, which are culturally, historically and socially produced and reproduced, by means of complex and multidimensional relationships (Engeström, 1999). Artefacts have possibilities for action that the user may or may not use. The creation and use of artefacts as tools for the production means that there are specific forms of human action. Primary artefacts are used directly in this production. Secondary artefacts are representations of actions modes with the artefacts (Wartofsky, 1979). The instrumentation and instrumentalisation process, from the instrumental genesis, associated with the construction of utilisation schemes in action mediated by artefacts (Rabardel, 1995) support the analysis teachers' action with the technological resources.

Supported in the work of Artigue (2002) focus on students, Drijvers and Trouche (2008) distinguished two utilisation schemes that Teixeira (2015) focus on teachers work associated with*:*

1) instrumentation, that relates to the management of the artefact when the student is initially confronted with the constraints and potentialities of the artefact that permanently condition his or her actions in order to solve a given problem — for example, turn on a calculator, adjust the contrast of a computer screen (focus in student) or distribute the number of computers available in the classroom for students (focus on the teachers), which they call usage schemas;

2) and instrumentalisation, oriented to perform specific tasks, and during which user personalize the artefact in ways that serve his or her purpose — for example, to study the limit of a function with the computer (focus on students), or create didactical exploration scenarios, focus on teachers which they call instrumented action schemas.

In the documentational dimension we discuss the interactions between mathematics teachers and curriculum resources.

To distinguish among these dimensions Hattie and Timperley's notion of *feedback* (2007) was used: "feedback is conceptualized as information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding. A teacher or parent can provide corrective information, a peer can provide an alternative strategy, a book can provide information to clarify ideas, a parent can provide encouragement, and a learner can look up the answer to evaluate the correctness of a response" (p. 81). They argue that feedbacks are one of the most powerful influences on learning and achievement. For these authors, effective feedback may reduce the gap between current performance and performance towards a goal or objective.

Pepin (2012) extend this concept to teachers' documentational work and distinguishes four levels of feedback: the task level (e.g. mathematical tasks characteristics), the process level (e.g. what do the tasks afford?), self-monitoring level (e.g. confidence about working with tool and mathematical tasks) and personal evaluation level (e.g. confidence to engage in further enquires).

DATA COLLECTION

In three in-service training workshops during the years 2009 and 2011 (totalling 95 sessions) led by two of us, teachers analysed the technological materials (CD-ROMs; web portals; applets built in dynamic geometry programs; programs in flash) that came with Portuguese mathematics textbooks from six different publishers. Voluntarily, 63 teachers from 24 basic and secondary schools participated in the workshops (table 1). These workshops may be attended by teachers as a means to progress in their career.

Table 1. Participant teachers per workshop.

Workshop A 2009	Workshop B 2009	Workshop C 2011	Total
21	22	20	63

Data resulting from the oral and written feedback documents produced by the teachers during their participation in the workshops was collected recording decisions and actions in three different moments: 1) during the analysis of the proposals contained in the technological materials, 2) during teachers activity as preparing tasks from textbooks or technological materials, and 3) after their actual application in classroom. Particular attention was paid to accounts of the activities developed by students, and the reflections on the teachers' own performance. Data also included written accounts of the sessions produced by the researchers.

DATA ANALYSIS

In the process of instrumental genesis (as the primary component of technological resources) Teixeira (2015) distinguishes two kinds of schemas:

1) *use schemes* resulting from the analysis of the possibilities and limitations of technological resources. These can be detected when teachers describe their contents. For example, when asked to describe the technological resources from a textbook for 9th grade, one group of teachers simply listed the contents (index) of the CD-ROM:

> Programmatic contents of the 9th grade, suitable to the current program; Didactic approaches: Introductory videos for all content; Interactive explanation; Interactive exercises; Tests at the end of each chapter; Global test. (Group 2, Workshop A)

This usage does not imply a teaching practice, but just an observation resulting from the first contact with the materials.

2) *instrumented action schemes* that result from the analysis of the possibilities and limitations of technological resources. These occur when teachers describe the contents and also show how they can build with them a didactic exploration scenario focused on the prescribed curriculum, the technology available in their schools, the characteristics of students in their classes, or a combination of at least two of these. The following is an example of an instrumented action scheme centred in technology. The teacher summarises the actions develop with the technological resources and explains how to overcome school limitations.

> In addition to the manual, I usually use the web site [of the publisher] to the view animations related to different content and some interactive applications. While existing animations are sort of a show-off the site, I usually use them to introduce the topics and then discuss in large group.
>
> I also have access to the eBook, where I can view the manual in electronic format, and flip through the book, looking for resources in the different pages. The projection of the manual is useful because I can use graphics or pictures to explain something related to what the students should notice and explain to students how to solve the exercises.
>
> To work, for example, with dynamic geometry software, at school there are some laptops that can be requested, but not always work correctly. Alternatively, I ask students to bring their laptops. (Group 4, Workshop C)

In the process of documentational genesis (the secondary component of technological resources) Teixeira (2015) distinguishes into:

1) *use schemes* that result from the construction of didactic exploration scenarios whose components are: available technological artefacts, the mathematical situation with the produced artefact (a task) and instrumental orchestration (Drijvers et al, 2010).

2) *instrumented action schemes* resulting from teachers' reflection on their teaching performance, including technological artefacts and artefacts produced.

To study the artefacts produced by teachers, the characterisation proposed by Brown (2009) was used, which defines three forms of interaction between teachers and curriculum materials: 1) offloading, the teacher just copies the proposed curriculum materials; 2) adapting, the teacher follows the suggestions in the course material, but adapts them to his or her context and preferences; 3) improvising, the teacher does not conform the suggestions made by the curriculum materials and follows his or her own ideas.

Table 2 summarizes the distinct ways in which instrumentalization was conducted by several groups of teachers using Brown's types of interaction in each workshop. Teachers organized into groups: either by school or by the same textbook or by the same mathematical content to be taught.

Table 2: Types of interaction between groups of teachers and curriculum materials by workshop.

	Workshop A	Workshop B	Workshop C	Total
Offloading	6	11	7	24
Adapting	5	0	2	7
Improvising	0	0	5	5
Total	11	11	14	36

Some teachers developed limited instrumented action schemes and their reflections about the didactical exploration scenarios were essentially descriptive of the actions developed in the classroom rather than reflective teaching experiences. In general many teachers chose a task from the textbook or from the technological resource and only in a few cases adapted or improvised a task.

Instrumented action schemas that occur in the instrumentalization process go further. They result from the analysis of the didactical strengths and limitations of the technological resources, and are imbued with actions centered on their use in class. Here, teachers show how they can build didactical exploration scenarios adapted to the prescribed curriculum, taking into account the resources available in their schools or the characteristics of students in their classes.

These schemas were gradually built by some participants. The following is an example of an instrumented action scheme. The teacher summarizes the actions developed with the technological resources, explains how to overcome school limitations, and hints at several didactical exploration scenarios. This group reflected on the didactical exploration scenario orchestrated and suggested how future uses of the materials should be conducted. The intention was to make 11th grade students listen to CD's oral presentations of mathematical content and things did not work as expected.

> As the colleague [teacher advisor] completed the first phase of the lesson, students expressed desire to revisit the CD because they did not understand anything. They were told they could handle the CD at will. Some students individually headed to the desk of the teacher, where the

> computer was located, and listen the CD [oral presentations] again. They became frustrated as they "could not understand anything".
>
> The teachers tried to calm down the students by encouraging them to focus on the presentation of the video, but once the work sheet was distributed, they literally panicked. Their stress was obvious. They could not even use the simple rule of three to convert degrees to radians. (…)
>
> This reaction from students was extremely surprising to us. (…)
>
> We met in the back of the room and decided to proceed with the next topic, the representation of an oriented angle, to check if they would understand it better. After viewing the presentation of the CD, students were able to answer correctly to their worksheet. But once we started watching them more closely they expressed concern about the previous content. So we decided that it would be better for one of us to present under the definition of radian and the rule of for three simple conversions from degrees to radians independently of the CD. (Group 10, Workshop A)

The limitations of the technological materials were not foreseen by the teachers and became apparent as they were put to use in class. Feedback from fellow teachers during the workshop was instrumental in highlighting these shortcomings.

In another example, teachers question the limitations of the materials and try to overcome them with insightful didactical exploration scenarios:

> As the materials do not present any interactive task, I will use the CD [of publisher B] - Mathematics 9 years to analyse student learning and the educational gains with this type of resource for a math class. For further study of this technological resource in student learning, I want to examine the work of a group of three students who will study some topics of the CD in special extra-curricular lessons and that will present the concepts in a [regular] mathematics class.
>
> The CD has some interactive videos and activities that will be explored with and by the students in units "Geometric solids. Areas and volumes" and "Circumference and polygons. Rotations". (Group 10, Workshop C)

This group also reflected upon the strengths and limitations of technological materials:

> The use of CD-ROM benefits student learning, when the presentation of concepts requires a visual support. However, the contents of the CD have proved insufficient whenever it was necessary to apply these concepts, since students were unable to solve the exercises, either on the CD or on the textbook without my support. Only displaying the CD has been clearly insufficient in obtaining any sort of learning. (Group 10, Workshop C)

> To work, for example, with dynamic geometry software, there are some laptops that can be booked in the school, but sometimes they not in good condition. Alternatively, I ask students to bring their own laptops. (Group 1, Workshop C)

CONCLUSION

The productions involving both usage schemas and instrumented action schemas show that teachers have developed particular schemas that allowed them to use the transformation of a technological artefact into an instrument. This process is not identical for all teachers. Some teachers developed limited instrumented action schemas. Some reflections about the didactical exploration scenarios in practice were essentially descriptive of the actions developed in the classroom rather than reflective teaching experiences.

However, all the teachers produced utilisation schemes, which showed us what happened:

Instrument = Scheme of utilisation + Artefact

The two processes of instrumental geneses (instrumentation and instrumentalisation) occurred; teachers took hold of the resources and evaluated their constraints and potentials, and at the same time integrated the materials in specific didactic settings within the context of their school. The transformations of the action of the artefact towards the teacher and towards the action of the teacher in relation to the artefact are complete. The two processes of instrumental geneses happened.

But it was not easy to see:

Document = Scheme of utilisation + Resource

In general teachers chose a task from the textbook or from the technological resource and only in a few cases adapted or improvised a task.

The contents of technological resources, their format, the characteristics of students, the prescribed curriculum and school technological facilities, in particular, the number of computers per student and the distribution of the resources in the class are considered in the schemes use by the teachers.

NOTES

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HOW TO PROFESSIONALIZE TEACHERS TO USE TECHNOLOGY IN A MEANINGFUL WAY – DESIGN RESEARCH OF A CPD PROGRAM

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The German Centre for Mathematics Teacher Education (DZLM) together with the Ministry of Education is in charge of developing, delivering and evaluating a long-term continuing professional development program (CPD) with respect to graphics calculators (GC). In this paper we describe the design of the CPD program and two associated research studies. The studies aim at examining conditions which must be considered when designing a CPD program, and at investigating the effects caused by the CPD program on teachers' beliefs and classroom practices as well as on students' competencies. We developed a questionnaire to measure teachers' beliefs related to GC and a survey about the integration of the GC into classroom practice. Furthermore, an achievement test was constructed to measure students' competencies that focus on areas where the literature expects the GC to be relevant.

Keywords: continuing professional development, CPD, graphics calculator, efficacy

INTRODUCTION

Integrating the graphics calculator (GC) in classroom practice is a challenge for teachers (Clark-Wilson 2014) and agreement exists, that teachers need professional development and support (Barzel 2012) in order to make appropriate use of the GC. In this paper we describe a research project aimed at developing, delivering, and evaluating a long-term continuing professional development (CPD) program for integrating GCs in mathematics teaching.

The project is situated within the context that since the beginning of the 2014 school year the use of GCs in upper secondary school is compulsory in North Rhine-Westphalia, the biggest German federal state. Till this time most of the teacher did not use graphic technology in their teaching although it was requested from the curriculum. But as long as there have not been centralized final examinations lots of teachers did not follow this request. Nowadays North Rhine-Westphalia has established together with other German states centralized final examinations. The German Centre for Mathematics Teacher Education (DZLM) together with the ministry of education are collaborating to support teachers to master this new challenge by designing a long-term professional development program which directly aims at teachers in upper secondary classroom. The research project comprises three parts: the design of the program, investigating the conditions for this CPD program on teachers' and students' level and research on the efficacy of the program.

We first give a brief overview of the theoretical framework before elaborating in more detail on research questions, methods and design of the CPD program. Finally, we present first empirical results and end with a prospective view.

THEORETICAL FRAMEWORK

First of all our theoretical framework comprises the idea of design research for a CPD program (Gravemeijer & Cobb 2006; Swan 2014). On a second layer we focus as theoretical frame on criteria concerning effective CPD as well as on the state of the art in the field of a meaningful use of the GC in mathematics teaching of elementary calculus.

Criteria for effective professional development

A lot of research has been conducted to reveal possible effects of CPD programs for teachers (e.g. Timperley et al. 2007) and agreement exists that these effects occur on different levels (e.g. Kirkpatrick & Kirkpatrick 2006). Whilst there is a consensus in the research community that different levels of effects exist, their number varies: Whereas Guskey (2000) defines five levels of effects, Lipowsky & Rzejak (2012) distinguish four levels of effects: Level 1: *Participant's reactions*, level 2: *Participant's beliefs and professional knowledge*, level 3: *Participant's use of new knowledge and skills in the classroom*, and level 4: *Student learning outcomes*.

Guskey's (2000) additional level describes "Organization Support and Change" and is positioned between the second and third level in the hierarchy above. Guskey's extra level specifies whole school changes as a result of a CPD initiative. Since our study does not focus on whole schools but on individual teachers and their students we chose to orient on the four level model.

From the literature various characteristics can be derived as criteria of efficient in-service teacher training (e.g. Loucks-Horsley et al. 2009; Garet et al. 2001). On the basis of a review of the current research with a special focus on mathematics, six design principles of professional development have been generated by the DZLM (Barzel & Selter 2015; Rösken-Winter et al. 2015): (a) *Competence-orientation:* Focusing on the participants' competencies which one want to procure or improve. This means mathematics content knowledge and skills on the one hand as well as mathematics pedagogical content knowledge and skills on the other hand. Furthermore, these expectations have to be transparently communicated. (b) *participant-orientation:* Centering on the heterogeneous and individual prerequisites of participants. Moreover, participants get actively involved into the CPD instead of a simple input-orientation. (c) *stimulating cooperation:* Motivating participants to work cooperatively, especially between and after the face-to-face phases, e.g. in professional learning communities (e.g. Weißenrieder et al. 2015). (d) *Case-relatedness:* Using examples which are practically relevant and which participants can identify with. (e) *Various instruction formats*: Switching between phases of attendance, self-study and e-learning. (f) *fostering (self-)reflection:* Continuously encouraging participants to reflect on their conceptions, attitudes, and practices. When taking these six principles seriously, this yields to the necessity to realize CPD initiatives in long-term formats as well (Rösken-Winter et al. 2015).

The benefits of graphics calculators

Mathematics educators and authorities believe that classroom practice should shift from computation to an emphasis on conceptual understanding and problem solving (Simonsen & Dick 1997). Research indicates that technology like GCs can play an important role in achieving this goal. Studies have shown that the use of GCs can improve problem solving and conceptual understanding (Ellington 2006) which holds in particular for the calculus classroom where various representations such as *graphical*, *numerical* and *symbolic* play an important role. Switching between these representations is supported by technological tools (such as a GC) and can improve students' conceptual understanding of functions (Penglase & Arnold 1996). Furthermore, the GC can foster the ability to connect multiple representations of algebraic concepts (Graham & Thomas 2000) and can support an increased understanding of a dual approach to problem solving, using both symbolic and graphical solution methods (Harskamp et al. 2000). Moreover, a GC can be a beneficial tool when promoting discovery learning in the classroom (e.g. Barzel & Möller 2001).

However, Kissane (2003, p. 153) pointed out that "Availability of technology is not by itself adequate, of course, to effect changes in the mathematics curriculum. A crucial mediating factor is the teacher, and curriculum developers ignore the real needs of teachers at their peril. Mathematics teachers need professional development directly related to graphics calculators if they are to be the main agents of reform, and ultimately directly responsible for whatever happens in the classroom." Hence, it is important to apply the characteristics of effective CPD as outlined above to the special case of GC to design an effective CPD program.

RESEARCH QUESTIONS AND DESIGN OF THE STUDY

The whole project addresses the following three research questions:

1. How can an effective CPD program for GCs be designed?
2. Which conditions and criteria have to be considered for designing a CPD program with respect to GC with a focus on elementary calculus?
3. Is the designed CPD program effective?

The first research question leads to a theoretical-based design of the program which should be redeveloped in further cycles of designing and researching. The project started in spring 2014 with a first draft for a concept and material for the CPD program. The CPD program is offered at three sites across North Rhine-Westphalia with 30 teachers participating at each site. It is structured into four modules spread over a half year period. Every module consists of a face-to-face one-day course, elements of blended learning, exchange in professional learning communities and phases of classroom practice between the modules.

To answer the second and third question we chose a classical pre-test-treatment-post-test design with two nonequivalent groups: Teachers participating in our CPD initiative (EG: experimental group) and those who don't (CG: control group). Out of 90 participants of the initiative 40 volunteered to take part in our research. The control group consists of 147 teachers, who were enlisted by a circular letter and an associated website. All teachers taught tenth grade students. We collected data from the teachers (EG: 40, CG: 147) in the program, as well as from their students (EG: 554, CG: 2585).

Figure 1 provides an overview over the whole project. In this paper we only focus on research question 1 and 2 and elaborate in more detail on the design of the CPD program and the methods used to answer the research questions.

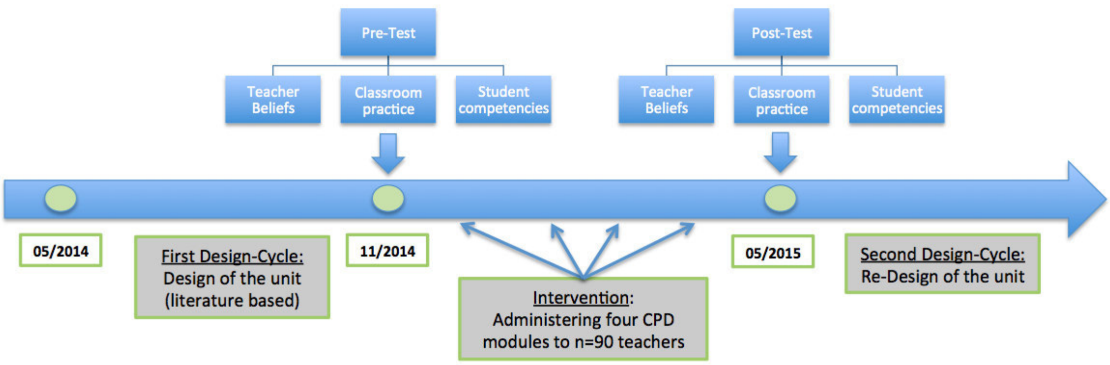

Figure 1. Project overview

Designing the CPD program

In the first design step the CPD program was developed by a group of researchers and experienced practitioners. The design was clearly driven by the design principles of effective CPD of the DZLM and by research results and experiences in the field of teaching calculus with technology (Zbiek et al. 2007; Barzel 2012). In the following we describe how the design principles were realised within the CPD program.

(a) *Competence-orientation:* The CPD course covers different dimensions of teachers' competencies, aiming at four main goals. The teachers should be able to use GCs in a flexible way, design tasks integrating GCs, organize the classroom in a technology based environment and develop appropriate formats and tasks for assessment with GC. The four modules were dedicated to these main aims: *Introduction into the work with GC – Designing tasks with an integrated use of GCs – Classroom organisation in a technology based environment – Assessment*. The design of these topics was based on research results. For example in the field of pedagogical content knowledge about functions, relevant concept images (vom Hofe 1995; Büchter 2008) and mathematical representations were presented to describe in detail the content. Systematic evidence is accessible on typical student errors, pre- and misconceptions, and ways of dealing with them effectively in mathematics lessons (Hadjidemetriou & Williams 2002; Barzel & Ganter 2010). All this was communicated and used for designing tasks and analysing students' solutions. The goals were made transparent for all participants, thus enabling teachers to clearly see the relation to their own teaching practice and increase their motivation while attending the program.
(b) *Participant-orientation:* First of all participant-orientation was ensured through a preliminary questionnaire regarding the teachers needs (with respect to content and didactical issues). All tasks used in the course are created in a way that they allow an immediate use in the classroom. Accompanying material and information about the task outline, possible solutions, typical errors and misconceptions, an idea how and where to integrate the task in the learning process and the relevant role of the technology. Furthermore, at the end of each course participants are actively involved in giving recommendations for content and methodology that should be included in the following meetings.
(c) *Stimulating cooperation*: Cooperation was especially stimulated by initiating professional learning communities with teachers from one school or neighboring schools. During the courses participants' work collaboratively within their professional learning communities on examples relevant for the classroom and discussing how to best implement them.
(d) *Case-relatedness:* All modules relate to practical aspects by discussing ideas based on the practical experiences of the teachers. Specific student results and examples are brought into the courses by the participants which form both a starting point for discussion and a context for application.
(e) *Various instruction formats:* Various instruction formats are used throughout all courses to ensure active participation. The CPD initiative includes phases of attendance, self-study and e-learning. Input, practical try-outs and reflection phases are alternating across the course.
(f) *Fostering (self-)reflection*: Participants are continuously encouraged to reflect on their conceptions, attitudes, and practices. Furthermore, participants are also encouraged to engage in self- and collaborative reflection on covered topics / material and possible transfer into their own classroom as well as on their own teaching or training practice.

Conditions and efficacy

The evaluation and research on the program was split up in two parts – conditions and efficacy of the program – both on teachers' and students' level.

On teachers' level we investigate teacher beliefs about mathematics' nature and the teaching and learning of mathematics as a key dimension of teachers' epistemological beliefs which can have profound implications on their classroom practices as well as on student performance (Stipek et al. 2001; Staub & Stern 2002). To determine these beliefs we used a test at the beginning of the CPD program with a set of 14 items from the TEDS-M study (e.g. Blömeke et al. 2014). Besides these general epistemological beliefs about mathematics, it is clear that beliefs related to the use of the GC have a profound impact on classroom practice (Molenje 2012). Teacher beliefs regarding the GC were measured using a questionnaire (Rögler 2014), which consisted of 23 items with Likert-type forced responses on a five point scale with 1=Strongly Disagree and 5=Strongly Agree. The questionnaire covers beliefs about the advantages of GC usage as well as common beliefs about disadvantages of the GC. For the advantages of the GC the following scales were used: (a) *beliefs regarding the connection between GC usage and discovery learning*, (b) *beliefs about the support of multiple representations through GC*, (c) *beliefs that the GC supports shifting teaching away from computational focus*. The scales referring to the disadvantages of GC usage were: (d) *beliefs about a negative impact of GC on basic computational and pen & paper skills* and (e) *beliefs about the GC and time constraints*, as there is a general concern that there is not enough time to cover the technology and the required curriculum, (f) *beliefs that the GC supports press & pray strategies*, which means students rely heavily on technology use without conceptual understanding. The last category deals with (g) *beliefs about whether students must master concepts and procedures prior to calculator use*.

To measure changes in classroom practice a questionnaire was administered covering the following categories: (a) *use of the GC for modelling tasks*, (b) *use of the GC for discovery learning*, (c) *use of the GC for problem based learning*, (d) *use of the GC as a graphing device*, (d) *use of the GC for multiple representations in the context of functions*, (e) *use of the GC as a checking device*. Additionally, we included a category covering the discussion of limitations of the GC in the classroom. Each category was covered by several items specifying the particular category. Since it is known that survey data is well suited of describing quantity but not as suitable for describing quality (Mayer 1999), the survey focused merely on the frequency teachers used the GC in these situations.

On the students' level we constructed a pre-test and post-test to measure competencies of students. The pre-test consists of 14 items on linear functions, quadratic functions and quadratic equations as relevant preliminary knowledge for the new content covered in grade 10, the first year of the upper secondary school. The post-test focuses on differential calculus since this is the main content in grade 10. Both tests are connected via anchoring items, which are identical items which appear at both times of measurement. On a more general view on competencies, we tried to cover all concept images of functional thinking (e.g. Büchter 2008) and demanded the ability to switch between multiple representations of functional relations. Outcomes were measured by a simple raw score which was the number of items solved correctly by a student.

FIRST RESULTS

As the whole program is an ongoing process, we can only report few first results on professed beliefs and student performance in the pre-test. For the beliefs we focus on the scales (a), (d) and (g), covering beliefs about discovery learning and GC, computational skills and GC and beliefs whether students must master concepts and procedures prior to calculator use. Reliability of the scales were good with Cronbach's alpha .88, .86, and .92, respectively.

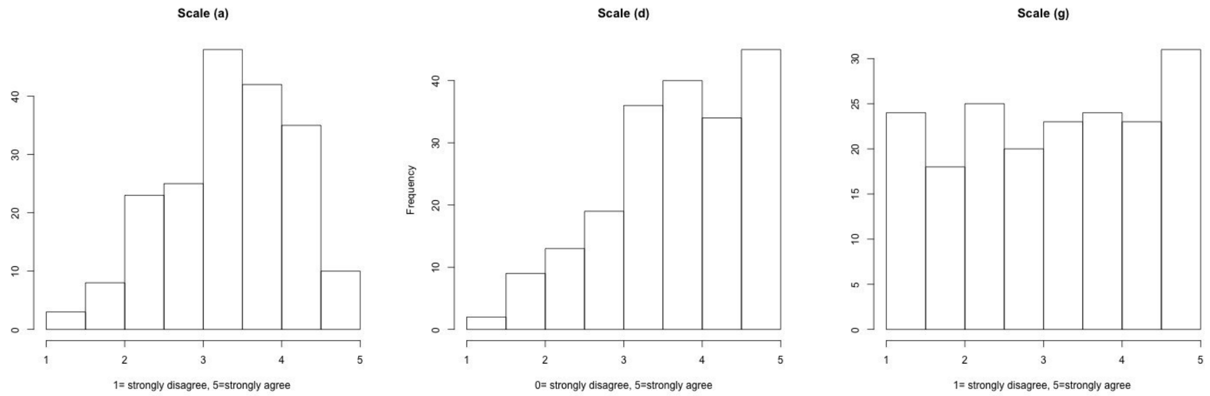

Figure 2. Histogram of the scales (a), (d), and (g), respectively from left to right

As it can be seen from the left histogram in figure 2, a large fraction of teachers belief that the GC can be a beneficial tool to support discovery learning. However, there are 18% of teachers with an average lower than 2.5 on this scale and hence do not share this belief. The middle histogram in Figure 2 reveals a clear concern of most of the teachers that pen & paper skills might be inhibited by the use of the GC. The right histogram in Figure 2 shows that teacher beliefs on (g) are quite heterogenous with a large number of teachers having quite extreme views to both sides.

The first student achievement test revealed some misconceptions and was able to quantify these. Figure 3 shows one item where option (b) represents an error Clement (1985) calls "treating the graph as a picture" which means "making a figurative correspondence between the shape of the graph and some visual characteristics of the problem scene". This option was chosen by 17.7% of the students. Furthermore, we also implemented items to diagnose a misconception called the "illusion of linearity" (De Bock et al. 2007) which is characterized by an improper linear reasoning in situations or processes which are nonproportional. We discovered that 9.5% up to 25.8% of the students (depending on the particular item) showed this misconception when graphing nonlinear processes. When a two-dimensional object is uniformly scaled and students should describe the behavior of the object's area, this amount was even higher: 75.2%.

OUTLOOK AND DISCUSSION

The preliminary results make clear that it is of crucial importance to take the preconditions with respect to teachers and students seriously. Results of the empirical study on prevalent beliefs should be included in the CPD course to initiate discussion about the different beliefs. When introducing concepts and content in the CPD program the teacher educators have to be aware of the beliefs towards the different aspects of GC and should choose methods to actively engage participants in reflecting on these beliefs. In addition, student competencies have to be considered when designing a CPD program in order to show teachers in detail where possible misconceptions are and how the integration of GCs can support to overcome these misconceptions. Further data analysis will focus on investigating connections between professed beliefs and classroom practice. After administering the post-test results on the efficacy of the CPD program can be obtained. This could give valuable insights whether teacher beliefs, classroom practice and student competencies have changed and which areas might be most affected. Based on these empirical data a redesign of the CPD program will take place with focus on integrating the empirical findings in content, methods and materials of the CPD program.

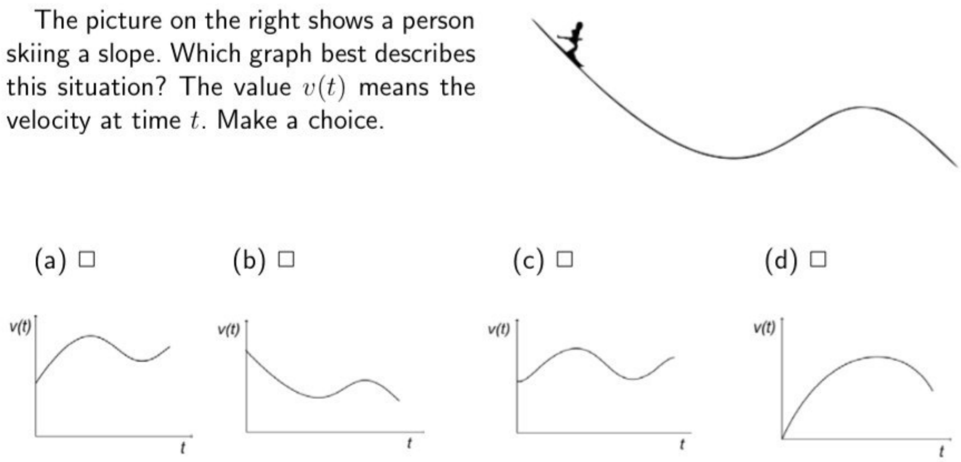

Figure 3. Example item for revealing the misconception "treating the graph as a picture" (translated, cf. Nitsch 2014)

NOTES

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INTERACTIVE RESOURCES FOR AN ACTIVE DESCRIPTIVE GEOMETRY LEARNING

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The author intends to promote a debate on the need to reorient Descriptive Geometry teaching practices in Portuguese High Schools, so that better responses to the present requirements are achieved, aiming so to improve students' capacities to understand and represent geometric concepts and its relations in three-dimensional space, through a better comprehension of what they represent.

To exemplify the benefits of exploring digital tools with educational purposes, the author presents some interactive resources created with GeoGebra, Rhinoceros and Grasshopper to complement Descriptive Geometry teaching, in order to assist the learning process from the student's perspective and illustrate the potentialities these software can offer to construct educational resources and expand teaching practices.

Keywords: Learning Styles, Descriptive Geometry Teaching, Dynamic Geometry Software, 3D Modelling software, Grasshopper.

Learning styles

Several studies report that some students react best to certain teaching methods than others and that learning is more elective and everlasting when subjects are approached in different ways and, particularly, when teaching strategies address different knowledge dimensions. The investigations of Richard Felder and Linda Silverman and, later, Richard Felder and Barbara Soloman with engineering teachers and students led them to define a classification scheme to understand the "preferences and tendencies students have for certain ways of taking in and processing information and responding to different instructional environments" (Felder, 2010, p.4). According to each personality, the reaction to a particular teaching strategy or even the subject involved, different students receive and process the information differently from others, the authors concluded. Aiming to recognize the preferences of the students in perceiving the information, they defined the following learning styles, which do not necessarily correspond to opposite categories, but to intuitive skills or intelligence levels that may even coexist in the same person:

> "Active and Reflective Learners: Active students learn best if given the opportunity to act and interact, that is, by doing something and discussing, applying and explaining it to other colleagues. Reflective students learn best if they are given the opportunity to think over and ponder, calmly and introspectively, upon the information received.
>
> Sensing and Intuitive Learners: Sensing students like to learn facts, data, principles and theories with a direct relation to situations occurring in real life, while intuitive students prefer to discover possible relationships between these facts (…).
>
> Visual and Verbal Learners: Visual students learn best if they can visualize images, demonstrations, graphs, drawings, pictures, movies, timelines, etc., while verbal students learn best through discussions and verbal explanations (anyone, in fact, will learn best if the subject is presented visually, because everyone is mostly a visual learner).

> Sequential and Global Learners: Sequential students prefer to learn subjects gradually and orderly sequenced, so they can find solutions to the problem, while global students prefer to absorb them almost randomly (…) so that the whole problem makes sense." (Viana, 2014, p.48).

Thorough investigations led the authors to the conclusion that traditional teaching methods do not address the majority of student's population, because they tend to privilege a passive behaviour from students, mostly focused on sitting steadily, taking notes and memorizing the information that in class is regularly presented in written form, through school books or in the classroom board.

Houghton (2004) also points out the emergence of active teaching methodologies in order to sustain a deeper learning from the students' perspective, as opposed to more traditional teaching methods, that support attitudes short in creativity and devoid of interaction. The educational strategies directed towards a deeper learning must necessarily include an active role for the students in a way they are able to understand the subjects taught by interacting with, sidestepping from the surface learning traditional teaching methods endorse. As Houghton points out, working abstracts concepts without a strong understanding of its basic principles will only contribute to a shallow and frustrating understanding of concepts (in his own words, to a surface learning), in which students memorize solutions for the complex problems that they are not able to understand.

Although the multiple learning styles theory in itself is open to debate, because some refute it as irrelevant (for lacking scientific data to certify them properly) while others consider it valid, it may be regarded as basis for a renewed conceptual framework that, according to "A good inventory to identify students learning styles" (2015), may help teachers "in understanding how students learn" inspiring "the pedagogical discussions surrounding learning strategies and instructional modes".

Teaching strategies for an active learning

> "In order to continue to use a new technology for doing mathematics we have to learn to use it in ways which transform mathematical activity, enabling us to do things which would not previously have been possible." (Sutherland, 2005, p.47)

To assist teachers to better adapt its teaching methodologies to the different learning styles found in the classroom, Felder and Silverman proposed some "Teaching Techniques to Address All Learning Styles" (Felder & Silverman, 1988, p.680). For the sake of conciseness, we will focus solely in one of the teaching techniques that can addresses to various learning styles and that, supported by the majority of subsequent investigations on these matters, best suits our train of thought: "Use computer-assisted instruction" (Felder & Silverman, 1988, p. 680).

As we will try to explain, the proficient exploration of digital technologies (and particularly, for the subject depicted, specific software dealing with geometric concepts and 3D modelling) can complement everyday educational practices, through innovative methods with a huge potential for broadening teaching methodologies that may inspire more consisting learning outcomes. If conceived to complement traditional teaching methods and well oriented, the use of computers in the educational context can attend to many different learning styles, not only for its visual attractiveness and interactive possibilities (which may increase the motivation for active, reflective, visual and sensing students to learn) but also because they can effectively support teachers practices.

Jaime Carvalho e Silva (2014, p. 3) sustains that schools should offer students the new tools that our technology in rapid evolution brings to our everyday life and the professional activities that "students will find someday in their adult life". Joel Klein, cited by Carlo Rotella (2013), states that if the use of the computer in the classroom is "not transformative, it's not worth it.", for it "can only make the hoped-for difference in how and what students learn if teachers come up with new ways to use it." On the same train of thought, Michael de Villiers (2006) referring the many pitfalls that befall in the introduction of computers in the teaching of Mathematics, emphasises the need for the "development of new skills" (2006, p.46) for the teachers that introduce them in the classroom. Reinforcing this inevitability, the International Society for Technology in Education (2014) states, as one of the standards and performance indicators, the need for teachers to develop "technology-enriched learning environments (…) and personalize learning activities to address students' diverse learning styles, working strategies, and abilities using digital tools and resources".

In conclusion, it is advisable that, for the majority of the school subjects, teachers explore digital technologies recurrently as pedagogical tools, taking full advantage of its possibilities, whenever possible, in innovative ways so that the teaching experience can be meaningful and, desirably, long-lasting in students' memories.

Digital educational tools should be explored in the classroom for the positive effects they offer, particularly when intended to give students the possibility to interact and to experience something pedagogically significant and different from the traditional educational context and from a mere "extension of paper and pencil geometry", that according to Sutherland (2005, p.4), many teachers often do, when using computers in the classroom. Quite the opposite, the exploration of computer-based technologies in this context should foremost assist students in their educational experience, while the teacher provides guidance whenever necessary to explain what is to be retained of it. But, as Michael de Villiers mentions:

> "Dynamic Geometry cannot offer a magical panacea for learning Geometry (…) simply by staring at the beautiful, moving pictures on the screen. Unless the learner or student critically engages or is carefully guided to observe and examine what is happening on the screen, very little learning may actually be taking place" (2006, p.48).

Thus the importance of guiding students in computer assisted activities assuring they are not led to think that software is more important than our own reasoning or that learning Geometry is no longer required. Some dynamic Geometry applets can be constructed beforehand so students explore them autonomously. This kind of applets might be very useful from the students' perspective, especially if they are instructed by the teacher to use them properly. It is also advisable that teachers tell students how traditional methods would not accomplish the problem's resolution in the way that software can provide without numerous intermediary steps (that still the teacher should point to as to avoid students to lose track of the underlying geometric concepts).

Computer-based technologies towards an active Geometry learning

> "If we agree that one of the educational goals is to provide youth with basic skills and competencies needed for later professions, what is the role of school Geometry curricula in this respect? Wouldn't one of its important goals be to prepare pupils for this shift of emphasis

imposed by the use of computerized tools? (…) How can Mathematics educators benefit from the tools available in order to enhance the teaching of Geometry?" (Osta, 1998, pg. 129)

Nowadays, the benefits provided by dynamic Geometry software in the educational context are unquestionable, for they allow teachers to hugely enhance their practice and students, through an "experimental laboratory" (Villiers, 2006, p. 1), to develop their mathematical reasoning with dynamic experiences that can become quite fulfilling, if properly oriented. In "Geometry Turned On!" King & Schattschneider (1997) explore the endless possibilities of dynamic Geometry software for the enhancement of Geometry didactics, providing an experimentation-based educational perspective that would otherwise remain inaccessible by traditional teaching methods.

The benefits brought to the teaching of Mathematics and its learning processes by the introduction of digital tools in the classroom can likewise broaden the horizons for Descriptive Geometry teaching (still taught in a small number of European countries) provided its exploration is oriented in a way it best supports the development of students reasoning and the comprehension of geometric concepts and its relations in space. Dynamic Geometry and 3D Modelling software (Fig. 1) can effectively support teachers' practices and particularly assist students, so they can better understand the underlying concepts depicted at their own learning pace, supporting the process of conceptually operating with geometric concepts and, in doing so, developing their spatial abilities and a desirable enthusiasm in learning Mathematics in a broader sense.

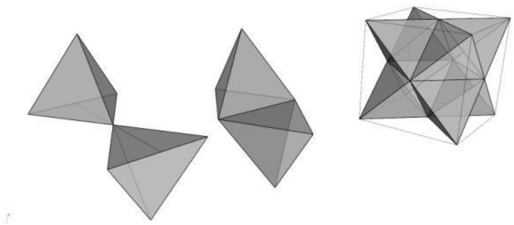


Figure 1 – Examples of non-polyhedra constructed with Rhinoceros

Descriptive Geometry computer-assisted teaching for a positive learning

In a recent interview conducted by Jakobsen & Matthiasen (2014), the participants concluded that students who have the possibility of complementing the practice of Drawing with Descriptive Geometry learning as a graphic representation method are better prepared for the practice of Architecture, Engineering and other areas concerned with representing space and operating with space modelling, because one of the purposes of learning Descriptive Geometry is, foremost, providing students with the ability to "think in space" (Jakobsen & Matthiasen, 2014, p.3).

In many European countries, the Descriptive Geometry subject has been discarded, although it stands as an important subject at the secondary level of education that can address students' spatial reasoning as a requisite of operating with geometric concepts in three-dimensional space. It is our belief that, in Portugal (where Descriptive Geometry is taught in the 10th and 11th grade), the current didactic methodologies and concepts taught should be object of deep adjustments in order to abandon the fierce connections that its teaching still maintains with somehow obsolete proceedings (that are no longer required in the professional context in which students will be integrated); articulate its concepts in deeper connexions with the concepts taught at the same grades in

Mathematics; and desirably, to provide a short period for students to explore themselves Dynamic Geometry and 3D Modelling software. Also, the awareness of students towards the relation between the three-dimensional "reality" and its representation (and vice-versa) should be enhanced and seriously considered the potential benefits that Descriptive Geometry learning can provide towards a more correct interpretation of the graphic representations of geometric objects and, inversely, the ability to represent them in the two-dimensional plane. In this matter, we consider that Descriptive Geometry concepts involving the Axonometric Projection System should be generalized to the students that learn Mathematics at secondary level, in an attempt to optimize the process of coding and decoding graphic representations (Parzysz, 1991, 578).

The importance of learning Geometry and (in the early instruction of every profession that deals with the modelling of space and its representation) Descriptive Geometry, should not be disregarded, for its evident importance in the development of students' geometric reasoning and their spatial abilities. Furthermore, we claim that the inclusion of the Descriptive Geometry in high schools curricula (assumed its necessary adaptation to present day requirements) should be reconsidered, given the significance of a very solid geometric knowledge for operating with three-dimensional concepts every expertise in 3D Modelling software (Fig. 2) must have.

> "The possibilities allowed by any good (…) software expand enormously the possibilities allowed by the ruler and compass on a sheet of paper. (…) This does not mean that this short set of instruments is not efficient from a basic learning perspective. But to go further today, we must use today's means." (Mateus, 2014, pp.54-55)

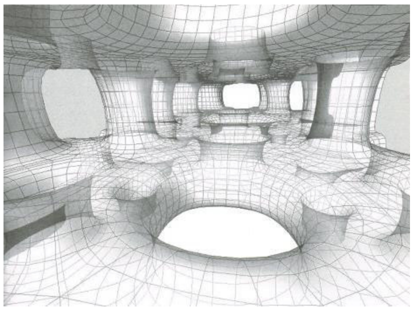
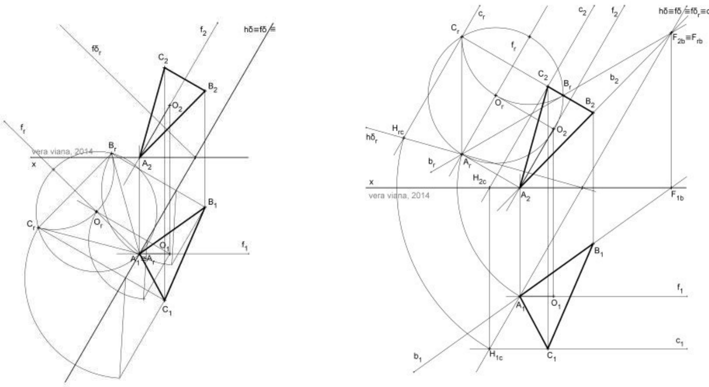


Figure 2 - Taichung Metropolitan Opera House - Toyo Ito.

Interactive GeoGebra resources for an active Descriptive Geometry learning

Taken from the personal website developed since 2007 by the author with the intent to share interactive constructions built with dynamic Geometry software, some applets are presented as possible educational strategies to address the different learning styles. These applets intended to support Descriptive Geometry didactics, are mainly focused in the students' perspective to support the classroom's learning outcomes and complement, in an innovative way, traditional teaching practices. Meant to address learning styles such as those of active, reflective, sensing, intuitive, visual and sequential learners, these applets can, as our teaching experience tells us, stimulate students to better understand the different results obtained from a predetermined situation.

Reflective and sequential students, for instance, may find useful to explore, while studying alone, step-by-step different resolutions of the problem exemplified in Figures 3 and 4. Teachers themselves can complement their educational practice exploring these constructions during classes.



Figures 3 and 4 – Two resolution procedures for a problem involving the orthographic projections of an oblique triangle. www.veraviana.net/diedpassoapassofig.html#EX151PAG129).

All applets are available in the website and some of them in a CD-ROM and/or PEN drive (since 2013) so that teachers can explore them according to their needs in the classroom and students, autonomously, at their own learning rhythm. It is our belief that, if the opportunity to explore these applets is provided in a way students can complement what has been learnt in the classroom, the opportunity to enrich the educational experience from the students' perspective is created in a way that the whole experience may be effectively productive.

With this intent, the example in Fig. 5 allows students to move the points that define a plane, thus understanding how its different locations determines the plane's orientation and, subsequently, its lines of intersection with the projecting planes. This experience may be productive for active, intuitive and visual learners.

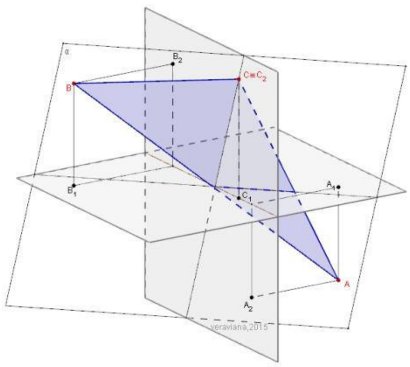


Figure 5 – Interactive 3D Axonometric projection of a triangle and its plane (http://www.veraviana.net/dieddinamicasfg.html#triangulo).

The applets can likewise complement dynamically the practice of teachers in the classroom and given the students the necessary guidance to understand what is shown, so that they are able to "search what remains constant in what is variable" (Xavier & Rebelo, 2001, p.4). The construction in Fig. 6 intends to dynamically demonstrate several possibilities from a specific graphic situation and its purpose is to assist students in a better understanding of the invariant properties of the geometric concepts involved (in this case, the truncation of the cube)

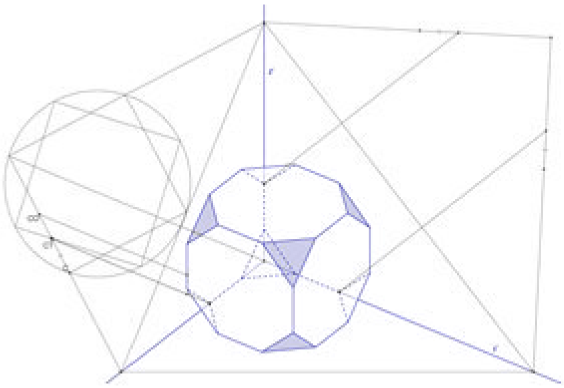


Figure 6 – Axonometric projection of a cube sectioned perpendicularly to the symmetry axis of each vertex (www.veraviana.net/arquimedianos.html#doisarquimedianos).

Most of the applets are available in GeoGebra Books, that distributes them by theme: Descriptive Geometry Applets (step-by-step) - http://tube.geogebra.org/student/bMNeCNYyn and Interactive Descriptive Geometry Applets - http://tube.geogebra.org/student/bcIg8exfg#.

Introducing Polyhedra with Rhinoceros and Grashopper

Our Descriptive Geometry teaching practices demonstrates that students became more interested in the learning process when geometric concepts are presented in different ways. When the subject depicting the orthographic representation of polyhedra is introduced, it is suggested that students manipulate real polyhedral models and, subsequently, infer polyhedral categories through its description and visualization of virtual representations of simple software such as Poly or Stella4D. Gutiérrez (1996, pg. 11) addresses the relevance of this kind of practice as educationally inspiring, because students tend to create mental images of solids in a better way and develop their abilities of mentally rotating similar geometric concepts.

Complementing this experience, 3D modelling software Rhinoceros and its plug-in Grasshopper can reveal themselves quite useful when a more effective visualization display is required to explore concepts dealing with solid geometry. With this intent, Figures 7, 8, 9 and 10 display virtual representations of right pyramids and right prisms that were constructed with the purpose of controlling parametrically the configuration and orientation of the base (or basis) and the solid's height.

As a classroom activity and with some guidance from the teacher, students can themselves move the sliders to control the orientation of the base plane, the number of base(is) edges and the solids height.

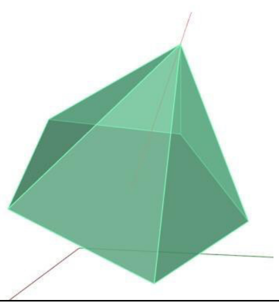
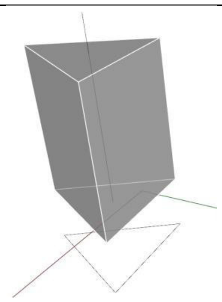
	

Fig. 7 - Pentagonal right pyramid	**Fig. 8 - Triangular right prism with its horizontal projection**

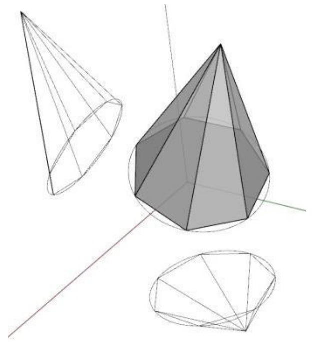
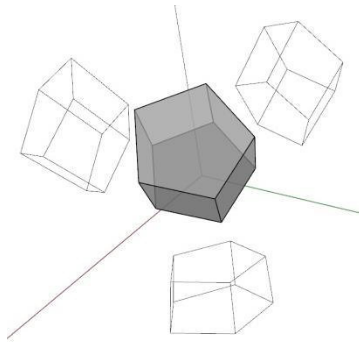
	

Fig. 9 - Heptagonal right pyramid with two orthogonal projections	**Fig. 10 - Pentagonal right pyramid with three orthogonal projections**

CONCLUSIONS

Considering the powerful digital tools nowadays available, there should be no reason for the educational environment not to explore them to create specific resources that can simplify the understanding of the concepts taught in school curricula, specifically those concerning abstract geometrical concepts which may be better understood by students through the exploration of interactive and dynamic constructions, step-by-step resolutions or virtual three-dimensional models.

It was our intention to justify the need to rethink some educational practices and to demonstrate that the exploration of digital resources by teachers and students is nowadays more emergent than ever.

> "To be a teacher does not mean simply to affirm that such a thing is so, or to deliver a lecture, etc. No, to be a teacher in the right sense is to be a learner. Instruction begins when you, the teacher, learn from the learner, put yourself in his place so that you may understand what he understands and the way he understands it." (Soren Kierkegaard, The Point of View for my Work as an Author, 1848)

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Theme: Students

IMPROVEMENT OF GIFTED STUDENTS' VISUALIZATION ABILITIES IN A 3D COMPUTER ENVIRONMENT

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We present the software Cubes & Cubes, designed to help students improve their visualization abilities. It presents tasks asking to draw the orthogonal projections of given sets of stacked cubes, or asking to build a set of stacked cubes corresponding to given orthogonal projections. This software allows teachers to pay differentiated attention to their pupils, in particular to mathematically talented students. We describe the strategies used by some mathematically talented students to solve tasks posed by Cubes & Cubes, and we analyze students' outcomes in terms of the amount of cognitive demand of their strategies.

Keywords: Spatial visualization; mathematically talented students; plane representations; 3d software.

INTRODUCTION

Research in mathematics education (summarized in Clements, 2013; Gutiérrez, Boero, 2006; Lester, 2007) has showed that the use of technology in mathematics classes facilitates the learning and improves the understanding of mathematical concepts, since technology offers opportunities to achieve better learning by engaging students in solving tasks. Furthermore, the suitable use of software can help teachers to organize a personalized learning in their classes. An aim of this paper is to present the software *Cubes & Cubes*, which helps students develop their visualization skills while solving 3-dimensional representation tasks. The controlled use of the software can be very helpful for teachers to pay individualized attention to their pupils in a class group, and to attend the learning necessities and develop deeper levels of understanding of all students.

Nowadays we can find students with different mathematical abilities in the same classroom. Teachers should take care of their pupils' different needs, but sometimes teachers do not have the necessary media to attend adequately their mathematically talented pupils. This can cause that those students do not develop their high mathematical capabilities as much as they could or, even, they could come to school failure. By *mathematically talented students* we mean those students having a mathematical ability clearly over the average students with their same age, school grade or learning experience. Gifted students are a extreme case of mathematically talented students. Authors like Freiman (2006), Greenes (1981) and Krutetskii (1976) have analyzed and described behaviour characteristics of mathematically talented students.

The other aim of this paper is to analyze different strategies used by mathematically talented students to solve visualization tasks with Cubes & Cubes. Namely, we aim i) to identify strategies used by students to solve different space visualization tasks, and ii) to analyze the tasks and those students' strategies to identify different levels of cognitive demand in their outcomes. We show different students' strategies to solve activities using the software Cubes & Cubes and we analyze the cognitive demand of those strategies. The results presented here are a part of a research project focused on the design of teaching units for ordinary classrooms that pay differentiated attention to mathematically talented students.

We use the model of *cognitive demand* (Smith & Stein, 1998) to organize tasks to teach mathematically talented students and to evaluate their problem solving outcomes. A mathematical task, activity or problem is classified into four levels of cognitive demand depending on the cognitive effort necessary for a student to solve it, which is narrowly related to the sophistication of the student's reasoning while solving the task. The levels, from the lowest to the highest, are labelled *memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics* (Smith & Stein, 1998). The cognitive demand model allows teachers and researchers understand students' answers from the viewpoint of the complexity of the mathematical knowledge used by students to solve tasks. For instance, it is possible to note that some tasks are solved by students experiencing a certain level of cognitive demand to get the answer, while students using lower levels of cognitive demand cannot solve those tasks.

This model allows teachers select tasks with an appropriate degree of challenge for their pupils. It has probed to be useful in analyzing mathematical problems by theoretically estimating the difficulty that supposes solving the problem for students (Stein, Grover & Henningsen, 1996). The model can also be used to evaluate the role of teachers selecting and implementing mathematical activities (Henningsen & Stein, 1997), as well as, to analyze the behaviour of students with different mathematical talent when they solve activities (García & Benítez, 2013).

CUBES & CUBES: A 3D SOFTWARE TO IMPROVE VISUALIZATION ABILITIES

The educational software *Cubes & Cubes* (Hoyos, Aristizábal & Acosta, 2014) aims to enhance the spatial visualization abilities of elementary and secondary school students. It allows them to handle solids made of unit-sized cubes (Figure 1). The solids can be rotated to visualize them from different positions, just by dragging the mouse/pad or pressing the key arrows, so the user can experience a tri-dimensional rotation of the solid. It also has a tools that automatically allows to observe the solid on the screen from its top, front and right side views (orthogonal projections), as showed in Figure 2, where the red arrow identifies the right side of the solid.

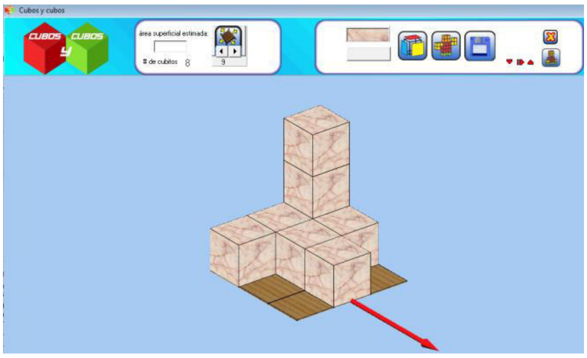


Figure 1. Cubes & Cubes environment to visualize objects constructed with unit sized cubes.

It is possible to build solids on different board sizes by adding and dropping cubes, and to paint the cubes with several colours and textures available. There is also the possibility of saving the solid on the screen and reload it, so teachers can prepare their own activities adapted to their specific pupils.

Cubes & Cubes offers several types of activities based on solids that can be built randomly by the software or loaded from files designed by the teacher. Those activities are specifically designed to help students understand important concepts related to spatial visualization, like orthogonal projection and orientation in space. In these types of activities the users have to rotate the solids to accomplish the task, helping them to develop their visualizations skills, to learn how to describe what a solid looks like from different views, and to learn how to get a 2-dimensional representation from a 3-dimensional solid in the space, and vice versa. The software also has the ability to evaluate the user's answer to all types of tasks. The different types of activities are:

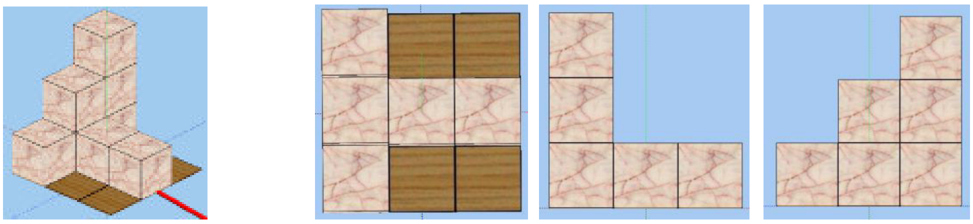
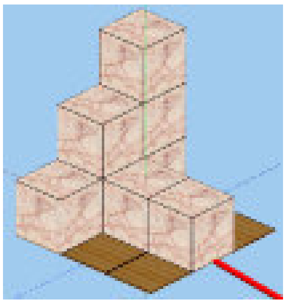
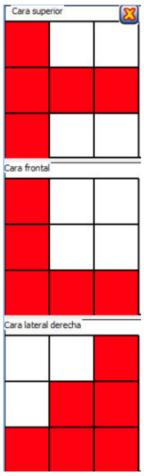


Figure 2. A solid and its views from top, front and right side in Cubes & Cubes.

- Draw on the screen the orthogonal projections (Figure 3a) or the numeric orthogonal projections (Figure 3b) of a given 3-dimensional solid. In numeric orthogonal projections, each cell of the projection shows the number of cubes the solid has in the row represented by that particular cell.



(a)



(b)

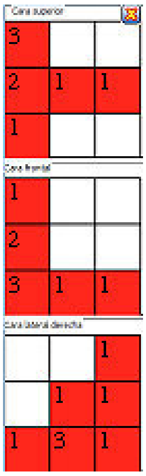
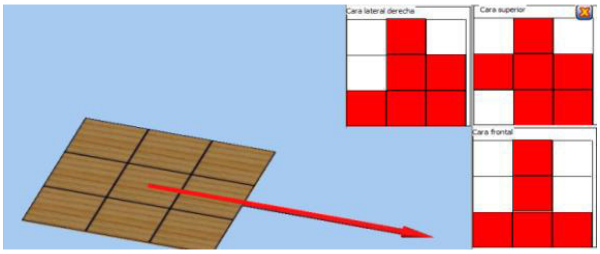


Figure 3. (a) Orthogonal and (b) numeric orthogonal projections of the solid from top, front and right side views.

- Build a 3-dimensional solid based on a given set of orthogonal projections (Figure 4a) or numeric orthogonal projections (Figure 4b). As the solution may not be unique in this type of tasks, this gives the teacher the possibility to open a discussion in the classroom, since different students may have built different solids from the same orthogonal projections.

(a)



(b)

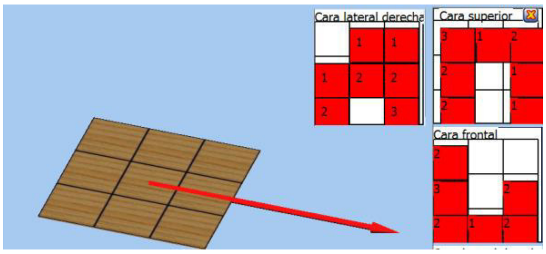


Figure 4. Build a solid represented by the given (a) orthogonal or (b) numeric orthogonal projections.

- Count the number of cubes used to build a given solid.

- Build a solid congruent to a given solid.

- Match the positions of two identical solids shown on the screen, so they look the same on the screen.

- Explore the concept of volume by calculating (Figure 5a) or estimating (Figure 5b) the number of unit-sized cubes necessary to fill in a solid.

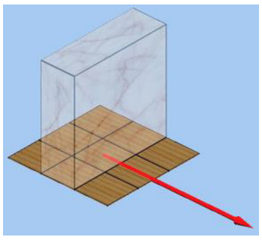
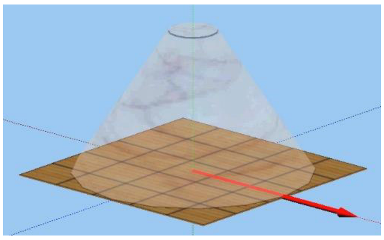
(a)  **(b)** 

Figure 5. (a) Calculate or (b) estimate the volume of a given solid by filling it in with unit-sized cubes.

METHODOLOGY

Participants and context

The subjects for this study were 40 mathematically talented students aged 10 to 12 years participating in a special out-of-school workshop conducted by the researchers. The classroom was organized in pairs of students, with one computer for each pair. The introduction of the experimental environment to the students was limited to make a short presentation of orthogonal projections, since they did not know about this kind of plane representation in geometry, and to show them how to manage the software Cubes & Cubes. We described the types of tasks they were going to be posed and, occasionally, the whole class worked out an example. We never showed the students any procedure for completing the activities nor explained how to solve them.

Data gathering instrument

Our source of data are the videos recorded by a screen capture software that also recorded sound, so we can see all the actions made by students on the screen and hear their talks. These data allowed us to identify the reasoning under students' decisions when choosing strategies to solve the tasks.

Activities

From the tasks supported by the software, described in the previous section, we posed to students the following ones, in this order: Draw the orthogonal projections of a solid; draw the numeric orthogonal projections of a solid; build a solid from a set of numeric orthogonal projections; and build a solid from a set of orthogonal projections. Previous research has proved that those types of activities have different difficulties for students (Gutiérrez, 1996), so we posed the tasks from the easiest to the most difficult one. For each type of task, we stated several problems differing on the complexity of the solids. These activities do not require any specific mathematical knowledge, but visual or analytical reasoning (Krutetskii, 1976) and visualization abilities to create and manage adequate mental images (Presmeg, 1986).

RESULTS

We have noted different strategies to solve each type of activities posed, and we have analyzed the cognitive features of these strategies to identify their level of cognitive demand. Below we describe and analyze the most interesting strategies used by students to solve each type of task.

Task: Draw the orthogonal projections of a solid

Strategy: Reproduction or copy. Some students discovered that the software has an option to automatically show each orthogonal projection of the solid on the screen, so they used it to copy the image given by the computer in the grid (Figure 6). This strategy only involves the careful reproduction of the image, and it is unambiguous, clear and direct. Students do not use any procedure, nor they need to use the meaning of orthogonal projections, they just copy what they see on the screen. Then, this strategy is typical of the memorization level of cognitive demand. The students who chose this strategy had not any problem to solve the task correctly but they did not improve their visualization abilities.

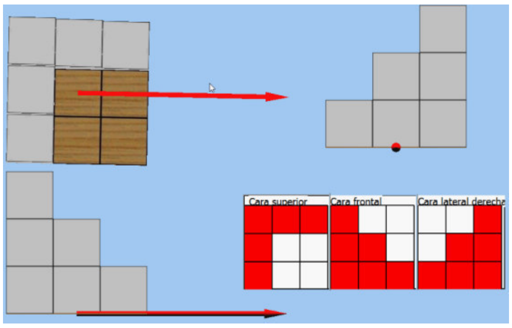


Figure 6. Task solved by reproducing the shapes of the solid's faces.

Strategy: Movement of the solid. Most students moved the solid on the screen to place it in a position they consider suitable to identify one of its orthogonal projections and draw it in the grid. Then, students moved the solid again, looking for another projection, and so on.

We have identified two mistakes made by students who used this strategy because they moved the solid to an inadequate position. An error was to show the top face placed in a wrong position (Figure 7a). The other error was to place the solid showing a wrong face (Figure 7b).

(a) **(b)**

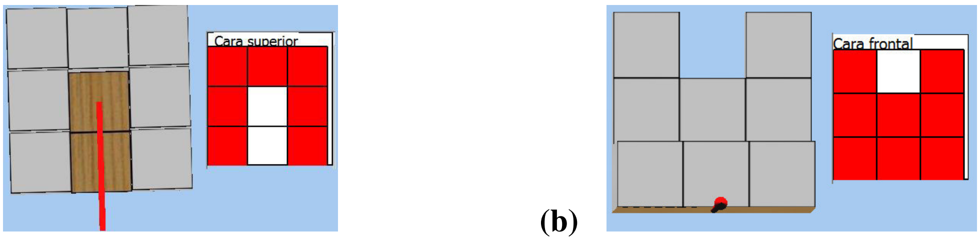


Figure 7. (a) Incorrect position of the top face. (b) Mix-up between front and right side faces.

This strategy is algorithmic but it requires from the students to know the meaning of each orthogonal projection and its corresponding face of the solid, to place it correctly on the screen. It has a limited cognitive demand for a successful completion, since it is only necessary to know what is an orthogonal projection, and it does not require neither reasoning nor connections. This strategy is focused on producing correct answers instead of on developing mathematical understanding. Then, it is typical of the procedures without connections level of cognitive demand.

Strategy: Still solid. Some students were able to visualize the three orthogonal projections keeping the solid still in a specific position. The position chosen is very important because it must let students imagine all the orthogonal projections. The students can successfully solve the task in different ways depending on the position of the solid. In general, the arrow has to point to the right and the figure should be slightly inclined (Figure 8).

This strategy requires some degree of cognitive effort. Although it may be used by most students, it cannot be applied mindlessly, since students need to coordinate different faces and visualize the solid from positions different from their real one. The students using this strategy developed their visualization abilities more than students using the other strategies. This strategy is typical of the procedure with connections level of cognitive demand.

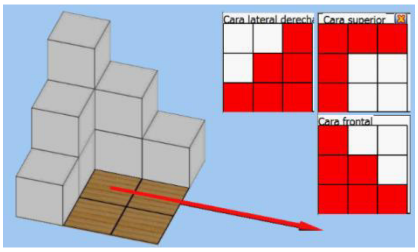


Figure 8. Orthogonal projections from a still solid.

Task: Draw the numeric orthogonal projections of a solid

The need to count the number of cubes in each row makes this type of tasks very different from the previous one. For instance, the first strategy used in the previous task is not useful now, since the orthogonal projections showed by the software do not allow count the number of cubes in each row.

We have found only two different strategies to solve this type of tasks, that are similar to the two last strategies described for the previous task. Some students moved the solid as many times as they considered necessary to see all the rows and count their cubes. This procedure is algorithmic and, if students follow it carefully, they do not have problems to solve the task. It does not require either reasoning or connections of elements, so it has a limited cognitive demand for successful completion. This strategy belongs to the procedure without connection level of cognitive demand.

Other students kept the solid still and used their visualization abilities to count the number of cubes in the rows. We have identified two different levels in this way of solution: Some students set the solid in only one position, while other students used several still position to complete each projection. This strategy requires a certain cognitive effort and it is necessary for the students to have developed their ability to visualize and understand the space. It is typical of the procedure with connection level of cognitive demand.

Task: Build a solid from a set of numeric orthogonal projections

Only 60% of the students were able to solve this kind of task, because it is more difficult than the previous tasks. The strategy used by most students consisted in building the solid observing first the top projection. Having made this step, we have found two different ways to follow up. Some students moved the solid to analyze the next orthogonal projection but forgetting that the solid has to fit both projections. This is an algorithmic strategy that does not establish the necessary connections between the projections and/or the solid's faces. The cognitive demand of this poor strategy belongs to the procedure without connections level.

Other students continued solving the task by moving the solid to see another projection and adding cubes to the solid while checking that it fitted both orthogonal projections. These students developed deep visualization abilities as they were able to connect the three numeric orthogonal projections and to solve the task correctly. This solving strategy corresponds to the procedure with connection level of cognitive demand.

Finally, a student used a great strategy, synthesized in Figure 9. The student analyzed the numbers in the orthogonal projections and identified analytic relationships that helped him to build the solid. This strategy required considerable cognitive effort, to establish non-algorithmic relationships among parts of the numeric projections, so it is typical of the doing mathematics level, the highest level of cognitive demands.

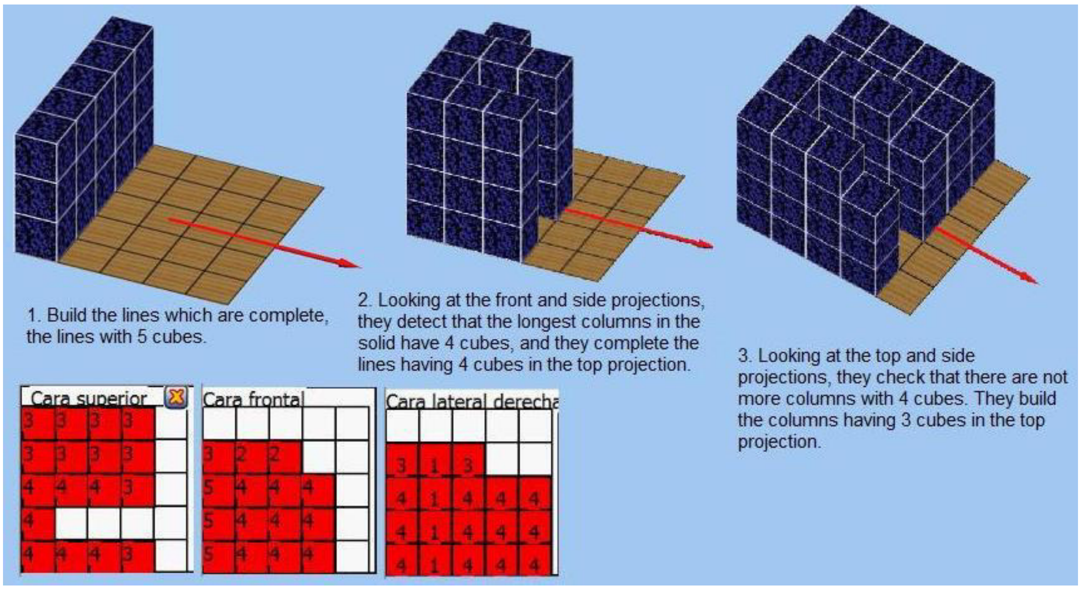


Figure 9. An optimal strategy to build a solid from a set of numeric orthogonal projections.

Task: Build a solid from a set of orthogonal projections

This task is the most difficult one. Only 40% of the students solved correctly this type of tasks. Similarly as for the first strategy showed for the previous task, there were students who tried to solve it by keeping in mind only one projection each time, so they were not able to build a solid fitting the three projections at the same time. Therefore, this strategy belongs to the procedure without connections level of cognitive demand.

Other students built first a solid looking at one projection. Then, they identified the cubes they had to add or remove to fit all the projections at the same time (Fig. 10). The students solved the task by a strategy requiring some degree of cognitive effort, since they had to link the different projections to the partially built solid, so its cognitive demand is in the procedure with connections level.

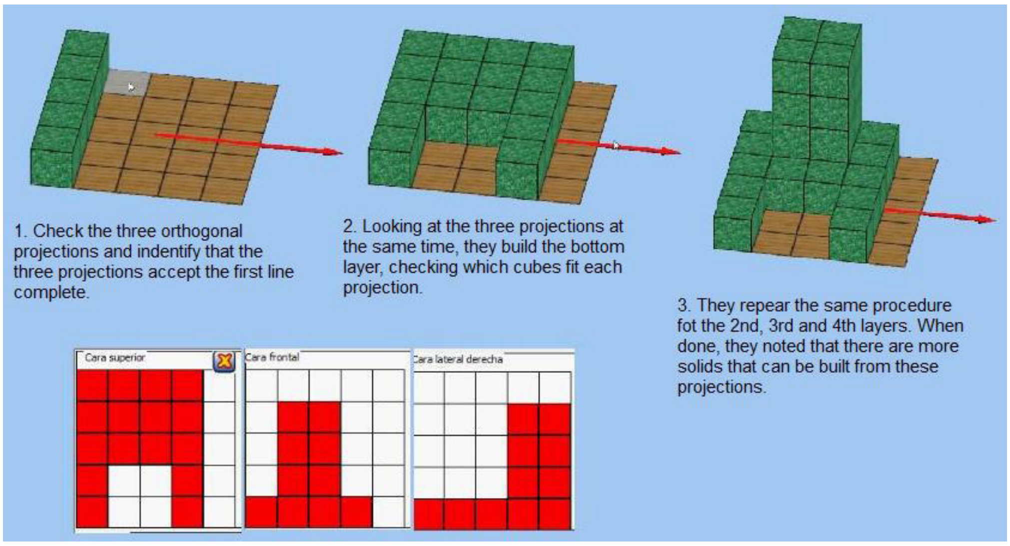


Figure 10. A strategy to build a solid from a set of orthogonal projections.

CONCLUSIONS

We have presented a software which is very useful for students to learn and improve visualization skills. It is easily adaptable by the teacher, allowing her to state easy tasks to students with more difficulties and, at the same time, to state challenging tasks to the mathematically talented students.

We have presented different mathematically talented students' answers classified according to their ways of reasoning and use of visualization abilities. We have analyzed these answers and identified styles of behaviour characteristic of the different levels of cognitive demand, proving that the model of cognitive demand is useful to discriminate among the different responses offered by the students.

Note:
The results reported are part of the R+D+I projects *Analysis of Learning Processes by Primary and Middle School Mathematically Talented Students in Contexts of Rich Mathematical Activities* (EDU2012-37259) and *Key Moments in the Learning of Geometry in a Technological and Collaborative Environment* (EDU2011-23240), funded by the Spanish Ministry of Economy and Competitivity.

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INTERACTIVE INTRODUCTION TO FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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Differential equations constitute a large and very important branch of modern mathematics. From the early days of the calculus this subject has been an area of great theoretical research and practical applications in several branches of science. Despite this importance the largest part of the students reveals strong difficulties to understand the theory of differential equations and its applications.

As a consequence of what was mentioned above we decided to create a new educational tool that accomplishes some of those goals, describing how this dynamic device can be used in the classroom when teaching the first approach of the first-order Ordinary Differential Equations (ODEs).

Keywords: Ordinary Differential Equations, Modelling, Growth and Decay, Computable Document Format

INTRODUCTION

Over the last decades we witnessed a huge development in the power of computation, simultaneously the electronic devices that perform numerical and symbolic programming evolved in power as they inversely evolved in size, making them so light and cheap that is possible nowadays for every student to carry his own laptop to the classrooms. This reality opens the possibility to create new approaches that might help them to understand the concepts associated with the subjects intended to teach in a more effective and fashion way.

We decided that would be a good idea to show and explore the interaction using both the symbolic programming capacity of *Mathematica*® and the graphical capacity. To reach our goals we take advantage of a tool recently developed by Wolfram Research, the Computable Document Format (CDF) [1], which enables the interaction between the digital document and the reader.

The appearance of this tool provides a complementary way of how to teach the mathematical concepts and, in our opinion, increases the probability of our success. In this context we pretend to use this tool to tackle some applications of the first-order differential equations – rate problems, following the ideas presented in (Coelho & Marreiros, 2013) and (Conceição, Pereira, Silva, & Simão, 2012).

In the section 2 we recall the most basic concepts of the theory of the first-order differential equations (see, for instance, (Ross, 1998)), as well as we present a description of the software created by us, showing the results obtained when applied to solve its most common applications in the section 3. Among the applications to these equations we choose the, so-called, rate problems (population growth, radioactive nuclei decay). Additional information about the construction and use of CDF file type can be found in (Ruskeepää, 2009) and (Wolfram, 1999).

1 This type of document runs with CDF-Player, which is a free software program that can be downloaded from http://www.wolfram.com/cdf-player.

MODEL

Basic Theory

A first-order ODE solved for the derivative is an equation connecting an independent variable x, a sought-for function $y = y(x)$ and its first derivative, that can be written in the form

$$y' = f(x, y), \tag{1}$$

where f is a real continuous function in some domain D of the $x \circ y$ plane.

The general solution of the differential equation (1) is a function

$$y = \varphi(x, C) \tag{2}$$

depending on one arbitrary constant C and such that (2) satisfies the equation (1) with any allowed value of C.

A particular solution of the differential equation (1) is a solution obtained from the general solution (2) with some defined value of the constant C.

Growth and decay rate problems

Let $y = y(t)$ denote the quantity of a substance at a time t, and suppose the rate, i.e., the derivative, $y' \equiv dy/dt$, at which the quantity changes is proportional to the amount of the quantity present at time t; i.e., we have

$$y' = ky, \tag{3}$$

where k is a constant of proportionality.

The differential equation (3) is an equation with variables separable; separating variables, integrating, and simplifying, we have that

$$y(t) = Ce^{kt} \tag{4}$$

is the general solution of the differential equation (3).

Usually, in this so-called "rate problems", there are given two initial conditions, which allow us to obtain the value of C and the constant of proportionality k. Let the following conditions be given

$$y(0) = y_0, \quad y(t_1) = y_1, \tag{5}$$

i.e., y_0 is the amount initially present and y_1 is the amount present at time t_1; $y_0, y_1 > 0$. Computing the constants C and k:

$$\begin{cases} y(0) = y_0 \\ y(t_1) = y_1 \end{cases} \Leftrightarrow \begin{cases} y_0 = C \\ y_1 = y_0 e^{kt_1} \end{cases} \Leftrightarrow \begin{cases} C = y_0 \\ k = \frac{\ln(y_1/y_0)}{t_1} \end{cases} \tag{6}$$

and inserting their values in (3) we get

$$y(t) = y_0 e^{\frac{\ln\left(y_1 y_0^{-1}\right)t}{t_1}}. \tag{7}$$

Note that if the quantity $y(t)$ growths, i.e., $y_1 > y_0$, then $k > 0$ (in this case, k is the, so-called, growth constant); if the quantity $y(t)$ decreases, i.e., $y_1 < y_0$, then $k < 0$ (in this case, k is the, so-called, decay constant).

Implemented software

The figure below shows the initial status of the layout of the implemented software.

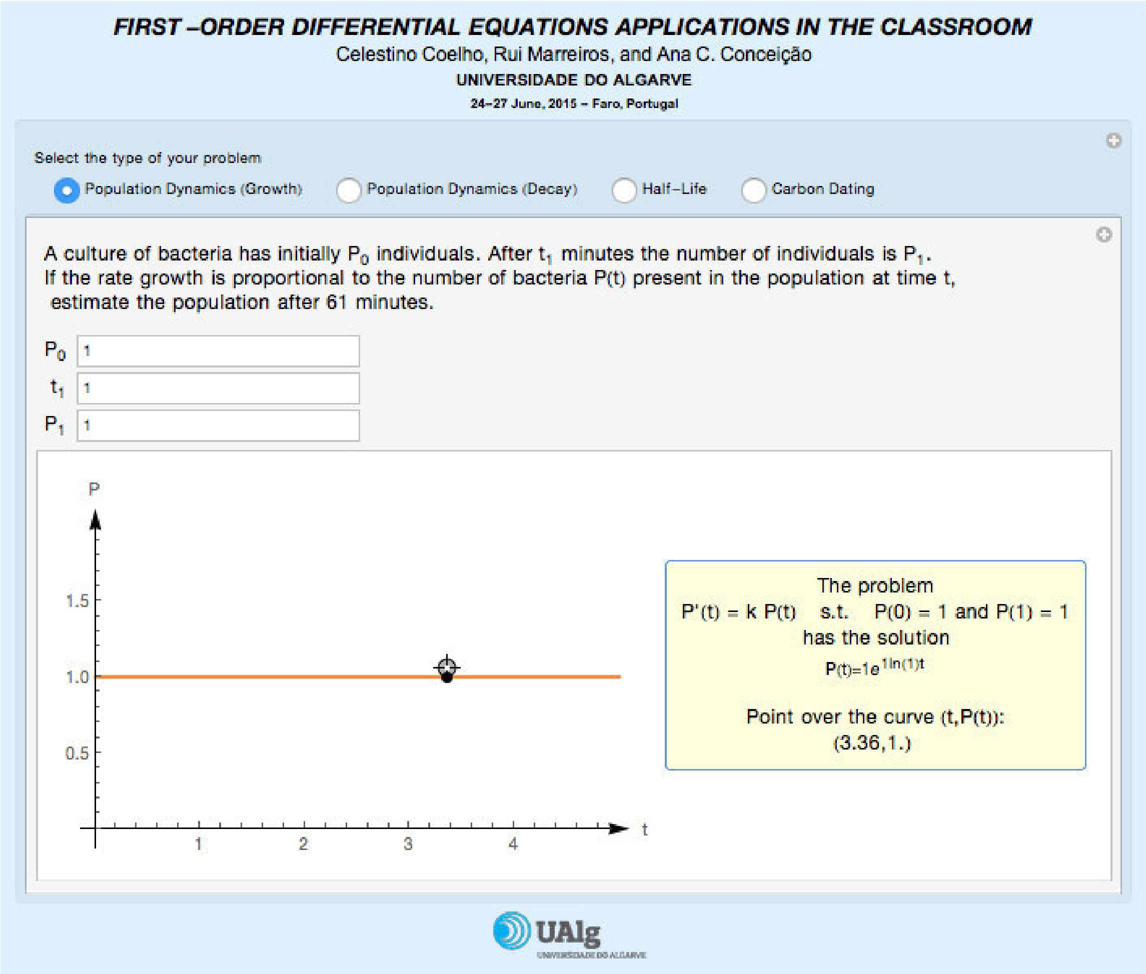

Figure 1. Structure of the software developed.

Analyzing the Figure 1 we see that the software is divided in two parts, the one at the top corresponds to the definition of the problem, where we found a list of radio buttons that enables the choice of the type of problem we intend to solve, followed by a region of input fields where the user defines the initial state of the population, that is, the number of individuals present at time t_0, P_0, a ulterior time, t_1, where a new information about the population is collected, as well as the number of individuals present, P_1. Notice that to get the answers to the questions asked it will be obligatory to grab the mouse and move the pointer over the plot to the left or to the right until we stabilize it over the point where we want to collect the information we seek. Sometimes it will be difficult, or even impossible, to get the exact coordinates of the point, what forces the teacher to be careful when choosing the question and the user when giving the answer, that is, if the answer has to be an overestimation or an underestimation.

One of the most important targets to reach constructing this software is the simplicity of its use. We want this software to be very easy to use, so, to change the fields in the input area, the user just needs to press TAB key (or use mouse) to jump automatically to the next input field. As the user changes the input field the output will be automatically updated, both graphical and analytical.

RESULTS

Population growth and decay

Example 1 (e-coli population growth)

A culture of bacteria (e-coli) has initially $P_0 = 3$ *individuals. After* 20 *minutes the population is twice the original value. If the rate growth is proportional to the number of bacteria* $P(t)$ *present in the population at time* t*:*

a) *find the number of bacteria* $P(t)$ *present in the population after* 61 *minutes;*
b) *what will be the time necessary to guarantee a population* 5 *times bigger than the initial population?*

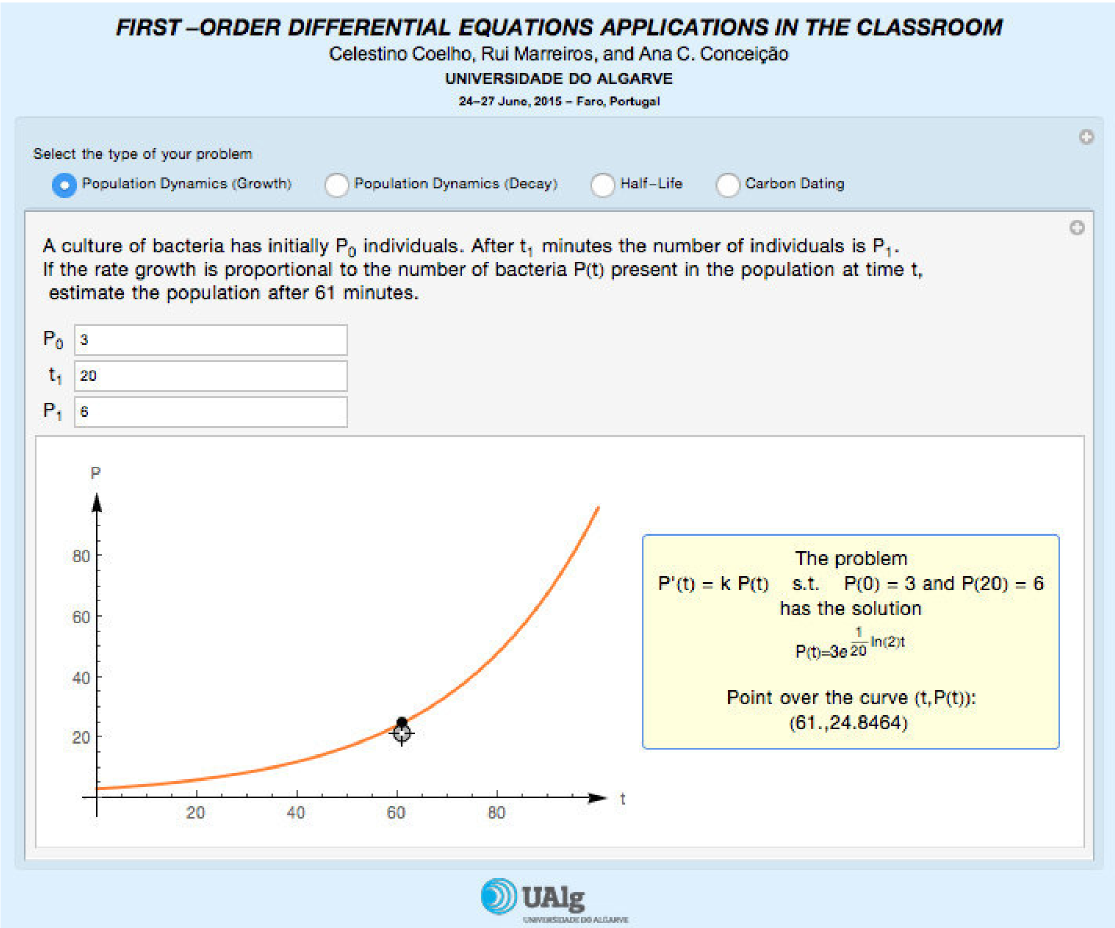

Figure 2. Tool application to solve a problem of population dynamics using a growth rate in e-coli growth population problem.

Figure 1 shows how to apply our tool to solve the questions exhibited in example presented before. If we attend to the input fields present in the software we see that, to get the solution to our problem, we have to input the number of individuals that are initially in the population, P_0, and the number that constitute the population after a certain elapsed time, t_1, denoted by P_1.

Example 2 (radioactive nuclei decay)

Consider an initial mass of the radium element, $M_0 = 100\ g$*, that is reduced in 1% after 23 year. If the rate decay is proportional to the amount present at time* t*:*

a) *find the mass present after* 2020 *years;*

b) what will be the time necessary to guarantee one half of the initial mass?

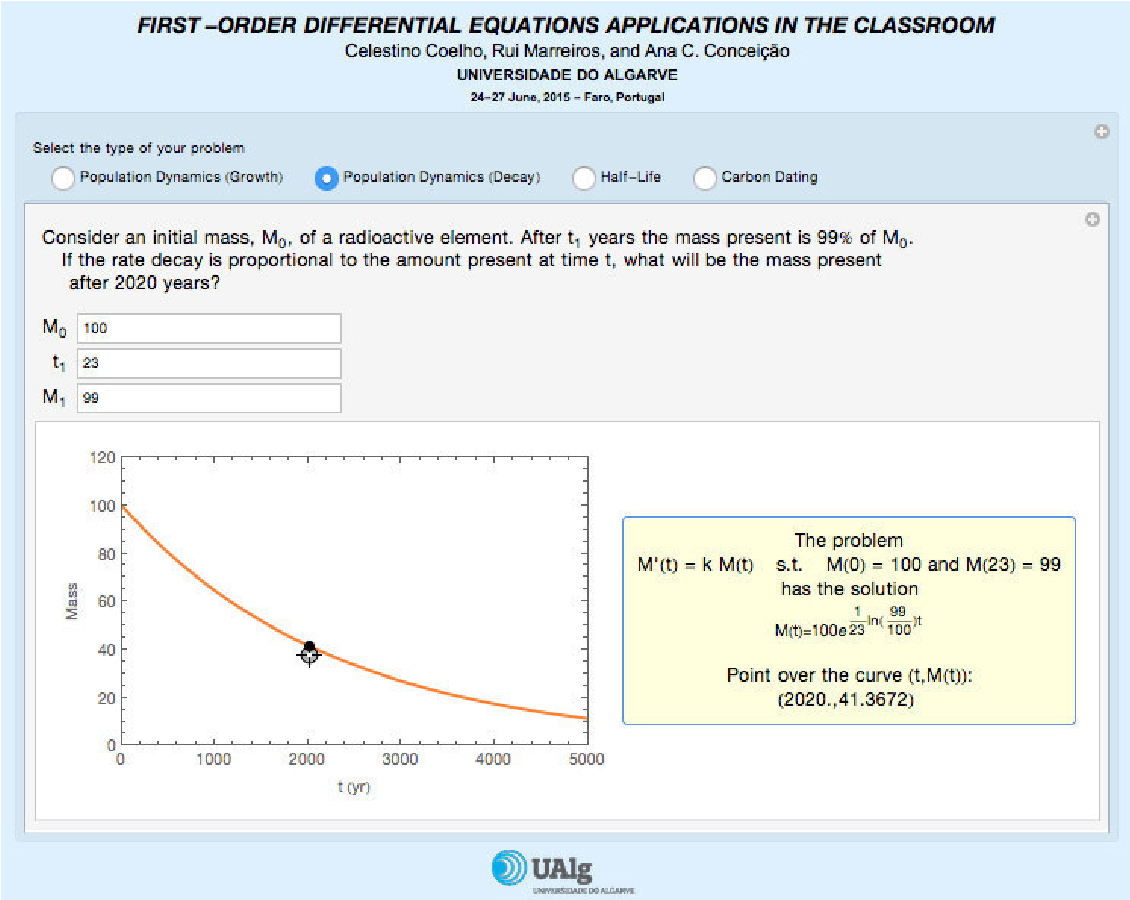

Figure 3. Tool application to solve the problem of radioactive nuclei decay presented in example 2.

The problem presented in example 2 is similar to the one shown in example 1, but in this case it is associated to a decay rate. In which concerns to the application of the tool it will be necessary to notice that to say that after 23 years the mass decreases 1% is equivalent to say that 99% of the mass is present after 23 years, as well as the fact that time here is counted in years. There are some subtle aspects, like the ones stated before, that require an additional attention of the user.

Half-life

This concept is often applied in Physics and provides a measure of the stability of a radioactive substance, corresponding to the time it takes for one-half of the atoms initially present in an amount of A_0 to disintegrate, or transmute, into the atoms of another element. The stability of a substance is directly related to the value of its half-life, that is, the larger the value of the half-life is, the more stable the substance is.

Example 3 (Half-life of plutonium)

A breeder reactor converts relatively stable uranium-228 into the isotope plutonium-239. After 15 *years it is determined that* 0.043% *of the initial amount of plutonium,* A_0*, has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.*

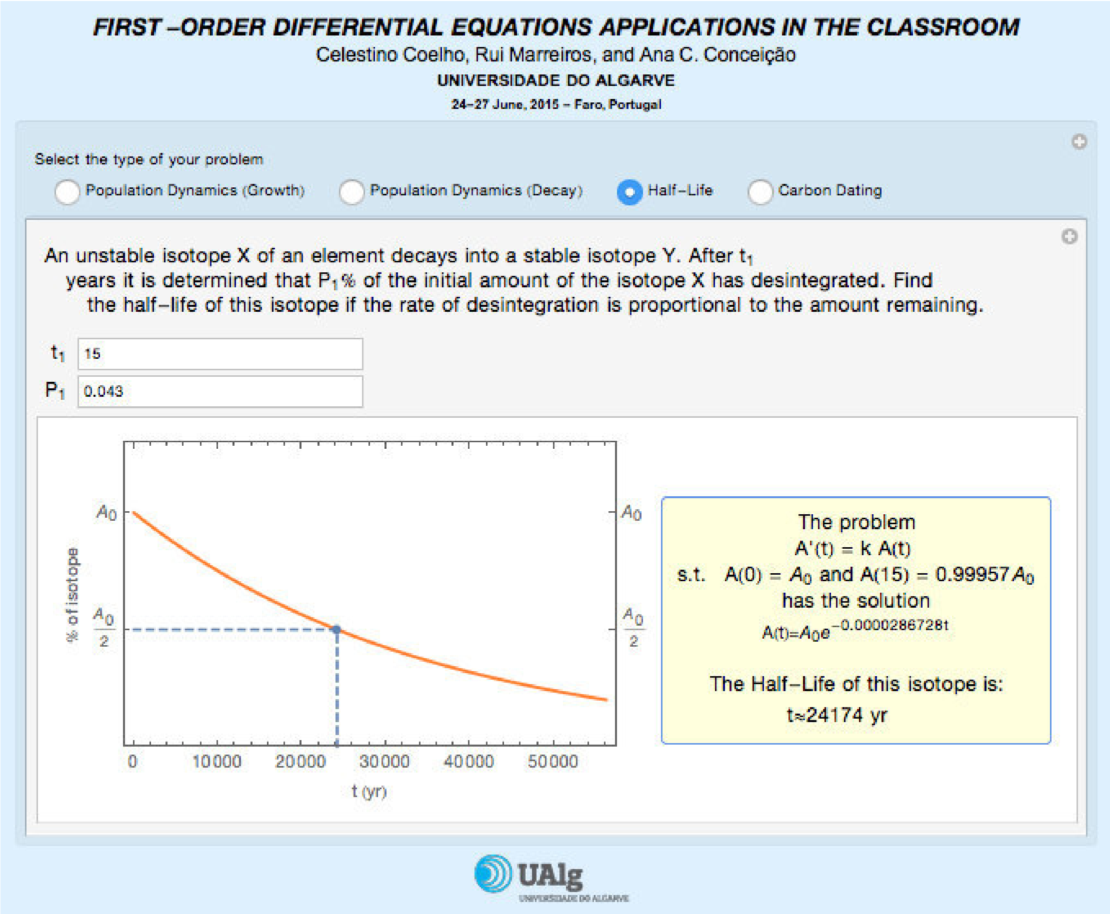

Figure 4. Tool application to solve the problem of radioactive nuclei decay presented in example 3.

Looking at Figure 4 we notice some slight differences between the layouts of the software in this case and the other two presented before. In fact, here we don't need to introduce the initial state because we are assuming that the percentage in that stage is 100%. Due to this assumption the user only need to give the time elapsed, t_1, and the percentage of the isotope that has been disintegrated after that period of time, P_1. In what concerns to the output we verify that there are also some differences worth mentioning. In the graphical solution the half-life of the isotope, which represents the answer to our problem, is automatically marked with a blue dot. The usage of additional dashed lines is made to give a more precise idea of the location of this point. Since the objective of this type of problems is to give the value of the half-life of the isotope it makes no sense to give freedom to move the cursor over the plot. This originates another subtle difference in the analytical output. Instead of showing the point over the curve, we present the value of the half-life of the isotope.

Carbon dating

The American physical chemist Willard F. Libby was the first to develop a theory of carbon dating. During the 50's Libby led a team of scientists at the University of Chicago, and, as recognition of the importance of this work, was awarded the Nobel Prize for Chemistry in 1960. Carbon dating is a particular type of radioactive dating, applicable in cases for which the matter to be dated was once living. The radioactive isotope carbon-14 (C^{14}) is produced at a relatively constant rate in the atmosphere, and like stable carbon-12 (C^{12}), combines with oxygen to form carbon dioxide, which is incorporated into all living things. When a living organism dies, its level of C^{12} remains relatively constant, but its level of C^{14} begins to decay with the rate

$$\frac{d[C^{14}]}{dt} = -k[C^{14}], \qquad (8)$$

where the decay rate is given by $k = 1.2097 \times 10^{-4} yr^{-1}$. This rate corresponds with the commonly quoted fact that C^{14} has a half-life of 5730 years. Since the fraction of C^{12} to C^{14} remains relatively constant in living organisms, at the same level as it occurs in the atmosphere, roughly

$$\frac{[C^{14}]}{[C^{12}]} \cong 1.3 \times 10^{-12}, \qquad (8)$$

we can determine how long an organism has been dead by measuring this ratio and determining how much C^{14} has radiated away. In practice, scientists measure this ratio of C^{14} to C^{12} in units called modern carbons, in which the living ratio that corresponds to the ratio of C^{14} to C^{12} in a living organism is defined to be 1 modern carbon. As it was mentioned this method is based on the knowledge of the half-life of C^{14}, which, in Libby's experiments was fixed in approximately 5600 years, while nowadays it is commonly accepted that this value is approximately 5730 years. This method has been used to date numerous things, being one of the latest the dating of fragments of woolen trousers excavated at Yanghai cemetery in western China, published just a few months ago, (Beck, Wagner, Li, Durkin-Meisterernst, & Tarasov, 2014).

Example 4 (Age of a fossil)

A fossilized bone is found to contain 20% *of its original amount of* C^{14}. *Determine the age of the fossil.*

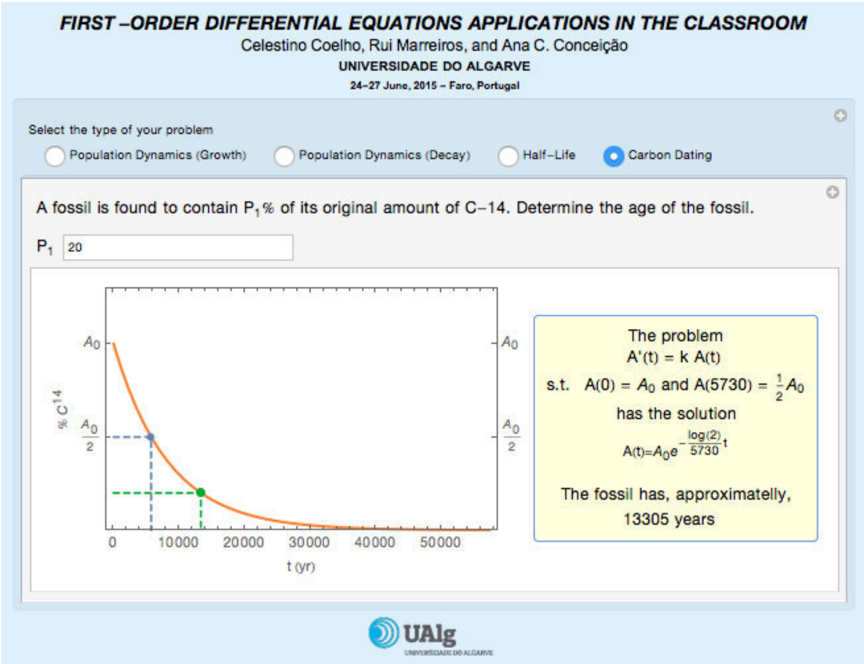

Figure 5. Tool application to solve the problem of carbon dating presented in example 4.

Concerning to Figure 5 there are three observations to be made. Since we are interested in the percentage of C^{14} *that is present in the object that is to be dated, the problem to be solved is always the one stated in the analytical output panel. This means that this type of problem has always the same solution. Due to this characteristic, the only input that should be given is the percentage of*

C^{14} that is measured in the object. And, finally, since we pretend to know the age of the object, we just need to solve the equation to find the value of t that corresponds to the percentage given. The answer to the problem is stated in the last line of the analytical output panel and marked with a green dot over the curve in the plot on the graphical output panel.

CONCLUSIONS

As we all know, most part of the students reveals strong difficulties to learn mathematical concepts, being one of the pertinent questions: “Where can we apply this?” For us, teachers, the introduction of new digital technologies represent a huge step on how we can help in answering this question. This kind of technologies can also be explored to explain, in a more appealing way, the mathematical concepts. The possibility of interaction with the software enables the test and enumerable number data sets, which, in most part of applications, is important to understand the behavior of the solution of the problem that is being solved. This intrinsic characteristic of the software represents a great advantage because different teachers with different sets of data can use it.

In the future we intend to use this software in the classroom testing it with our students and collecting the feedback of their use, hopping that their feedback will be important to develop this kind of software to other application cases to ODEs as well as other subjects. Other future goal is to present this software in workshops, for other teachers, and in seminars, for a more general audience.

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STUDENTS LEARNING ALGEBRA WITH APPLETS

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The transition from arithmetic to algebra is a process that involves complex reasoning and it is a topic where the students present many difficulties. Many studies show that the teaching and learning of mathematics may be potentiated by the use of technology. On this paper we intended to show how the use of an electronic tool might help students solve algebraic equations. Students deal formally with this kind of task in 7th grade for the first time and it is possible identify how the tool mediates the learning process. The theoretical framework is based in the activity theory and the formulations of David Tall about the advanced mathematical thinking and the proceptual view of the mathematical concepts. Based in a qualitative approach and using an interpretative methodology, we observed two groups of students working with an applet during the process of solving algebraic equations. This work gives us evidences about the procedural and conceptual thinking developed by students and the role of the tool during this process. Analysing the performance of students with applets it is possible observe how the semiotic potential of the artefact mediate the learning process.

Keywords: Mediation, Proceptual thinking, Solving equations, Applets.

INTRODUCTION

Algebra is one of the topics that are presented across the curriculum of the Basic and Secondary School in Portugal. The transition from arithmetic to algebra is actually formalized in the beginning of the third cycle of the basic school, when students have about 13 years old. The teaching of this topic is a challenge to teachers and the learning process is complex for many students. In this paper we present a group of students that learn about algebraic equations with support of technological tools. The teaching experience was based mainly on the use of applets for different purposes. Some applets are used to introduce the notion of variable, problem solving with the purpose of developing algebraic reasoning. Another group of tasks were based on the use of applets that act as games, were students may consolidate their previous learning. Others are based in algebra balance scales to develop the notion of equilibrium and consolidate the algebraic principles to be used in the solving equations. In this paper we analyse students' performance with an applet that presents a group of algebraic equations to be solved in the algebraic form in which they need to define the strategy to solve it. This task was performed at the end of the study when the students are proficient in the process of instrumental genesis (Rabardel, 1995).

THEORETICAL FRAMEWORK

Teaching and learning that take place in environments using technology often involve a complex mathematical thinking. Sometimes this kind of thinking is seen from the cognitive point of view and has been designated by advanced mathematical thinking (Dreyfus, 2002; Tall, 2007). The processes of representation and abstraction allow students to move from the level of elementary to advanced mathematical thinking and when used in this sense are often mathematical and psychological processes at the same time.

The use of symbols is a key feature in the development of this kind of thinking and can be viewed with a double meaning, introducing some ambiguity between the procedure and the concept. This combination of procedural and conceptual thoughts is called *proceptual thinking* (Gray & Tall, 1994), and it is characterized as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is this proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation.

Activity Theory initiated by Vygotsky and developed by Leont'ev, assuming its system of collective activity (object oriented and mediated by artefacts) as the unit of analysis, has been developed over three generations. Was initially based on the idea of mediation introduced by Vygotsky in his triangular model that becoming the triad subject - object - mediator artefact, with the separation between the person and the social environment on the background (Engeström, 2001). In the second generation, centred on Leont'ev work, the unit of analysis is no longer individual and now includes links to other areas involved in a collective activity system, focusing now on the interrelationships between individual objects and communities. Based on the collective activity system and giving special attention to the role that the mediator artefacts play in the relationship between subject and object, it becomes crucial to address the concepts of instrumental genesis. The instrumental genesis (Rabardel, 1995) involves two processes, instrumentalization and instrumentation, that enable the development and evolution of the instruments. The idea of mediation has been referred to the potentiality that a specific artefact has to foster the learning process. Rabardel (1995) distinguishes two kinds of mediation; the *epistemic mediation* where the instrument is a medium that allows to know the object and the *pragmatic mediation* where the instrument is the means of a transforming action directed to the object. With the use of technological artifacts it seems to be crucial consider the notion of semiotic mediation (Bussi & Mariotti, 2008) to enhance mathematics teaching and learning. In this context it is important take into account the semiotic potential of the artefact that involves two semiotic links, one between the artefact and the personal signs that emerging from its use and the second between the artefact and the mathematical signs evoked by its use and recognizable as mathematics by an expert.

In summary we intended to understand how technological tools mediate the learning process, based in the semiotic mediation within the activity theory and recognize the proceptual thinking that is potentiated by these tools.

METHODOLOGY

This study is based in a qualitative approach and uses an interpretative methodology to understand how students develop their algebraic reasoning using technological tools. The topic of algebraic equations was introduced to students using applets and supported by documents produced by the teacher that reproduce the main frames of the applet and help students on the registration of the fundamental procedures presented in the applet. During the learning process students alternate between this kind of tasks and others that are presented in documents produced by the teacher with the goal to consolidate prior learning. In this paper we discuss the work of two groups of students (two students per group) working in computers (one group per computer) with the applet *Solving equations with balance-strategy: game* (http://www.fi.uu.nl/wisweb/applets/mainframe_en.html.) This task was performed at the end of the teaching experience with the goal of consolidate the process of solving equations.

Participant students in this teaching experience attended 7th grade and integrate a class of 26 pupils. Due to a lack of computational material we selected 4 students to participate in this work. Students were chosen taking into account their availability, their previous academic performance (considered to be medium by the teacher of the class) and prior parental consent. The researcher worked with these 4 students for 10 lessons, with three of these lessons attended by all students in the class. Manuel and Gustavo composed the first group. Manuel is a student that, initially, reveals some difficulties in mathematics and uses mainly a procedural approach to the concepts. Gustavo develops a conceptual understanding of the concepts studied and reveals a proceptual reasoning when solve the tasks proposed. The second group is formed by Isabel and José. Both students were proficient in mathematics and demonstrated to have a proceptual reasoning.

Data was based on the records of students' actions on the screen and the dialogues they established. Documents produced by the students were collected and field notes were taken by the researchers.

During the teaching experience, when the uses of applets are sought, these students alternated between the use of the applet and the algebraic resolution through the manipulation of the algebraic expressions. The interchange between these to ways of representation, potentiated by the tool, helped students to develop their proceptual understanding.

DATA ANALYSIS

The data analysed here is based in the use of the applet "Solving equations with balance-strategy: game" assessed in the site of the University of Utrecht in the connection with the Freudenthal Institute where the WisWeb applets are located. This applet generates an equation that the student must solve by applying the principles of equivalence. Operators are available on the top of the bar (figure 1), and the student must choose it to perform the same operation on both sides of the equation.

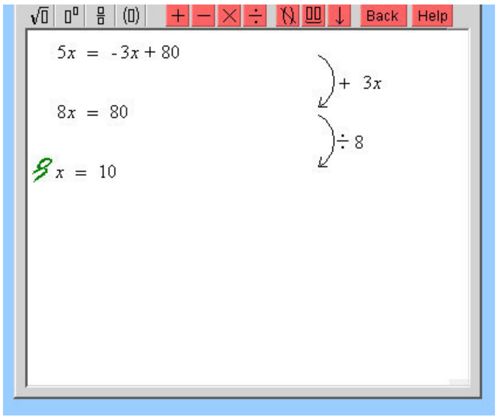

Figure 1 – Window of the applet "*Solving equations with balance-strategy: game*"

This applet asks students to solve twenty equations, and the degree of difficulty increases as they work through the task. Our main objectives were to consolidate previous knowledge and identify the thinking processes that students develop in the interaction with the tool and identify learning situations that reinforce the semiotic potential of the artefact. It is intended that students submit their resolutions on a sheet. From their solutions students should be able to explain all the procedures performed. At this stage students can start the task by solving the equation using the applet or using first algebraic procedures. If students make a mistake the applet marks it with a red mark. Even if the student changes his resolution that mark remains. This feature has made the resolution process to be carried out carefully by both groups.

Prior to this educational experience, students from both groups had developed some tasks related to the transition form arithmetic to algebra and equations solving, in order to systematize the principles of equivalence of equations. These tasks have been introduced using other applets and the algebraic procedures using pencil and paper.

The didactical methodology used was that of each group member; alternately, solve each of the proposed equations. Students were clarifying some doubts among themselves, had no difficulty in the first thirteen equations, and they performed the procedures correctly. However we can find different kinds of thinking reflected in their solutions on paper and in the dialogues among them. When solved the equation presented at figure 1, Manuel and Gustavo presented the following written answers (figure 2):

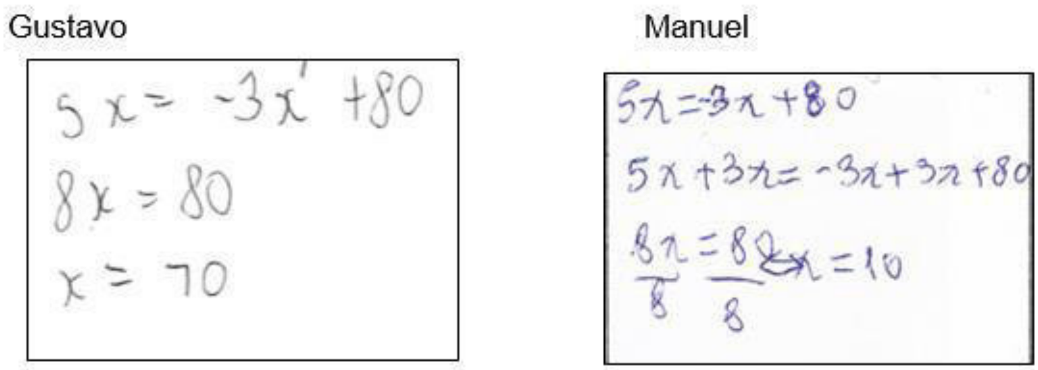

Figure 2 – Solving the 5th equation.

While Gustavo did not need to enter all procedures to explain his solution, Manuel choose to indicate all intermediate steps, explaining to Gustavo that this option was a more clear way to solve the equation. It is clear that Manuel is still in a transition from procedural to proceptual thinking, while Gustavo has already reached this level. For both students the applet worked as a learning mediator helping them to develop the two types of mediation referred by Rabardel: the epistemic mediation because the applet allows them to identify the equations and its solutions and the pragmatic mediation that allows them to transform the equations in order to solve them.

The solution presented in figure 3 was more challenging because it was the first situation where expressions with parentheses appeared. Students found the operation to be implemented by the applet, but not risked their running as they did not verify its viability algebraically. So they choose to establish the distributive property algebraically and after they advanced in solving the equation with the applet.

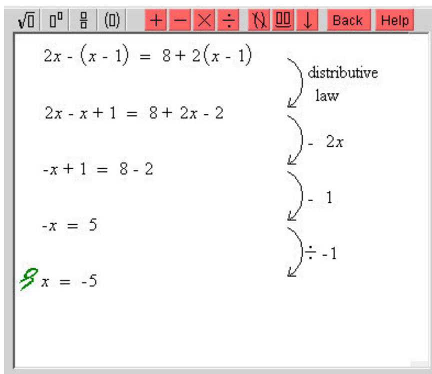

Figure 3 – Solving an equation with parentheses.

The group of Isabel and José shows this idea in their solutions (figure 4), presenting all the steps of the process. The dialogues between them and the computer screen recordings show that there were a strong interaction between algebraic procedures and the use of the tool.

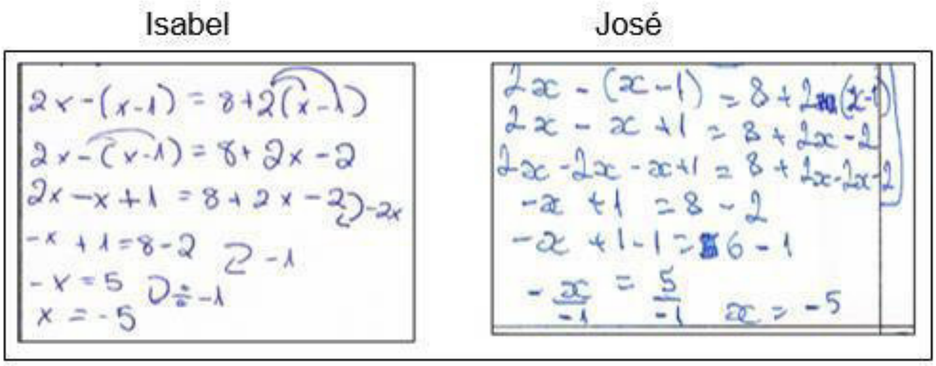

Figure 4 – Solution made by Isabel and José.

In that sense non-routine tasks are performed in the interaction between the tool and the algebraic processes, allowing them to achieve the concepts under study without teacher intervention. This solution shows us that to perform more complex tasks students used their procedural thinking as support to develop conceptual thinking. In a written solution Isabel seems to be closer to a procedural thinking by pointing out all the procedures performed without displaying all intermediate calculations. She supported his solution in the representation of the applet that worked as a semiotic mediator of this process. José is limited to perform the various steps involved in the process being able to explain how carried out the various steps of solution.

The last equations presented by the applet involved fractional coefficients (figure 5). This topic was not treated during teaching because it is not part of the curriculum at this level of education.

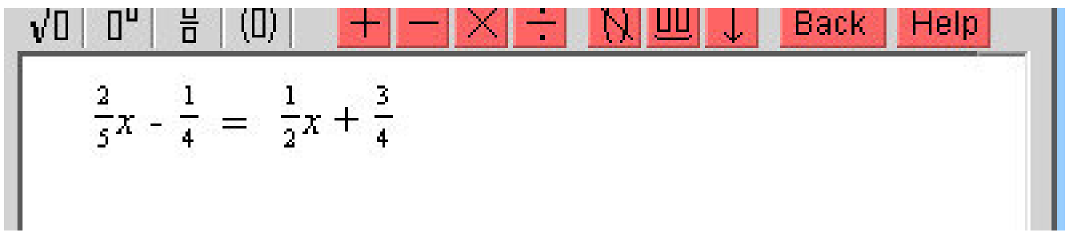

Figure 5 – Equation presented by the applet.

The students used their knowledge of operations with fractions and carried out operations in algebraic form. Only later they used the applet to confirm the solution.

The dialogues established between José and Isabel show us that both have opted initialy by the algebraic resolution of the equation. They solve algebraically the equation using different procedures. José began to use the balance-strategy suggested by the applet. Only after simplifying part of the expression found the same denominator to operate with similar monomials (figure 6).

José showed a good performance in algebraic calculation, mediated by the applet, while summoning other mathematical knowledge needed to solve the task. So he was able to assemble proceptual knowledge that allows him to solve more complex equations than those that would be expected for this level of education.

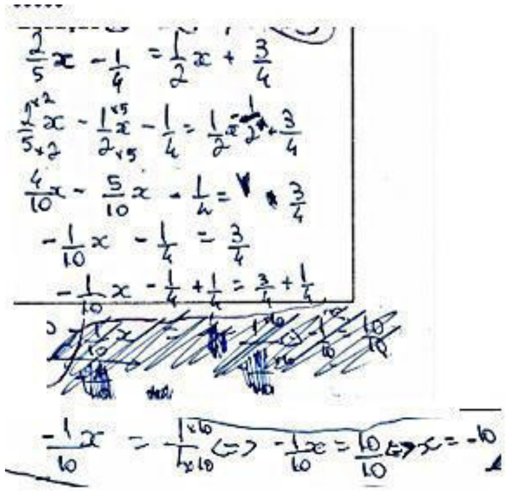

Figure 6 – Solution presented by José.

Isabel began to use knowledge she had about fractions and reduced all fractions to the same denominator (figure 7) by obtaining all the coefficients with the same denominator used to simplify the balance-strategy suggested by the applet, thus solving the equation.

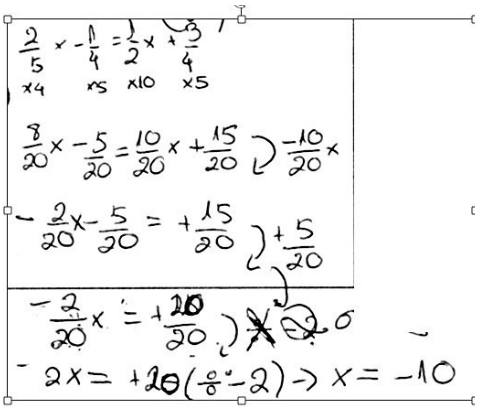

Figure 7 – Solution presented by Isabel.

In this way Isabel showed a good algebraic performance mediated by the semiotic representation that the applet provided him. This stage of the study showed that Isabel has a well developed proceptual thought, which allows her to return to the processes that are behind it, whenever this is necessary.

CONCLUSIONS

In this paper we showed how an applet-based learning environment could help students develop algebraic reasoning. Students made the transition from arithmetic to algebra involved in this technological environment and we discuss here their performance in solving first-degree equations. Activity theory helps us to understand the development of the mathematical objects, mediated by artifacts (applets and technological tools) immersed in a cultural and social context.

The use of the applet appeals to the use of the principles of equivalence in solving equations and students use simultaneously the applet and the algebraic resolution of paper and pencil. The applet worked many times as a learning mediator helping students to develop the two kinds of mediation referred by Rabardel (1995): the epistemic mediation once the applet allows them to identify the equations and its solutions and the pragmatic mediation that allows them to transform the equations in order to solve them. Depending on the familiarity with the equation presented, students use a procedural or conceptual thinking. When dealing with a similar equation to others already studied predominates proceptual thought, where students manipulate the equation as a mathematical object. When the equation presented sets up a new situation (eg. equations where the coefficients are fractional) predominates initially a procedural thinking. This thought moves rapidly toward a conceptual thinking based on the interaction that students develop with the applet.

We can thus observe learning situations where the semiotic potential of the artefact becomes evident (Bussi & Mariotti, 2008). While the tool provides them with the creation of personal meanings to perform the task, the teacher identifies the mathematical meanings expected by himself in students oral and written productions.

Notes

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BECOMING MATHEMATICAL SUBJECTS BY PLAYING MATHEMATICAL INSTRUMENTS: GIBBOUS LINES WITH WIIGRAPH

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In this paper, we bring together the vision of the body in the inclusive materialism of de Freitas and Sinclair (2014) with the vision of playing mathematical instruments offered by Nemirovsky et al. (2013). We pursue these approaches to study an activity that involved a grade 9 class in graphing motion through the use of two Wiimotes, the remote controls of the Nintendo Wii. The students were asked to face some tasks that were designed with the aim of introducing the Wii as a mathematical instrument into the classroom. We propose that the students' ways of talking, moving and feeling, which are prompted by playing the instruments, are the knowledge that they are creating and the subjects that they are becoming, which we call mathematical instrumental subjects.

Keywords: Embodiment, Graphs, Mathematical Instrument, Movement, Wii

INTRODUCTION

In his ICME 12 lecture, Paul Drijvers (2012) discussed the crucial issues of why digital technology *works or does not* in mathematics education and on which factors its success or failure may depend. "It works" referred to improving mathematics learning, providing effectiveness and motivation, and empowering teachers to better teach mathematics. Nowadays, learners are third generation digital natives, surrounded with smart touch, multi-touch and haptic technologies. This naturally prompts us to confront with these issues, even from a socio-cultural perspective. On the other hand, Aldon (2012) pointed out that "knowledge is everywhere, within reach of a click" and one of the big issues is that students "don't need to know because they know where the knowledge *is* available; or more precisely, they believe that they do not need to know because they believe that the available knowledge *is* understandable." (p. 5; our emphasis). This assumption, with its focus on availability, tends, we believe, to look at a static taken-for-granted knowledge, residing in the Platonic world or given in an *a-priori* Kantian form, both ways pulling away from the social and the material and loosing the ethical contingency of learning with the tool. Talking about aesthetic experiences in mathematics with digital tools, Sinclair (2014) sees "[f]eedback, exploration, precision, expressiveness" as features of the technology that are important in these experiences, but she also argues: "in these environments driven by the hand or body, the human is constantly reinscribing herself into the idealized, abstract mathematics." (p. 168). This is not just to say, as she writes, that the new technology "does not simply help support learning, or even change the way learning may happen—it changes the mathematics", making "the learning of mathematics somewhat of a moving target" (p. 169), as Hoyles's work had suggested was the case. Of course, digital technology-based mathematical activities, beyond engaging the visual, the auditory, the mobile and the kinaesthetic, change the mathematical concepts under investigation and, in so doing, necessarily change the mathematical discourse around them. It is to say that they make visible the way in which they summon to the body, they inevitably mobilise "the person doing mathematics—the way that person moves and thinks and feels" (Sinclair, 2014, p. 172), demanding ontological consequences that attend to the fundamental materiality of the concept and to the entanglement of knowing and becoming. Instead of exploring whether and how learners give sense to the available knowledge (as

Aldon's position silently suggests), we are much more interested in looking at *how* the new ways of talking, moving and feeling *are* themselves the knowledge that students 'make available' and the subjects they are becoming: *mathematical subjects*, as de Freitas and Sinclair (2014) would put it.

Our aim is to pursue this vision and to discuss its consequences for a particular digital technology-based classroom activity, taking advantage of some lines of flight offered by de Freitas and Sinclair (2014)'s inclusive materialism. In this activity, we have introduced uses of the Nintendo Wii as a mathematical instrument (Nemirovsky *et al*., 2013). So a second theme in this article will be how these experiences motivate mathematical investigations that engage the students in 'playing' Wii devices (the play metaphor is explained below) and in becoming mathematical subjects.

In the next sections, we will first identify some insights from the inclusive materialism for our work in this article, which mainly focuses on proprioception and embodiment, and we will trace the influence of the "playing mathematical instruments" idea on our use of the Wii in the mathematics classroom. We will then discuss an example that we use to face the following research question: How kinaesthetic experiences, driven by technology use, affect the mathematical subjects that the students are becoming and the way that they mobilise the mathematical concepts?

MATHEMATICAL SUBJECTS & MATHEMATICAL INSTRUMENTS

The inclusive materialist perspective de-essentialises the traditional view of the body, rethinking its borders as malleable and unstable, only provisionally individuated. De Freitas and Sinclair (2014) suggest that, rather than study the learner who interacts with the dynamic diagram on the screen "as an enclosed body that knows, acts or feels independently from the mouse, the screen or the digital interface more generally", we should look for "how bodies are assembled through activity." (p. 15). So the body is no longer confined to the container of its skin, nor fixed or finished at the surface, but extended beyond its contours and incorporated into the world, as "an assemblage of human and non-human components, always in a process of becoming that belies any centralizing control." (p. 25). The body is made of the indeterminate set of heterogeneous relations between the learner, the diagram, the mouse, the screen and the digital interface. Similarly, it is not just the isolated human learner drawing a circle with an inert compass, instead the tool becomes part of the learner, continually re-creating the boundaries that constitute her body.

Taking on this materialist view, we conceptualise learning as inseparable with engaging with the material surrounding, rejecting the conceptualist idea that tools have confined properties of their own and redistributing agency across the learning situation. The centre of the activity is no longer given to the learner and the learner's body is "centrally implicated in the constitution of itself and society" and granted with "some measure of agency and power in the making of subjectivity." (de Freitas & Sinclair, 2014, p. 40). Distributing agency across the situation helps us theorize the role of technology in learning mathematics differently. We draw attention to how the technology does not just change the mathematics but also the doer of mathematics, offering new ways of talking, moving and feeling to the learners. Instead of focusing on whether the technology works or does not, as in Drijvers' intent, we focus more on *how* the technology *works with*in the classroom. "How it works" refers to the ways of thinking and communicating that students experience as intrinsic in their experience *with* (not simply of) the technology. These ways connote the technology-based activity, giving the environment a creative impulse and guaranteeing a place for the social and the material contingencies of mathematical experiences. We can also look at how tools are implicated *in* the assembling of meaning in powerful agential ways. The emerging assemblage of students, technology and mathematics is always in motion, in a process of becoming. "Human bodies are constantly encountering, engaging and indeed amalgamating with other objects; the limits of our body are extended through these encounters" (de Freitas & Sinclair, 2014, p. 26). Mathematical subjects are constituted this way: "the mathematical subject comes into being (is always *becoming*) as an assemblage of material/social encounters." (p. 85). Technology carves out new subjectivities for the learners, who *are* the mathematics, through ways of thinking and moving that are occasioned by the human-technology assemblage.

Drawing on Sinclair et al. (2013), de Freitas and Sinclair make the cases of dynamic geometry software (DGEs) and motion detectors as digital technologies that attempt to mobilize mathematics temporalizing mathematical behaviour: "In a DGE, for example, a triangle is not a representation of the abstract triangle, nor is it an example of a particular triangle; rather, it is *all* and *any* possible triangles, which the user can make by dragging the vertices that determine it." (p. 90). The seeking hand, tentative and awkward at first, enters a process of learning to move. For motion detectors, "the real time feedback of the tool is what makes the graphs on the screen dynamic and responsive to *all* and *any* possible motion, which can be performed with either the user's body or an object. New ways of thinking are offered through the experience of this sensorimotor feedback" (p. 91). The seeking body also learns to move. In these situations, both the hand and the body drive the mathematical experiences through which "humans are constantly reinscribing themselves into the idealized, abstract mathematics." (p. 91). With this line of flight, our discourse becomes one about agency and subjectivity, mainly focused on the material encounters between technology, students and mathematics in assembling meaning. Here the idea of mathematical instrument is relevant.

Nemirovsky and colleagues (2013) define a mathematical instrument "as a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics." (p. 376). They use the term instrument, and not for example tool, because it intentionally connotes the culture of music, where one cannot speak of "a violinist's expertise as something divorced from the quick movements of her fingers over the strings and the trained dance of her eyes across a musical score" (p. 377). In a parallel way, one cannot talk about mathematical expertise divorcing it from the "skillful motoric and perceptual engagement with the tools of the discipline." (p. 377). Briefly speaking, Nemirovsky and his colleagues study, in the context of a science museum exhibit, emergent fluency with a mathematical instrument as a way to access certain kinds of mathematics. The language of *playing* mathematical instruments is used to conceptualize mathematical instruments analogously to musical instruments: "[f]luent use of a mathematical instrument allows for culturally recognizable creation in mathematical domains, just as musical instruments enable practitioners to produce distinct kinds of music that members of musical communities acknowledge." (p. 373). Emergent tool fluency is described in terms of perceptuomotor integration, a phenomenon that occurs when the perceptual and motoric aspects of using a mathematical instrument become intertwined. For the authors, perceptuomotor integration is constitutive of mathematics learning processes and is common to using a wide variety of tools, not only that considered in their study.

In our study, one original aim was to design tasks to explore the possibility of introducing a digital technology (that is not born with didactic purposes) as a mathematical instrument in the classroom.

The technology in question is the Nintendo Wii (with the Wii Remotes, the Balance Board, etc.) and the software WiiGraph (presented in the next section). Another aim was to examine the ways that the students, through the tasks, "played" the instruments gaining fluency, and the kinds of mathematical subjects they were becoming in doing so. Pursuing the two theoretical perspectives as complementary helps us to centre on the mathematical subjectivities that are constituted while using the Wii devices as mathematical instruments. New ways of talking, moving and feeling, and new encounters, are implicated in the process of playing the instruments. Using a metaphor preserving the idea of the human-technology assemblage, we might say that our interest is on how the learners' enter a process of becoming *mathematical instrumental subjects*.

WIIGRAPH: GRAPHING MOTION WITH THE WIIMOTES

WiiGraph, the mathematical instrument that is the focus of this study, is an interactive software application that leverages Nintendo Wii Remotes ("Wiimotes") to detect and graphically display the location of users as they move along life-size number lines. WiiGraph has been developed and released by Ricardo Nemirovsky, Coram Bryant and Bohdan Rhodehamel at the Center for Research in Mathematics and Science Education (CRMSE) of San Diego State University. Wiimote is the Wii Remote controller for the Nintendo Wii console, a small remote control with very few buttons that incorporates accelerometer and optical sensor technology supporting motion sensing capability. WiiGraph uses two controllers, possibly of two different colours, but also needs the sensor bar for the Wii, which communicates with the Wiimotes via infrared technology. Once the Wiimotes are connected via bluetooth to the software installed on a computer, it is sufficient to place the sensor bar in front of a large space for interaction, where embodied exploration through the movement of the two Wiimotes can start. Of course, a display area is also needed, being it the computer screen or a projection screen, so that the interaction can be shared within the classroom. The only constraint for commencing a graphing session is that each controller is pointed toward the sensor bar, so that its camera is directed at the sensor and a diffuse circle, matching a specific color chosen for the Wiimote (default colours are pink and blue), appears on the Graph Form (the control panel plus a graph area). When the diffuse circle is not present on the graph area the graphing does not function because the software is not able to detect the position/distance of the Wiimote from the bar. With the circle visible, WiiGraph will instead produce a real time graph of the same colour. As written in the user guide, WiiGraph provides several graph types, challenges, and composite operations for users to individually and collaboratively explore, including shape tracing, maze traversal, and ratio resolution. The graphs are configured in the graph area according to selected graph type, operations, ranges, time periods and targets (see Fig. 1a). On the control panel, one can choose other options, like for example play, pause or refresh an experience, but also hide or reveal a specific curve. The Line Graph type, without operation, allows for depicting the two distance versus time lines ($a(t)$, $b(t)$) corresponding to the Wiimotes' movement in front of the sensor bar, where a and b are the positions of the controllers. The thin pink and blue graphs are shown on the same Cartesian plane in the fixed time interval and they correspond to individual users (see Fig. 1b).

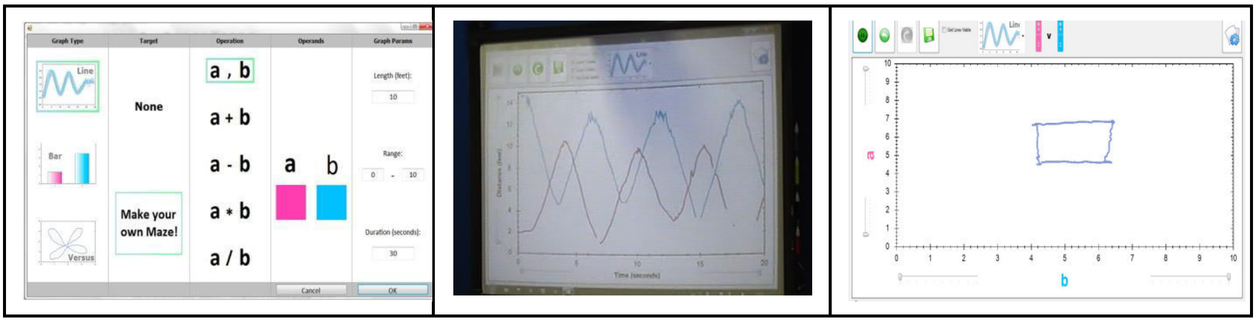

Figure 1. (a) Graph setup window; (b) lines after a Line Graph session; (c) Versus Graph session

When an operation is selected, for example the sum *a*+*b*, this type adds to the previous lines a new thicker and darker distance versus time line, which is the result of the operation at any given point, in this case (*a*+*b*)(*t*). Very intriguing is the "Make your own Maze!" target, which can be used both with or without operation option. In this situation, a target maze can be built with a certain number of inflection points, width, tension and layout (giving the graph a certain degree of complexity) and becomes a challenge target for the two individual users. At the end of the experience, a score is associated to the users and measured with respect to the maze's degree of complexity. Another type that we used in our research is the Versus Graph type, which plots an ordered pair of the distances of each user over time (the creation of the ordered pair is implicit). One of the most interesting challenges for the Versus Graph involves the creation of plane shapes, like rectangles, diamonds, circles (e.g. Fig. 1c). While in the Maze session the users are competitive, challenging each other to get the best score, this other activity gives the students the opportunity of working together and being collaborative to reach a common goal. Attention in this paper focuses only on Line Graphs, which our students used in the first part of the study as described in the next section.

METHOD AND ACTIVITY

This study is part of a wider research that has been developed for a Master Degree thesis at the University of Torino, in Italy. A grade 9 class of 30 students and their regular teacher was involved for about two months in the project. The main goals of the project include the design of tasks that use the Nintendo Wii in teaching mathematics and ongoing research on how proprioceptive and embodied aspects related to playing the Wii are entangled with the social/material contingencies of learning about temporospatial relationships. Two main sessions were planned, in which the students alternatively worked individually and in groups of three, and participated in collective discussions guided by the two authors, who were both present in the classroom. All these phases were filmed, two groups of three students during group work. The first session concerned the use of WiiGraph and two Wiimotes to work specifically on the concept of function in graphing motion; the second session centred on the use of a Balance Board (BB) along with the software DarwiinRemote, which allows reasoning on the strategies adopted by a person who plays a game with the BB, and the corresponding motion trajectories of her centre of gravity throughout.

In this paper attention is on the first session and specifically on the use of the Graph Line type. At this moment, the students still had limited experience with WiiGraph about moving in order to get specific graphs. In the classroom setting, we projected the WiiGraph window on an IWB and we used a blue and a pink Wiimote, in order to easily match the graph colours. The students could move two at a time, each holding a Wiimote toward the sensor bar, and reason on the information furnished by the real time graphs depicted in the graph area of the software. For the activity, which is the focus of this study, the class was divided into groups of three students who were challenged to face the following task: Think of a way to obtain two line graphs with three "humps" without being one superimposed to the other. Two students from distinct groups were asked to move.

In the episode below, Alessandro and Emanuele have just moved and obtained the two intersecting lines shown in Figure 1b. This entails a new task: having exactly three humps, with no additional pieces, and one graph shifted vertically with respect to the other (a mathematical translation).

ANALYSIS AND DISCUSSION

The students are talking about dividing the given 20-second interval by 3 to have equal humps, and to know "the time of a hump" (Lucrezia). But the researcher shifts attention to the way of moving:

1	R:	The idea, say, is to have three different time intervals, three distinct intervals but equal long in time [*Mimes the interval*], in which what do I do?
2	Emanuele:	A complete hump
3	R:	How do I move? It [*Indicates WiiGraph's graph area projected on the IWB*] makes a complete hump. I do something else, you [focus] do something else
4	Emanuele:	I do, I start from the beginning [*Indicates the physical start*], I arrive at the end [*Indicates the physical end*] and then I go back to the start [*Indicates the physical start again*] in 6 seconds and a half

Even if Emanuele expresses an idea for how to move in the interaction space, some students return to the division. Federico, for example, suggests that they could reduce time to 18 seconds because "it's easier", or make a hump in 6 seconds and the other two in 7. Alex focuses on the back and forth movement that gives a hump, thinking of dividing 20 seconds by 6 instead of 3. It is at this point that the researcher moves the focus away from numbers again toward the two movements:

9	R:	All right, you want to be super precise. But for me, if we try, maybe we're able to well understand if we're good, if we can be good, if we're able to do what we aim to. Who moves? Them again? [Some say: Yes] Are you ready? [*Alessandro and Emanuele come in front of the sensor with the Wiimotes*] I understood how he moves; I didn't understand how Alessandro moves
10	Emanuele:	According to my idea [*Moves to his start, while Alessandro says: I've tried to imitate him*], I'd do this way [*Starts moving forward, keeps the Wiimote pointed to the sensor with his right hand*] in 6 seconds and I'd come back [*Moves backward*]. But he'd do the same [*Moves to the position from which Alessandro should start moving*] from here [*Stops at his previous end*] while I'm there [*Indicates his start*], this way [*Moves forward, now the Wiimote in his left hand, keeps pointing his right arm back to the floor, looks back*], and he'd come back [*Gazes at the IWB*]. Two humps like these [*Opens his right hand thumb and index finger to form a distance in the air in front, mimes the making of two 'parallel' lines with humps*; Fig. 2a] will show up
11	R:	I heard a No, who said no?
12	Lucrezia:	He [*Refers to Alberto*] did

13 Alberto: Shouldn't the humps be opposite? [*Mimes two opposite humps in the air*]

14 R: What do you mean with opposite?

15 Alberto: One upward and the other one downward... [*Mimes the two curves, goes to the interaction space*] Simply, he [Alessandro] had to start from here and go to, there [*Points to the starting and end positions on the floor*], so distance reduced [*Looks at the IWB, mimes a decreasing piece of curve*]. Instead, where did you start [*Refers to Emanuele*]?

16 Emanuele: I started from there [*Indicates his start*] and arrived halfway [of the space]

17 Alberto: So they both came out this way [*Mimes twice the first hump in the air*]

18 R: Maybe you [Emanuele] told it differently before

19 Alberto: Yeah, before you told that you started from here [*Points to the intermediate position*] and went backward [*Moves to repeat the movement*]

20 Emanuele: No, I told... He [Alessandro] started from here [*Indicates the intermediate position*] and went forward [*Mimes the first part of movement with his right index finger pointed to the floor*], I start down there and go here [*Mimes his initial and end positions with the index finger pointed to the floor*]

21 R: Well, in this case what does it happen?

22 Alberto: In this case, yes, two humps like these [*Mimes the making of the humps with his right hand thumb and index finger open*; Fig. 2b] take shape

23 R: So your conjecture [*Refers to Emanuele*] is that it comes out what?

24 Emanuele: [*Goes to the IWB, taking also the second Wiimote that was on the desk*] That two humps like these come out [*Positions the Wiimotes close to the graph area on the IWB, mimes the making of the first hump moving the controllers parallel to each other as if they were graphing the lines*], like these [*Goes on moving, changing from the first to the second hump*; Fig. 2c], like these [*Mimes the making of the initial part of the last hump for both lines*; Fig. 2d], with the lines like these

25 R: All right, can we say that they're two shifted lines?

26 Ss: Yeah, parallel

Figure 2. (a, b) The 'distance' gesture shared (c, d) The Wiimotes making the lines on the graph area

The researcher's request for an explanation of the way in which Alessandro has to move entails new ways of talking, moving and feeling by the students. Emanuele first moves back and forth in the farther half of the interaction space, showing how he would move to obtain a hump. While walking, he keeps the Wiimote pointed at the sensor bar as if WiiGraph was working, even with the same tension in his body. When Emanuele starts talking about the way that his mate should move, he changes the reference space to the closer half as if time was re-started from zero. He is playing

Alessandro's role now, in walking back and forth in the new space, pretending to be his mate walking contemporary to him. The parallelism between the movements (of the two boys, of the two Wiimotes as well: "the same", "this way") is actualized through the right arm indicating the farther reference space, where Emanuele previously experienced his movement. The arm is always kept in a rigid position, pointed with the same slant towards the floor. Emanuele is imagining to produce both the movements, the one identified by his 'being' Alessandro, the other one by the target positions on the floor given by his pointing arm. It is as if he had the need for being in two distant positions at the same time. The new 'distance' gesture soon appears in the air space, shared by Emanuele and Alberto. It contracts the fixed distance just created and expands it to the virtual points that would originate the two graphs ("two humps like these"). The parallelism between the motion experiences is conveyed from the positions on the floor to the shortly distant positions in the air, where the way of moving the two open fingers actualizes the two 'gibbous' lines in a plane that appears to be vertical respect to the floor ('gibbous' means for the lines to have humps). But unexpectedly, Emanuele moves again, from this vertical plane to the graph area on the IWB as if they were the same plane. He grabs the other Wiimote (held by Alessandro before the discussion) and directs to the IWB for re-actualizing the graphs in the space in which the software would depict them. Significantly, he puts the controllers very close to the surface of the IWB but surprisingly moves them without touching the surface. In a way, he positions himself between the interaction space (the previous movements) and WiiGraph (the mathematical lines). He also adds a rhythm to the 'pace' of the three humps ("with the lines like these"). It is the rhythm that before set the pace of the moving fingers that generated the gibbous lines in the air, particularly present in Alberto's gesture. It gives a way of feeling that is expressed even in words, by the repetition, three times (just as three have to be the humps), of "like these", the feeling lived during the imagined motion experience in front of the sensor.

CONCLUSIONS

The episode opens and closes with the controllers: they are used in two different manners to talk about and actualize the same thing. The first manner is the usual one that is entailed by the bodily experiences with WiiGraph (moving by keeping the Wiimotes pointed at the sensor). The second manner has instead to do with the mathematical modelling of these experiences, through the graph lines that are depicted by the software, and entails the unusual way in which the students imagine to make 'the work of the software' (moving the controllers at the IWB to make the two lines). But the students are no longer simply moving or imagining to move in the interaction space, they are doing this *with* WiiGraph. The controllers, the software, the sensor bar are all implicated in the assembling of meaning, along with the lines and the steps. The tools are not inert but constantly interact with each other and with learners' bodies. Interactions with the graph are necessarily embodied: at the very least, it is necessary for human perception and the body to coordinate with the graph. These bodily and kinaesthetic experiences, such as pointing to positions on the floor with the arm, moving at a certain pace in the interaction space, miming particular humps in the air with two open fingers, moving the Wiimotes parallel on the graph area using both hands, are also important in the learning process in that they are shaped by material encounters such as the design of the task in a visual and proprioceptive sense. In other words, Emanuele's (or Alberto's) interactions with the graph are not merely initiated by the student, but are compromised between his body and his surrounding material world. Through these encounters, body boundaries are re-created and the assemblage of students-technology-mathematics emerges as the moving body in the classroom.

The mathematics is not really just standing there but it is much more implicated in the moving legs, arms and hands. According to de Freitas and Sinclair, the gestures are boundary-drawing devices, which reconfigure the world shifting the divide between the real and the mathematical, between matter and meaning. Emanuele's hand gestures of miming the three humps are not merely iconic representations of the lines. Rather, they are part of the emerging human-technology assemblage that starts giving rise to new functional thinking and fluency in playing the tools. The students enter a process of becoming mathematical instrumental subjects with the mutating assemblage. The body movements located in the physical space are just part of a more complex movement: the tools and concepts are mobile and full of potentiality, but the *most freedom* of movement belongs to thought. The episode sheds light on the on-going movement of thought, which actualizes the virtual and of which the sensory-motor is the actualization (through gesture, voice, feeling), where the virtual is that dimension of memory that fuels the creation of the new (de Freitas & Ferrara, 2014). This movement of thought marks the indeterminate whole of material entanglements of the humans and non-humans components that are implicated in doing mathematics in this situation. Focussing on it, we can reconceptualise learning as occurring in the unscripted relations amongst human, technology and mathematics, calling attention on the embodied nature of mathematical thinking and learning.

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THE CHALLENGE FOR MATHEMATICS TEACHER EDUCATORS: LEADING STUDENTS TOWARD TEACHING IN A TECHNOLOGICAL ENVIRONMENT

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The purpose of our research was to learn how pre-service teachers [hereunder – "students"] who studied as school pupils without technological tools adopt technology in their teaching. The research population consisted of 20 students from two academic years. All the students participated in two courses, a didactics course "Teaching Mathematics" and a pedagogy course. During the didactics course the students were asked to present instructional activities on a chosen mathematical topic that included usage of some technological tool, while during the pedagogy course the same students were encouraged to use those tools in their teaching. In both courses the students were required to specify the added value of using the chosen technological tool, i.e. how it led to qualitative changes in teaching. We found, that the students of the two courses integrated various technological tools, such as digital presentations and dynamic mathematics software in the prepared activities as well as in their practical teaching. We also found that the students, at an earlier stage of their education, believed that the integration of technology could enhance their pupils' learning, while at a later stage they realized the benefits of using technology to stimulate pupils' interest in learning as well as for more effective lesson planning.

INTRODUCTION

A growing number of studies indicate that technology has become an integral part of teaching and learning for younger generations (Olive et al., 2010; Oldknow & Knights, 2011; Cheung & Slavin, 2013). In this context, it has become extremely important that pre-service teachers [hereafter – "students"], who themselves have had no experience of studying in a technologically-enhanced environment; learn how to teach using new advanced technologies. Students usually attempt to rely on their own experience accumulated as pupils. At the same time, they attempt to adopt the theories they acquired during their training. Thus, in practice they frequently find themselves in a state of indecision. As three students in one of our classes noted in their assignment:

> Of course, most of us belong to the constructivist educational approach of today, but we are affected by the conservative educational approach that prevailed in our times. Therefore, we try to take the best of each of those approaches (February, 2013).

In school mathematics, digital technologies include a wide range of dynamic resources providing visualizations and enabling active learning by pupils, i.e. experimenting with different mathematical objects and receiving immediate feedback, exploring a given object or relationship and then validating solutions simply by dragging the resulting objects, thereby corroborating or to refuting a conjecture and then, if necessary, moving on to another conjecture (Drijvers et al., 2010). In viewing these benefits, numerous recent studies have indicated that integration of technology into mathematics classes has a significant impact on mathematics teaching by enhancing the acquisition of mathematical learning (Monaghan, 2001; Hollerbands, 2007). Moreover, it was observed that the use of technology in mathematics classes may motivate students to learn mathematics (Nuggent et al., 2006), enhance pupil participation and interaction (Ruthven & Lavicza, 2011), and contribute to effective lesson management (Perrotta, 2013).

At the same time, several studies argue that not all students are confident in the use of technology, nor are they all convinced of the benefits of computer-aided teaching (Trouche, 2005). D'Souza and Wood (2004) found that students frequently mistrusted software and felt more comfortable with traditional methods; namely, they preferred using pen and paper, because this was more reliable and easier. Thus, using various technological tools and learning how to work with them become a real challenge for students and their educators (Desimone, 2009). Recent studies have widely discussed the two types of barriers to integrating technology into teaching: first-order barriers, such as environmental unavailability and teachers' lack of knowledge, and second-order barriers, such as teachers' lack of belief in the potential of technology (Angeli & Valandis, 2009, Ertmer, 2005, So & Kim, 2009).

Changing pedagogical beliefs towards the constructivist approach while integrating technology obliges teachers to modify their beliefs, which might be irreversible, and thus the change is difficult to implement (Ertmer, 2005; Kim, Kim, Lee, Spector & DeMeester, 2013; Ellis, 2014). Furthermore, as a result of integrating technology, even the teachers' way of "doing mathematics" may change - from the belief that in mathematics there are only correct or incorrect statements to the belief that mathematics may mean the process of solving a particular mathematical problem, while refining the understanding and clarifying the correct mathematical ideas (Sachs, 2014). Therefore, teacher educators should foster students' skills of critical thinking and analysis so that they may plan a lesson in line with the constructivist approach while integrating technological tools (Handal, Campbell, Cavanagh, Petocz & Kelly, 2012). Minor, Onwuegbuzie, Witcher and James (2002, p. 116) suggest that students should learn "to examine and confront […] beliefs and values they hold regarding various aspects of the practice of teaching". Deaney and Hennessy (2011) discuss the importance of looking for an adaptive approach to harnessing technology that can address a wide diversity of individual differences encountered in a very mixed class of students. Salomon and Ben-Zvi (2006) propose that when introducing technology, instructors must ensure that this integration has added value, i.e. it leads to qualitative changes in teaching.

In the current research, we traced changes in students' attitudes toward the usage of technology in their teaching at different stages of their education. More precisely, we tested the students' choice of technological tools as well as the assumed benefits of technology integration as expressed in their instructional activities prepared as part of a didactics course as well as in their lessons during their practical work in school as a part of a pedagogy course (for more details about the above courses, see Methods).

METHODS

Participants and setting

The research was conducted over two years - 2010-2011. The 20 participants of the first phase were second-year students in the educational program for secondary-school mathematics teachers in the Mathematics Teaching Department at Achva College of Education in Israel (15 women, 5 men; Mage = 27.5 years; age range = 19–40 years). These students were enrolled in a didactics course during the second semester of 2010. The participants of the second phase were 15 students from the first group taking a pedagogy course (11 women, 4 men; Mage = 25.9; age range = 20–35 years) during the academic year 2010 – 2011.

During the didactics course, the students acquire knowledge about different methods of teaching while planning various instructional activities on a chosen mathematical topic. Each student was required to prepare activities that included some technological tool and present it in class. During the pedagogy course, students were required to plan a lesson. They were encouraged to integrate technological tools during their teaching as part of the practical work in school. In addition, the students of the two above courses were asked to explain their reasoning for choosing the tool as well as its added value (e.g. the features of the tool that improve the learning/teaching experience) for the given activities/lesson.

Data Collection

The data were collected via artifacts and an anonymous open-ended questionnaire at the end of each academic year. The artifacts included students' instructional planning activities and reasoning in the didactics course and students' lesson plans and reflections in the pedagogy course. The questionnaire consisted of two items, as follows: (1) In your opinion, what are the benefits of integrating technology while planning the activities (didactics course) / when teaching (pedagogy course) with respect to your pupils? (2) In your opinion, what are the benefits of integrating technology while planning the activities (didactics course) / when teaching (pedagogy course) with respect to you as a future teacher?

DATA ANALYSIS

The qualitative analysis: All the data were analyzed in order to extract categories that referred to the benefits of using technologies in mathematics lessons. We used the phenomenographic approach (Marton, 1986) to data analysis, in which subjects' utterances are grouped according to similarities and differences among their explanations. Student responses were regularly analyzed for commonalities and three principal categories were established. Namely, we found that the students refer to two categories with respect to pupils: (1) improvement in learning and (2) enhancement of pupils' motivation to learn, and one category with respect to themselves as teachers: effectiveness of lesson management. Further classification into sub- categories appears in the Results section.

The quantitative analysis: For the quantitative analysis, frequencies of each technological tool chosen by students were calculated and tabulated. Frequencies and percentages of every established category referring to the benefits of using technology were assessed separately for each course, and χ^2 tests were conducted to compare the rates of each derived category between the two courses.

RESULTS

Table 1 presents the numbers of the students that used each type of technological tool as a preferred one, separately for the didactics and pedagogy courses.

Course / Type of tool*	Didactics course	Pedagogy course
Mathematics software	2	11
Applets	7	4
Digital presentation	19	4
Materials from the internet	2	1
No technology tool	0**	6

Table 1: **Technology tools chosen by the students in the two courses**

* Some of the students mentioned using more than one technological tool in the same activity/lesson.
** In the didactic course the students were required to use some technological tool.

The results indicate that the most popular technology tool among the students of didactics course was digital presentation, while most of the pedagogy course students chose dynamic mathematics software.

In the table below, the frequencies and percentages of students who indicate the benefits of using technology in teaching according to each derived category and sub-category are shown separately for the didactics and pedagogy courses. A χ^2 test was performed in order to compare the above percentages within the sub-categories for each course.

Categories / Total no. of students	Didactics course 20	Pedagogy course 15	χ^2 test by course
Improvement in learning:			
Improvement of pupils' understanding of the material	12	4	
	60%	27%	$\chi^2 = 3.84$, $p = 0.05$*
Development of pupils' self-learning abilities	11	5	
	55%	33%	$\chi^2 = 1.62$, $p = 0.2$
Formation of mathematical thinking	12	4	
	60%	27%	$\chi^2 = 3.84$, $p = 0.05$*
Enhancement of pupils' motivation to learn:			
Stimulation of pupils' interest in learning	9	14	
	45%	93%	$\chi^2 = 8.89$, $p = 0.0029$*
Creation of an environment that encourages learning	4	2	
	25%	14%	$\chi^2 = 0.27$, $p = 0.604$
Promotion of fun	10	0	
	50%	0%	$\chi^2 = 10.5$, $p = 0.0012$*
Effectiveness of lesson management:			
Dealing with diversity among pupils	8	7	
	40%	47%	$\chi^2 = 0.16$, $p = 0.693$
Visualization of learning material*	6	10	
	30%	67%	$\chi^2 = 4.64$, $p = 0.0312$*
Lesson planning*	8	12	
	40%	81%	$\chi^2 = 5.6$, $p = 0.018$*

Table 2: The rate of computer benefits estimated by the students by type of course

The results corresponding to the significant differences between the two courses ($p \leq 0.05$) are marked by *.

The results obtained with respect to the three derived categories indicated as follows:

Improvement in learning — The percentage of students who believed that using technology in teaching had benefits in terms of pupils' learning was significantly higher among the students of the didactics course (for two of the three sub-categories, namely for "formation of mathematical thinking" and "improvement of pupils' understanding of the material").

Enhancement of pupils' motivation to learn —The percentage of students who believed that using technology in teaching had benefits in terms of "stimulation of pupils' interest in learning" was significantly higher for the students of the pedagogy course, while for the sub-category of "promotion of fun" it was higher for the students of the didactics course.

Effectiveness of lesson management —The percentage of students who believed that using technology in teaching had benefits in terms of "effectiveness of lesson management" was significantly higher among the students of the pedagogy course (for two of the three sub-categories, namely for "visualization of learning material" and for "lesson planning").

DISCUSSION

In the present study we attempted to follow the changes in our students' choice of technological tools as well as in the benefits of integrating those tools in teaching according to their own estimations made at different stages of their education.

The interest in this subject is based on our more than 15 years' experience of integrating technology in teaching. During this period we, as instructors, underwent meaningful changes concerning the usage of various mathematics software as well as digital presentations in our mathematics teaching (Gorev & Gurevich, 2015; Gurevich & Gorev, 2012). We observed that the integration of technology in teaching improves the students' learning: promotes understanding of the material; contributes to the development of mathematical thinking and enhances the students' motivation to learn. Based on this experience, we aimed to adapt our students, who are future teachers, to a high-tech environment.

We compared both the technological tools chosen by the students and their beliefs in the benefits of the chosen tools across two courses. In the didactics course, the students learned to plan the activities, whereas in the pedagogy course they practiced planning and teaching lessons at school. The results of the present study show that in the didactics course the students preferred to use digital presentations as the main tool for their activities, while in the pedagogy course the same students preferred dynamic mathematics software. We suggest that the above finding might be explained by the fact that the students of the didactics course, in the second year of their training, are not yet sufficiently confident. Hence, they prefer planning the required activities using digital presentation for greater self-confidence. Conversely, the pedagogy course students, who were in the third year of their training and were accompanied by the pedagogical instructor both in their lesson planning and during their classroom practice, dared to implement open mathematics software in their teaching. At the same time, it should be noted that six students from the pedagogy course did not integrate technology into their lessons. This might be due to the fact that not all the schools where the practice takes place are suitably equipped for this yet.

With regard to the results related to the benefits of using technology estimated by the students themselves, interesting and significant changes were found within the two courses explored. The results indicated that the percentage of students who found benefits of using technology in teaching in terms of improvement of understanding of the material as well as pupils' formation of mathematical thinking was significantly higher among the students in the didactics course than among those in the pedagogy course. We suggest that the students of the pedagogy course simply were not able to see improvement in pupils' learning since during their practice they met them only once a week, while those pupils continued studying mathematics twice more each week with their regular mathematics teacher. Thus, the students were unable to follow any changes in the pupils' comprehension of the material. With regard to the students of the didactics course, their relatively high estimations of the role of technology in terms of improvement in learning might be a consequence of their own positive experiences as learners at the college. At the same time, we found that the percentage of students who specified the benefits of using technology in teaching in terms of stimulation of pupils' interest in learning, visualization of learning material and lesson planning was significantly higher among the students in the pedagogy course than those in the didactics course. These results indicate the successful experience with technological tools in teaching lessons during their practice in school. Moreover, the above results illustrate students' ability to generate a change in their teaching by using modern technology.

Thus, based on the results of the current study obtained during, didactics and pedagogy courses for mathematics teaching, we suggest that instructors should coordinate all their efforts in order to help students to prepare for teaching in a modern technological environment. The contribution of the current study is the presentation of an example of professional development of student teachers while integrating technological tools.

We believe that the experience presented might allow educators to help student teachers to overcome psychological, didactic and technical barriers, as well as develop their beliefs in the power of technology in teaching.

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DESIGNING INTERACTIVE REPRESENTATIONS FOR LEARNING FRACTION EQUIVALENCE

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This paper describes a study that investigated the use of a tool in an exploratory learning environment (ELE) designed to support students' understanding of equivalent fractions. The study, part of a larger project, involved 67 9-11 year old students in England. It addressed the question: How does a partitioning tool support students' conceptual understanding of equivalence? Data were collected through observations, students' written work in an equivalence task, and a written self-reflection on learning at the end of their time in the ELE. Results showed that using the partitioning tool with an area representation was instrumental in challenging some students' preconceived ideas about equivalent fractions and that the students were able to develop situated abstractions about fraction equivalence.

Keywords: equivalent fractions, exploratory learning environments, design, partitioning, fraction representations

INTRODUCTION

It is widely accepted that fractions are a complex and difficult aspect of mathematics education. An early emphasis on whole-number constructs using additive structures foregrounds the multiplicative structures used with rational numbers, while this whole-number bias (Behr, Harel, Post & Lesh, 1994; Schmittau, 2003) appears to be a barrier within many fraction equivalence misconceptions. As a result, students often inappropriately apply whole-number schemes, instead of seeing a fraction as a quantity or quotient relation between two numbers, and default to seeing it as two separate numbers (Murray & Newstead, 1998). Furthermore, fraction tasks (especially computerised ones) are often limited in scope and their approach is typically instructional and procedural.

Using a Design-Based Research methodology (Design-Based Research Collective 2003), we have developed an exploratory learning environment, Fractions Lab, that allows students to interact with various fractions representations, add or subtract them, and check their equivalence. In this paper, we explore how our design decisions related to fraction equivalence tasks enabled students to build upon their intuitive thinking about fractions (Mack, 1990) and challenged them to reflect on the feedback they received.

FRACTIONS LAB

Fractions Lab is an exploratory learning environment that acts as a stand-alone program or as a component of the iTalk2Learn project's (www.italk2learn.eu) intelligent tutoring system. It aims at fostering conceptual knowledge, which we define as implicit or explicit understanding about underlying principles and structures of a domain (Rittle-Johnson & Alibali, 1999). The focus of this type of knowledge lies on understanding why, for example, different mathematical principles refer to each other and on making sense of these connections. Conceptual understanding of equivalent fractions, for example, includes students being able to make connections between fraction representations by understanding what is the same and different within them (Lesh et al., 1983) and showing that a fraction represents a number with many names (Wong & Evans, 2007).

Fractions Lab adopts a holistic approach, encourages student-directed activity, adopts a constructivist stance to learning and assumes an active role for the student (Ben-Naim, Marcus, & Bain, 2008). Tasks that would support students' conceptual development within the Fractions Lab environment were also developed to support students to address common misconceptions and conceptual understanding. Students are given tasks that, for example, ask them to (i) construct three fractions equivalent to 3/5, (ii) find the odd fraction out, (iii) make a fraction equivalent to ¾ that has a denominator of 12, and (iv) explain whether another student is correct or incorrect when they state "2/6 = 1/12 because 2x6 = one 12".

FRACTIONS LAB DESIGN DECISIONS

We took several design decisions that involve the use of virtual manipulatives as fraction representations (area, number line, sets of objects, liquid measures and symbols) (Lamon, 2012) and complementary tools. In this paper we discuss three graphical representations (rectangle, number line, measuring jug, because sets were unavailable in this iteration) and the partitioning tool, a tool designed to support students' understanding of equivalent fractions.

In Fractions Lab, students construct their own fractions using the virtual manipulatives within the learning environment. Typically, virtual manipulatives include additional features and options that develop what a physical manipulative offers and they can also represent situations that are not even possible with physical manipulatives (Moyer, Bolyard & Spikell, 2002, Reimer & Moyer, 2005, Steen et al, 2006; Suh & Heo, 2005). When using teacher-led virtual manipulatives involving fraction equivalence, Suh & Heo (2005) found students linked graphical and symbolic representations, experimented and tested hypotheses (using trial and error in a non-threatening environment), spent longer on task, and appeared to model the fluid nature of their thinking.

We designed Fractions Lab to enable students to perform actions on representations that they would not be able to with physical manipulatives. For example, it is possible to establish a relation between a part and a whole by partitioning a rectangle, changing the denominator and numerator while leaving the original whole intact, something that was not previously possible without destroying the original (Olive & Lobato, 2007).

Tools

Students can manipulate each of the representations by using various tools to partition, add or subtract fractions, while executing tasks that challenge common fraction errors and misconceptions. We designed the tools to enable a student to offload what would be otherwise a heavy and error-prone cognitive burden (Pea, 1993). Additionally, students are able to undertake "profoundly different" (Nardi, 1998) actions that "enable us to act, perceive and reason beyond our natural limits" (Nunes, 1997:30). Clarebout, Elen, Johnson & Shaw (2002) suggest that support tools should be embedded into a software task and that students should be able to choose to what extent they use them. However, students must enter into a relationship with the tools to "afford the user expressive power: the user must be capable of expressing thoughts and feelings with it. It is not enough for the tool to merely 'be there', it must enter into the user's thoughts, actions and language" (Noss and Hoyles, 1996:59). In this paper we focus on just one of the tools available in Fractions Lab: the partitioning tool.

The partitioning tool

The partitioning tool was designed to support students' understanding of equivalence. Figs. 1, 2 and 3 show the stages of three different representations being partitioned. It was our aspiration that students would use the partitioning tool to split up the parts of the representation already shown to reflect the original fraction made and, with carefully-designed tasks, notice patterns within the changing fraction symbol.

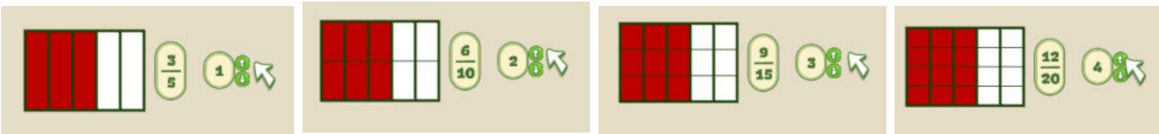

Figure 6: 3/5 in a rectangle, partitioned three times. The fraction symbol changes to reflect the new fraction name.

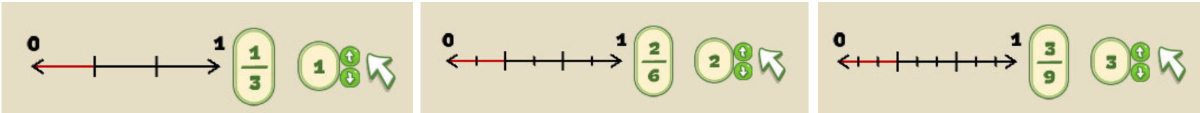

Figure 7: 1/3 on a number line, partitioned twice. The fraction symbol changes to reflect the new fraction name.

Figure 8: 3/4 in a measuring jug, partitioned three times. The fraction symbol changes to reflect the new fraction name.

METHOD

We validated our design decisions based on an analysis of a series of ‘design experiments’ (Cobb *et al.*, 2003) that provide support for the efficacy of Fractions Lab and the lessons learned for interactive fraction representations. In the iteration reported in this paper, three representations were available to the students. These representations were area, number line and liquid measures.

We worked with 32 Year 5 (9-10 year old) and 35 Year 6 (10-11 year old) students from one school near the end of their academic year, visiting each cohort once over a period of two days in June and July. All the participants and their parents gave informed consent and permission for the study was obtained from the school's staff. The purpose of the study was to evaluate the impact of the tools and the effectiveness of the tasks on students’ conceptual understanding of fractions equivalence, addition and subtraction. Prior to using Fractions Lab the students had not met the notion of partitioning a fraction to find an equivalent.

During each visit, each student worked with Fractions Lab for a duration of 15 - 30 minutes, undertaking a small and varied selection of tasks, such as those outlined above. 18 Year 5 students completed a task to familiarise them with the partitioning tool. All students (n=67) completed a ‘reflection on my learning’ questionnaire afterwards. We captured all students’ work on the screen and recorded their speech. We periodically intervened to gain further insights into the students’ thinking-in-change or to aid their reflection on the task.

FINDINGS

First, we present a case study of a student completing one equivalence fraction task in the ELE, to demonstrate their thinking-in-change about fractions. We follow the case study with data from the partitioning tool task and the reflection questionnaire, demonstrating the impact Fractions Lab had more generally on some of the students' thinking about equivalent fractions.

1. Case Study

We first present a case study of a student completing the task *Explain whether another student is correct or incorrect when they state "2/6 = 1/12 because 2x6 = one 12"*. Initially, the student (who we will call George) believes the statement is correct. The case study demonstrates how, over 12 minutes 45 seconds, and with feedback from Fractions Lab, George's thinking changes.

1	George:	2/6 = 1/12 because two sixes equals twelve. [Constructs 2/6 and 1/12 and uses the 'compare tool' to check]. Oh. This [computer] is wrong.
2	Researcher:	Why do you think it is wrong?
3	George:	The computer is wrong because I know that 1/12 equals 2/6. Look, um, 2/4 = 1/8.
4	George:	I know one twelfth equals two sixths [pause] or is it I'm wrong? I don't know ... I don't know which one it is. [Period of trial and error. The student finds that 2/4 and 4/8 are equivalent].
5	George:	Yay! It worked!
6	Researcher:	Why do you think it worked?
7	George:	I know what I did wrong ... need to have one twelfth equals twelve [pause] fourths, no, no, no I'm confused. [Continues to explore why 2/4 and 4/8 are equivalent].
8	George:	So what makes this one work that the other one doesn't? [pause] [points to denominators] Two 4s equals four 8s. What's four 8s? 32. So I need to make a thirty two[th].
9	Researcher:	Why don't you try out 2/6 and 1/12 to find two fractions that are equivalent?

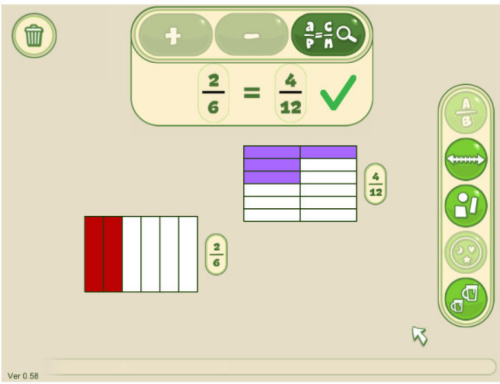

Figure 4: Screenshot of George's work at line 10

10	George:	[Makes 2/6, then partitions 1/12 into 2/24. He changes the denominator to 26 and the numerator to 4. He changes the number of partitions until he makes a denominator of 10]. Wait. [Makes a denominator of 12 then 14]. Wait, is it 12? Let's try that one. [Makes a denominator of 12 (so the fraction is 4/12). Checks the two fractions are equivalent]. Yay!

11	Researcher:	Why do you think that is correct?
12	George:	What is four twelves? Forty eight. [Pause] I know what I did. It's got to make up the same number. So they're a bit the same. They are in the same times table. One's 4 and one's 2, so two 2's equals 4 and then two 6's is 12.

George uses the rectangle representation throughout, which is representative of other students' working. We suspect that this is due to students being most familiar with area representations (Duval, 2006), but it may also be because of the central position of the area button in Fractions Lab. George held the misconception "x/y = 1/xy" strongly. Initially he believed the computer was wrong (line 1) and tested his conjecture using 2/4 = 1/8 (line 3). Through trial and error he found that 2/4 and 4/8 were equivalent (line 4) and his thinking is perturbed (lines 7-8). He begins to make a link between the numerators and denominators of the two equivalent fractions (line 8). After using the partitioning tool he tests 4/12 to see if it is equivalent to 3/4 and finds that it is. He demonstrates intuitive multiplicative reasoning by stating that the numerator of each equivalent fraction and denominator are "in the same times table", and we can see that he appears to be overcoming his use of additive structures and whole-number bias by stating the relationship between the two fractions using multiplication.

2. Partitioning tool task

18 Year 5 students completed a familiarisation task using the Partitioning Tool. They were required to make a fraction, note which representation they used, write the resulting fractions of three partitions and explain in their own words, for another child, how partitioning works. 16 students used the area representation, the number line and liquid measures representations were each used by one student. Table 1 shows the types of students' responses.

Explanations related to...		
multiples or multiplication (e.g. it times by the number you partition by)	7	39%
doubling (e.g. double the numerator and denominator)	4	22%
splitting the number (e.g. partitioning works by the number being split up)	2	11%
addition (e.g. on the top number you add 5 and on the bottom number you add 10)	2	11%
splitting / cutting up the rectangle to make smaller pieces (e.g. cutting it up into smaller pieces)	3	17%
Total:	**18**	**100%**

Table 1. Partitioning Tool task: Students' explanations of how the tool worked

No students referred to a procedural method (e.g. "times the top and bottom by the same number") but instead focused on the action of the partitioning tool on the representation. 15 (73%) of the 18 students commented on the change on the symbol as a result of the action. The remaining students (17%) focused on the representation (rectangle) itself.

3. Written reflections

All the students (n=67) were asked to reflect on their learning using Fractions Lab. There was no clear difference between the responses of the Year 5 and Year 6 so the data are combined. Because the questions allowed for open-ended responses, many students focused on aspects beyond the scope of this paper, such as fraction size, addition or subtraction.

All students were asked how each of the representations helped them to understand fractions. 17 of the 67 students made statements related to equivalence. These are presented in Table 2. When considering *how each of the representations helped students to learn*, a quarter referred to equivalence. More students referred to rectangles supporting their learning of equivalent fractions compared to the number line and liquid measures. This may be because not all the students will have undertaken equivalence tasks with those representations.

Indicative student statements	Totals:
General statements about finding equivalence, e.g. • The equivalent. • It helps me actually visualise the fractions as decimal numbers. • How to show equivalent fractions. • How to find equivalent fractions.	6
Visual comparison, e.g. • Which fractions are the same because it colours in the rectangle. • Equivalent fractions. Put a fraction above another to see if they were the exact same. • How fractions look the same but have different numbers 1/2 = 2/4.	5
Partitioning, e.g. • They could be partitioned. • How partition works. • To partition instead of times by two.	6
	17

Table 2. Student responses related to equivalence

DISCUSSION

The main objective of our work is to challenge students' pre-conceived ideas of how fractions are represented and how Fractions Lab can create an environment for students to develop situated abstractions about equivalent fractions. Prior to using Fractions Lab the students had not met the notion of partitioning a fraction to find an equivalent.

The case study provides a window into one student's thinking-in-change, as he challenges his own misconception about equivalent fractions. George appears to have entered into a relationship with the representations and tools in Fractions Lab which gave him expressive power, entering his thoughts, actions and language (Noss and Hoyles, 1996). It also shows how George's whole-number bias (Behr et al, 1983; Newstead & Murray, 1998; Schmittau, 2003) remained dominant throughout but that he noticed a relationship between the numerators and denominators of the equivalent fractions that we argue is intuitive multiplicative reasoning: "They are a bit the same, they are in the same times table".

The case study supports the findings of Reimer & Moyer (2005), Suh & Heo (2005) and Steen *et al.* (2006), demonstrating how the use of a virtual manipulative (in this case a rectangle representation with a partitioning tool) provided immediate feedback that allowed George to self-regulate his thinking and make amendments, undertaking a task that would be difficult if not impossible with physical manipulatives. Although paper-based or other tasks with tangible objects provide different affordances, the potential of Fractions Lab includes its dynamic nature and particularly the direct connection between graphical and symbolic representation. Furthermore, it demonstrates how

George's misconception is so strongly held that he tries out another statement (2/4=1/8) because he thinks the computer is wrong. However, when the feedback is not what he expects, his assumptions begin to be challenged, and turbulence in his thinking can be observed.

The students' written comments explaining to a friend how the partition tool works supports the findings of Reimer & Moyer (2005) and Suh & Heo (2005) as the majority of students (78%) explicitly commented where a link between the graphical representation and its symbol could be observed.

Despite the open-ended nature of the question, we were surprised when 25% of the students stated that using the representations helped them to understand equivalent fractions. While it is feasible that most students may not have used the number line or liquid measures for an equivalent fractions task, it is also worth exploring further if the way the rectangle is partitioned differently to the other representations also has an impact.

CONCLUSION

We have emerging data that students' interaction with Fractions Lab, and in particular the partitioning tool, provokes them to think conceptually about equivalent fractions. Some appear to be able to capitalise on their intuition, and sometimes to challenge it, discouraging them from simply calculating an answer procedurally. Developing virtual manipulatives that enable students to witness what happens dynamically as they create and partition a fraction appears to have the potential to enhance their conceptual understanding of fraction equivalence by challenging their whole-number bias.

The next step in our research is to evaluate the newly-introduced sets representation to Fractions Lab and to systematically evaluate the effect partitioning using all the representations has on students' conceptual understanding of fraction equivalence. We are also interested in how Fractions Lab may further support students to bridge the gap between their additive reasoning and their multiplicative reasoning.

Acknowledgment

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DIGITAL ASSESSMENT-DRIVEN EXAMPLES-BASED MATHEMATICS FOR COMPUTER SCIENCE STUDENTS

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Repeated formative, diagnostic assessment lies at the heart of student-centred, assessment-driven instruction, providing students and teacher with a continuous stream of information on the mastery of course topics. When integrated into an e-learning environment, formative assessment can make that information instantaneous, which is a crucial aspect for intelligent feedback in student-centred instruction. A digital mathematics course for computer science students is presented in which repeated formative assessment providing intelligent feedback and learning through worked-out examples have been fully integrated into a mathematics e-learning platform. We give examples of the benefits of this approach and discuss what still needs attention based on data from an intensive four week mathematics course.

Keywords: assessment-driven learning, interactive mathematics documents, formative assessment, intelligent feedback

DIGITAL ASSESSMENT-DRIVEN AND EXAMPLES-BASED INSTRUCTION

The classic function of assessment is that of measuring students' knowledge and skills in the form of an aptitude test. For completion of the learning process, we expect students to demonstrate mastery of the subject. In traditional summative assessment feedback is hardly more than a mark that comes available only after finishing all learning. This type of assessment has long been recognised as the most influential factor in shaping what and how students in higher education choose to learn (e.g., Ramsden, 2003). The alternative form of assessment, formative assessment, has a completely different function, namely informing student and teacher about the current level of competence and the necessary steps to make progress. This information helps better shape teaching and supports learning. Practice testing and diagnostic testing are two most common forms of formative assessment. They are especially useful during or prior to the learning because the feed-back on a test constitutes the main sustenance for further learning. It is crucial that feedback is promptly available, preferably immediately, and that it is presented in an intelligent, thus effective way. At this point, digital testing comes on the scene: one cannot imagine instant feedback from formative assessment without using a computer.

Numerous digital environments for assessment-driven learning of mathematics are known, such as MyMathLab, WebAssign, and WeBWorK. In secondary and higher education they are often used as online homework systems in combination with a textbook. Their main characteristic is that each step in the learning path begins with a question that a student is supposed to answer. If (s)he does not master the step, the environment gives a hint, offers a worked-out example, or guides the student through a step-by-step solution. Hereafter the parameter-based implementation of the problem allows the loading of a new version of conceptually the same problem for the student to demonstrate the newly acquired mastery.

Explanations and expert's worked-out examples provide a learner problem solutions to study and emulate. Research (see, for example, Renkl & Atkinson, 2002) has shown that worked-out examples are key to initial cognitive skills acquisition in many well-structured domains such as programming, physics and (arguably) basic mathematics. Moreover, learning from worked-out examples, or, more precisely from example-problem pairs, is an effective mode of learning preferred by novices. Sweller and Cooper (1985) mention that engaging in solving a similar problem immediately after example study may be more motivating for students than starting a new problem, because it is more active. Cognitive Load Theory (Sweller, van Merriënboer, & Paas, 1998; Sweller, Ayres, & Kalyuaga, 2011) has underpinned the research results and shown that worked-out examples can be more effective than learning by problem solving. Insight has been provided under what conditions worked-out examples promote or inhibit learning. For example, Kalyuaga (2007) has pointed out the so-called expertise reversal effect: presenting detailed guidance to experienced learners who do not need this anymore may hinder their performance. A computer is generally considered an excellent medium to provide each individual learner with as many good quality examples as needed (upon demand), at the required level of detail and matching the learner's cognitive level. Fading of worked-out steps in explanations and fading of elaborate feedback in digital exercises that accompany worked-out examples can be implemented. In our opinion, many of the instructional design principles associated with effective examples-based learning could hardly be implemented in mass instruction without the assistance of computer technology that offers its users tools to learn in a more autonomous way.

In this area, designers of intelligent tutoring systems set themselves the hardest task imaginable: they attempt to build computer environments that mimic human tutoring as much as possible by tracing the cognitive state of the user on the basis of models of domain-specific human methods and strategies, by generating context sensitive feedback via comparison of the user's actions with built-in solution paths, and by adapting the contents to the user's level (Koedinger & Corbett, 2006; Mitrovic, Martin, & Suraweera, 2007). Our goals are more modest: we do not base our work on elaborate student models, domain models and instructor models, nor focus on tutoring via computer generated hints and customized instruction. Although we also base our support of a student trying to solve an exercise on an analysis of the given (intermediate) answer in the way a teacher or tutor would do, we do not take into account the route via which the student came to the answer and we only give feedback in the form of indicators of the level of correctness of the answer, by provision of hints of suggestions for well-known common mistakes, and occasionally by offering the opportunity to go through the problem in smaller steps. The customized feedback is hardcoded.

The digital learning environment that we used in the mathematics course for second-year computer science students in this study was SOWISO [1]. It meets the requirements for implementing assessment-driven and examples-based instruction. Even more, it extends the features for personalized assessment-driven learning with generous possibilities to provide feedback while a student is answering an online question, not only by telling that there was a mistake, but also what the mistake precisely was. Help is available in the form of hints that adapt to answers of the students. If a student gets stuck, (s)he can retrieve the solution with a fully worked-out explanation, or (s)he can ask for guidance by splitting the original problem into sub problems. All exercises are parametrically randomized. This provides a student an almost endless pool of equivalent exercises to practise with. Furthermore, SOWISO is a multimedia authoring environment based on the mathdox technology (Cuypers, Cohen, & Verrijzer, 2010). This enables the creation of interactive mathematics texts, which are HTML5 compliant and can be viewed by standard web browsers on laptops, tablets and smart phones. An interactive mathematical document is here considered to be a collection of mathematical pages containing theorems, proofs, examples, exercises, and the like, delivered to the user through hypermedia. Every chapter in a course may consists of theory with embedded interactive elements, dynamic examples, interactive exercises with targeted feedback, and (diagnostic) assessments. Last, but not least, in SOWISO teachers and students can monitor the progress made during the course via learning analytics tools at various levels of details and can inspect the gradebook to see the competency level reached so far.

CASE STUDY: NUMERICAL RECIPES COURSE FOR COMPUTER SCIENCE

We describe the course settings, exemplify the implementation of intelligent feedback in exercises, and we present course results, our experiences, and student evaluations. In particular we are interested in the following two questions: (1) Do the intelligent feedback design and the assessment-driven examples-based instructional design fit the desires and needs of the students who work more or less autonomously with the digital instructional materials; (2) Does the instructional approach lead to satisfactory course results and to teacher and students' satisfaction?

Course settings

Our empirical contribution focuses on the Numerical Recipes course in the second year of the Computer Science bachelor programme at the University of Amsterdam. Sixty-seven students (64 male, 3 female) participated in this course. The course design was assessment-driven examples-based instruction with intelligent feedback design to acquire mathematics cognitive skills. The course was taught for the first time in January 2015 and took four intensive weeks with 24 hours of contact time per week and no other courses scheduled in parallel. Attendance of students in the class sessions was obligatory and registered. During these sessions, the students were autonomously going through on-line instructional materials within the SOWISO environment, including answering randomized exercises and making obligatory assignments, and they were implementing standard algorithms from numerical analysis in the Python language. The lecturer (first author of this paper) and one tutor were present during the sessions to answer questions and to give explanations and advice, both to individual students, groups of students, and the whole class, when appropriate.

The mathematics core of the course consisted of a cognitive skills part and a mathematical application part. The cognitive skills part concerned: (1) basic calculus, which was partly a repetition of supposedly acquired secondary school knowledge and skills; (2) calculus of several variables; and (3) a short introduction to ordinary differential equations. The mathematical applications part of the course, in which students' work consisted of understanding and implementing numerical algorithms, consisted of the following subjects: (1) representation of number on computers and its consequences; (2) numerical root finding; (3) polynomial interpolation; (4) numerical differentiation; (5) numerical integration; (6) visualization of functions of several variables; (7) edge recognition in digital images; and (7) numerical solving of an initial value problem via basic numerical methods. The main idea behind the applications part was to motivate students by letting them experience how mathematics is used in computer science practice and to prepare them for follow-up courses like computer vision. It was also meant to gradually get away from pure assessment-driven examples-based instruction for initial cognitive skills acquisition towards problem solving as mathematical expertise of students grows. For the two programming tasks that students had to hand in for marking, we stressed that not only the programming code would be evaluated, but also their report on how they had analysed the problem situation, had turned it into a Python program code and had used their code for mathematical experiments to gain insight in the numerical aspects of the algorithms and their programs. Students did these tasks in dyads so that they could discuss the numerical aspects with each other and write their conclusions in the report. Working in small groups aimed to raise the quality of mathematical thinking, problem solving, and exploring numerical algorithms.

Treatment of each mathematical subject ended with a randomized digital assessment for which the pass mark was 8 (out of 10). Despite this summative nature, the main purpose of the tests was formative: students could objectively verify their progress with or mastery of the mathematics subject in relation with the targets set by the instructor. The reason for the high pass mark level was that students had ample opportunity to practise, using the instructional materials available in the online platform. It is rooted on the main principle of mastery learning that almost anything can be learned provided that enough study effort is put into it. The formative assessments were short (on average three questions) but there were many: in total, students had to pass thirty-three tests. Students had five chances for meeting the pass level, but in order to reduce the level of annoyance of constantly repeating a test with a chance of new mistakes, students could prior to submission see which questions they had already correctly answered and which answers needed to be improved to meet the pass level. Students were told at the beginning of the course that only in case they pass all tests, they would meet the course requirements and get a final mark, regardless of their positive results on the two summative mathematics examinations during the course and on the two programming tasks that they had to hand in for marking. On the other hand, a carrot was dangled: Seventy percent of the examination questions would be taken from the formative assessments or have great resemblance with these items. The two mathematics examinations each counted for 20% in the final course mark; the two programming tasks each counted for 30% in the final mark.

Implementation of intelligent feedback

One of the benefits of our assessment-driven examples-based instructional design is the frequency and quality of the feedback to both students and teacher. At the end of each short formative assessment, the score and the mistakes are displayed and the student can then access the solutions to the questions that they failed to answer correctly. At this point, the student is encouraged to study the theory and examples more thoroughly and to discuss the misunderstanding with peers, tutor or lecturer. The initiative is in the hands of the students. To what extent the failed exercises and test items can be resolved by the student without any help of a tutor or lecturer depends on the quality of the feedback. More precisely, the quality of feedback depends on how much of the expertise of the mathematics teacher, i.e., his/her Technological Pedagogical Content Knowledge (TPACK, [2]), can be built into a digital learning environment such that students can receive effective feedback when they make a mistake or do not give yet the best possible correct answer. We refer to this as intelligent feedback.

In education, feedback is inextricably bound up with learning processes (see, for example, Mory, 2004). Hattie and Timperley (2007) state that effective feedback must answer three major questions asked by a students and/or by a teacher: “Where am I going?” (What are the goals?), “How am I going?” (What progress is being made toward the goal?), and “Where to next?” (What activities need to be undertaken to make better progress?). Each feedback demand can work at four levels (focus of the feedback): (1) task level: how well tasks are understood/performed (FT); (2) process level: the main process needed to understand/perform tasks (FP); (3) Self-regulation level: self-monitoring, directing and regulating of actions (FR); and (4) Self level: personal evaluations and affect about the learner's actions (FS).

Hattie and Timperley (2007) make some general statements on the effectiveness of (combinations of) feedback types, including that FS feedback is least effective, simple FT feedback is more effective than complex FT feedback, FT and FS do not mix well ("Well done, that is correct" is less effective than "Correct" only), and that FT is more powerful when it's about faulty interpretations, not lack of information. In summary, the feedback effects of cues and corrective feedback are deemed best. Furthermore the authors state that one should be attentive to the varying importance of the feedback information whilst performing a task.

The feedback design of the formative assessment in our digital learning environment is based on the basis feedback principles described above and it is an integral part of our pedagogical scenario. In this paper we restrict ourselves to 'local feedback', i.e., to feedback that is based on the entered response to a particular question or task. Considering the goal of feedback, different forms of local feedback can be distinguished. Often more than one of these forms is present in a single feedback response. Some examples:

- *conceptual elucidation:* explanation or comment on a particular student action (comments on the accuracy of the student's answer, instructions of how to input mathematical expressions, etc.);
- *clarifying remarks:* supportive comments on content (hints, references to theory, etc.);
- *evaluative information:* information about total score, time spent on the task, and so on;
- *affective statements:* motivational encouragements such as indications of correctness of intermediate results; recommendations such as suggestions for further practice.

We speak of intelligent feedback when it is based on an analysis of the student's answer or action and when it goes beyond a simple correct/incorrect sign or a mark. For example, an incorrect answer is annotated with a short explanation of what might have gone wrong. Also, a hint might be given or a student gets a second chance. Another example is the highlighting of a correct, but mathematically inferior answer (e.g., $\sqrt{4}$ is usually simplified to 2). Intelligent feedback is in our case a detailed reaction based on (1) expert knowledge about the mathematical content; (2) a model in which common mistakes and actions are rubricated; and (3) knowledge about learning and instruction so that the most suitable feedback can be chosen.

The initial feedback design is based on the lecturer's/designer's TPACK. We present a prototypical example, in which the task is to integrate the monomial $\frac{1}{5}u^3$. A possible SOWISO session is shown in Figure 1 and a selection of the feedback design for this particular example is listed in Table 1.

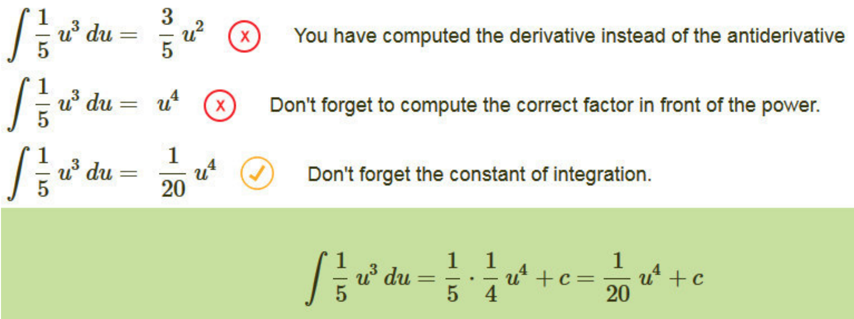

Figure 1. The feedback whilst solving a problem and the display of the solution.

Answer	*Feedback*	*Explanation*
$\frac{1}{20}u^4 + c$	Well done	Correct answer
$\frac{1}{20}x^4 + c$	Did you use the correct variable?	Wrong independent variable
$\frac{1}{5}\left(\frac{1}{4}u^4 + c\right)$	Correct, but simplify further.	Unusual correct answer that can be simplified
$\frac{1}{20}u^4$	Don't forget the constant of integration	No constant of integration
$\frac{1}{20}u^4 + \frac{1}{5}$	This is a particular solution, but other values than $\frac{1}{5}$ are also possible as constant of integration.	No general solution
$\frac{3}{5}u^2$	You have computed the derivative instead of the antiderivative	Reading error?

Table 1. Feedback design for computing an antiderivative.

Because we do not know in advance which formula a student types, we need a computer algebra system as background engine to eliminate the problem that a mathematical expression can be written in many different ways while the semantic meaning remains the same (for example, $(x+1)^2$ and x^2+2x+1 are equivalent expressions), to process a given response and to find a match of given input with appropriate feedback. This kind of feedback can be constantly given during the process of completing a task. Giving merely a signal that one is still well on track may often suffice or even be the most effective and motivating. A screenshot of such an interactive session is shown in Figure 2.

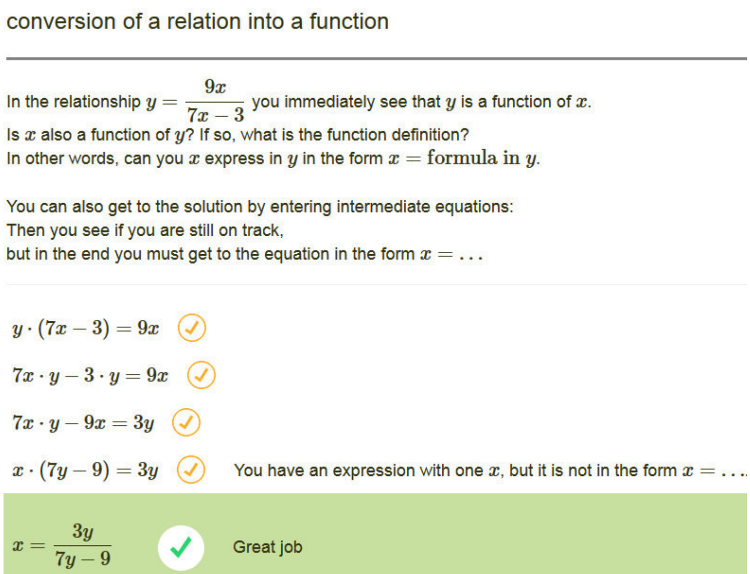

Figure 2. Screenshot of a computer session in which a mathematical task is performed.

It is important to realize that equivalence of learner's input with the answer preferred by lecturer or instructional designer is not good enough to label already the answer as correct: when asked to compute $\frac{1}{4}+\frac{1}{6}$, the answer $\frac{5}{12}$ is certainly correct and maybe $\frac{10}{24}$ is acceptable, but one can frown one's eyebrows with the answers such $\frac{1}{12}+\frac{1}{3}$, $\frac{1}{4}+\frac{1}{6}$, and $\frac{30}{72}$. Thus, it is important that automatic simplifications can be dealt with and that one can give appropriate feedback for different cases at different stages of instruction. In SOWISO, this is called the implementation of positive feedback, i.e., feedback to a solution that is in principal correct, but not in the format considered as a good.

Both examples show the main workflow of assessment-driven examples-based design: (1) the computer program creates a question which contains a (randomized) mathematical object (e.g., a formula or figure); (2) the student decides what to do next (use a hint, look at a solution, make an attempt, go through an example); (3) the student manipulates the created object in most cases (e.g., answers the question or interacts with the figure); (4) the computer program automatically establishes properties of the student's action (e.g., using a computer algebra system); (5) the computer program assigns outcomes, including intelligent feedback; and (6) the student interprets the generated outcome and feedback, and returns to step (2), namely decides what to do next.

For formative assessment it is crucial that almost all exercises are randomized in some way. This can be randomization of values used in a question, expected answer, feedback and worked-out solution, but also randomization of formulas and other mathematical objects. In SOWISO each randomized exercise itself can be used as a randomized example, so that a student can go through as many examples as needed. Figure 3 shows two instances of one and the same randomized example.

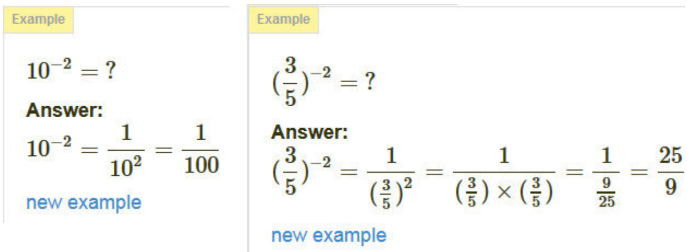

Figure 3. Repeated worked-out examples.

Course results

Data of student activity were collected from logging of students' actions, their performance in the digital learning environment at assessments, from evaluation reports of the students (set up in the form of a digital questionnaire, with an unsolicited response percentage of 66%), and from assessment papers and computer homework.

Students worked hard during the four course weeks: 67 students made in total 23334 exercises and opened 22389 theory fragments, which constitutes a total of 45723 activities. This is on average per student about 350 mathematics exercises and 334 theory fragments during the course. Figure 4 shows the distribution over course days. The three sharp peaks can be related to mathematics examinations that took place the next day. But the graph shows a rather constant level of activity during week days. From the graph on the right-hand side of Figure 4, it can be seen that the students worked not only during the sessions, but also in the evening hours.

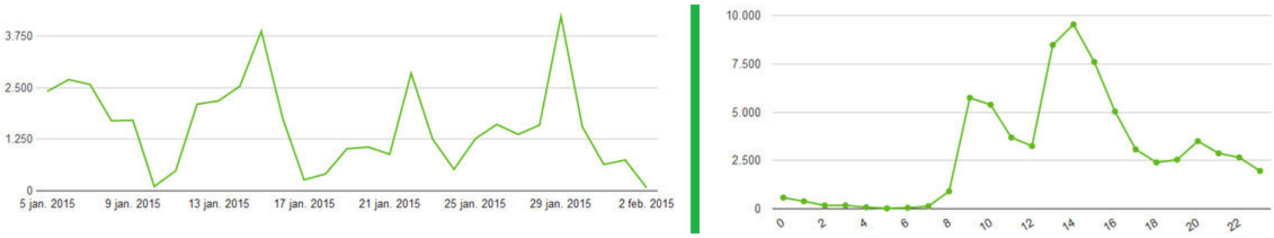

Figure 4. Distribution of activities over course days and hourly activity level.

The experiences of students with the course were positive and 90% of the students passed the course. In a questionnaire we asked them to mark the course between 1 and 10. The average mark (μ) was 7.1 with standard deviation (σ) equal to 1.4. On a 5-point Likert scale students agreed (μ=4, σ=1.1) that the chosen course design appealed to them and was sufficient to pass the course (μ=3.8, σ=1.0). The students found the course contents interesting (μ=3.7, σ=0.7), had enough prior knowledge (μ=4.2, σ=0.7), had learned a lot during this course (μ=3.6, σ=0.9), and had enough time for digital exercising (μ=4.4, σ=0.7). Students agreed that the SOWISO environment was easy to use (μ=4.0, σ=0.9) and that the feedback in the questions was good (μ=3.6, σ=1.0). They appreciated very much that they could practise and look at worked-out examples before doing the short assessments (μ=4.4, σ=0.5). They also valued the interactive mathematics texts (μ=3.8, σ=0.9). The only issue for improvement of the course seem to be the clarity of the programming tasks (μ=2.8, σ=1.1) and the expected outcomes of these tasks (μ=3.0, σ=1.1), which was evaluated neutrally by the students.

CONCLUSIONS

The intensive use of a digital environment for assessment-driven examples-based learning of mathematics had a large impact on academic performance of the bachelor computer science student in the Numerical Recipes course. Ninety percent of the students passed the exam. However, in a student-centred course it is not yet sufficient when the teacher is convinced of the benefits of the course set-up. In the Numerical Recipes course, the students regulated their own learning process, made their own choices on the intensity with which they exercised. From the survey it can be concluded that they were (or became) convinced of the usefulness of the digital environment and the instructional approach. On the one hand, by our course rules of engagement we more or less forced them to attend the face-to-face session and participate. Also, knowing that the mathematics examinations would contain similar questions as in the short formative assessments may have raised the students' confidence. Anyway, the study success the Numerical Recipes course was in comparison with other courses unexpectedly high.

More research is needed to find out if only procedural knowledge and skills have been mastered, or also problem solving skills have been developed, how long the acquired competencies last, and whether also transfer of mathematical competencies has been reached. Anyhow, this course has proved to the students that they are able to master mathematical knowledge and skills needed by computer scientists.

NOTES

1. www.sowiso.nl

2. www.tpack.org

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SOLVING PROBLEMS ON THE SCREEN: DIGITAL TOOLS SUPPORTING SOLVING-AND-EXPRESSING

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The main goal of this research is to understand the problem solving activity with everyday digital technologies, within the context of a beyond school mathematical competition, SUB14. Following a qualitative approach, we observed the activity of a young student, Marco, when solving a geometrical problem, aiming at describing and analysing his activity. The results show that the digital inscriptions presented in the statement trigger his visualization abilities which, in turn, assume a relevant role in understanding the problem, selecting a technological tool, planning and implementing a strategy. Based on the case of this student-with-computer we discuss the role of the everyday digital tools in the activity of solving-and-expressing the problems of the competition.

Keywords: Everyday life technology; Humans-with-media; Mathematical problem solving; Solving-and-expressing; Visualization.

INTRODUCTION

Research concerning out-of-school mathematical activities is remarkably scarce and while the interest from the research community regarding problem solving as a research topic has been decreasing (English, Lesh & Fennewald, 2008), there is still much to learn about the role and impact of everyday digital tools in the development of mathematical thinking, particularly, in problem solving processes (Santos-Trigo & Barrera-Mora, 2007). Hence, our study aims to unveil the kind of non-routine problem solving activity with the digital tools that young participants choose to use within the frame of a web-based mathematics competition. In particular, we propose to look at the role of digital tools in the case of a participant solving and expressing the solution to a geometry problem.

A mathematical problem solving competition

SUB14® is a beyond school mathematical problem solving competition that addresses 12-13 years old students covering the southern area of Portugal. The Qualifying phase consists of 10 problems, each one posted online every two weeks. Participants must send their answer as well as a complete and detailed explanation of the reasoning process through their own email accounts or with the tools available at the website. The organizing committee assesses every answer and replies with a specific feedback, either praising or requiring a revision. The selected participants attend the Final phase, held at the University of the Algarve (Carreira, 2012).

THEORETICAL FRAMEWORK

This section highlights the theoretical notions that support the description of the non-routine problem solving activity with digital tools, within SUB14. By assuming the inseparability between the subject and the tool to solve the problems, we claim that the selection of a particular technology is grounded on a symbiosis between the perception of its affordances and the mathematical abilities of the solver, pondering the role of visual thinking for solving and expressing the problems by means of such tools.

Solving-and-expressing with digital tools

The mathematical problems posed at SUB14 seek to intellectually challenge the students, and since these problems are not aligned with the curriculum, solving them involves the development of a *productive way of thinking* about the challenging situation (Lesh & Zawojewski, 2007), often resorting to informal knowledge and incorporating descriptive elements in the approach.

This problem solving activity is seen as a synchronous process of mathematization and expression of mathematical thinking (Carreira, Jones, Amado, Jacinto & Nobre, in press), which means that answering to a particular problem includes finding the solution and reporting the whole process. Solving the problem and expressing the reasoning are strongly interconnected activities that gain sight in the presence of digital tools that support the expression of mathematical thinking. As the rules of SUB14 explicitly demand the submission of 'digital solutions', the participants consciously ponder the digitally expression of their mathematical reasoning. Hence, it seems appropriate to consider the illustrations, diagrams, or the use of colours, as they become part of a script of the mathematical thinking processes:

> …descriptions, explanations, and constructions are not simply processes students use on the way to "producing the answer", and, they are not simply postscripts that students give after the "answer" has been produced. They ARE the most important components of the responses that are needed (Lesh & Doerr, 2003, p. 3).

Since this activity inevitably includes a distinctive description of the mathematical thinking produced, due to the use of digital tools, we seek an understanding of the user and the tool as a unit and focus on the ways in which this entity solves-and-expresses mathematical problems. Borba and Villarreal (2005) argue that the processes mediated by technologies lead to a reorganization of the human mind, hence, mathematical knowledge arises from this symbiosis of individuals and the tools. This synergetic relationship brings about a new entity that the authors name as *humans-with-media*: a metaphor for describing the influence of the digital tools that are used to communicate, produce and represent mathematical ideas, in the transformation of the mathematical thinking. Thus the mathematical thinking produced by humans-with-paper-and-pencil is qualitatively different from the one produced by humans-with-spreadsheet or by humans-with-dynamic-geometry-software (Villarreal & Borba, 2010).

The differences seem to lie on the perception of the affordances (Gibson, 1979) of the tool, hence the contrast between distinct conceptual models that support achieving the same solution may be rooted in the symbiosis between the mathematical aptitude of the solver and the perception of the affordances (Greeno, 1994) of the tools chosen for solving-and-expressing the solution.

Visualization and problem solving with digital technologies

Solving mathematical problems usually requires the use of diagrams, drawings, illustrations (Lavy, 2007; Pitta-Pantazi, Sophocleous & Christou, 2013; Presmeg, 1986), since they support the *visualization* of the concepts underlying the context (Zimmermann & Cunningham, 1991). This is strongly evidenced in the productions of the participants in SUB14, whose solutions usually include a variety of *inscriptions* (Presmeg, 2006). We consider *visualization* as the ability to construct images of mathematical notions, either mentally, using paper and pencil or technological tools, and their efficient use as a mathematical way of thinking and knowing (Hershkowitz, 2014; Presmeg, 1986; Zimmermann & Cunningham 1991).

It has been found that visualization and problem solving skills are strongly related, especially when tackling non-routine problems (Wheatley, Brown & Solano, 1994). Besides supporting the initial stages related to understanding the situation or the organisation of the mathematical notions embedded in figures or statements, visualization also supports the transition from a contextual to an abstract kind of thinking (Lavy, 2007), that is, the process of mathematization. Presmeg (1986) has proposed mathematical problems that could be solved either by visual or non-visual strategies to high school students. She concentrated on the *visualizers*, i.e., those who preferred to use *visual methods* when there was a choice. Several other studies (Pitta-Pantazi, Sophocleous & Christou, 2013; Rösken & Rolka, 2006) have sought to understand the traces of visualizers as opposed to the verbalizers, which prefer analytical methods for solving mathematical problems. For instance, Kozhevnikov, Hegarty and Mayer (2002) concluded that the visualizers with high spatial abilities are successful in problem solving because their preferences impel them to make schematic inscriptions of the relationships between mathematical objects.

Even though it is not our intention to discuss these categorizations in depth, it is timely to consider the distinction between the *spatial visualizers*, i.e., those students with high ability to process information regarding spatial relations and manipulate complex spatial images, and the *object visualizers*, i.e., those with high ability to deal with visual and pictorial properties of objects (Blazhenkova & Kozhevnikov, 2010). The spatial visualizers usually resort to flexible spatial images, manipulate dynamic images, are successful in tasks that require the mental transformation of objects, are capable of analysing an object part-by-part, which enables them to manipulate spatial images and engage in a diversity of transformations (Kozhevnikov, Kosslyn & Shephard, 2005).

Presmeg (2006) has pointed out the need for further investigation regarding visualization in mathematics education and, specifically, for further understanding about the affordances of digital technologies that transform the mathematical thinking and the learning of mathematics. In fact, there are numerous digital environments that enable the visualization of certain mathematical properties during the construction of figures and their transformation, allowing reflecting upon them and using them to communicate. Arcavi and Hadas (2000) claim that the inherent dynamism of digital tools may influence the "habit of transforming (either mentally or by means of a tool) a particular instance, in order to study variations, visually suggest invariants, and possibly provide the intuitive basis for formal justifications of conjectures and propositions" (p. 26). Thus, the characteristics of a visualizer may influence his/her choice of the digital tool, as well as the recognition of its most relevant affordances, for engaging with visual methods.

Understanding mathematical problem solving with digital tools, in the context of SUB14, requires looking at students-with-media developing productive ways of thinking visually mediated by digital tools that trigger, support or enhance visual mathematical thinking.

RESEARCH METHODS

This study followed an interpretative perspective informed by a combination of theory and data within a qualitative approach (Quivy & Campenhoudt, 2008). Our focus is on the understanding of the processes of problem solving with digital tools, in the context of the competition SUB14, so we report the case of a participant, under the pseudonym of Marco, since he usually resorts to a variety of technological tools to solve the problems of the competition and also due to the quality of the explanations or justifications he presents.

An extensive collection of data took place throughout one year. This paper deals with the analysis of a segment of such empirical data, namely the observation and video recording of Marco's problem solving activity in his home environment, after the formal consent of his parents. We selected three problems addressing different mathematical notions and posted them on the SUB14 website, shortly before the observation, which was carried out by the first author. Marco was asked to choose and solve one of those problems and simulate, as closely as possible, his usual problem solving activity following the rules of the competition. He was also asked to explain his processes, so the observation was complemented by clarification questions.

The data organization and transcription was carried out using NVivo, where other traces of Marco's activity were registered: gestures, switching between computer tools, or the sequence of operations in each software. The descriptive and analytical case of Marco results from the analyses of data in light of the theoretical framework, aiming to enable a deeper understanding of the role of digital tools in solving-and-expressing mathematical problems.

SOLVING-AND-EXPRESSING ON THE SCREEN: THE CASE OF MARCO

This section reports the solving process followed by Marco, starting from the selection of the problem, the quest of a suitable approach, the production of the answer and its submission to the competition.

Visual exploration of the figure

Marco starts by carefully analysing the three problems on the competition's website and choses to solve the problem 'Decorative Drawing' (Figure 1) since it is his favourite.

The picture shows a decorative drawing that will be used in the construction of a stained glass window. The equilateral triangle has a height of 12 cm. The circles are all tangent to the triangle and also each small circle is tangent to the large circle.
Which is the radius of the smaller circle?

Don't forget to explain your problem solving process!

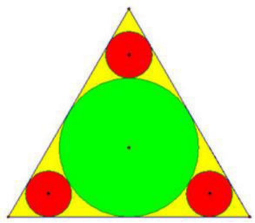


Figure 9. Statement of the problem 'Decorative Drawing' chosen by Marco

When asked about his reasons, he explains:

Marco: It has to do with triangles and stuff like that, also in the 7th grade I got 100 [%] in both tests…

Researcher: In geometry?

Marco: Yes, I studied congruent triangles and that…

His choice seems grounded on the immediate recognition of the mathematical notions that may be necessary to solve the problem and, simultaneously, in how familiar and self-confident he feels. Marco focuses his attention on the problem, reading it again, and explains: "I'm trying, I'm still trying… to see how to do this. Hum… since the triangle is equilateral… if I reach the circle in the centre, I might get to the others. I'm…" Then, in silence, he stares at the screen.

The understanding of the situation seems to take place as Marco interprets the figure, impelling the outline of an initial approach. He then starts to explore, visually and on the screen, several decompositions of the triangle: sliding his finger down the screen he 'traces' a line that bisects the lower right angle, and then a new vertical line passing through the upper vertex:

Marco: How should I put it? It's like it divides in half. Dividing from each vertex to the midpoint of the opposite side and try to figure out a term… if I could… but I'm still trying to see how I'll do it…

Marco continues to develop some attempts in devising a visual method of approaching the problem and, after a while, he points out a slightly different perspective:

Marco: It has 12 cm. At the middle of the triangle it is not 12, for sure. But it could be 4. Dividing these parts… [After using his thumb and index to measure the radius of the circle, he moves 3 times along the height of the triangle]. Yes, maybe. Because they are tangent . . . I can tell they are the same length. The only problem is that it doesn't say a thing about it…

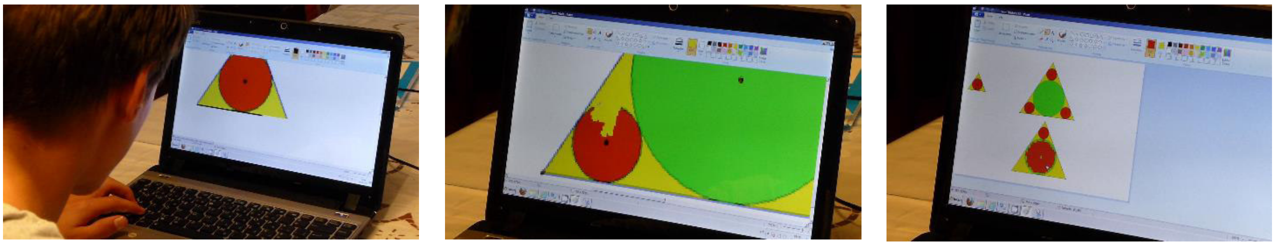
He seems to be aware that the centroid does not divide the height of the triangle in half, since he proposes that the radius of the larger circle is 4 cm, which results from a visual intuition supported by a rudimentary measurement using his fingers. Even though he concludes that the radius of the large circle is 1/3 of the height of the triangle, he realizes that such statement requires a solid proof.

Marco then notes that he has thought of different approaches – "now I'm thinking, there are many [ways] but… I don't know if they will work, that is the problem" – and decides to decompose the triangle into two figures, obtaining a smaller triangle at the top and a trapezoid. He then posits a new conjecture: "if we draw a triangle here . . . this is an enlargement of the other triangle. If it is 12, 12 divided by 3, [equals] 4. That means the radius is 2. Maybe the radius of the smaller circle is 2".

These different approaches correspond to mental manipulation and transformation that have not yet been materialized by Marco, besides 'drawing' with his finger on the screen. The vague descriptions are complemented by the 'interaction' with the figure on the screen: Marco points, 'measures' distances, covers areas as if they ceased to exist. He is developing a visual method for solving the problem, analysing the affordances of decomposing the figure, and mentally simulating the transformation – cutting, reorganizing, changing colours – hence disclosing his intention of arranging a 'graphical editing' of the figure as being indispensable to obtaining the solution.

Transforming the figure to obtain the solution

The production of an answer, that he has not yet completely uncovered, takes place through the implementation of the strategy visually planned (decomposing the figure), by means of the graphical editing of the figure. Using the software Snipping Tool, Marco defines a rectangular area in order to crop the top of the triangle, obtaining a smaller triangle with a central red circle and saves the file. Using a similar process he creates another file holding the original triangle and inserts them on the Paint window. He tries to overlap them but he perceives a technical problem: both the images have a white background so it is not possible to place one over the other, as he seems to have planned.



i) Draws the lower side of the triangle. ii) Paints te red circles in yellow. iii) Paints the central circle in red.

Figure 2. Three steps of the image processing

This unforeseen problem takes Marco into a slightly different direction, as depicted in Figure 2: i) he increases the view of the desktop so that he is able to draw, with accuracy, a line at the bottom of the figure to be considered a 'triangle', which is not a simple pictorial issue; ii) working on the original triangle, he uses the 'eyedropper tool' to identify the exact shade of yellow covering the background of the triangle and then uses it to colour the red smaller circles; finally iii) using the eyedropper once again, he captures the shade of red in order to paint the central circle. The purpose of the images processing is to show that the smaller triangle is, indeed, a reduction of the original triangle: it is even true because Marco can transform the original one to 'look like' the smaller triangle. When asked about how often he invests in the graphical display of his solutions, he replies that this is not a habit but, in this case, he seems worried: "one can tell [the imperfections]". However, such attention has another motive: "to further demonstrate how this one would look like if it were an enlargement of the other one", i.e., the graphical editing assumes a fundamental role in his strategy. Besides illustrating his thinking in the most reliable way at his disposal, these figures have become a visual mathematical argument since they should convince the SUB14 judges.

After editing the images he saves the files and proceeds by opening Calc, the spreadsheet of OpenOffice. As Marco explains, he usually identifies the number of the problem on a cell at the upper left side of the sheet, pastes in the images (in case there are) and writes down his resolution process on the right side (Figure 3). Marco continues explaining that he found a "similarity between the central circle and the other smaller ones", hence considering that the smaller triangle is a reduction of the initial triangle, in the ratio 12:3, although he fails to prove that they are similar. Thus, by assuming that the diameter of the larger circle is 1/3 of the height of the larger triangle, Marco explains that the smaller circle will have a radius of 1/3 of the height of the smaller triangle, that is, 1/3 of 4.

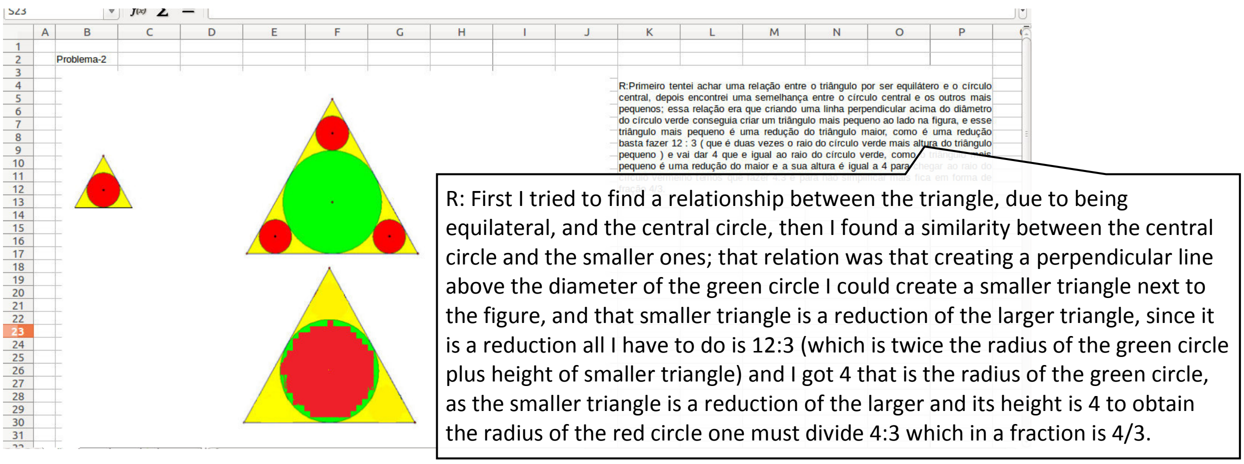


Figure 3. The solution sent by Marco (print screen) with a translation of his written explanation

Although it is naturally based on the latter reasoning, this solution is in contradiction with Marco's last hypothesis: the length of the radius could be 2 cm. However, it seems that the systematization of visual information and its connection to elementary mathematical facts and procedures, afforded by these digital tools, lead Marco into achieving the solution of the problem. This is a strong evidence of the role of the digital tools that not only support the understanding of the problem, but also trigger the implementation of a strategy as well as its effective reporting.

Moreover, this case discloses the challenge in establishing clear boundaries between solving the problem (i.e., the processes followed in obtaining the solution) and the construction of the answer (i.e., the file to be submitted), since the mathematical thinking is developed in a continuum and is refined whilst the explanation is being produced. Marco intentionally chooses the digital tools based on the recognition of their affordances in order to accomplish his plan through processes of transformation and exposing his reasoning. The mathematization of the situation occurs alongside with the expression of mathematical thinking; due to the mediational role of the technological tools, Marco resorted to textual inscriptions and illustrations specifically designed for this purpose. Hence, solving-and-expressing is a way of accounting for the mathematization processes of Marco.

FINAL REMARKS

This paper aims at describing the activity of a student while solving a geometrical problem with digital tools, simulating as closely as possible his usual participation in SUB14. This paper opens a window on the kinds of mathematical thinking and problem solving skills that youngsters are capable of putting forth in challenging situations beyond school, entangling scholar and informal knowledge.

The case presented exposes a balance between the spatial visualization skills of Marco, the characteristics of the problem and the tools that he chooses to use: the selection of the problem reveals his preferences for geometrical problems in which he is allowed to use his digital skills that afford the implementation of visual methods in order to manipulate and transform the figure in relevant ways to achieve the solution – this may be interpreted as his "habit of transforming" (Arcavi & Hadas, 2000) visual elements into usable mathematical tools. The initial figure and the ones constructed later through decomposition and reconstruction triggered the generation of conjectures and mathematical arguments. Visualization played a paramount role in every phase of the synchronous process of mathematization and expression of thinking, which means that his is an instance of the solving-and-expressing activity proposed by Carreira et al. (in press). Moreover, Marco's activity exemplifies the aspects of mathematical thinking processes described by Lesh and Doerr (2003) in the sense that the constructions and the explanation he provides are not mere postscripts included once the solution is found. Instead, those inscriptions are crucial elements within his work that assume a double role: they simultaneously support the finding and the reporting of the answer.

Another very important aspect of Marco's activity is the fact that he 'moves' from the website of the competition, where the problem is displayed, to the Snipping Tool, to Paint, and to Calc, without leaving the screen, that is, without using any other type of tool or written support. This case also illustrates particular features of the collective entitled 'human-with-media' (Borba & Villarreal); in other words, we are looking at the activity of an entity that emerges from a symbiosis between a student and his computer, who is not only mastering the digital tools but is also capable of putting them into practice, by recognising their affordances in the development of this solution. The problem solving activity of Marco-with-computer is unveiled through the visual interaction with the inscriptions on the screen and their transformation, the manipulation of the figure in light of the mathematical notion of similarity and, furthermore, in the digital expression of the process leading to the solution.

We would also like to highlight the relevant role of home-technologies, which at the surface look as deprived of mathematical affordances, in the problem solving activity developed in this competition (Carreira, 2012). The most common frameworks that aim to describe mathematical problem solving were developed in formal learning settings where paper and pencil were the predominant tools. Still, the real world problem solving activity, highly infused with digital tools, requires a framework with a broader scope, capable of supporting the peculiarities of the tools and their affordances in terms of the mathematical thinking developed for achieving elegant and efficient solutions.

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CATO – A GUIDED USER INTERFACE FOR DIFFERENT CAS

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CATO is a new user interface, written in Java, and developed by the author as a response to the significant difficulties faced by those engineers and students who only sporadically use computer algebra systems (CAS).

The usage of CAS in mathematical lectures should be an integral part of mathematical instruction. However, difficulties arise for those students who have classes that meet only once or twice a week, and therefore use CAS only irregularly. Such difficulties are compounded when two different CAS must be used in instruction.

The author has developed a guided user interface (GUI) which translates commands into the languages of different CAS. His intention in so doing has been to develop intuitive operability for CAS. For example, it is common in existing CAS that commands with more than one parameter have their own input windows with commentated input rows for every parameter. In response, the newly developed surface CATO orders the parameters, and uses the right brackets and separators.

Below, the author demonstrates the usage of CATO with Maxima and with the Symbolic Math. Toolbox of MATLAB.

Keywords: GUI, guided input, CAS, Maxima, MATLAB

PREVIOUS APPROACHES TO, AND CONSIDERATIONS FOR CA USER INTERFACES

The demand for better designs of user interfaces for computer algebra systems is almost as old as the systems themselves. Kajler has described and developed his ideas for a perfect user interface in various works (Kajler 1992 & 1993a), and elaborated these in further works (Kajler 1998; Kajler and Soiffer 1998).

Kajler has postulated that well-designed computer algebra interfaces should afford intuitive access. As such, they should enable the entry of commands with more than one parameter in a two-dimensional fashion. This prevents syntactic and structural errors. In addition, all templates and masks should follow the convention of operating from left to right.

Intuitive interfaces should also apply conventional mathematical notations, and decouple the surface from the computer algebra systems. The interface should be serviced independently, and regularly developed and updated. Ideally, it should understand a range of computer algebra systems.

Kaljer has responded to his own demand for such a surface for different systems with his development of CAS/Pi (Kajler 1993b). He wrote (Kajler 1998, pg. 151): "…, it is desirable to produce a portable interface that handles lexical, syntactical, and functional differences between different CAS."

CATO AS A SIMPLIFIED USE OF CAS

The author has used various computer algebra systems in his mathematics courses at the University of Applied Sciences (HTWG) in Konstanz, beginning with Maple, followed by Mathematica and finally Maxima. His experiences of the occasional usage of CAS in mathematics courses are complex; he has observed that the experience is not necessarily positive for all students. Only a small number of students are fascinated by the usage of a CAS, and independently explore its potential. A larger proportion of students accept CAS, but only learn those commands they consider necessary to prepare for upcoming examinations. Enough students treat the learning of the vocabulary, syntax and grammar required for the usage of CAS as an unwelcome burden, and consequently give up on mathematics. So the majority never see the opportunities provided by computer algebra as a mathematical aid.

There are many reasons for the usage of a computer algebra system in teaching mathematics (see, for example, Barzel, B. 2012, 2013; Brawn, M. & Steel, C.D.C. 2013; Driver, D. 2008; Greenhow, M. & Zaczek, K. 2011; the first section of Langtry, T., Nicorovici, N. & Zinder, N. 2011 or general Stacey, K.; Chick H. and Kendal, M., 2004). In the author's experience it makes sense to teach mathematics integrating occasional usage of computer algebra systems. He has developed an interface to use a computer algebra system without diverting from mathematics, called the **C**omputer **A**lgebra **T**aschenrechner (Calculator) **O**berfläche (surface) or CATO for short.

THE GUI CATO

The user interface of CATO appears on the screen immediately after start-up. Two text areas and several buttons and combo boxes are visible. To use CATO, students should first select the computer algebra system to use (see fig.1). All computer algebra systems denoted here have to be installed and referenced during the installation process. CATO can then instantiate a connection to the chosen computer algebra system, and creates the output of the version string of the computer algebra system to the second text area, reserved for outputs. During a CATO session, students can switch between the different systems and can use old inputs, if the commands they use are available in the newly selected computer algebra system.

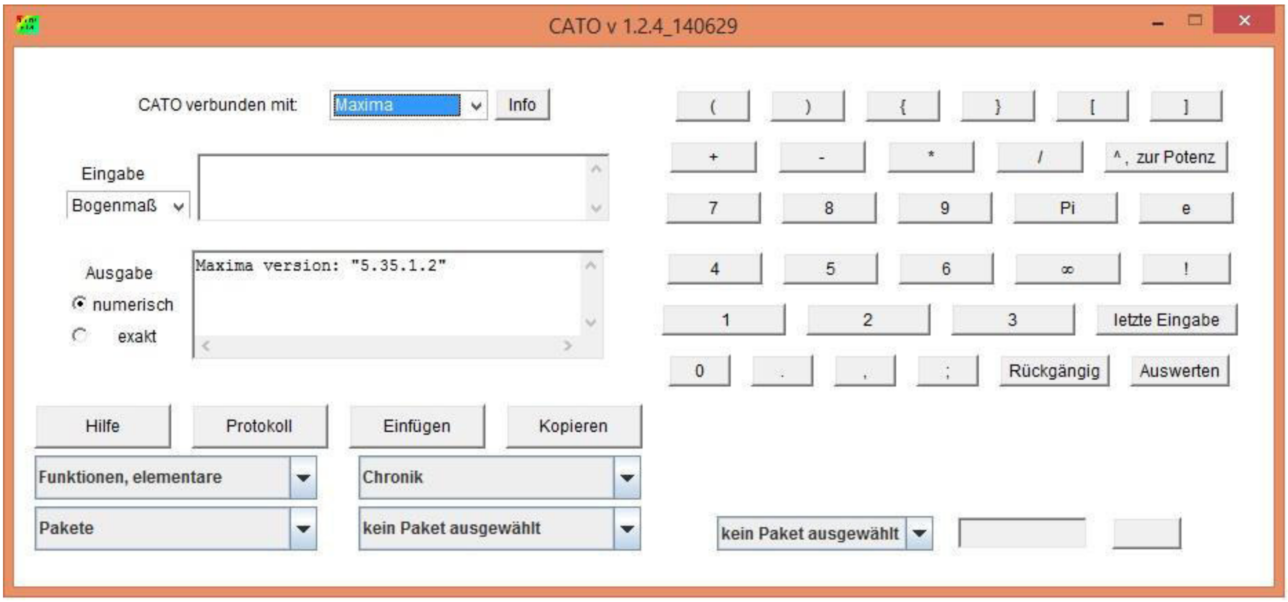

Figure 1. The GUI CATO

Now CATO is ready to execute mathematical commands. CATO can be used as a simple calculator, and students can enter simple commands, such as "sin (4.5)". Students may either click the button "*Auswerten*" (evaluate) or press the return key. Nearly all simple, one-parameter functions can be entered manually using the keyboard. Students have of course to know the names of the commands in order to manually enter them.

All commands can be selected via packages and menus (see below). The commands are grouped in packages or sub-packages. The packages group related commands together. Students who do not know which command to use can query CATO help, which is accessible via a button on the left side of the screen, labelled "*Hilfe*" (help), see fig. 1.

CATO Help is an HTML document, and therefore independent of both the computer algebra system and the operating system used. Furthermore, CATO Help is always available, even when CATO is not running. CATO Help is also offered online on the author's website (Janetzko 2007a). CATO Help is structured as follows: There is an alphabetically ordered list of all commands and their synonyms on the left side. Most of the space is given to the respective help text. All help text is constructed in the same manner: The name of the command appears first (black), followed by the packages containing this command (green), and finally the computer algebra system in which the command is available (red). The free text that follows contains first an abstract description, and then several comprehensible examples. More descriptions of CATO Help and CATO itself can be found in Janetzko 2007b, 2013, 2014a.

ONE CAS IS NOT ENOUGH

The author uses CATO in the courses Mathematics One and Two in the Bachelor of Electrical Information Technology degree programme. The University of Applied Sciences does not offer a general "Introduction to Computer Algebra" course. Consequently, the author introduces students to the use of CATO in his first lecture, after describing what a computer algebra system is, and why it makes sense to use it. Students gain practical experience using CATO in computer lab exercises. They can choose between two systems, the free system Maxima and the Symbolic Math. Toolbox of MATLAB; the Department of Electrical Engineering and Information Technology has classroom licences and licences for all students.

In Mathematics Two, the author teaches ordinary differential equations, and the first steps of statistics. Maxima cannot determine the inverse Laplace transformation of the Laplace transformation of a non-differentiable function; therefore, for these purposes the author uses CATO with the Symbolic Mathematical Toolbox of MATLAB. But the Symbolic Math. Toolbox of MATLAB has no random generators for probability distributions. In those cases where these functions are required the author uses CATO with Maxima.

SOLVING AN ORDINARY DIFFERENTIAL EQUATION (ODE) WITH LAPLACE TRANSFORMATION

For example, we consider the following ordinary differential equation for an undamped system with force:

$$\ddot{y}(t) + 4 \cdot y(t) = f(t),\ y(0^+) = 0,\ \dot{y}(0^+) = 0,\ f(t) = \begin{cases} 0 & \text{für } t < 5 \\ \frac{t-5}{5} & \text{für } 5 \le t < 10 \\ 1 & \text{für } t \ge 10 \end{cases}$$

The source f(t) is known as ramp loading. For the Laplace transformation we write f(t) with the unit step function

$$\ddot{y}(t) + 4 \cdot y(t) = u_s(t) \cdot \tfrac{t-5}{5} - u_{10}(t) \cdot \tfrac{t-5}{5} + u_{10}(t),\ y(0^+) = 0,\ \dot{y}(0^+) = 0$$

Let L(y) be the Laplace transformation of the function y(t); we replace $y(0^+)$ and $\dot{y}(0^+)$ by their values, simplify the term and have to find the inverse Laplace transformation of:

$$L(y) = \left(\frac{1}{5} \cdot \frac{1}{p^2} \cdot e^{-5 \cdot p} - \frac{1}{5} \cdot \frac{1}{p^2} \cdot e^{-10 \cdot p}\right) \cdot \frac{1}{p^2+4}$$

Now students are asked to confirm the solution step-by-step with CATO and computer algebra. To do so, they first need the Laplace transformation of f(t), the ramp loading function. In CATO, they select the package "Analysis III" using the combo box on the lower left (see fig.1). All sub-packages from this package are downloaded into the combo box directly second to the left. There students can select the sub-package "*Integraltransformationen*" (integral transformations), and find the command "*Laplacetransformierte*" (Laplace transform) in the third combo box right to the second one. Selecting this option opens the window for this command (see fig. 2).

Figure 2. The window for the command "*Laplacetransformierte*".

In the first input row, students should enter the ramp loading function. So they select the package „*benannte Funktionen*" (named functions) in the combo box on the lower left in CATO (see fig.1). They then select the command "*Sigmafunktion*" (unit step function) in the combo box second to the right. The first input row of the command window of the "*Laplacetransformierte*" (Laplace transformation) now reads "*Sigmafunktion*(" and type: "t-5)*(t-5)/5 -" .

Now students select the command *"Sigmafunktion"* for a second time, and can then type: "t-10)*(t-10)/5". They click the button *"weiter!"* (next), type "t", click the next button *"weiter",* type "p", click the last button *"weiter!"* (next) and then the button *"alle Eingaben abgeschlossen"* (all inputs are entered). The complete command is written in the input row of the CATO surface. Now they can either click the button *"Auswerten"* (evaluate) or press the return key.

To get the inverse Laplace Transformation, students follow the same steps. They select the package "Analysis III", then select in this sub-package *"Integraltransformationen"* (integral transformations) the command *"Rücktransf. bei Laplace-T."* (inverse Laplace transformation) and finally arrive at the following window (see fig. 3).

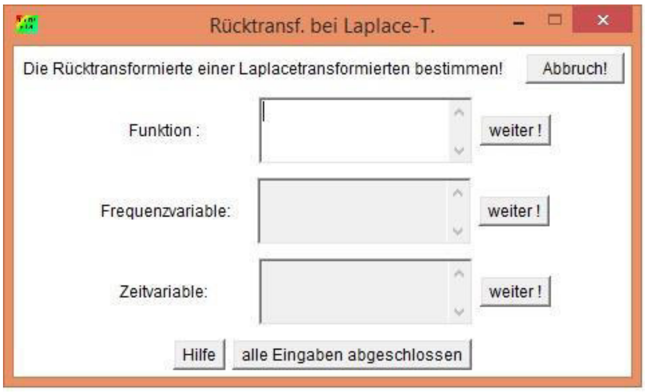

Figure 3. The window for the command *“Rücktransf. bei Laplace-T.”*.

To type the input for the first row, students use the command *„Exponentialfunktion”* (exponential function) in the package “Analysis I”.

WORK WITH THE LAW OF LARGE NUMBERS

As the lecturer I might want to calculate a both-sided symmetric confidence interval for a random variable which is described by N(μ,σ²) with the confidence level of 1-α. I choose a sample X_1, X_2, … , X_n, and so $T = \frac{\bar{X}-\mu}{S} * \sqrt{n}$ has a Student t-distribution with ν = n – 1 degrees of freedom. A confidence interval for the parameter μ is the interval $\left[\bar{x} - t_{1-\frac{\alpha}{2}}(\nu) * \frac{S}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}(\nu) * \frac{S}{\sqrt{n}}\right]$, and $t_{1-\frac{\alpha}{2}}(\nu)$ is a quantile of the Student t-distribution with ν degrees of freedom.

The law of large numbers has the consequence that after the calculating of a large number of confidence intervals with level 1-α, α*100% of the intervals do not include μ.

Now I want to present an example in class, using the values μ = 0, σ = 1, n = 30, 1-α = 0.9. So I estimate 20 confidence intervals, hoping that two intervals do not include μ = 0.

In CATO I select the package *“Definitionen”* (definitions), using the combo box on the lower left (see fig. 1). In the combo box second to the left I can then select the command *“Definition, Ausdruck”* (definition, term), and get the following window (see fig. 4):

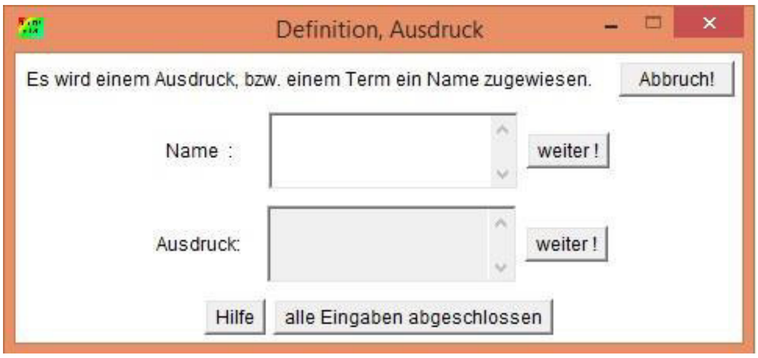

Figure 4. The window for the command “Definition, Ausdruck”.

In the first line I insert *„liste1“,* click on *“weiter!”* (next!), and when the cursor is in the second line, I must select the next command on the CATO surface: In CATO I select the package *“Statistik”* (statistics), using the combo box on the lower left (see fig. 1). In the combo box second to the left, I can then select the sub-package *“Normalverteilung”* (normal distribution) and find the command

"Zufallszahlen, normalverteilt" (random numbers, normally distributed) in the third combo box in the next window to the right. Selecting this command, the following window appears (see fig. 5).

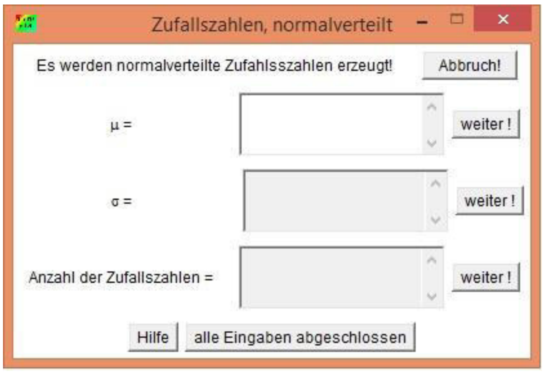

Figure 5. The window for the command *"Zufallszahlen, normalverteilt"*.

On the top we insert "0", click *"weiter!"* (next!), in the second row we type "1", click the second "weiter!" *(next!)* and in the third row enter "30". Clicking the last *"weiter!"* (next) and *"alle Eingaben abgeschlossen"* (all inputs are entered) closes the window *"Zufallszahlen, normalverteilt"* (random numbers, normal distributed). The complete command is now in the second input row of the first window. There, I click *"weiter!"* (next!) and *"alle Eingaben abgeschlossen"* (all inputs are entered); the complete command is now in the input window of CATO. Now I can either click the button "*Auswerten"* (evaluate) or press the return key to execute the command.

The one-parameter command *"Mittelwert"* (arithmetic mean) is found by selecting *"Statistik"* and the sub-package *"deskriptive Statistik"* (descriptive statistics). After selecting *"Mittelwert"* (arithmetic mean) the input window of CATO shows *"Mittelwert* (in the input row of CATO, I type *"liste1*)" and the command can be executed.

Finally, I estimate the quantile of the Student t-distribution by selecting first *"Statistik"* and then the sub-package *"t-Verteilung"* (Student t-distribution). There I can choose the command *"Quantil der t-V."* (quantile of the Student t-distribution) and get the following window (see fig 6):

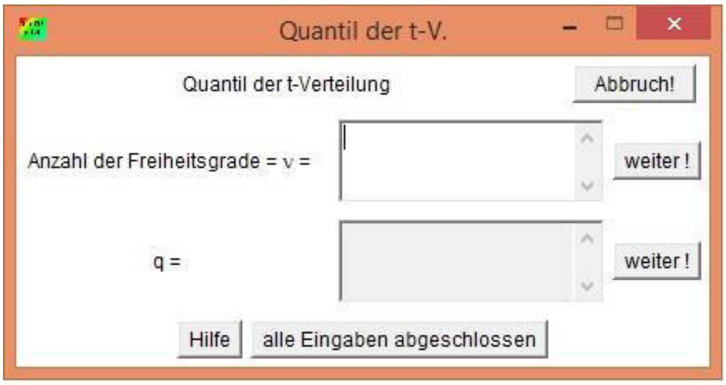

Figure 6. The window for the command "*Quantil der t-V.*"

CONCLUSION

The interface presented to the user remains the same whether a command is entered using Maxima or the Symbolic Toolbox of MATLAB. When CATO is linked with the Symbolic Toolbox of MATLAB, students need not declare symbolic variables. If the selected command needs more than one parameter, students always get a guided command window of the same kind; the system CATO puts the parameters in order, and uses the right brackets and the right separators. In those cases where a command is available in both computer algebra systems (Maxima and the Symbolic Toolbox of MATLAB), both the command window of CATO and the order of the input rows remain the same. Students notice no differences, but get the same guide. The differences occur in the design of the output.

STUDENTS' FEEDBACK

The author does not have the opportunity to teach classes using both CATO and CAS, or alternatively classes without CATO, or without CAS: Using CATO, the lecture was modified, examples and exercises with large numbers of calculations were shortened, because otherwise the mathematical part of the instruction would have become too complex. CATO is used to finish the calculation of the examples, and the students are always able to follow the application of CATO, entering the same commands on their own, if desired. They know that they should not learn specific commands but rather the usage of CATO.

CATO connected with a computer algebra system is accepted as an auxiliary tool from the beginning. During lectures, there are often questions from students regarding CATO that result from independent self-study using CATO (these questions come from both beginner and intermediate students). Shortly before the final exams, there are often complaints about the very limited number of commands offered by CATO, or the sometimes cryptic results from *Maxima*. These complaints are more generally raised by weak students. Of course, students can and should use CATO in their exams, to integrate, to determine amperage of electrical circuits, or to calculate probabilities.

Feedback from students has improved the usability of CATO: They requested that CATO allow grouping of all important commands for the next written exam into a new package, exportation of this package, and then re-importation into any computer for the exam. This is now possible in the latest version, CATO v1.2.4 (Janetzko 2014b).

CURRENT STATUS OF CATO

Currently there are, respectively, approximately 500 commands for Maxima, 500 commands for Mathematica (version 4.0 or higher), 400 commands for the mathematical toolbox of MATLAB, 300 commands for Maple (version 9.5 or higher), 300 commands for MuPAD 3, 200 commands for Yacas, and 100 commands for MATLAB. In addition, there are more than 50 CATO internal commands.

The connection to MATLAB has the notable feature that it allows the Symbolic Math. Ttoolbox to be automatically selected whenever available. Furthermore, an external application interface has to be installed to use MATLAB, resulting in the requirement for version R 2009 or higher.

The CATO code consists of 20500 lines of Java, excluding the command database. CATO has also been tested on Mac OS X and Linux/Ubuntu.

Free trial versions of CATO can be downloaded from the author's website at any time, http://www.computeralgebra.biz.

The concepts of the German-language version of CATO can be easily transferred to a version of CATO in another language.

ACKNOWLEDGEMENTS

I would like to thank the IT Service Centre of the HTWG Konstanz for making CATO available in the PC room every semester, from the beginning, and the Department of Electrical Engineering and Information Technology of the HTWG Konstanz for their encouragement.

BIOGRAPHICAL NOTES

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YOUNG CHILDREN'S ANGLE SIZE ESTIMATION IN DYNAMIC GEOMETRY ENVIRONMENT

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This paper examines young children's thinking about angle size in a dynamic geometry environment during angle estimation tasks. Children used two types of routines frequently, firstly comparing angle measure with position of clock hands, and secondly repeating a small turn over and over in order to get to a bigger turn. Findings suggest that children used internal and external referents for angle size estimation. Children's gestures, motion and environment played an important role in their thinking.

Keywords: Angles, primary education, technology

INTRODUCTION

Concepts of angle and rotation are central to the development of geometric knowledge (Clements and Battista, 1992). Although angle is a basic concept that is used by humans in analysing their spatial environment, it can pose challenges to learners, even into secondary school due to its multi-faceted nature (Mitchelmore & White 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years (Spelke, Gilmore and McCarthy, 2011). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (11-12 years old). The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to explore angle learning at the k-1 grade levels. We have chosen a DGE (Dynamic Geometry Environment) approach in order to support the dynamic, angle-as-turn conception (see Kaur & Sinclair, 2012).

In this paper, I report on an exploratory study conducted with a split class of kindergarten/grade 1 children (ages 5-6) working with angle using *The Geometer's Sketchpad*. No unit of angle measurement is introduced to children at this early stage. I am interested in finding out how children estimate the amount of turn needed to reach at a particular position, what kind of thinking is involved in estimating an angle and what spatial factors they consider while estimating angles especially in the absence of any measuring unit. These questions form the basis of this research study.

CHILDREN'S UNDERSTANDING OF ANGLE

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Henderson & Taimina, 2005). Freudenthal (1983) describes 'angle as turn' as "the process of change of direction" (p.327). Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore & White (1995) suggests that angles occur in a wide variety of physical situations that are not easily correlated. Despite the excellent knowledge of all situations, specific features of each situation strongly hinder recognition of the common features required for defining the angle concept.

Mitchelmore and White (2000) further investigated the angle conceptual development among 192 students from grade 2 to grade 8. They found that children could identify *2-line angles* where both arms of the angle are visible, (e.g. corners of a room, road intersections, pair of scissors), as early as in grade 2. On the other hand, although young children seemed to understand very well a number of *1-line angle* situations (Doors, widescreen wipers, ramps etc.) and *0-line angle* situations (turning of doorknobs, wheels etc.), even by grade 8 many students couldn't interpret these situations using angles. The difficulty in case of 1-line and 0-line angle situations can be due to requirement of imagination of the one or both missing arms/parts.

Research has reported about the young children's difficulties in understanding the turn as an angle as well as connecting static angles to turns (Mitchelmore, 1998; Clements, Battista, Sarama & Swaminathan, 1996). Clements et al (1996) in their study with third graders formed a hypothesis that students learned about turn measurement by integrating two schemes, turn as body movement and turn as number. They further conclude that changes in orientation are harder to understand than changes in position because they are less salient, as young children don't naturally connect static angles to turns. Students also think that the length of the arms is related to the size of the angle (Stavy and Tirosh, 2000). Students tend to think that 'the longer the rays, the greater the measure of the angle'. In the previous work (Kaur, 2013), it is found that DGE-based instruction of angle proved helpful in focusing children's attention on the quantity of turn rather than on the length of line segments even in case of paper pencil based angle comparison tasks. Bustang, Zulkardi, Darmawijoyo, Dolk, & van Eerde (2013) found that visual field activities involving vision and constructing spatial representations helped third graders developing their understanding of the concept of angle as well as their initial understanding of the notion of vision lines and blind spots. The present study focuses on relatively younger children (kindergarten/grade1) and uses the dynamic geometry environment based tasks. The 'trace' feature of *Sketchpad* offers a visible, geometric record of the amount of turn and helps children to see the process of turning explicitly as compared to visual field activities.

THEORETICAL PERSPECTIVE

In previous research, Sfard's (2008) 'commognition' approach is found suitable for analysing the geometric learning of students interacting with DGEs (see Kaur & Sinclair, 2012; Kaur, 2015). For Sfard, thinking is a type of discursive activity. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). According to Sfard, mathematical objects are discursive objects and constructed personally by the students. Differences in the realization of a signifier (a word that acts as a noun in the mathematical discourse) can be observed through one's word use, visual mediator, routines, and narratives. A realization is a perceptually accessible thing so that the narratives about the signifier can be translated into the narratives about its realization. Sfard (2009) further recognises the importance of gestural communication as it ensures all interlocutors "speak about the same mathematical object" (p.197). Gestures are essential for effective mathematical communication.

Given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow 2006), and their potential to communicate temporal conceptions of mathematics (Núñez 2003), it is necessary to focus not just on the words or the visual mediators that the children use, but also on their gestures. This is not just true for young children, of course, as Kita (2000) has pointed out that the production of a gesture helps speakers organize rich spatio-motoric information, where spatio-motoric thinking organizes information differently than analytic thinking (which is used for speech). Given the visual and dynamic nature of the learning environment of this study, gestures are likely to be an important component of the classroom communication about angles. I am particularly interested in investigating how the students might move between different word uses, routines and visual mediators and to examine the informal language they use to talk about the size of angles.

METHODOLOGY OF RESEARCH

We (research team and class teacher) worked with kindergarten/grade1 children (aged 5-6) from a school in a rural low SES town in the northern part of British Columbia. There are 22 children with diverse ethnic backgrounds and wide range of academic abilities. We designed lessons related to angle along with the classroom teacher, who has a Masters degree in mathematics education and has been developing her practice of using DGEs for a couple of years. The teacher and children worked with angles in different ways using Sketchpad for six lessons in a whole class setting with an IWB (Interactive Whiteboard). Each lesson lasted approximately 40 minutes and was conducted in a group with the children seated on a carpet in front of a screen. Lessons were videotaped and transcribed.

Dynamic Sketches

Different sketches were used to explore the concept of angle with the children during six angle lessons (details in Kaur & Sinclair, 2012).

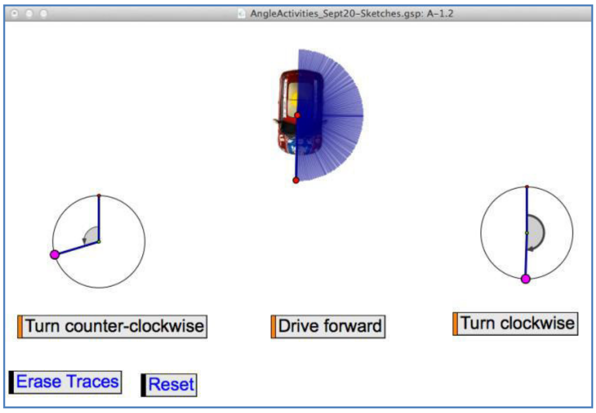


Figure 1a: Driving angle sketch

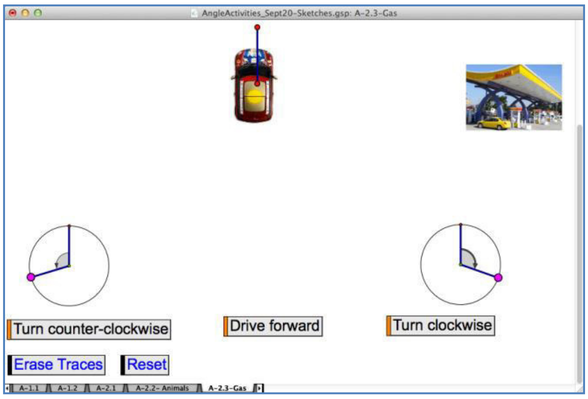


Figure 1b: Driving angle sketch with gas station

The *driving angle model* sketch (figure 1a) shows both a static as well as dynamic angle. It includes a car that can move forward as well as turn around a point. The turning is by controlled two small dials (each of which has two arms and a centre) - one dial allows clockwise turn and other counter-clockwise turn. Five action buttons (**Turn counter-clockwise**, **Drive forward**, **Turn clockwise**, **Erase Traces** and **Reset**) control the movement of the car. Pressing **Drive forward** button once moves the car forward to a distance equal to length of the line segment that goes through the centre of car. The traces of turn offer a visible, geometric record of the amount of turn. In this sketch, the turn of the car is associated with the amount of angle adjusted in the small dials, which act as steering wheels. So, the very first goal of the instruction was to make children explore and understand the turning of the car and its association with the dial.

In the first and second lesson, children worked with the 'driving angle model' (figure 1a), so they became familiar with the turning of the car and its association with the dial. Also, the teacher explained them the meaning of clockwise and counter-clockwise turn using a clock. There was no instruction about the measurement or units of angles. In the third lesson, they were posed with a problem where car runs out of gas and they needed to help the car to reach at a gas station (figure 1b). During this lesson, the position of the gas station was changed at various places on the screen. Here, I report on children's attempt to take the car to the gas station.

CHILDREN'S ATTEMPT TO TAKE THE CAR TO THE GAS STATION

The teacher showed the car sketch (figure 1b) on IWB and asked how the car could reach the gas station.

Louis: Turn this way <*gesturing as in fig. 2a*>

Teacher: Umm… Lily?

Lily: You have to turn it that way <*pointing towards clockwise* >

Teacher: Good, on turning. So, you used the word 'Turn that way'. What kind of a turn is that? Can you think of a different way to describe that turn? <*Some children raise their hand*> Pam?

Pam: Counter clockwise or clockwise.

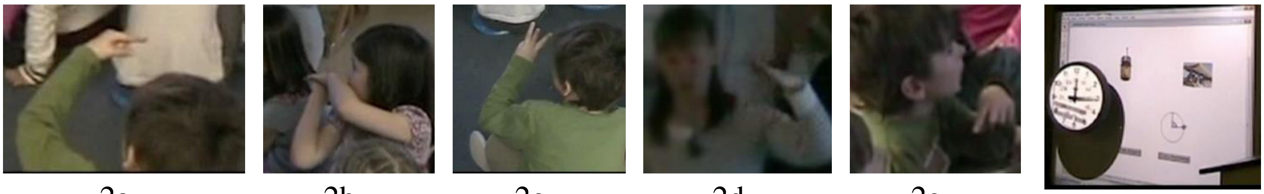
Teacher: We can use the words clockwise or counter clockwise. Umm... Kim?

Kim: Turn clockwise (*also gesturing turn as in fig 2b*).

Teacher: What about the amount of turn? Can you think about ways to describe the amount of turn?

Louis: Three (gestures three with fingers as in fig. 2c) three.

As soon as the teacher posed the question, the children immediately recognised that they needed to turn the car. Initially, Louis and Lily used the words "turn this way" and "turn it that way" along with the pointing gestures. Pointing gestures and utterances together communicated the children's thinking. Absence of either one would have failed to communicate *which direction* (conveyed by gesture and words "this way" or "that way") the children wanted to *turn* (conveyed by utterances) the car. Upon asking about another way to describe the type of turn, Pam's proposal to use words 'counter clockwise and clockwise' for describing the direction of turn indicates children's readiness to use the vocabulary words that were introduced in the previous lesson. Louis, Lily and Kim used the gestures with their hands and arms, which suggests the use of the embodied visual mediators in their communication to depict the process of turning. The turning of the car is transformed to the turning of arms in case of Kim along with the utterance "turn clockwise", which suggests the use of dynamic thinking. These dynamic gestures communicate temporal relationships (change in position of car). When the teacher asked about the amount of turn needed, Louis suggested 'three' along with a three-finger gesture (fig. 2c).



2a 2b 2c 2d 2e 2f

Figure 2: Snapshots of various gestures

The teacher thought that Louis was suggesting that car is needed to **Drive forward** three times in order to reach the gas station. She again asked about the amount of turn while making a turn motion with her left arm (as in fig 2d). Louis reaffirmed that the car needed to turn 'three', saying, "That's what I am saying". The teacher asked Louis to explain more about what he meant by "three".

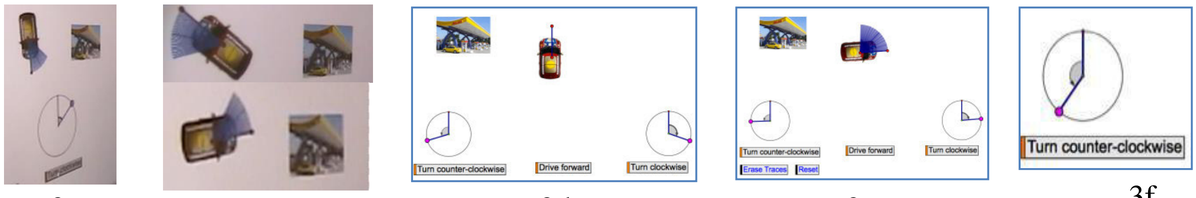
Louis: If you turn this way three (*gesturing clockwise turn with index finger and stoping at approximately quarter turn position, see figure 2e*) amounts of turn.

Louis's gesture (fig. 2e) along with the words "if you turn this way three amounts of turn" shows his confidence in what he was thinking about the turn. In order to understand Louis's response, the teacher started to adjust the angle in the clockwise dial by dragging the red point and asked Louis to stop her when turn gets to three. As soon as the red point reached to about a quarter turn (see clockwise dial in fig. 2f), Louis said, "stop". At this point, the teacher hypothesised that Louis was referring to the clock for the turn. Louis confirmed that he was visualising the clock by saying "That is just what I mean it was". Teacher explained this to the whole class by showing the clock (fig. 2f). In this episode, Louis was locating the position of the car with respect to clock (an external frame of reference) and he was aligning the specific position of car's steering wheel with respect to the position of hands in the clock. The clock acted as a visual mediator for forming the above routine. Louis's technique is based on the mechanism of metaphor. He inserted the new signifier "amount of turn" into his familiar discursive template "clock". This spontaneous metaphorical projection helped him to engage in the new type of talk and evoked an awareness of what may be proper as an utterance about the amount of turn.

Other children demonstrated a different approach. When the car was at a particular position as shown in figure (3a), the teacher asked Kim about how to turn the car.

Kim: I want to turn all the way (gesturing the turn with hand around the car) over there (pointing towards gas station)

Kim pressed the **Turn clockwise** button six times in order to get the car to the position as shown in figure (3c) and then pressed the **Drive forward** button. She observed the change of car's position on pressing the **Turn clockwise** button once and then pressed a second time and then a third, fourth, fifth and sixth time. Thus, Kim's routine consisted of repeating a small turn over and over in order to get to a bigger turn. In this routine, the amount of turn needed for the car is determined without any external referent, because after the initial two small turns, she recognised the pattern and turned the car again and again until it reached the required direction.



3a 3b,c 3d 3e 3f

Figure 4: Snapshots of various car positions relative to gas station

Louis's routine of comparing angle measure with the position of clock hands, was shared widely in class, as evidenced in a follow up one-on-one interview with several students. For example, I asked Gia to make the car reach the gas station (shown in figure 3d). She proposed that a counter-clockwise turn would be smaller in this case and adjusted the angle in the counter-clockwise dial, but she pressed the **Turn clockwise** button. So the car reached the position shown in figure (3e). I asked how much turn she needed to turn car towards the gas station (from position as shown in fig 3e).

Gia: (Thinking for about 7 sec) Six

Gia adjusted the angle to six by dragging the red dot as shown in figure 3(f). This shows Gia's spatial reasoning about half turn using clock as visual mediator. Clearly this is the example of children's reasoning when the car was not facing upward. This shows that Gia was able to estimate angles using the clock as referent even in the case when the car was not facing upward (in the standard 12 o'clock position).

DISCUSSION AND CONCLUSION

The above episode provides evidence that children utilised a variety of resources, including language, gestures and visual mediators in estimating angles in DGE. For estimating the amount of turn, they made use of external (such as clock, gestures of turning arms, hands) and internal referents (thinking of many small turns for one big turn). This whole process suggests two routines of angle estimation among young children working in a DGE (1) comparison of angle measure with position of clock hands, and (2) repetition of a small turn over and over in order to get to a bigger turn. Thus, these two routines resulted in two realizations (1) amount of turn can be described with the reference to the numbers in the clock, and (2) the smaller turn can be repeated until desired bigger turn is achieved. The emergence of arm turning or hand turning gestures might be due to dynamic functionalities offered by DGE, where children used dragging the point to adjust angle and observed the process of turning of car in a particular direction.

The use of the clock for talking about angle size was unexpected for the teacher. It is worth noting that teacher used the clock in the previous lesson to explain the meaning of clockwise and counter clockwise directions of turning. Louis not only paid attention to the directionality, but the other properties of the clock such as numbers were also transformed to the sketch. Louis realized that the amount of turn could be described with reference to the numbers in the clock. Thus, in the process of teaching, while we may want to draw children's attention to particular components of an artefact, the children can transfer the other elements of the artefact as well. The use of the numbers in the clock as unit of angle measurement by Louis is quite interesting. In the measurement system, the different angle units are: degrees (deg), gradians (grad), radians (rad), turns (turn). There are 360 degrees or 400 gradians or 2π radians or 1 turn in a full circle. Thus, angle measurement is absolute, as a full circle will have a constant angle regardless of place or time. The clock metaphor used by Louis is quite similar to the prevalent angle units, where a full circle has twelve “amounts of turn”. Louis estimated that the (quarter) turn needed by the car is about “three amounts of turn”. This new non-standard measurement unit of angle used by Louis is also absolute in nature. Louis emerged as leader in associating position of clock hands with amount of turn, the evidence of later use of this technique correctly by other children supports the ideas of sociocultural theory, where students develop a shared understanding. In further research it would be interesting to analyse the change in children’s routines after instruction of benchmark angles in next lesson.

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USING COMPUTER IN TEACHING MATHEMATICS AND ITS EFFECTS ON MOTIVATION AND LEARNING OUTCOMES OF STUDENTS IN A PRIMARY SCHOOL

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The study investigates the effects of use of computer on motivation and learning outcomes, and how useful of different software is perceived by students. Computer is used frequently in the experimental group: Cabri-3D shows cross sections of 3-D shapes, Excel generates graph of linear algebraic equation, and tools of Word construct graphs. Students present project with PowerPoint and video clips on Media Player. They learn materials from the Internet and You-Tube. Students' feedback in the survey reveals that use of computer makes mathematics lessons more interesting attracts students' attention, and learning atmosphere is more positive and enthusiastic. Opinion from students randomly selected for interview also supports the effects of motivation and engagement in learning. Students point out that learning outcome is improved, in the quality of the worksheet. Some think those who are poorly motivated, inattentive, low in rank, and messy in work benefit more.

Keywords: attention, engagement, learning outcomes, learning atmosphere, motivation

INTRODUCTION

The new curriculum outlined in the report by the Curriculum Development Council (2008) aims to provide quality education to students in Hong Kong schools. It adapts from the model proposed by Wiggins and McTighe (2005) that combines classroom processes with desired outcomes. Students develop generic abilities in school that are related to attaining specified targets in the curriculum. These general abilities enable transfer of knowledge and skills to new specific situations in future. These include collaboration, communication, creativity, scientific and technological skills, critical thinking, using information technology, numeracy, problem solving, self-management and study skills. The curriculum model advocates turning the desired outcomes into questions that are subject of students' discussion and investigation. Students can develop skills of explaining, interpreting, application and empathy, and endow them with a sense of perspective and self-knowledge. Classroom learning is learner-centred with topics around a theme, and involves inquiry-based learning and project learning.

At the school the researchers work, it is equipped with advanced computer network. Classrooms and special rooms have desk-top computer and interactive whiteboard. There are two computer rooms and a language laboratory. Students can use the computer room and library for learning during recesses and after school. Students can access to Planetii website for on-line mathematics learning and practice. Students can continue independent learning of mathematics at home. Students are given one or two mathematics projects using computer each year. On-line Planetii assessments are administered three times each year. Students welcome the multiple-choice questions since higher marks can be attained. The school encourages teachers to use computer in teaching. Regular use of Active Inspire flip charts has been implemented in the mathematics department. However, heavy teaching loads and pressure from administration for content coverage do not spare time for teachers to prepare worksheets and use computer to teach mathematics frequently. Computer and software are not utilized in most lessons. Some mathematics teachers thought instruction by discussions and explanations of notes and examples to be more effective in students' learning. They believe that class activities with real objects and authentic examples in everyday life experienced by students are better for acquisition of skills, scaffolding to construct new knowledge and understanding mathematical concepts. A national survey in United States found that only 11% mathematics teachers in secondary schools were frequent users of computer (Becker 2000). It was argued that only education reform with improvements in teacher training, curriculum, assessment and school's capacity for change could result in the use of technology as an effective learning tool (Roschelle, Pea, Hoadley, Gordin and Means, 2000). Teacher's philosophy and pedagogy also affect the use of computer. It was found that teachers who adopt the constructivist teaching philosophy and whose pedagogy involve depth and understanding of the topic are more enthusiastic in using computer (Becker, 1994).

LITERATURE ON EFFECTS OF USING COMPUTER TO LEARN MATHEMATICS

Lepper (1985) asserted out that computer-based learning activity could lead to increased student engagement on academic task because the activities provided intellectual challenge to motivate students to find a solution for the problem. It stimulated students' curiosity to resolve an incongruity. It provided a sense of independent control and mastery over the environment and provoked sustained, intense effort. Means and Olson (1995) found that nearly all intensive computer-using classes reported effect of increased motivation. For some cases it was improvement for students' effort at learning the specific subject-matter. But for others students perceived a sense of accomplishment gained from working with computers. Sandholtz, Ringstaff and Dwyer (1997) reported evidence of increased student engagement in academic work. They pointed out that ordinary students would stay behind to discuss with teacher about content and assignment. Others arrived early and stayed after school to work on the computers. They suggested that the most effective ways in engaging students could be using the computer as a tool for exploration and project-based work. However, Lepper and Chabay (1985) cautioned about the use of computer for less motivated and capable students who might have less satisfactory learning outcomes.

Research findings support the positive effect of use of computer to learn mathematics with improved learning outcomes. In a national study of Educational Testing Service in the United States, Wenglinsky (1998) reported that technology helped academic achievement of some student groups. If computer was used for higher order thinking on mathematics and applications that encouraged deep reasoning, students increased learning. On the contrary, applications that tried to make repetitive skill practice more entertaining decreased performance. The results were mixed; there was substantial relationship between the positive use of technology and academic achievement for 8^{th} graders but only negligible for 4^{th} graders. Becker (1994) argued that the teaching of computer-using teachers were said to be exemplary not for evidence of higher test scores or greater intellectual competence. Students' frequent use of computer resulted in higher order thinking. He explained:

> 'Instead, our attribution of exemplary teaching practice was an assumption that important academic outcomes would result from systematic and frequent use of computer software for activities involving higher order thinking such as interpreting data, reasoning, writing, solving concrete, complex, real-world problems, and conducting scientific investigations.' (p.288)

On the topic of learning functions, Schwarz and Bruckheimer (1988) studied the transfer of knowledge in a computerized environment. The researchers studied children searching for solutions by different approaches: some searched through graphic representation first, while others used the algebraic representation first. They looked at the three cognitive levels of functional thinking: the numerical level, the functional reasoning level, and the dynamic functional reasoning level; at which the concept of function was understood and searching was efficient. They concluded that children focussing on graphing before algebra led to a higher level of functional reasoning. The accuracy and convergence procedures were transferred from graphing to algebra. Confrey and Smith (1992) argued that understanding can be built through multiple representations and contextual problems. Technology offered students access to various types of representations, and to manipulate functions as graphs. Kllogjeri (2010) asserted that Geogebra could meet the needs and trends of young people in this generation, who were potentially game-driven and curiosity problem-driven. He explained that the visual double representations linked the geometric and algebraic properties of the mathematical object: the interface had a graphics display window and the numeric representation was shown in the algebraic window. This attribute helped students forming concepts on functions and properties of algebraic equations, and performing operations like multiplication of fractions. Students dragged the right square over to get the overlapped parts and found the solution. He thought that for primary students to make sense of formal concept definitions, they must have the experience of linking with the concept image. He explained:

> 'So, by acquiring visualization ability students form visual reasoning and get the right information in performing the algorithm and understanding and owning a mathematical concept.' (p.684)

RESEARCH DESIGN

An experimental study was carried out in 2013 for remedial class students. Twelve students in Grade 5 learnt geometry of 3D-solid cross-sections using Cabri-3D software. In contrast the students in the control group learnt by cutting and viewing the cross-sections by dissecting potatoes and turnips. Students in the experimental group gave positive feedback on using computer in teaching of mathematics. They were engaged in learning because it was more interesting and motivating to learn. The main attribute of the software was that it gave vivid animations and graphic representations by moving and rotating either the 3-D solid or the inclined plane. Students said that the experience was very real and instantaneous with movements of the mouse, the cross-sections of triangle, square, rectangle, trapezium, pentagon and hexagon changing before them.

The present study in 2014 to 2015 investigates the effects of using computer in teaching mathematics in a Hong Kong primary school.

The research questions are:

1. What are the students' opinions on the use of computer on motivation in learning mathematics?

2. Does the use of computer have any effects on students' learning outcomes?

3. How do students perceive the usefulness of different software in learning mathematics?

Five classes from Grade 3, 4 and 6 taught by the researchers are selected to be the experimental group. Students experience learning and doing activity with computer in mathematics lessons.

Teachers and students use computer frequently in the teaching and learning. Teachers in Grade 6 build Excel spreadsheet to represent x and y variables of a linear algebraic equation. Teachers write simple programme with symbols to instruct the operations in different cells. By manipulating the input x to get output of y to the algebraic equation: $y = 4x + 2$ and $y = 5x - 2$, the computer generates the line graphs. The visual and graphical representations help students understand the abstract concept of function. Teacher shares information and pictures with students in Grade 4 on the topics of line symmetry and tessellated shapes. Information and materials on the main points are learnt from Internet websites and viewing You-Tube video clips with Media Player. Teacher also instructs with tools in Word to construct bar charts. Students analyze and identify differences, calculate total and average, and predict future trends. Activity worksheets are completed and submitted to teacher by e-mail. Teacher gives instruction and guidance to Grade 3 students on the project on capacity. Students present project with video clips of an experiment and discuss the topic with PowerPoint. Teacher also instructs with tools in Word to construct block graphs.

METHOD OF DATA COLLECTION

Three methods of data collection were employed: by direct observation, questionnaire survey, and semi-structured interview. Lessons of the five classes in the experimental group were recorded on video. Observations and reflections on teachers' instructions and students' learning performances and behaviours were made from viewing the playback of the video clips. Students' engagement, attention span, questioning and responding, and on-task academic learning time were included as performance indicators of motivation and learning outcomes. A self-completed questionnaire survey was carried out for all the students in the experimental group and 129 questionnaires were returned. The aim was to gather feedback from all students taking part in the study. Quantitative data was collected from five-point Likert type rating scale on ten questions. The mean scores for each of the ten questions were calculated and compiled. Questions 1 and 5 are set for effects on motivation: 'using computer makes learning more interesting' and 'the learning atmosphere is more positive and enthusiastic'. Question 3 is set for engaging students in learning: 'texts and graphs from software attract more of my attention'. Question 2 is on learning outcomes: 'using computer in mathematics improves the quality of worksheets and quiz scores'. Question 7 is about interaction and feedback: 'there is sufficient time and opportunity for asking questions and interaction'. Other questions asked students' perceived usefulness of different software: Word (Question 4), PowerPoint (Question 6), Media Player (Question 8), Excel (Question 9), and Cabri-3D (Question10). Ten students of mixed ability and representative of the cohort in the experimental group were randomly selected to give feedback to gather qualitative data. The aim was to solicit more in-depth and succinct opinions to answer the questions and explore the underlying reasons. Four questions were formulated for the semi-structured interviews which were recorded on video. 1. Does teaching with computer help you learn mathematics? Which kind of students will benefit most? 2. How does teaching with computer help learning mathematics? 3. Some students think they don't have enough chance and time to ask questions when using computer. How can we solve the problem? 4. Which software do you think is most useful to learn mathematics? From data gathered from the observation, survey and interview the researchers could compare and contrast the effects of using computer to learn mathematics. They could check which software was perceived to be most useful. They could look into the reasons of motivating and engaging students to learn, and how to foster students' acquisition of skills and understanding in mathematical concepts.

FINDINGS

Reviewing the video recordings, it could be observed that most of the time students were motivated and engaged in learning in the mathematics lessons. Students' learning performances were very good; they paid attention to teacher's explanation and demonstration and completed the activity worksheet. Some students asked questions if they could not follow or understand. Many students raised their hands and responded to teacher's questions. Use of computer to learn mathematics attracted students' attention and engaged them in learning activities for 25 minutes.

The results of the survey on the effects of using computer to learn mathematics are in Table 1.

Q1	Using computer makes learning more interesting.	4.0
Q3	Texts and graphs from software attract more of my attention.	3.8
Q5	Learning atmosphere is more positive and enthusiastic.	3.6
Q2	Using computer to learn math improves the quality of worksheet and quiz score.	3.3
Q7	There is sufficient time and opportunity for asking questions and interaction.	3.2

Table 1. Mean rating score of effects on learning (N=129)

It can be seen that rating scores are high at 4.0 and 3.6 for questions on motivation (Q1, Q5). Rating score is also high at 3.8 on attracting students' attention in learning (Q3). But rating scores are low at 3.3 on improving the quality of worksheets and quiz scores (Q2), and 3.2 on sufficient time and opportunity for asking questions and interaction (Q7).

Q4	Tools in Word improve the skills of presenting data in block graph and bar chart.	3.8
Q6	Notes on slides of PowerPoint help remembering the main points.	3.8
Q8	Video clips by Media Player help better understanding the procedures of experiment.	3.6
Q9	Excel helps to learn graphical representation of a linear equation.	3.5
Q10	Cabri-3D helps to learn the cross sections of 3-D solids cut by an inclined plane.	3.6

Table 2. Mean rating score on computer software (N=129)

The results for perceived usefulness of different computer software are in Table 2. Rating scores are high for Word, PowerPoint, Media Player and Cabri-3D, but lower for Excel. Word is perceived by many students to be useful for understanding the skills for organizing and presenting statistical data in block graph and bar chart. Similarly the notes on the PowerPoint slides is also perceived by many students to be useful for better remembering the main learning points of the topic discussed. There are some students who give moderate rating on Media Player for editing video clips to explain how an experiment is carried out. Similarly some experience using Cabri-3D think it gives visual representations and animations that help understand the cross-section geometry of 3-D solids.

Review of transcripts of teacher and students' conversation in the interview on first part of Question 1 revealed that six out of ten respondents mentioned positively about the motivation effects, attract students' attention or engaging students to learn. The followings are excerpts of the responses:

Teacher: Does teaching with computer help you learn mathematics?

Student 1: Students not interested in mathematics may be attracted by computer and pay attention more.

Student 2: Use of computer in learning mathematics is interesting and will be useful when I grow up. The class work using computer to draw the chart and send to teacher by e-mail is wonderful. The computer makes the students interested in drawing chart.

Student 3: Completing the class work using the computer is better and the lesson is more interesting.

Student 4: Students not attentive or keen to learn are attracted by computer and enjoy learning. It is fun to use the computer.

Student 8: Students can learn better and more because the lesson is so interesting.

Student 9: Students can learn mathematics better by using computer to make graphs and present projects.

On the second part of Question 1 students had more diverse views of who would benefit most. Seven boys thought *students poorly motivated, inattentive, low-achieving, and messy in work benefit most*. Three boys asserted the attentive, brighter, interested, and keen to learn students benefit most.

Seven students responded to Question 2 positively and thought learning outcomes have improved in terms of quality of the worksheet, but not the test scores. They gave thoughtful insights and plausible reasons on how learning has achieved. The followings are excerpts of the responses:

Teacher: How does teaching with computer help learning mathematics?

Student 2: It helps improve the quality of the worksheets. The texts and lines are typed and drawn in perfection.

Student 3: Drawing bar charts with the computer, the colours are brighter and the lines are straighter. Students can learn and imitate.

Student 4: We can learn more using computer, we draw graphs neatly.

Student 5: Some students draw the outlines of the graph messily without ruler. Computer can draw the graphs neatly and perfectly.

Student 6: With Word and the computer, there is no need to use the ruler to draw. It is so need and tidy, the borders are straight. Computer improves the quality of the worksheet. Technology saves time and the graphs can be drawn quickly, leaving more time to answer the questions. But Word cannot answer the questions in quiz, which is in pencil and paper.

Student 7: We can learn faster and more on mathematics. Graphs be drawn quickly and can spare more time doing computation and answering questions.

Student 10: Use of computer improves my block graph and calculations. I can finish drawing quickly and has more time to answer questions. Students who draw crooked lines and do not use ruler will benefit. With the computer the lines are straight and the graph is neat.

One high-achieving student (Student 1) said in the interview that he did not think using computer helped him learning mathematics in drawing bar-charts with Word. It was same for him whether teacher used PowerPoint or discussed the topic with examples on the board. But he appreciated the use of Internet and You-Tube to get more information and second opinion on some questions.

Some students gave constructive suggestions in responding to Question 3 about more interaction and asking students more questions. They pointed out that teachers should give steps on printout notes and let students do worksheets as homework. Instead they thought teachers should go around the classroom providing help to solve students' problems, explain and discuss word problems on the board. For Question 4, students' responses on the most useful software were Word and Powerpoint.

DISCUSSION AND CONCLUSION

From observation of students' learning performances, feedback gathered in quantitative and qualitative data, most students supported the use of computer to learn mathematics. Students in the survey and the interview rated and mentioned consistently the positive effect on motivation, attract their attention and engage them in learning. They perceived different software to be useful in learning mathematics. Some low-achieving students pointed out the vivid and graphical animations engaged them in learning. The plausible reason could be that visual representations helped them understand abstract concept and the hand-on experience organized their long-term memory in learning. Computer generated graphs with straight borders, perfect outlines of block graph and different shading and patterns of bar chart. It saved time for students to construct the graphs so they had more time to analyze the data, identify the differences, calculate the total and average, and predict the future trend. It helped some students learn in presentation by imitating the neatness of the examples. Respondents thought that students who were poorly motivated, inattentive, low-achieving, and messy in work would benefit more. Students also liked additional information and materials of line symmetry and tessellating shapes from websites on the Internet and video clips from You-Tube which supplemented the textbook. The effect on improving the learning outcomes was less obvious and low in students' ratings in the survey. Respondents in the interview *pointed out that learning outcome was improved, as evident in the improved quality of the worksheet. Perhaps from their experience, improvement with higher test scores could come by only if their computation was accurate and manage to solve complicated word problems. In reality numeracy and algebra made up over 70% of the marks in quizzes and assessments. Measure, drawing and data handling only got about 30% of the total.*

One problem raised from the low rating in the questionnaire was insufficient time and opportunity to ask questions and interact with teachers. Some respondents in the interview asserted that students could ask questions if they did not understand during lessons or recesses. But teachers could improve the lesson and make learning mathematics with the use of computer more effective by interacting more with students and giving them more opportunity and time to ask questions. They should print notes and steps, ask students questions, discuss and explain examples on the board. They should give corrective feedback more; go around the classroom helping students solve their problems and difficulties. Teacher should give reassurance and praise when students make good progress. Finally computer software could not apply in all topics and in every lesson in mathematics. Understanding and finding solution of more complicated word problems and algebraic equation with representation of symbols still required effective teachers' explanations and discussions with students on the board.

LIMITATIONS AND ACKNOWLEDGMENT

Due to constraints of time and resources, the main study is small-scale and lasts for a short period of two and a half months. The population of subjects in each grade level in the experimental group is too small for statistical significance. The research findings may be valid only in the learning environment of the researchers' school in Hong Kong. Nevertheless, the investigation serves the purpose of action research for teachers to improve their teaching of mathematics by using the computer more frequently. The objectives of the research are to foster students' motivation in learning, to attract students' full attention, engage students in learning, to improve the classroom learning environment, to enhance students' learning outcomes. Good practices raise the effectiveness of instruction methods and pedagogical strategies in the teaching and learning of mathematics. It will be interesting to follow up the present one to investigate the effects if the researchers could undertake a large-scale and longitudinal study in future. The researchers wish to thank the Diocesan Boys' School Primary Division for providing the support and resources in the studies from May 2013 to February 2015.

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WEEKLY ONLINE QUIZZES TO A MATHEMATICS COURSE FOR ENGINEERING STUDENTS

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A set of weekly optional online quizzes, on Moodle, was applied to the 104 students of the Multivariable Calculus course (MC). Quizzes were scored up to two extra points to be added if the student scored more than 9 points (out of 20) on the exam. All the students got the same questions (there were not generated different questions). The students could resubmit the answers without penalty. There were usually several sub-questions embedded on each item. This study measure these quizzes effectiveness: students' adherence, the effect on the student's amount of study, the effect on the student's awareness of their own level of understanding, the effect on the outcome scores, and whether the students found it a fair and useful assessment tool.

The approval rate highly increased in this semester. This success cannot be directly attributed to the quizzes, however there are some indicators that they gave a positive contribution.

Keywords: Online Quizzes; Calculus; Mathematics; Higher Education; Engineering.

INTRODUCTION

In today's pursuit, to find how to take advantage of technology to improve learning, the use of online quizzes arrives as a powerful tool. According to Gibbs (2000) student assessment is an effective way to increase understanding. Online quizzes force students to spend more time working productively outside of class, this is valuable especially to procrastinators, see Tuckman (1998). Since online quizzes provide immediate feedback it allows students to be aware of their understanding for each topic and their general level of understanding.

Many studies had emerged around the world on the use of quizzes to support mathematics teaching in Higher Education with many different results, there is still a need for evidence of effectiveness. Moreover, there are many different strategies when applying quizzes and investigators still look for the best combination: mandatory versus optional; contribution to the final grades or not; using in background a Computer Algebra System that generates a slightly different question (new instance) for each student, or not and every student gets the same question; the periodicity of quizzes are widely variable; penalty for submitting the answer more than once, or not; questions with only one final question or with multiple embedded questions.

Being online has advantages and disadvantages. Is useful for not spending precious class time with it. Has the advantage that the answers are automatically corrected which allows working with big quantities of students with little effort. But leads to the problem of unfairness. Is important to assess if there is unfairness, by students cheating, and if students feel treated unfairly. If not all students had easy access to a computer with Internet it would be also an ethical issue (it was not a problem with those students since they belong to a graduation about computing).

Follows a sample of the diversity of approaches and its findings about the use of online quizzes to Mathematics at Higher Education.

In Australia, Lim, Thiel and Searles (2012) applied six quizzes during the semester to Mathematics 2A course to about 120 engineering and mathematics students. This assessment was compulsory and contributed 20% to the total grade. Pass rate did not increased, relatively to previous semesters.

Blanco and Ginovart (2009) created a big set of Moodle quizzes on mathematics for Mathematics 1 and 2 of Catalunya Politècnica Universitat, Spain. The quizzes were used in many different ways. When used in computer lab sessions, students' results were not predictive of students grade at the course: there was no correlation between quizzes and course grades. However, in a questionnaire about the quizzes, more than 80% students rated quizzes as a positive activity; more than 70% students state that quizzes helped them to understand some topics covered in lectures; around 45% felt that performing quizzes made them more interested in the subject.

Siew (2003) administered six quizzes at the Linear Algebra course that count 20% to the final grade. Quizzes use Maple in background, giving questions with different values each time the question is launched. A penalty is assigned when a student resubmits an answer and the solution is only available after the due date. According to 18 over 21 students the quizzes contributed to their understanding of the subject; and for 20 over 21 students the feedback on quizzes was useful to learning. Student's scores at the course were higher in this year than in the previous years.

Shorter and Young (2011) made a comparison of three assessment methods: (1) daily in-class quizzes, (2) online homework, and (3) project-based learning. They found "daily in-class quizzes" as the most predictors of students learning (dependent upon post-test grades) for 117 undergraduate students on a Calculus course.

Myers and Myers (2007) assessed a statistics course with around 65 students in two semesters with two different strategies. First strategy: on a semester students get two exams, one in midterm and the other in the end of semester. Second strategy: in other semester students made a test every two weeks. The second strategy produced better results.

In Portugal, the Instituto de Engenharia de Coimbra, with the project eMAIO (https://lvm.isec.pt/lvm2/login/index.php) offer up online quizzes to support learning Mathematical Analysis 1, these quizzes are not required nor count to grades (Caridade, 2012). Instituto Superior Tecnico has, at least since 2007, "Módulos de Apoio à Formação Básica em Matematica" (http://modulos.math.ist.utl.pt) to improve the freshmen performance in mathematics concepts taught at high school. In the Faculdade de Ciências, Universidade do Porto, João Nuno Tavares leads a team that produced the project "Apoio ao Aluno da FCUP: Temas de Matemática Elementar" (http://cmup.fc.up.pt/cmup/apoiomat/ index.html) which, also with quizzes, aims to improve the understanding of high school concepts.

THE STUDY

The researcher created a set of optional weekly online quizzes, on Moodle, to the Multivariable Calculus course (MC). MC is taught in theoretical-practical classes, 6 hours a week, to Electronic, Telecomunications and Computers Engineering students at the Instituto Superior de Engenharia de Lisboa from Instituto Politécnico de Lisboa. Moodle (version 1.9.8+) is the Course Management System used in the Institute. The researcher was the teacher responsible by the course during the second semester of 2013/14 and created weekly online quizzes available on Moodle with some questions on the subject explored in classes in the previous week. There was three MC classes each with a different teacher: two classes took place during the day and the other during the night. I taught one during the day.

Quizzes rules/philosophy were presented to students in classes and in the Moodle: (translated)

"Mini-tests on Moodle: Every Wednesday at 9pm will be available for 24 hours a mini-test on Moodle.

To the students who get a grade higher than 9.0 points in tests or exams will be added a grade between 0 and 2 points proportional to the average of the ten best grades on mini-tests (out of 14 mini-tests that will be made).

The philosophy behind the mini-tests is that students keep up-to-date with the subject. There will be few questions and with a similar difficulty to class and tests/exams questions. Often students think they manage the subject and only realise that it is not true when they get the first assessment... and it is a bit late. So, we want you to realize, from the first moment, the level you are reaching... we hope that: if it is a good level it motivates you to keep going... if it is a bad level it motivates you to study more.

Of course, you may copy all mini-tests answers... but... don't fool yourself! Mini-tests are useful to you... to find out the level you are reaching...if you copy all the mini-tests without understanding nothing, the most probable is that you don't reach the 9.0 points required and the mini-tests became useless! This did not mean you should not discuss with your colleagues… of course you should... discussing we learn a lot... but be aware of the level you would reach alone... of course, in the end, the natural is that all students get 100% in all mini-tests... and will get a bonus of 2 points for having done it... it's fair!"

There was no penalization for submitting more than once, this intends that students try to answer by themselves and, if it is not correct, try again and again, until they understand the subject. If there was penalization, students would try to get the correct answer from colleagues before submitting the quiz and it could lead them to try less by itself.

Students get immediate feedback about their answer. Quiz indicates if their answer is correct or wrong but do not indicates what is the correct answer. Quizzes ("mini-tests") could naturally been named "Homework" but we thought "mini-tests" would give more relevance, more importance to it.

THE QUIZZES

The quizzes were produced through the "Moodle activity": "test". It allows the introduction of images and mathematical symbols using LaTeX (see Figure 1).

Whenever it was possible, we used numeric or short answers instead of multiple-choice answers since in multiple-choice answers, with a few tries, students got the correct answer. The most used type of questions was "embedded answers" since it allows to embed more than one question and those questions may be chosen from all the different kinds of questions: numeric, short answers, multiple choice, true or false, … It allows to evaluate the student through the path and not only the final result (see Figure 2). Introducing the path can be more formative.

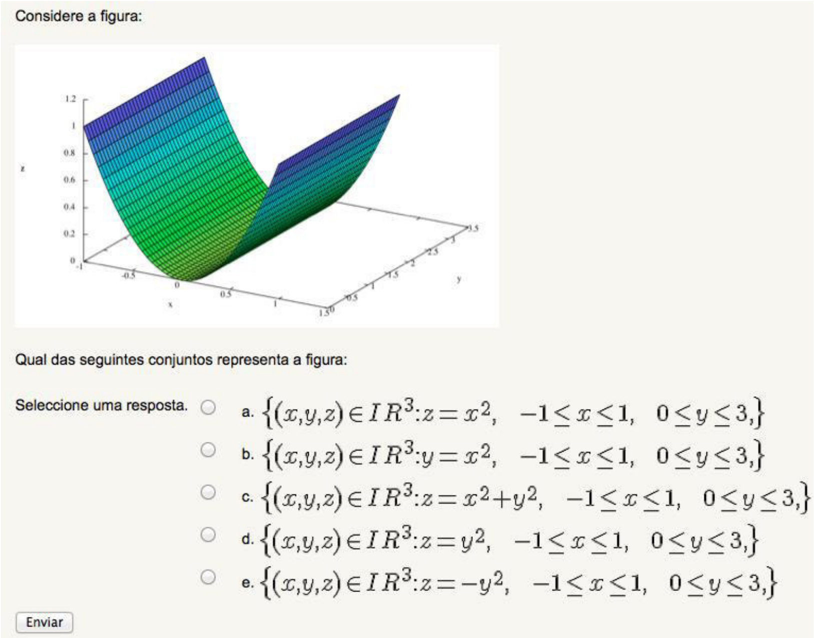

Figure 1. A multiple-choice question including a figure and mathematical text. (Translation: "Which of the sets represent the figure:"; "Select one answer."; "Send")

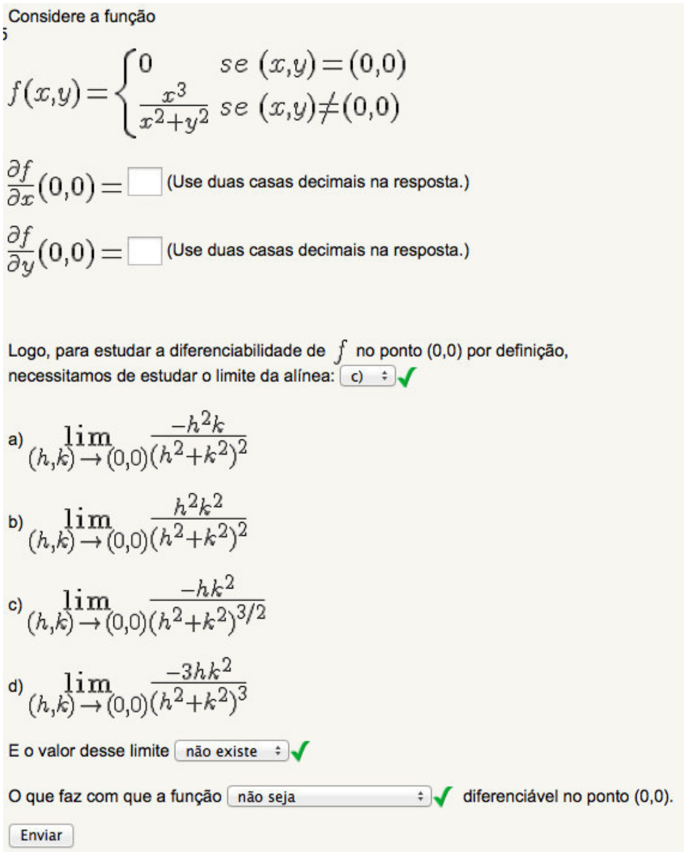

Figure 2. A question with multiple embedded questions along the path. (Translation: "Consider the function"; "Use two decimals in the answer"; "Then, to study by definition the differentiability of f on the point (0,0) we need to study the limit in subparagraph: c)"; "The value of that limit | don't exist |"; "That makes that the function |is not| differentiable on the point (0,0)")

With some creativity it was possible to evaluate all subject parts (see Figure 3).

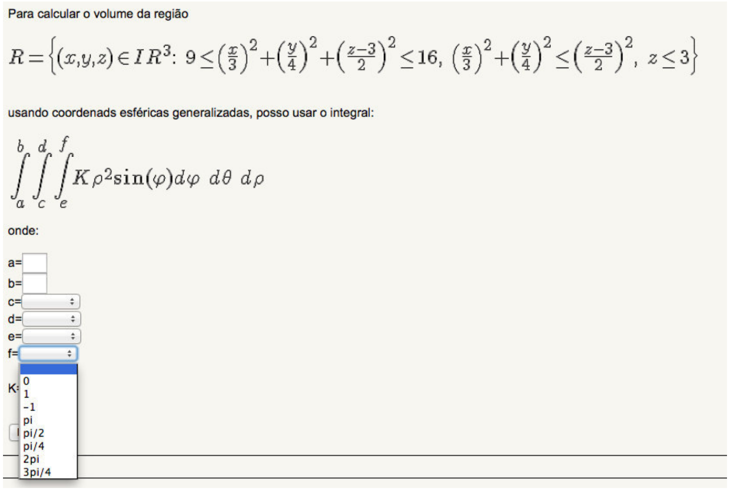

Figure 3. An inventive way to evaluate triple integrals. (Translation: "To calculate the region volume"; "using generalized spherical coordinates, I may use the integral:"; "where:")

The possibility of creating questions with different instances for each student was taken into account but it would take much more time to create questions and students also know how to solve a problem with a constant, say *k*, instead of a number, so it didn't seemed worthwhile.

METHODOLOGY AND DATA

Summarizing, the strategies chosen to apply these quizzes were: quizzes were applied online, weekly, were optional, students get extra grades for doing it, all students get the same question (not generated different ones), students may resubmit without penalty, questions usually have several sub-questions embedded.

The research question is if these quizzes (with the strategies chosen to apply them) are effective? Which will be split into some questions:

(1) Students adhered to quizzes?
(2) Quizzes made students study more?
(3) Quizzes made students get higher grades?
(4) Quizzes aware students of their real level of understanding?
(5) Students perceived quizzes as useful?
(6) Quizzes were perceived as a fair tool of assessment?

Those questions are relevant because: (1) If students did not adhere to quizzes they had been worthless. (2) When students study more they learn more (usually), they are up-to-date in the subject (it makes to be easier to them to understand the upcoming concepts), they are not letting MC to study only other courses that are requesting their attention; (3) Higher grades are usually reflect of higher understanding which is the main goal; (4) Being aware of their own level of understanding make students adequate the effort to get the desired level and usually it makes students work more. (5) Students perception of the usefulness of quizzes seems a strong indicator of its real effectiveness. (6) Since the questions are equal to all students it could generate unfairness, is important to assess if the strategies adopted had the desired effect of not producing relevant unfairness on students grades.

To address those questions were studied student's grades during nine semesters and applied an anonymous online questionnaire addressed to all students of the course, follows the results.

The pass rate (pass students over subscribed students) was much higher on that semester: S2 in 2013/14 (see Table 1). The rate of pass students over assessed students was 54/79=68%.

	2010/11		**2011/12**		**2012/13**		**2013/14**		**2114/15**
	S1	S2	S1	S2	S1	S2	S1	**S2**	S1
Subscribed students	101	200	128	153	90	123	80	**104**	56
Students who passed	27	38	31	41	20	23	12	**54**	10
Average grade of passed	11.7	11.8	12.3	11.7				**13.9**	12.4
Passed/Subscribed	27%	19%	24%	27%	22%	19%	15%	**52%**	18%
Professors	A+…	A+…	A+…	A+…	A+B	A+C	D+E	**F+G+H**	D+F

Table 1. Course data since 2010

Students' adherence to quizzes was high. Total of subscribed students were 104, one quiz got attempts from 94 students. From the 79 students that went to any "regular" assessment, 70 took at least one quiz, 44 took the maximum grade and 61 got more than 80% in quizzes grades.

The anonymous questionnaire on Moodle was addressed to all students. The sample of students who answered the questionnaire was reasonable. From the 104 students subscribed to the course, all subscribed at Moodle and 65 answered the questionnaire. Moreover, splitting the students by their grade at the first test (the questionnaire was applied before the second test) the number of students answering the questionnaire with a given grade correlates to the number of students in general who got that grade (see Table 2). Pearson correlation coefficient is = 0.6.

Grades in first test 0-10 points	**To all students**	**To the students who answered questionnaire**
More than 7.5 points	15%	32%
Between 5 and 7.4 points	43%	38%
Between 4 and 4.9 points	17%	11%
Less than 4 points	25%	18%

Table 2. Percentage of students who got some grade at the first test in the whole sample versus among the students who answered questionnaire.

Questionnaire answers of the most relevant questions are presented in Table 3.

Usually answer to quizzes?	always 38 58%	mostly 22 34%	sometimes 2 3%	rarely 3 5%	never 0 0%	N 65
How do you answer to quizzes? [1]	alone 12 18%	alone, then colleagues 33 51%	together 19 29%	copy 0 0%	don't answer 1 2%	N 65

1 Recall this questionnaire is automatically anonymous, be honest!

Without quizzes, I've studied…	less 44 69%	the same 14 22%	more 6 9%			N 64
The quizzes were…	useful 60 92%	indifferent 4 6%	useless 1 2%			N 65
Without quizzes, I've scored…	higher 0 0%	the same 14 23%	lower 48 77%			N 62
Quizzes generate unfairness?	no 63 98%	yes 1 2%				N 64

Table 3. Answers to questionnaire most relevant questions.

The questionnaire answers to "Select ALL the statements that you agree with:" were: (N=65)

(a) Quizzes remind me to study the subject every week.	55	85%
(b) Quizzes show me there are things I thought I knew but I didn't.	48	74%
(c) Quizzes help me to have a better perception of the level I'm reaching.	47	72%
(d) I learn new things answering to quizzes.	33	51%
(e) Quizzes have no interest.	0	0%
(f) I do not care for quizzes, I just copy the results.	0	0%
(g) I do not care for quizzes; I not even copy the results.	1	2%
(h) Quizzes stress me too much.	3	5%

There were some open questions but its answers did not get nothing new, only reinforce the topics addressed before.

REFLECTIONS

The much higher success rate cannot be assigned to the quizzes since it was not the only different variable in the semester. (1) It changed the responsible teacher: in the data available, there was three different responsible teachers and the results of this semester were much higher than the results with the two others responsible teachers. (2) Contents were the same in all semesters but the responsible teacher in that semester gave more emphasis to concepts applications and it could have increased student's motivation. (3) Exam exercises difficulty was not smaller, on the contrary (exercises like in Figure 3 would be considered as too difficult for the others responsible teachers and never would take place in an assessment, as often happened in that semester). (4) There were three teachers in that semester and in the others there were two, but usually one of those two got two classes what makes the same number of students by class.

Since the quizzes were online answered and so students may copy from each other, there was a concern if students feel that quizzes allow unfairness. In questionnaire only one student posed the possibility of quizzes being unfair, and even that student do not stated that he felt it as unfair but that maybe some student could feel it as unfair because the student may not have a group of colleagues with whom discuss the answers (the researcher reflection is if having a group of colleagues it not a social competency that worthwhile to measure in students).

The feedback informally given by students to the teachers was that quizzes had a relevant role making students do not neglecting MC over other courses and be aware of their performance. Teacher's feeling was that quizzes were, in fact, strongly important.

CONCLUSIONS

In this study a set of quizzes was applied to a course of Multivariable Calculus with 104 engineering students. They were applied online, weekly, were optional, students got extra grades for doing it, all students got the same question (not generated different ones), students could resubmit without penalty, each question usually had several sub-questions embedded. The goal was to fight the low pass rate by making students: study more, be always up-to-date, feel requested also by this course (not only by the others), be engaged in a working environment, and be aware of their real level of understanding since the beginning.

This paper evaluates these quizzes (with the strategies chosen to apply them) effectiveness. Grades of students during 9 semesters and the answers to a questionnaire allowed addressing the effectiveness split into the six questions of the following paragraphs.

Students adhered to quizzes? – although the quizzes were optional, its adherence was really high. One of the quizzes was answered by 94 of the 104 of subscribed students. All the quizzes had high rate of attendance. Among the students that went to any "regular" assessment, almost all took a quiz and a large percentage got high average grades on the quizzes.

Quizzes make students study more? – More than two thirds of students stated in questionnaire that without quizzes they had studied less and that quizzes remind them to study every week. Moreover half of respondents state that learn new things answering to the quizzes. It also had the effect of making pressure on students to do not study only to the other courses (because students are often requested by other courses to make works, and if they are not also regularly requested by this course they leave it to study later... and often became lost in the subject).

Quizzes make students get higher grades? – the pass rate in that semester increased highly, it was roughly the double of the remaining eight semesters studied. Average grade of passed students was also higher. This success cannot be directly attributed to quizzes, there were other changing variables in that semester, and however, this is a positive indicator.

Quizzes contribute to students awareness of their level of understanding? – On questionnaire, 75% of students agree that quizzes help them to have a better perception of their level and that, facing quizzes, they realize that they did not knew some parts of the subjects that they thought they knew.

Students perceive quizzes as useful? – According to questionnaire, 92% of students classified the quizzes as useful and no student stated that they have no interest.

Quizzes are perceived as a fair tool of assessment? – No student stated that had copied the answers to quizzes, besides of being remembered that the questionnaire was automatically anonymous. Only one student questioned the possibility of quizzes being unfair so we may conclude that, in general, students perceive them as fair.

Summarizing, there are good indicators that quizzes applied with this set of strategies were effective: had strong adherence, make students study more, increase students grades, contribute to students awareness of their level of understanding and were considered as fair and useful.

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RIEMANN INTEGRAL: DIDACTICAL MEDIATION WITH GEOGEBRA SOFTWARE ARTICULATED WITH USUAL PRACTICES WITH 1ST YEAR GRADUATE STUDENTS IN MATHEMATICS TEACHING

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In the article we present the results of an experimental teaching aiming to answer the question: how effective is a didactic mediation of concepts on Riemann integral, using Geogebra software, articulated with usual practices, by students, based on their mobilized and available retrospective knowledge? The goal was to experiment a teaching and learning modes of the Riemann integral of real functions of a real variable, using Geogebra software as instrument, articulated with usual practices. The study was based in Anthropological Theory of the Didactic - TAD by Chevallard and theory of instrumentation by Rabardel. It was a qualitative study in the form of case study, having appealed to some aspects of didactic engineering: design and a priori analysis of tasks; a posterior analysis and internal validation. The experiment showed that the blended computer and usual practices processes promote construction of knowledge by students to the Riemann integral.

Keywords: Didactical mediation. Riemann integral. Anthropological Theory of the Didactic. Theory of Instrumentation/instrumentalization.

INTRODUCTION

The Riemann integral is one of the fundamental concepts of Calculus and Mathematical Analysis, with wide application in important sectors of social activity, industrial and economic as well as in many sciences and mathematics itself. Furthermore, this concept causes some problems in the teaching and learning in high school and / or college.

The studies by Araujo (2002), Viana (1998), Leng (2011), just to mention a small part, show a diversity of problems on teaching and learning Calculus. Reported problems highlight the lack of understanding of the basic concepts on the subject, often causing failures of students.

We conducted this study to answer the following question: how effective is a didactical mediation for building and learning the concept of Riemann integral using the Geogebra software articulated with the usual practices? And we aim to experiment a modality of teaching and learning of the Riemann integral concept of real functions of a real variable, incorporating in the process Geogebra software articulated with usual media and practices.

From this objective, the theoretical framework of the research, which helped us in formulation and delimitation of the problematic is presented in the two following sections.

ANTHROPOLOGICAL THEORY OF THE DIDACTIC

The Anthropological Theory of the Didactic - TAD - describes the mathematical activity in the set of human activities regularly developed, describing the mathematical knowledge in terms of praxeological organizations or praxeologies $\wp$ whose basic notions are the notions of types of tasks T, techniques τ (ways of solving the tasks), technology θ (a rational discourse aimed to *justify*, *explain* and to *produce* techniques) and theories Θ (aimed to *justify*, *explain* and to *produce* technologies) that allow to model the social practices in general and the mathematics activity, in particular (Chevallard, 1999, 2014). From this point of view, according to the author, the praxeology $\wp$ consists of a practical-technical block $\prod$ (praxis) [type of task/technique], which corresponds to a know-how, and a technological-theoretical block (*logos*) [technology/theory] which corresponds to a knowing. The notion of task presupposes a relatively precise object, for which there is some available technique with a technological-theoretical surrounding more or less explicit. In most cases, a task (and the type of associated tasks) is expressed by a verb evoking an action, what exists to do, for example, integrating the function $f(x) = lnx$ between x = 1 and x = 2 is a task that can be justified with technology of integration by parts and as theory, the Fundamental Theorem of Calculus.

The following definition is the most important result on Riemann integral:

> Let f be a real function of a real variable in interval I = [a, b]. We say f is *integrable in I* (in Riemann sense) if and only if there is a number, denoted by $\int_a^b f(x)dx$ and called *integral* of f over I, such that, for any $\varepsilon > 0$, there is a partition P_ε (P_ε generally depends on ε) so that, $P \supset P_\varepsilon \Rightarrow \left|\sum(P,f) - \int_a^b f(x)dx\right| < \varepsilon$, understood that the inequality always holds if any arbitrary number from the set of numbers $\{\sum(P,f)\}$ is replaced by number $\sum(P,f)$. (Labarre, 2008, p. 147).

We recall that the partition P of real interval I = [a, b], with more than one point, is a finite subset of points $P = \{t_0, t_1, t_2, \ldots, t_n\}$, such that $a \in P$ and $b \in P$, and $a = t_0 < t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n = b$. The relation $P \supset P_\varepsilon$ implies that the partition P refine the partition P_ε. In Labarre's (2008) view, the refinement process of I is the key idea on Riemann integral. We completely agree with this view and we believe that the Geogebra software is particularly good to perform a such process. We also recall that the definition above does not distinguish the type of partition P, whether regular or irregular. And, because of our limited capacity on computer programming, we used regular partition, and we think that there was no loss of generality.

According to Chevallard (2002), the set of conditions and restrictions that allow the mathematical development (ecology of a mathematical praxeology), ie, conditions and restrictions that allow the production and use of tasks in institutions, depends on the ostensible objects (perceptible to the human senses and capable to be handled, such as sounds, graphics and gestures) and non-ostensive objects. We assume the Geogebra software as an ostensible object that allows materializes and handle a mathematical knowledge.

The non ostensive objects are those, such as ideas, intuitions or concepts institutionally existing, but cannot be seen, perceived or shown by themselves. The non ostensive objects can only be evoked by an appropriate manipulation of certain associated ostensive objects. For example, to find the integral $\int_0^1 x^2dx$, we have to evoke some ideas, principles and laws of integration that can't be seen, but they drive the solver's actions to get the result. In the next section we present the theory of instrumentation.

INSTRUMENTATION THEORY OF RABARDEL

We chose to study some aspects of Instrumentation/Instrumentation Theory from the perspective of Rabardel (1995), considering it important for the analysis of the relationship between researcher and students in their effort to interact with the tasks and with computer during the discussions in experimental sessions.

Rabardel (1995) states that instrument replace some intellectual functions from others, rebuilds and reconstructs the whole structure of behaviour. We feel that using Geogebra software, changes the practices of teaching and learning Riemann integral, and the possibilities of abstraction and generalization of this concept.

The intermediary position of the instrument makes it as a mediator of the relationship between the subject and the object. It is an intermediary world, whose main characteristic is to be adapted both to the subject, as to the object. This adaptation occurs in material terms and in terms of cognitive and semiotic properties according to the type of activity in which the instrument is inserted or is intended to be inserted. Thus, two types of mediation are identified:

- a mediation from object to the subject, described as an epistemic mediation, in which the instrument is a means enabling the user to know the intended object;

- a pragmatic mediation, from subject to the object, in which the instrument is a means for transforming action (in a broader sense, including control and regulation) directed to the object.

The instrumental elaboration by the user is thus, addressed, both for himself (this is the dimension of instrumental genesis called instrumentation), and for the artefact (the instrumentalization dimension).

Instrumentation processes are related to the emergence and evolution of the use of schemes and action mediated by the instrument: its constitution, its functioning, its evolution by accommodation, combination, coordination, inclusion and mutual assimilation, the assimilation of new artefacts to the set of the schemes already existing.

The *instrumentalization processes* are related to the emergence and evolution of artifactual components of the instrument: selection, consolidation, production and institution of functions, deviations and catacreses, assignment of property, artifactual transformation (structure, function, etc.) that prolong the creations and achievements of the artefacts whose limits are difficult to determine. In the sequence, we highlight the research method.

THE RESEARCH METHODS

Based on the established problematic and built theoretical framework, we developed the methodological framework precisely based on the two previous theoretical assumptions and limited to some elements of didactic engineering (Artigue, 2010), with regard to the design and *a priori* analysis of tasks for experimentation; the experimentation; the *a posteriori* analysis and internal validation.

The experimental teaching was carried out in two sessions. The 1st session took place on 15th June 2013, and the second one, on 19th June 2013. Because of space available for this article, we present only some parts of experimental discussion of the session 2, and some discriptive general indications on what did happen in the session 1. In this respect, in the first session we discussed, using the function f: I = [-2, 2] $\rightarrow$ IR, with the following formation rule: $f(x) = 2\sin(2x) + 3$, several basic ideas conducting to the integrability of a function f in the Riemann sense: partition P of I in n subintervals; the refinement of P, continuity of f in I and in subinterval $I_i = [t_{i-1}, t_i]$ of I, with i = 1, 2, …, n; upper and lower bounds, the existence of supremum (maximum) and the infimum (minimum) of f in I and in each subinterval $I_i = [t_{i-1}, t_i]$, and monotonicity of f in I and in each subinterval I_i.

Eight first year graduate prospect teachers attended the both sessions. The teaching experiment took place before the students formally studied the topic. Note that we assume the Geogebra software as an ostensive object which materialize the mathematical ideas in classroom (Chavallard, 2002), allowing the process of constructing of the techniques and technologies (properties and theorems) of target notion, and as an instrument which mediates actions and structures the behaviour of teacher and learner (Rabardel, 1995, 2005). Also, as we stressed before, the Geogebra software is integrated into the usual practices.

The second session was aimed to build the notions of Riemann lower and upper sums, the notion of oscillation of *f* in I and the corresponding Riemann integral. Bellow we present some extracts of this session.

At beginning, the researcher writes on the whiteboard the programming steps of the task in Geogebra the students could follow:

Function: *f(x) = if [0 <= x <= 1, x ^ 2];*

Ends of the range: *a = 0, b = 1*;

Slider: *k = [0, 32], step 1;*

Number of rectangles: *n = 2 ^ k*;

Base of rectangle = *(b - a) / n*;

Lower sum *(Si) = lowersum (f, a, b, n)*;

Uppersum *(Ss) = uppersum (f, a, b, n)*.

After this script, the researcher distributes the form with the tasks for students to work on in pairs:

Group: **19/06/2013**

Task 1: Let us consider the function f: IR → IR, defined by $f(x) = x^2$. Consider the interval I = [0, 1] of the domain of *f*. Using Geogebra, let's approximate the area under the graph of *f* in the interval I, for underestimating and overestimating. As we obtain these areas we fill the table bellow.

Table caption:

n – number of subdivisions (subintervals);

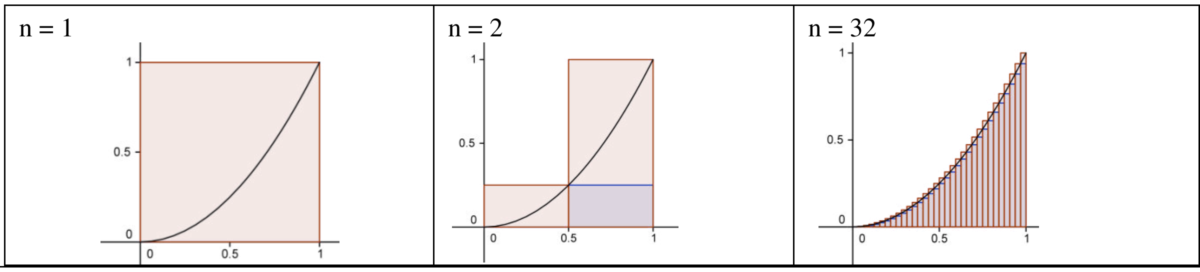
$\Delta x = \frac{b-a}{n} = \frac{1}{n}$ - length of each subinterval $[x_{i-1}, x_i]$ in *I*

$S_i = \sum_{i=1}^{n} \Delta x f(x_{i-1})$ - lower sum (sum of "inscribed" rectangles to the graph of *f* in I).

$S_s = \sum_{i=1}^{n} \Delta x f(x_i)$ - upper sum (sum of "circumscribed" rectangles to the graph of *f* in I).

$\omega_i = S_s - S_i$ – oscillation of the function *f* on interval *I*.

As an example, we fill for n = 1, n = 2, and n = 32. Continue filling the table for other values of n.



n	1	2				32					
$\Delta x = \frac{1}{n}$	1	0,5				0,0313					
$S_i = \sum_{i=1}^{n} \Delta x f(x_{i-1})$	0	0,125				0,3179					
$S_s = \sum_{i=1}^{n} \Delta x f(x_i)$	1	0,625				0,3491					
$\omega_i = S_s - S_i$	1	0,5				0,0312					

Reflections

a) What happens to the monotony of the following elements?

n ____

Δx ____

S_i ____

S_s____

b) The answers to the question a) are related. Try to find the meaning of these responses when compared to the number and sizes of rectangles involved in different interactions.

c) What must be the exact area of region under analysis? Why?

d) What must mean this exact area for this situation?

e) Assuming that we do not have Geogebra, develop an algebraic way that allows us to determine the same result.

Figure 1: form of tasks for students

After receiving this form, the students worked on it in pairs, with help of Geogebra software.

RESULTS

Next, we include some passages of the discussion, highlighting some sequences of occurred dialogues.

Tutor: Now is to fill that. Then we draw conclusions. For example, is just to see all the elements which are there. We have lower sum, upper sum. We have delta, we have n. We have oscillation. Well, I'll leave this, at this side here (the researcher refers to the script that he leaves on whiteboard corner) to be all clear. We have here (refer to the Figure1, for n = 1) means that the area (in the chart) is divided into one part only. As we were saying, to determine the upper sum, as our partition is on this interval ([0, 1]), it means that, if I want the lower sum, I take the lower end.

Students: We have one (width of rectangle) times f of zero. Then it will give this area, this line here (one student points the line [0, 1]). One student performs: $s(f, P_1) = 1 \cdot f(0) = 1 \cdot 0 = 0$.

Tutor: In fact, it is that lower sum that we have over there. And the upper sum?

Students: It will be one times f of one. (They write $S_1(f, P_1) = 1 \cdot f(1) = 1 \cdot 1 = 1$).

Tutor: Then it will be that area (writes down as the students did: $S_1(f, P1) = 1 \cdot f(1) = 1 \cdot 1 = 1$). So this is how you will fill this form here. The first part is already fulfilled, for example. For n = 1, the delta, which is it?

Students: One.

Tutor: The delta is the length of this division ($\Delta x = \frac{b-a}{n}$). So we have S_1, on the first column, already filled on. You will continue, for n equals to 2, which delta, which S_i, lower sum, which is the upper sum, which is the oscillation.

Student: The oscillation!

Tutor: Yes. The oscillation is the difference between S_s and S_i, […]

The students work in computers, in their respective groups

Group 2: Delta becomes constant. ... Only n changes.

At this stage the students have the numerical results and corresponding graph, as shown at the Figure 2.

Tutor: […] The size of rectangles, what happens? And, as consequence, how is it S_i and S_s? How is it about lower sum? Upper sum, in those conditions? There are consequences. There are some consequences. Then which are the consequences? You have to talk about n, about delta, talk about lower sum, about upper sum […].

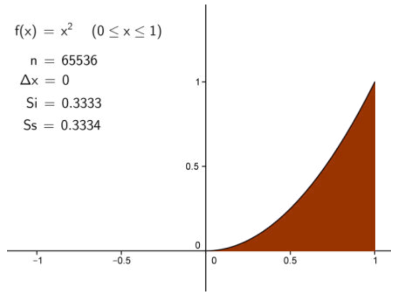


Figure 2: upper and lower approximating areas under graph of function *f*: [0, 1] → IR, *f*(x) = x²

Tutor: What will be the area of this part here, under the graph (of $f(x) = x^2$), on interval I = [0, 1]? You are already seeing, is not it? Each one must try to show what must be the exact area.

Up to this moment the students have the tables filled, as it is shown at a small part at Figure 3, accompanied with the corresponding graphic image, at each interaction, as in Figure 2:

n	4096	8192	16384	32762 (~~262144~~)	65536	131072
$\Delta x = \frac{1}{n}$	.0,0002	0,0001	0,0001	.0	0	0
$S_i = \sum_{i=1}^{n} \Delta x f(x_{i-1})$	0,3332	0,3333	0,3333	0,3333	0,3333	0,3333
$S_s = \sum_{i=1}^{n} \Delta x f(x_i)$	0,3335	0,3334	0,3334	0,3334	0,3334	0,3334
$W_i = S_s - S_i$	0,0002	0,0001	0,0001	0,0001	0,0001	0,0001

Figure 3: Leading processes to the Riemann sums and integral, using Geogebra software.

Tutor: Yes. What is the exact area; it is what I'm asking for, what you are seeing from there (in Figure 3). [...] How is it Si, and Ss in this case? Don't you have any idea what should be the exact area … or is there a group that has an idea about what should be the exact area?

Student (Micas): Yes. I have some ideas: (the students respond to the questions a), b) and c) on reflections section of the form):

a) What happens to the monotony for each of the following elements: n, Δx, S_i, S_s?

Student: The values of n are growing up, increasing; the values of Δx are diminishing, decreasing; the values of Si are growing up, increasing; the values of S_s are diminishing, decreasing

b) The answers to the question a) are related. Try to find the meaning of these responses when compared to the number and sizes of rectangles involved in different iterations.

Student: In regard to the size of rectangles involved in different iterations, in case of number 1 we have big rectangle, when value of n increases, the rectangles diminish their size, and the values of upper sum diminish as well, while the values of lower sum increase.

c) What must be the exact area of region under analysis? Why?

Student: With the increase of n, the exact area of the region in analysis will be the lower sum (0,3333). Because each time we increase the value of n, the part of upper sum tends to vanish.

Tutor: Well, this is the Mica's response, I don't know what others groups say?

The extract above gives some clues about how the discussion was developed and how the key ideas of convergence emerged in the processes of instrumentation (constructing of schemes of knowing) and instrumentalization (institution of function to the artifact), in the use of a computer as a instrument. We leave bellow some comments and conclusions about this teaching experiment.

COMMENTS

We believe that the strategy used to study the construction procedure of upper sum and lower sum approximations using computer resource has been effective, it is achieved very good approximate results for the exact area. At this level the software was used as an ostensible object (Chevallard, 2002), the "materializer" of the mathematical processes to build the mathematical techniques and technologies (ways of solving mathematical questions and their explanations), according to Chevallard (1999, 2014) and as an instrument, mediator of actions to the target mathematical object (Rabardel, 1995, 2002). The graphical, numerical and algebraic visual images actually show what is wanted to be constructed: the ideas of approximation (the refinement of the partition) and its implications to the quality of approximation, inducing to the inferential reasoning for limit processes. We have the perception that the computer aided enough to characterize the target mathematical object by providing graphical visual information and numerical approximations.

We can formulate our perception from the Rabardel (1995) perspective noting that the computational resource has expanded the possibilities of representation of the actions of students and the object of activity. To better understand this statement, just imagining how it would be possible, in tradicional modes to obtain 32,768 rectangles in a range from 0 to 1, with about 0.00003 cm wide? The graphical result is effectively suggesting that the area under the graph of the function must be 0.3333. So, for us, this approach is dramatically different from traditional handlings on Riemann integral. The limit process arises naturally and meaningfully. Therefore, according to Rabardel (1995), the computational tool played, in this phase of activity, a real role of cognitive tool. We consider as cognitive resource at the disposal to the students, the feedback the machine gives as the student works on her. So, we stress that the Geogebra software structured the behavior of students and the environment of learning: the *milieu* of learning.

On the other hand, we noticed some disturbing influences to the correct learning process, arising from information provided by the computer in relation to its limited capability to present the correct decimal places of the result and the overlap of the upper and lower sums that led the student Mica, as we see in his speech, in which he concluded that the upper decreasing area was vanishing. The student seems to have confused the upper area with the oscillation, as this is what really disappeared in the approximation process of the upper to the lower sum.

Overall, we believe that the strategy was effective in the way that it provided a construction of some elements of meaning (approximate sums) leading to the formal definition of the definite integral as generalization of process that was graphically well produced.

We still believe the articulation between the results (tables and figures) obtained through the computer and the theoretical analysis was consistent, because according to Rabardel (1995), the students show they have adapted themselves to the epistemic, pragmatic and heuristic functions of the instrument in the sense that it has provided an understanding of the task, its transformation to obtain the result, guidance and control of their actions. We interpret these processes as meaning learning, ie, the construction of knowledge by students when it is considered in the perspective of Chevallard (1999), as a work with a particular question to produce a satisfactory answer. So the learning corresponds to put in place the target notions in the exercises or in the problem solving processes. We base our claims on the students' behavior, both in computer use, in articulation between the result obtained by computer and on filling the table and in their interventions on the results obtained. These actions match with our expectances presented at *a priori* analysis (although they are not presented in this article).

It is our feeling that the strategy for the experiment was fully interpreted: articulate the computational resource with the usual practices in introduction to the Riemann integral, although we didn't use the idea of the norm of a partition, but it helped us to produce the generalization.

CONCLUSIONS

The notion of Riemann integral was effectively built with the didactical mediation with computational resources. The intervention stressed the central concepts of the definite integral, which are the concepts of partition refinement and numerical approximation (Labarre, 2008). We emphasize that for the students who participated to the study, it was the first moment for them to attend on the subject, as this content is not studied in high school. However, their performance was good, at least in relation to the type of interventions and arguments used to justify their ideas, as we have seen above.

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SAMING, REIFICATION, AND ENCAPSULATION IN DYNAMIC CALCULUS ENVIRONMENT

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This paper discusses the use of dynamic geometry environment (DGE) for facilitating three Sfardian processes for creating new discursive objects: saming, reification, and encapsulation. In particular, I provide an analysis of communication about a DGE sketch involving three high school students who had been enrolled in a calculus course where DGEs were consistently incorporated into the lessons. Findings suggest that the students used a combination of saming, encapsulation, and reification for exploring calculus ideas. Moreover, encapsulation was needed in order to develop an object-level discourse around the area-accumulating function. This paper raises implications about teaching mathematics with technology, particularly by attending to communications and changes in communications about functions as an ordered pair and a graph.

Keywords: Dynamic Geometry Environment, Calculus, Classroom Discourse, Algebra and Algebraic Thinking, Geometrical and Spatial Thinking

INTRODUCTION

The notion of encapsulation as a process of abstraction has been widely undertaken in mathematics education (see for example, Dubinsky, 1991 and Gray & Tall, 1991). This line of work takes on a dualistic view of learning calculus—that is, thinking is conceptualised as mental processes occurring in an individual's head while the body does what the mind says. Recent advances on the embodiment of mathematics have also contributed to theorizing the learning of calculus. In their seminal work, Lakoff and Núñez (2000) investigate cognition as a physically-embodied phenomenon. They took up the idea of abstraction very differently, as they argued that the foundations of mathematics rests on a sizable collection of crucial conceptual metaphors.

The invention of graphing and dynamic geometry technology has opened new possibilities for learning calculus in dynamic, embodied ways. It allows the user to interact in a physical way by pointing, selecting and dragging objects onscreen to extend the embodied context of calculus. Although the theory of Embodied Cognition considers how calculus may be understood cognitively through the individuals' bodily-based metaphors, it pays little attention to learning as arising through social interaction in communicative settings—a non-dualistic and participationist view of learning. A dualistic view of thinking suggests that signs like language and gestures are products of mental action. In a non-dualistic view, however, signs are *not* indicators of mental actions; they are considered as essential component of the thinking process (Sfard, 2008). Signs may be considered in a broader sense that includes words, gestures and actions with artifacts. Moreover, the participationist lens is especially relevant in the context of this paper which takes place in pair-work activities in simulated classroom settings.

The current study examines students' discourse in the learning of calculus in technological enhanced environment in non-dualistic terms. In particular, I investigate students' communication while they explored a chosen calculus concept in a dynamic geometry environment. The purpose of this paper is to provide insights on the processes of saming, reification, and encapsulation in mathematical thinking in a dynamic calculus environment: How are saming, reification, and encapsulation evidenced in a dynamic calculus environment, and what are the roles of these processes in the learning of calculus?

THEORETICAL FRAMEWORK

Sfard's communicational framework (2008) is based upon the social dimensions of learning and highlights the communicative aspects of thinking and learning. For Sfard, *thinking* is part and parcel of the process of *communicating*; her approach highlights the way in which thinking and communicating stop being "expressions" of thinking and become the process of thinking in itself. Sfard (2008) proposes four features of the mathematical discourse, *word use, visual mediator, routines, and narratives*, which could be used to analyse mathematical thinking and changes in thinking. These features will be used for analysing the use of language, gestures, and dragging in one's the mathematical discourse in the present study. For example, as a student engages in a mathematical problem, her mathematical discourse is not limited to the vocabulary she uses. Her hand-drawn diagram and gestures can be taken as a form of *visual mediator* to complement word use. According to Sfard (2009), gestural communication ensures all interlocutors "speak about the same mathematical object" (p.197). Gestures are essential for effective mathematical communication: "Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p.198). Routines are meta-rules defining a discursive pattern that repeats itself in certain types of situations. For example, learners may use certain words, gestures, or dragging actions repeatedly model a discursive pattern, such as looking for patterns and what it means to be "the same".

Sfard (2008) conceptualises learning mathematics as a change in one's mathematical discourse. Differences in the realization of a signifier (like "quadratic function") can be observed through one's word use, visual mediator, routines, and narratives. There are three mechanisms for the production of *compound discursive objects*, which results in greater range and depth of the realization: *saming, encapsulation, and reification*. The process of *saming* can be seen as the act of calling different things the same name. *Encapsulation* is the act of assigning a noun or pronoun (signifier) to *a specific set* of discursive objects, so that some of the stories about the members of this set that have, so far, been told in plural may now be told in singular. Finally, *reification* involves replacement of talk about processes with talk about objects.

PARTICIPANTS, DATA COLLECTION AND TASK

The participants of the study were two pairs of 12th grade students (aged 17 to 18) enrolled in a calculus class in a culturally diverse high school in Western Canada. The participants were self-selected in a class of 25. All of them were regular partners during assigned pair-work activities and were described by their teacher-researcher (also the author) as motivated and comfortable working with each other.

The study took place at the end of the first trimester of the school year in the participants' regular calculus classroom, outside of school hours. At the time, the participants have just finished learning key concepts in differential calculus where the iPad-based DGE, *Sketchpad Explorer* (Jackiw, 2011), was consistently incorporated into the lessons. Therefore, the participants were experienced with exploring and discussing, in pairs, calculus concepts such as derivative, derivative functions and related rates through dynamic sketches at the time of study.

The task used in this study invited the students to discuss a sketch presented in *Sketchpad Explorer* that they had not previously seen. For the purpose of comparing students' routines during the exploratory activity, they were given a sketch containing five pages all related to the concept of area-accumulating functions. This concept was new to the students at the time of study, as they had just completed the differential calculus component of the course and one lesson on indefinite integrals. The participants were asked to "explore the pages, talk about what you see, what concepts may be involved" in each page of the sketch and then to move onto the "Try" page of the sketch where a problem was posed. They were asked to solve it on a dry-erase whiteboard and were told that their teacher would check in with them from time to time to make sure that they understood the procedure and discuss their progress. Each student-pair took around 30 to 45 minutes to complete the task. In total, 115 minutes of video data were collected in the study.

Design of sketch

The design of the sketch mainly features three functionalities offered by *The Geometer's Sketchpad*: the *Hide/Show* button, the *Dragging* tool, and the *Trace* tool. These functionalities have the potential to evoke mathematical relationships that would have been difficult to capture in static diagrams. The first four pages of the sketch contain the same *Hide/Show* buttons to allow the "Function f", "Bounds", "Area under f", and "Trace of A" to be shown or hidden conveniently. Each page displays a different function when the "Show function f" button is activated: a constant function on Page 1 (Figure 1a), a linear function of degree-1 on Page 2, a quadratic function on Page 3, and the sine function on the Page 4 (Figure 1b). The student-pairs may explore the area under these functions ('*A*') both numerically and geometrically by dragging the points '*a*' or '*x*'. For example, Figure 1a shows that the area under the function *"f(x)=1"* is *"A=4.70"* when the bounds are set to *"a=0"* and *"x=4.70"*.

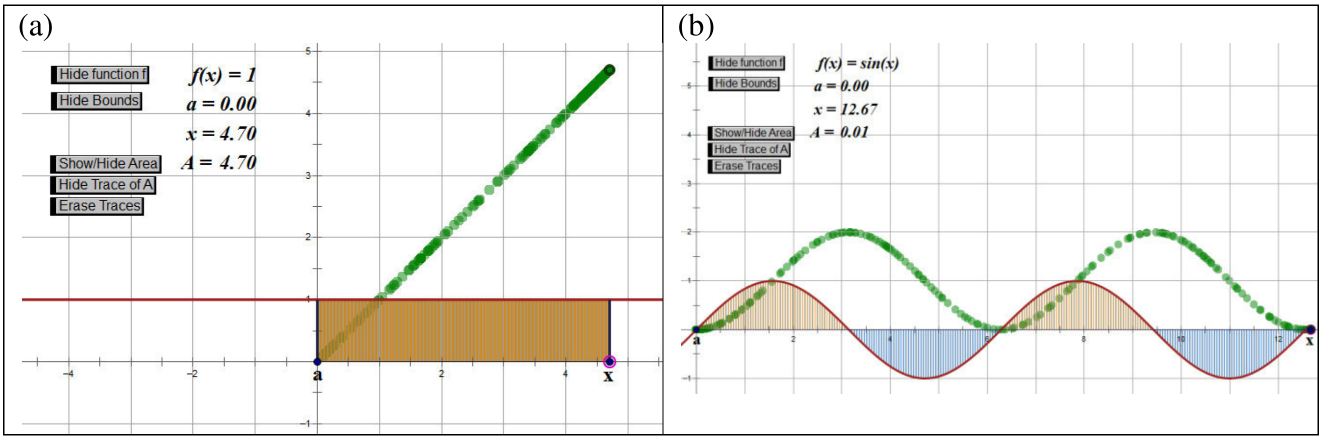

Figure 1(a): A sketch conveying the area under a constant function with all buttons activated and the '*x*' dragged from zero to its current positions. (b): The area under a sine function is colour-coded with orange (for positive area) and blue (for negative area).

As the user drags the points '*a*' or '*x*' along the *x*-axis continuously, the value of '*A*' and the shaded area change correspondingly. Performing this kind of dragging actions mediates a functional dependency between variables by enabling the user to contrast between what is independent ('*x*') and dependent ('*A*') (see Falcade, Laborde, & Mariotti, 2007). Furthermore, the present sketch illustrates how the dynamicity of dragging may connect numerical with geometrical representations of calculus concepts, since dragging '*a*' or '*x*' simultaneously changes the *numerical* value as well as the *geometrical* representation of '*A*'.

The *Trace* tool can be used to generate a set of "green traces", which represent the graph of the area-accumulating function *A(x)* for the chosen '*a*' and '*f(x)*'. It is used to mediate the relationships between '*x*' and '*A*' dynamically. When the button "Show Trace of A" is pressed, a green point appears on the page at (*x*, *A*). This point is not draggable which implies that it is not an independent object. More importantly, the green point leaves behind traces of its previous positions as one drags '*x*' along the *x*-axis, creating green traces in the shape of the corresponding area-accumulating function. Since the only way to vary the green point is to drag '*x*' (or '*a*' which would result in a vertical translation the green point), this conveys the idea that the graph of the area under '*f*' is dependent on '*x*' and hence *area is a function of 'x'*.

As the students had not yet been exposed to the definite integral or the function $A(x) = \int_a^x f(t)dt$ in their regular classroom, the goal of this sketch was to introduce the idea of "area as a function", and this can be achieved when the students are able to relate the "green traces" as the graph of area under '*f*' from '*a*' to '*x*'. It is anticipated that the notion of "negative area" may pose a problem for students because their understanding of "area" may be limited to a geometrical one that is non-negative. To support students' visualisation, positive and negative areas were coloured-coded differently with the area above the *x*-axis coloured orange and the area below the *x*-axis coloured blue (Figure 1b).

DATA ANALYSIS

The researcher had initially arranged J and K, and R and L to work in pairs for the task. After about 30 minutes of working with the task, the researcher asked the two pairs to switch partners so that they could "chat about" what they explored with their new partner. The following is a 95-second episode taken from K and R's discussion over what they had "found" during their explorations with their respective partners.

(Underlined Transcript = utterance spoken simultaneously with gestures; Double-underlined Transcript = utterance spoken simultaneously with dragging; (?) = rising intonation)

K: Did you find it?

R: Yeah.

R: What did you find?

K: This graph is the derivative of this graph (see Figure 2a, b). <K taps "Page 2"> This graph is the derivative of this graph.

R: Oh.

K: So, the sine graph is the derivative of negative cosine 'x' right? <K drags 'x'>

R: Oh. Ok. That explains it.

K: <K taps the "Try" page> And the… cosine graph… is the derivative of sine graph right? So… it's like this. The sine graph.

R: Ya. Ok. <K chuckles>

R: Well, we were, we were actually looking at area(?) So ‘x’ and ‘y’. And then <R tabs “Page 1”> So here, both ‘x’ and ‘y’ are positive, the area is… <K drags ‘x’> going up. And then, ‘x’ is negative <R drags ‘x’>, but ‘y’ is positive, so the area is negative. <R taps “Page 2”> It works the same for all of these. That’s how we graphed ours. <R drags ‘x’> If they are not… like the same sign, then it’s going down (see Figure 2c). <K nods>

Figure 2: K’s gestures as she uttered, “This graph (Figure 2a) is the derivative of this graph” (Figure 2b). R’s gestures as she uttered, “then it’s going down” (Figure 2c)

At the start of the transcript, R and K took turns to discuss what they found in their previous exploration. Although both students said that they have “found” something, their discourse indicated two very different realizations of area-accumulating functions. K began by providing two examples; she used “Page 1” and “Page 2” of the sketch to state the relationship between f(x) and the green traces, that the former is the derivative of the latter. She also used gestures to realize both functions geometrically, while she uttered, “*this* graph is the derivative of *this* graph” on both pages. In Sfard’s terms, her gestures were taken as *visual mediators* and instances of *actual realization*, as they were performed in the presence of the signifiers. In her third example, she turned to the “Page 4” of the sketch which showed the sine function. Unlike her previous communication incorporating gestures, here she used dragging to show the movement of the green traces. Since K and her partner had previously set ‘$a=\pi/2$’, the green traces were in the shape of: $A(x) = \int_{\pi/2}^{x} sin(t)\, dt = -cos(x)$. Hence, K asks R rhetorically, “So, the sine graph is the derivative of negative cosine ‘x’ right?”

It can be observed that R did not see what K saw in the sketch. What she had “found” seemed to be something different, evident in her statement, “Oh, that explains it,” and the expression “oh” in two occasions. After a long pause at the end of K’s last remark, R explained that she was “actually looking at area” followed by a gesture in which the thumb and index finger mimicked the act of measuring something horizontal (as she uttered ‘x’) and then something vertical (as she uttered ‘y’). She referred to the green point as “going up” on “Page 1” as she dragged ‘x’. The verb use of “going up” suggests that she was thinking of area-accumulation as the process of plotting the ordered pairs (x, A(x)). Although she saw (x, A(x)) as a reified, discursive object, she did not see the set of all (x, A(x))—the graph of A(x)—as a compound discursive object. In contrast, K’s use of the singular form “this graph” suggests that her realization of area-accumulating function was the set of all ordered pairs. She was referring to the “graph” as a new discursive object through *encapsulating* all ordered pairs (x, A(x)). Later, R used the process of *saming* to show that her reasoning of the sign of ‘x’ and ‘y’ “works the *same* for all of these. That’s how we graphed ours”. She ended with, “If they are not… like the same sign, then it’s going down.” Altogether, R had explained how to determine the movement of the green trace by looking at the sign of ‘x’ and ‘y’. This seemed to be what she meant by the *same* for all the pages.

After this discussion, the researcher (“O”) came to check in with the pair K and R. She prompted the students to explore the meaning of ‘a’, since they had not talked about it yet. The students explored the sketch silently for about three minutes and did not seem to make any progress. J (who was K’s original partner) joined the discussion with O, K, and R. The students were exploring the effect of dragging ‘a’ on “Page 1” when the researcher asked the students to try dragging ‘x’ upon dragging ‘a’. At this point, the sketch was showing a set of vertical green traces (obtained by their previous dragging of ‘a’) and two sets of parallel green traces (obtained by their previous and most recent dragging of ‘x’). Figure 3 shows a sample screenshot of the sketch at this point of their discussion. K and R noticed that there were two sets parallel green traces upon the researcher’s prompt. The researcher wanted the students to see that no matter where they set ‘a’, the graph of $A(x)={}_a\int^x 1dt=x+C$. In other words, the two sets of green traces are both antiderivatives of ‘f’ that differ by a constant ‘C’. Then, the following discussion unfolded.

O: Ok. So… If they are parallel… Let’s see, if they are parallel, does that still… Like your original one was ‘a=0’, right? What, what would that look like? Does it look similar to that one? Those two?

R: Ya, it looks the same.

O: It’s the same... same what? Same…

K: Oh…

O: So they are all going to be like that.

K: They are going to be like this, but the y…intercept gonna be different.

O: Ok. Ya. Do you guys agree? They are all gonna be like... And so… what about the derivative thing? Does it still work? Or not? You said the, the derivative thing.

K: It still works. Except that… <long pause> It does.

O: It does? So you mean these ones right? The derivative of these ones will still be...

K: ‘Cause, for these, these two graphs, the y-intercept is ‘c’ right? Constant. They will be cancelled.

O: Do you want to say more? Tell them what you think.

K: Like for this graph, it will be ‘y=x+4’ or something like that. And this graph, ‘y=x+8’ something like that right? And the derivative of those graphs will be ‘y=1’ right? ‘Cause, constant will be cancelled out.

O: Constant will be what? Constant will be what?

J,K: Zero. <K writes the numeral ‘0’ on the whiteboard>

J: So do I set ‘a’ somewhere else? <J drags ‘a’ then ‘x’ slowly> Ya right.

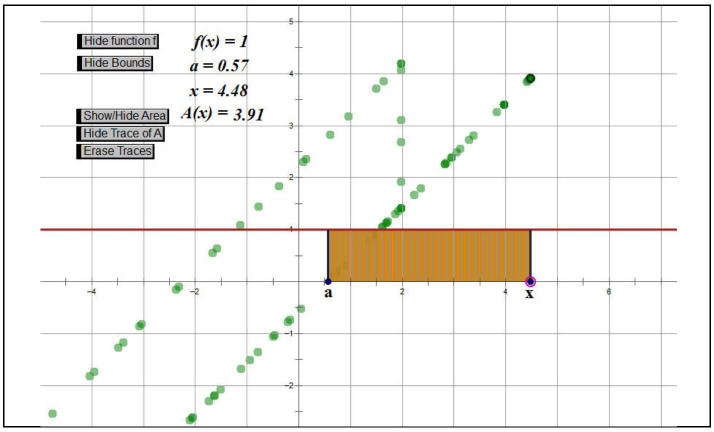

Figure 3: Sample screenshot of a sketch showing two sets of green traces, both representing the antiderivatives of f(x)=1.

In the above transcript, the researcher asked the students to recall what the green traces would have looked like with the "original one", where 'a=0'. R responded, "it looks the same", and K added that, "they are going to be like that". Using her hand gestures to visually mediate slope, K was able to explain that the two graphs would be parallel, "but the y…intercept gonna be different." The researcher revoiced K's comment and asked the students to think about whether "the derivative thing" would "still work". K said it would still work because, "These two graphs, the y-intercept is 'c' right? Constant. They will be cancelled." This was a turning point in the discussion for J and K, as they both seemed to have found that the relationship that they previously found continued to hold true for all values of 'a'. After this transcript, they continued to explain to R by writing equations y=x+4 and y=x+6 on the whiteboards. For example, J explained that the red line was the "dy/dx", followed by K's "always the same".

The episode shows the importance of the process of encapsulation for the learning of calculus. Throughout the discussion, the researcher's realization of the area-accumulating function was one where the set of ordered pairs (x, A(x)) was encapsulated into a graph. She frequently asked the students to compare the two sets of parallel green traces and the "original one", y=x. J and K took much time in order to shift their discourse. In the beginning, they did not attend to the shape of the green traces. When prompted, K began to talk of the two sets of green traces as "they". The use of the pronoun in the plural form suggests that K was thinking of the green traces as an encapsulation of two sets of (x, A(x)). This seemed to allow her to compare the shapes of the two "graphs" and used *saming* to refer to the "graphs" algebraically, by writing equations like y=x+4 and y=x+8 on the whiteboard.

On the other hand, R only made one comment in the discussion: "it looks the same". It was possible that her realization of area-accumulating functions influenced her participation in the discussion. The previous analysis shows that R's realization was around discrete ordered pairs (x, A(x)). These ordered pairs were reified in the sense that they were "timeless" stories about relations between objects. However, she did not see the graph of A(x) as a singular object by encapsulating the set of all ordered pairs (x, A(x)). Unlike K who frequently mentioned "this graph", J never used the word "graph" as a noun to signify the area-accumulating function. Her only use of the word "graph" was in "that's how we *graphed* ours"—a verb. Not able to see area-accumulating function at the object-level may have affected R's learning and her participation in the discussion with J, K, and the researcher.

DISCUSSION

The data analysis provides insights on students' developing discourse about the area-accumulating function as an *ordered pair* (x, A(x)) and as a *graph*. The students' discourse revealed that, although they both "found" something, their discoveries were significantly different realizations of the area-accumulating function. R's discourse was a reified one; she talked of the area under f as the green point, the *ordered pair* (x, A(x)). She used verbs in present-continuous tense to describe the movement of the green point as "going up" and "going down", which shows that she was thinking of the green point as a singular, as *an* ordered pair. On the other hand, K consistently used the "graph" as a noun to refer to the set of all green traces and the verb "is" to state the relationship between f and the "graph". Accompanied with her two statements, "this graph *is* the derivative of this graph," she used gestures to realize the shape of the respective "graphs" on the page. These gestures and word use suggest her realizations of both functions as mathematical and geometrical objects. Thus, K not only reified the process of area-accumulation as the ordered pair (x,A(x)), she also encapsulated the set of all ordered pairs into a new discursive object, a *graph.*

The analysis supports Sfard's (2008) theory that encapsulation and reification are two different processes for creating new discursive objects. While some cognitive theories (Dubinsky, 1991; Gray & Tall, 1991) make no distinctions between the two processes, this study shows the merit of distinguishing between the two. The analysis of R's discourse shows that reification does not always lead to encapsulation. Hence, combining the two processes as one, as in Dubinsky (1991), may be problematic. R's difficulty with making connections between 'f' and the "green traces" may be attributed to her reified discourse of area as an ordered pair but not as an encapsulation. This may have inhibited her participation in the subsequent discussion. Conversely, K's ability to see the "green traces" as a single graph may have facilitated her learning.

This paper makes a case for studying students' communication and changes in communication in the learning of calculus. In particular, I argue that attending to students' use of saming, reification, and encapsulation may help address difficulties in learning, especially in terms of dynamic aspects of calculus. A number of studies reported students' difficulties in creating graphical representation of a function's derivative and area-accumulating function (Tall, 1986; Ubuz, 2007). The results of this paper shed light in this area, suggesting that encapsulation is needed to think about derivative and area-accumulation as functions (graphs) and to work with relations between them at an object-level. The teacher's questioning and gestures to signify the set of ordered pairs as an object and the use of DGE may help facilitate the process of encapsulation. The *Hide/Show* buttons allowed the students to talk about their ideas gradually, one button at a time, while the *dragging* affordance enabled them to attend to dynamic relationships and connect algebraic with geometric representations of calculus. In tune with previous studies on DGE-mediated student thinking (Falcade, Laborde and Mariotti, 2007), the students may have communicated about area and derivatives geometrically and dynamically by exploiting these functionalities. As the students saw the green traces "grow" on the screen, the *Tracing* tool potentially contributed to the visual mediation of *area as a function* (see Ferrara, Pratt, & Robutti, 2006). Furthermore, the touchscreen-based DGE seemed to offer a haptic environment for learners to interact with dynamic relationships with their fingers, where the nature of gestures is re-conceptualised (Sinclair & de Freitas, to appear). Future research should consider examining the role of touchscreen dragging for facilitating the process of encapsulation.

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THE SUPPORT OF THE SPREADSHEET IN THE LEARNING OF THE TOPIC QUADRATIC EQUATIONS

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In this paper we analyse the role of the spreadsheet in the development of algebraic thinking of grade 9 students in the study of 2nd degree equations. The research aims are (i) to understand how students approach their tasks in the spreadsheet, i.e., which representations they use and how they coordinate them, and (ii) the extent to which using spreadsheets in conjunction with working with paper and pencil, influences the development of algebraic thinking, particularly in the use of more formal (algebraic) language. Data analysis focuses on the productions and dialogues of a student while solving tasks with the spreadsheet in the classroom. The spreadsheet, articulated with working with paper and pencil, was important to the student's developing of her algebraic thinking in this topic, boosting the use of a more formal language.

Keywords: algebraic thinking; spreadsheet; mathematical representations; quadratic equations; formal methods

INTRODUCTION

The algebraic language "is the only system that offers opportunity to logically investigate, justifiy, generalize and prove mathematical hipotheses" (Panasuk, 2010), p. 253). However, for a large number of students this language remains an obstacle when they are expected to use it. The spreadsheet is widely recognized as a very useful tool for introducing algebra and for developing algebraic thinking (e.g., Dettori et al., 2001; Rojano, 2002) as it promotes an understanding of relations of dependence between variables and encourages students to present algebraic methods rather than arithmetic methods (Rojano, 2002). Our perspective of algebraic thinking stresses the distinction between using algebraic notation and understanding algebraic structures, separated by a gap that is often underestimated. We suggest that this gap can be gainfully filled by working with suitable spreadsheet activities (Nobre, Amado & Carreira, 2012).

In this paper we analyse the role of the spreadsheet in the development of algebraic thinking of a grade 9 student during the study of 2nd degree equations. We focus on the mathematical representations that she used and seek to understand how the work on the spreadsheet, in conjunction with paper and pencil, influences her development of algebraic thinking, particularly in the use of a more formal language. This paper is part of a wider research project involving the study of various aspects of the development of algebraic thinking, like the learning algebraic formal methods, problem solving and mathematical representations. In this paper our focus is on mathematical representations.

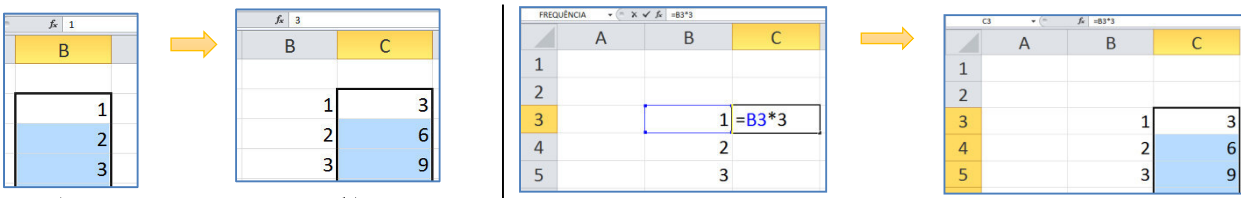
MATHEMATICAL REPRESENTATIONS

Working with different types of representation appears to enhance learning, in that it allows students to develop a better understanding of mathematics (Clements, 1999; Goldin, 2002; Panasuk, 2010; Tripathi, 2008). Tripathi (2008) stresses that a comprehensive picture of a concept only begins to emerge when the object it refers to is viewed from different perspectives. Thus the use of different representations can be understood as a variety of lenses which provide different perspectives and allow a broader and deeper understanding of a concept. The author notes that a discourse around the use of multiple representations can enrich the classroom culture and help students to actively participate in the learning process. In addition, she argues that students' representations and their ability to transfer ideas from one representation to another are indicators of their understanding. And Abrahamson (2006) also suggests that classroom discussions are useful to help students understand the mathematical ideas associated with different representations.

Duval (2014) points out that one must identify a given object by means of different semiotic registers and examine, in particular, the transformation of representation registers. Following this recommendation, we analyse the transformations of representations used by a student, namely treatments (transformations within a register) and conversions (transformations that result in a representation in another register). According to Duval (2003), it is the conversion of representations of a mathematical object for another register that enables the construction of knowledge: "The originality of mathematical activity relies on the simultaneous mobilization of at least two registers of representation, or on the possibility to change at any time of register of representation" (p. 14). Yet, this is not a simple and immediate activity for most students. In general, when students work on a problem, they think of several registers at once, even when they favour just one register in the final production. This requires a continuous activity of conversion, which typically is not produced explicitly but implicitly and spontaneously.

In the work with paper and pencil, we consider the following registers of representations: natural language, numerical notation system (NNS), algebraic notation system (ANS), and pictorial and graphical representations. In the spreadsheet environment we consider: natural language, input of numerical values, formulas (variable-cell and variable-column), graphical representations and formatting means such as the colour highlighting of specific cells (Haspekian, 2005). In the study of representations in the spreadsheet we use the notions of conversion and treatment from Duval, which can be extended to representations created in this computational environment.

In figure 1 we show examples of treatments on the spreadsheet: the construction of a numeric sequence without using a formula or using a formula. The first case shows a linear sequence being constructed by dragging the handle of a set of cells. The same sequence may be produced by introducing a formula that establishes a relationship between the two columns. In the second case the introduction of the formula makes explicit the relationship between the two columns.



a) b) c) d)

a) Construction of a numeric sequence.

b) Construction of a sequence establishing a (non explicit) relationship with the previous column.

c) Writing the formula to establish the relationship.

d) Construction of a sequence explicitly connecting the two columns (variable-column).

Figure 1. Implicit treatments (a e b) and explicit treatments (c e d) on the spreadsheet

Another relevant affordance of the spreadsheet is the conversion of tables into graphical representations.

THE STUDY OF SECOND DEGREE EQUATIONS

The Portuguese curricular guidelines determine that students solve quadratic equations by resorting to the square root operation, to the zero-product property and to the quadratic formula. The students must also be proficient in factoring, as well as in using the special cases of binomial multiplication.

For the teaching of this topic a set of tasks was outlined, as presented in Table 1. The work on the spreadsheet was always accompanied by some questions to be solved with paper and pencil in order to promote an interaction between the use of these two resources.

Table 1. Tasks and main resources used throughout the study of the topic

Tasks	A3 Diagnosis	B3 The age of the brothers	C3 Factoring	D3 The bouncing ball	E3 The experience in the laboratory	F3 The quadratic formula	G3 Problems
Resources	PP	PP/S	PP	PP/S	PP/S	PP	PP

Note: PP – Paper and pencil; S- Spreadsheet

In some of the problems solved with the spreadsheet, the use of graphical representations was suggested to promote a better understanding of the meaning of a quadratic function and its solutions. This is in line with Vaiyavutjamai and Clements (2006), who claim that students sometimes are capable to obtain the solutions of an equation like $(x-a)(x-b)=0$ but do not understand its meaning or what they stand for. Didis et al. (2011) corroborate this idea by stating that students have difficulty in understanding the meaning of symbols in 2nd degree equations On the other hand, they suggest that teachers should encourage students to use different techniques for solving 2nd degree equations as a way to improve students' understanding of those equations.

RESEARCH METHODS

In this study we aim to understand the contribution of the spreadsheet for students' learning of 2nd degree equations. As we intend to obtain a holistic view of the researched phenomenon, we elected a qualitative research methodology. Given the nature of the study, the methodology draws on an interpretative paradigm because we want to study the phenomenon in all its complexity and in its natural context (Bogdan & Biklen, 1994). In this research we use a case study design, where the first author, assumes the dual role of teacher and researcher. The experience took 11 lessons of 90 minutes in a class of 9th grade with 24 students, aged between 14 and 18 years. Aida was chosen as case study, she has 14 years old and usually does not express difficulties in learning mathematics. Documentary data hold a relevant place as a prime source for obtaining information on the researched phenomenon. In the classroom, we collected the students' productions, recorded the computer screens, audio-recorded their dialogues and took field notes from participant observation. The data analysis mainly involved content analysis (Bardin, 1977).

RESULTS

In the work of Aida on 2nd degree equations, a special attention was given to the transformation of representations carried out by the student. From this analysis two charts were produced: (i) one referring to treatments performed in NNS and ANS on the work with paper and pencil (PP) and on the generation of numeric sequences and variable-columns on the work with the spreadsheet (S), (ii) the other referring to conversions of representations to the NNS and ANS, as well as to graphical representations (figure 2). The treatments and the conversions are grouped according to the different types of task (problem solving (PS), exercises/other (E/O).With paper and pencil, we observed an increase of treatments in ANS as result of a more intense work on manipulation of 2nd degree expressions and equations. With the spreadsheet, the treatments involved in the construction of numerical sequences and variable-columns did not present a significant increase. As for conversions, in the work with paper and pencil, initially the representations in NNS were most prevalent through calculations in assigning values to unknowns in algebraic expressions. Task B3, partially solved on the spreadsheet, was fundamental for converting to ANS with paper and pencil. In tasks D3 and E3, Aida performed conversions on the spreadsheet from the table to graphical representation that allowed perceiving the variation in parabolas with different types of concavity.

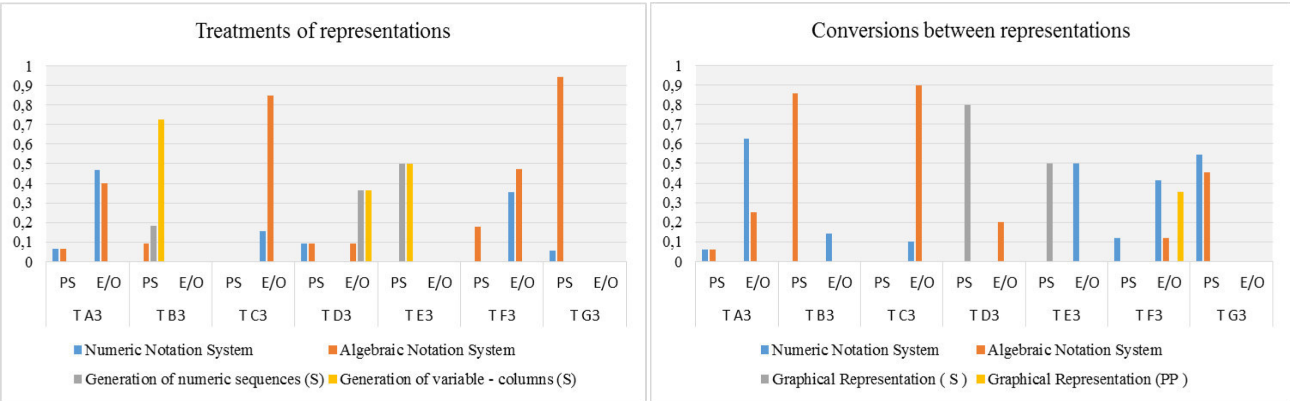


Figure 2. Summary of conversions and treatments (in percent) used by Aida

In the following, we present some excerpts of the work developed by Aida, by focusing on three tasks that she solved with the spreadsheet. Based on the student's productions, we highlight aspects that have proved particularly relevant in her use of formal language.

In the task B3 it was proposed to solve a problem with the spreadsheet, as follows:

Carlos, Ana and Ricardo are three brothers. Ana is one year older than Carlos and one year younger than Ricardo. One day, Ana was making operations with the numbers corresponding to their ages and said to her brothers: I compared the product of your age with the square of my age and found something very interesting! See if you can also find it! What may Ana have discovered?

Aida started by identifying the relationship between the ages of the three persons, selected the age of Ana as an independent variable (column E) and generated a number sequence with an increment of 1 unit. Then she used the formulas "=E5-1" for the age of Carlos, and "=E5+1" for the age of Ricardo, thus generating variable-columns, dependent on the age of Ana. Later she entered the respective formulas for the "product of the brothers' ages" and for the "square of the Ana's age", also generating variable-columns (figures 3 and 4).

These treatments on the spreadsheet provided by the dragging of cell handles led Aida to realise the variation in values in the different columns and to discover the relationship between the ages of the brothers, as she noted: "... if we add one unit to the product of the ages of the brothers, we obtain the age of Ana". Afterwards, it was intended that students explained algebraically what they had concluded in the previous question. Therefore, it was required that students made the conversion of the relationship found in the spreadsheet into algebraic notation. Aida did not show explicitly the relationship between the ages of the three brothers as she did not include the difference of the ages between them, as may be seen in Figure 5.

Product of the brother's ages — Square of Ana's age

Carlos	Ana	Ricardo	Produto das idades dos irmãos	quadrado da idade da Ana
-1	0	1	-1	0
0	1	2	0	1
1	2	3	3	4
2	3	4	8	9

Figure 3. Extract from Aida resolution on the spreadsheet

Carlos	Ana	Ricardo	Produto das idades dos irmãos	quadrado da idade da Ana
=E5-1	0	=E5+1	=F5*D5	=E5*E5
=E6-1	1	=E6+1	=F6*D6	=E6*E6
=E7-1	2	=E7+1	=F7*D7	=E7*E7
=E8-1	3	=E8+1	=F8*D8	=E8*E8

Figure 4. Formulas created on the spreadsheet

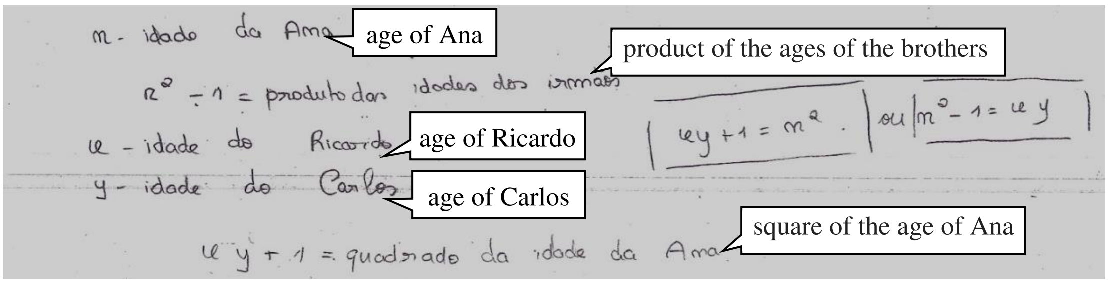


Figure 5. Answer of Aida to task B3

The student shows a caption with the meaning of the variables and writes equalities, which contain a mix of natural language and algebraic expressions. Finally, she presents two equivalent equations in response to the question. No student in the class expressed the expected algebraic relation, so it was necessary to encourage all students to rethink their answers, based on a solution from one of them. After some questioning about the relations between the three ages, and how they were expressed on the spreadsheet, the students arrived at the equality $(a-1)(a+1)=a^2-1$. Then the teacher inquired the class about this equality. Aida was one of the students who did not understand it at the outset.

It was then proposed that the students analysed the situation where the brothers would have an age difference of 5 years and Aida decided to use the spreadsheet. She found that the difference would then be 25. Finally, it was suggested the situation where the age difference would be *k*. The students no longer turned to the spreadsheet:

Teacher: If the age difference between them, rather than one, or rather than 5, is equal to *k*, what will happen?

Aida and Carlos: a^2-k^2 [simultaneous answer].

Others: $a-k$ times $a+k$ equals a^2-k^2.

Pedro: Just replace 5 by *k*.

The discussion led the students to the generalization of the condition they had found earlier. The solution of this task encouraged the students to do the conversion of the formulas used on the spreadsheet to algebraic language. Although Aida showed some lack of clarity in writing the required expression, this difficulty disappeared with the whole class discussion. In addition, the variation of the age difference helped Aida as well as other students in realizing that the initial expression could be generalized.

In task D3, also to be solved with the spreadsheet, the students were asked to represent the trajectory of a ball that was thrown vertically upwards at various moments and bounced several times on the floor. Several algebraic expressions for the height of the ball were given as functions of time: $A(t) = -20t^2 + 160t$, $B(t) = -20t^2 + 120t$, $C(t) = -20t^2 + 80t$, and $D(t) = -20t^2 + 40t$. The first question asked for the simulation on the spreadsheet of the first bounce of the ball, and also for its graphical representation. Aida began to insert the values for the time through a numerical sequence of increment 1. In the column for the height of the ball, she entered the calculation "=-20*0^2+160*0" in the first cell, the calculation "=-20*1^2+120*1" in the second cell, the calculation "=-20*2^2+80*2" and finally the calculation "=-20*3^2+40*3". Therefore, in each cell she changed the formula and the value for time. Her representation was indicative of a still unclear understanding of the task, since she was using the algebraic expressions of the functions for the several balls being thrown upwards. After some clarification from the teacher, Aida generated a variable-column for the height of the ball, as shown in Figure 6. Then the student created a graph-plot with the spreadsheet.

Tempo (time)	Altura (height)
0	0
1	140
2	240
3	300
4	320
5	300
6	240
7	140
8	0

Tempo (time)	Altura (height)
0	=-20*D4^2+160*D4
1	=-20*D5^2+160*D5
2	=-20*D6^2+160*D6
3	=-20*D7^2+160*D7
4	=-20*D8^2+160*D8
5	=-20*D9^2+160*D9
6	=-20*D10^2+160*D10
7	=-20*D11^2+160*D11
8	=-20*D12^2+160*D12

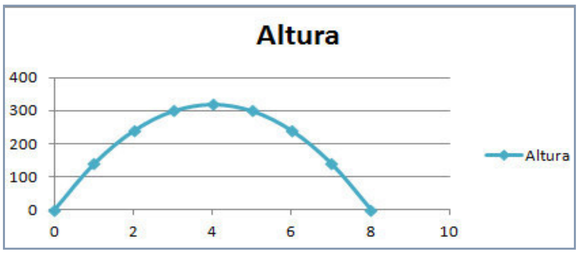


Figure 6. Table and graphical representation obtained by Aida for the first throwing of the ball

This conversion of the table to the graphical representation in the spreadsheet allowed the student to relate the table values with the graphical representation. This was her first contact with the graph of a parabola. The subsequent questions combined the work done on the spreadsheet with written work on paper, namely to find the maximum height of the ball and the time when it was reached, followed by a question on the time elapsed until the ball touched the ground.

Another question asked for the t values that would satisfy the condition $A(t) = 0$ and for its meaning in the context of the problem. Aida did not show difficulty in interpreting the condition. During the discussion, the teacher also asked the students to justify algebraically the values of t obtained. Unlike most students in the class, Aida quickly solved the equation correctly using the zero-product law, as shown in Figure 7. The teacher eventually reinforced the connection

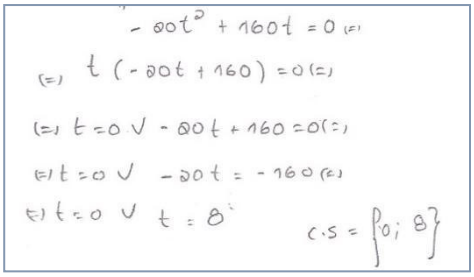


Figure 7. Answer of Aida

between the representations that the students had before them (the numerical table, the graph and the algebraic representation) to facilitate the understanding of the meaning of the solution of the equation:

Teacher: You have already answered this question by looking at the table and observing the graph. Now, you have an algebraic resolution... In your graph where does the parabola intersects the x-axis?

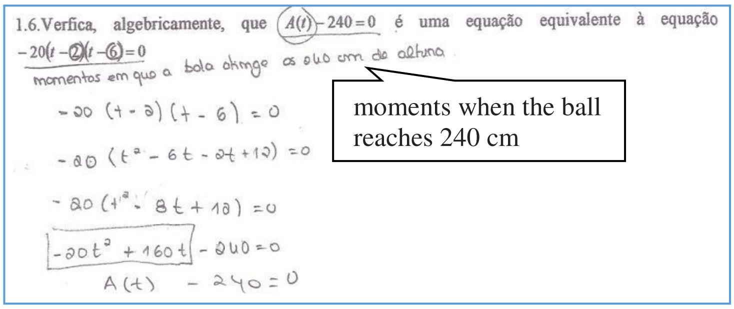


Figure 8. Answer of Aida

Students: At 0 and 8.

Teacher: So this means that 0 and 8 are the solutions of the equation you have here.

The next question asked the t values so that $A(t)=240$. After that, the students had to show, algebraically, that the equation $A(t)-240=0$ is equivalent to $-20(t-2)(t-6)=0$.

Many students showed difficulty in solving the question and some initially could not do it. Aida managed to solve it until the penultimate step (Figure 8). Given the difficulties of a few students, the teacher took one of their solutions on the board and explained that it is possible to write a quadratic equation as a product of factors, making its solutions visible, on the form $c(x-r_1)(x-r_2)=0$, where c is a constant and r_1 an r_2 are the roots.

Task E3 was carried out using both the spreadsheet and paper and pencil. In this case, an algebraic expression represented the temperature of a substance as a function of time and its graph was a parabola. Aida did not show difficulty in solving this task. As in the previous case, the determination of the output of a given input was requested and vice versa, and also their interpretation within the context of the problem. In this task, Aida used a procedure on the spreadsheet similar to that she had already used in task D3 and she noticed that this time the parabola was facing upwards (figure 9). There were also questions about the value of the initial and final temperature of the substance, for interpreting the values in the table and in the graph created on the spreadsheet, which Aida had no difficulty in answering.

horas	Temperatura
0	11
1	6
2	3
3	2
4	3
5	6

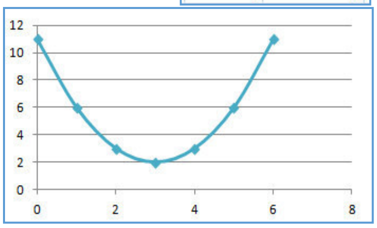


Figure 9. Results of Aida

From task F3 on, the students learned and applied the quadratic formula. An interview with Aida, involving solving problems and exercises on the topic, was carried out at the end of the teaching experiment. Aida resorted to conversions to ANS for writing 2nd degree equations and for converting to NNS in calculations associated with the assigning of values, and in particular, with the use of the quadratic formula. Throughout the study, we found that problem solving was an essential activity that promoted conversions to ANS, especially in translating problems into 2nd degree equations, which is important on the learning of this topic.

FINAL REMARKS

The proposed tasks encouraged the establishment of connections between the work with the spreadsheet and the work with pencil and paper. The spreadsheet allowed a first approach to the graphical representation of a quadratic function. This representation, in parallel with the tabular representation, and with representations in natural language, allowed Aida to understand the meaning of solving a 2nd degree equation even before the formal learning of the algebraic method, as suggested by research on the role and effectiveness of multiple representations (Clements, 1999; Goldin, 2002; Panasuk, 2010; Tripathi, 2008).

It is also worth noting the results obtained by Aida with pencil and paper, using formal algebraic language, especially in solving equations. The kind of connections that were encouraged and motivated was essential to the development of her algebraic thinking in line with the findings highlighted by Dettori et al. (2001) and by Nobre, Amado & Carreira (2012). The work in these two environments provided her the basis for understanding the meaning of factoring a 2nd degree equation based on its roots. According to Vaiyavutjamai and Clements (2006) and Didis et al. (2011), the students should solve tasks that lead to the understanding of meanings rather than merely to applying memorized procedures. The conversions performed both on the spreadsheet and from the spreadsheet into algebraic notation were pivotal to the development of the algebraic knowledge of Aida, in line with Duval's (2014) ideas.

We conclude by emphasizing that the environment provided by the spreadsheet proved suitable for the development of the algebraic thinking of Aida, in her learning of 2nd degree equations, without the constraint of using symbolic algebra from the very beginning. Moreover, the conversions carried out contributed to her understanding of the meaning of the symbolic algebraic work carried out with paper and pencil.

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THE ROLE OF PEER AND COMPUTER FEEDBACK IN STUDENTS LEARNING

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In this paper we present an episode taken from a two years teaching experiment involving the regular use a Dynamic Geometry Software with a class of 7th graders. The learning context is centred on student-student-computer interactions. The overall aim of the research is to understand the role of the emerging peer and computer feedback in students' learning. We adopt a qualitative and interpretive research methodology. The data presented are related to an episode involving the actions of a pair of students while solving a geometry task with GeoGebra. The empirical data were analysed and interpreted by applying a model of feedback phases combined with a model of strategies devising. The results point to a strong connection between the role of peer and computer feedback and the forms of devising and revising strategies to solve the problem.

Keywords: Peer Feedback; Computer Feedback; Strategies; Geometry; Geogebra.

INTRODUTION

The integration of the computer in mathematics teaching and learning is one of the important research areas among mathematics education researchers as it may be seen in the fullness of released articles, particularly in the vast set of papers that have been presented in ICTMT over the years. In general, there is a broad awareness about the numerous possibilities of developing learning within classroom environments that enable students to do Geometry with the use of digital tools.

The Dynamic Geometry Software (DGS) environments, under the right conditions, empower students learning. Dynamic geometry software establishes a Euclidean geometry model that provides feedback through the "dragging" feature, such as checking the accuracy of constructions or theorems. It is possible to change parameters so that the invariant relations are noticeable, or use the measurement of lengths or angles so that the "results" are observed as invariant patterns of measures (Laborde, 1998).

The DGS can help to develop a learning social environment (Vygotsky, 1962). Students can work together with DGS through discussion, collaboration, and feedback, when they are involved in constructing mathematical meanings (Arzarello & Robutti, 2010). When students are working in pairs with the computer continuing interactions are likely to take place, and this includes not only interactions between students but also their interaction with the computer. Feedback between peers can emerge, in the form of dialogues, while the images appear on the computer screen resulting from the students' need of justifying or clarifying their conjectures (Yu, Barrett & Presmeg, 2009). In this context the computer feedback emerges as *a lever* to peer feedback. In the existing literature, little is known about the characteristics of emerging feedback and its implications for learning.

The overall aim of the present research is to understand the role of the emerging peer and computer feedback in student's learning. To accomplish that purpose a teaching experiment with a class of students was implemented. The experience was carried out for two consecutive years with the same class (during grade 7 and grade 8) and was centred on Geometry tasks with the use of Geogebra, having students working independently in pairs on the given tasks.

FEEDBACK

Whether regarded as one of the most powerful influences for learning and performance (Hattie & Timperley, 2007), or the lifeblood of learning (Smith, 2007), or as the most effective form of educational intervention (Hattie, 1999; Wiliam, 2007), feedback is one of the characteristics of the teaching/learning process most misunderstood (Cohen, 1985; Black & Wiliam, 2010).

In this article we propose a notion of feedback as a dialogical and relational process. Here feedback takes the form of a dialogue opened to all participants in the classroom. In this context, the role of the teacher is to stir dialogue with and among the students (co-constructive dialogues between peers) and the role of the learner is to be actively involved in the process (Askew & Lodge, 2000).

THE TEACHING EXPERIMENT

With the aim of "experience, firsthand, students' mathematical learning" (Steffe & Thompson, 2000, p.267) we implemented a teaching experiment. The experiment consisted in creating an exploratory environment where students work in pairs with Geogebra tasks. This methodology allows students to exchange views with each other, clarify doubts and share information (Ponte et al., 2007), has computer and peer feedback emerges. These tasks were prepared for students' engagement in developing mathematical knowledge. In the process of solving the task, multiple strategies arise from each pair of students. These strategies are materialized through students' actions, especially those on the computer. Those two clusters, active strategies together with peer and computer feedback, make the focus that will lead our purpose of better understanding the contribution of feedback to students learning. The figure below tries to clarify the model implemented (Fig. 1).

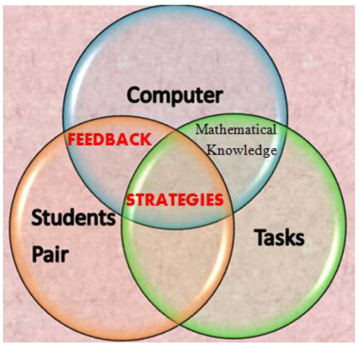

Figure1. Teaching Experiment Model

METHODOLOGY

In this study, the teacher (first author) assumes the dual role of teacher and researcher. The research follows a qualitative and interpretative methodological approach (Bogdan & Biklen, 1994; Gay, Mills & Airasian, 2006). The class consisted of 22 students aged between 12 and 14 years.

In this experiment the computer was gradually introduced along 33 tasks through four subtopics of Geometry in a two-year teaching program, allowing students to get familiar with the technology and become Geogebra natural users. Students worked in pairs, with a computer for each pair.

Data collection involved participant observation, audio and video recording of lessons, and students' productions. We used audio recordings of conversations between peers, video recordings of students' actions in the computer and we collected the Geogebra files produced by the students.

In this paper, we describe and analyse one episode related to a construction task. For the data analysis on the feedback role, a descriptive model of feedback phases adapted from Kollar & Fischer (2010) was used (Fig. 2). The model comprises the phases of:

Performance – A student interacts with the computer performing an action in a given task. This action may result from a strategy established with a partner or not.

Feedback Provision – Following the student's action, visual feedback provided by the computer appears which may be complemented by possible oral feedback from a peer. This peer feedback focuses on the thinking process; it may reveal doubts, discordance or approval, indications or simple findings resulting from actions, procedures or reasoning.

Feedback Reception – The reception of feedback by a student may lead to the emergence of a response (generating feedback to feedback) and thus leading to a cyclical process, termed as interactive dialogue, whose aim is to clarify actions, procedures or reasoning, which can be used as a lever to the process of co-regulation.

Construction Process Revision – A new search for strategies starts, opened to the participation of each student in order to enhance a new action. This step consists of proactive (feedforward) and reactive strategies. Reactive strategies are those in which a student responds to visual feedback while manipulating geometric figures. Proactive strategies are those in which the student determines which actions to take before carrying them out, feeding forward the process. While in the reactive strategies the student is more dependent on the feedback provided by the computer, in the proactive strategies the student starts from mathematical properties and uses the computer to implement a specific plan (Hollebrands, 2007).

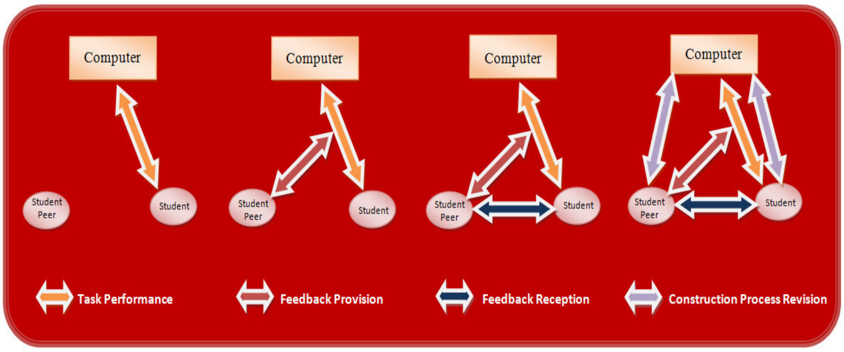

Figure 2. Model of the feedback phases.

For the data analysis on development of strategies, a model of 3 kinds of reasoning modes was used:

Imitation - Occurs when the student acts in an equal way as a result of a previous exposure to a particular stimulus repeated. This is not a mindless copying of actions. The imitation strategies presuppose some understanding of the structural relations of the task involved (Vygotsky, 1987). Particularly when the feedback of the computer was perceived as a good response to a task, the student tends to repeat the action or set of actions taken. These strategies may emerge even when the present task is more complex, and serves to accomplish an intermediate objective.

Association - Occurs when the student acts by association with analogous experiences and previous similar observations. Thus, these types of strategies are different, but similar in some way, of the ones implemented before. There are three possibilities for the purpose of implementing this kind of strategies: if the emergent feedback from the strategy implemented before was perceived as a good response to a task or objective, then similar strategies may contribute to achieve a different goal; if the feedback was perceived as a bad response to a task or objective, analogous strategies can contribute to retry to achieve the goal not reached before, and there is the possibility of combining various strategies used before.

Interaction: Occurs when the student acts due to teacher or peer interaction and/or with the computer in order to carry out substantially different strategies from the ones attempted previously.

THE ANALYSIS OF AN EPISODE

The following episode took place during the second time (the initial stage of the experiment) that involved the use of GeoGebra in the classroom. Here, we will describe and analyse students working on a task related to the construction of different types of triangles. This episode focuses specially on a pair of students whose fictional names are Andre and Lukas. However, sometimes there is the exchange of ideas and discussion with other pairs and also with the teacher, usually as a last resort. The task had several questions related to scalene, isosceles and equilateral triangles, as well as right-angled triangles construction and classification. When starting the task, Lukas took on the work on the computer but later Andre suggested doing it by turns throughout the various questions. However it was mainly Andre who manipulated the computer. One of the questions asked students to construct a right-angled triangle that would not deform when its vertices were dragged. The question was illustrated with a scalene right-angled triangle at vertex A.

Andre takes the lead of building the triangle after Lukas having stated that it must have a 90 degrees angle. His first action is to mark on the plan a 90 degrees angle using the “angle with given size” tool (This was the first time they used this tool). With this tool, they get a triangle ABA’ right-angled at B and isosceles (Fig. 3).

This is how this tool works and the resulting three points together with the small square (indicating right angle) constitute the visual feedback of the action performed by the student who decided to use this affordance of the software.

Figure 3. Angle with given size

Lukas immediately reacts noticing that the 90 degrees angle should have the vertex at A (he was referring to the figure shown in the worksheet) and offers himself to perform another construction but Andre doesn’t accept it and deletes the construction.

He starts it all over again by marking three points A, B and C using the “polygon tool”. By moving the vertices he tries to make a 90 degrees angle. In order to understand how the angle was being made, Lukas asks Andre some questions. Lukas wants to be sure that the angle becomes unchangeable. They discuss about it and Lukas understands that the angle actually changes its size when the vertices are dragged. After the computer visual feedback, Andre accepts that he didn't solve the problem, agreeing that it was always supposed to be 90 degrees.

Based on the initial figure made with "the angle with given size" tool, Lukas suggests that Andre should go back and repeat the initial procedure of marking a 90 degrees angle using two of the vertices of the polygon previously made. Afterwards he drags one of the previous polygon vertices so that side AB sets up on one of the sides of the right angle as figure 4 shows.

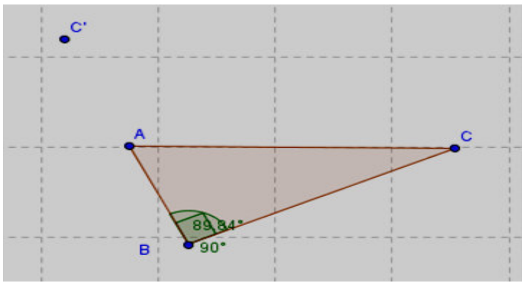

Figure 4. Setting up with side AB

Lukas keeps raising questions about the result and insists that he wants to be sure of the angle size. Andre answers that it is fine and well done but Lukas undoes Andre's last action on the computer and repeats the procedure of dragging the vertex.

Lukas: Wait, I am going to see it.

Andre: No. It is OK.

Andre: Well, you made the triangle again.

Lukas: It's 89.84. It's not 90!

Andre: Why did you delete the other?!

Lukas: Come on, Andre!

Andre: Look, this is how you should do it. Use your head.

Andre measures the angle and obtains 90.42. He tries to drag again and he manages to get 90 degrees. At this point both students could see that they didn't solve the problem because the angle kept changing.

Lukas: I have an idea. May I? [He takes the computer and chooses to show the axes].

Lukas: Andre, why haven't we thought of this? [He constructs a right-angled triangle plotting the vertices on points of the grid].

Lukas: The shape is the same (referring to the triangle given in the worksheet).

Andre: No, it isn't.

Lukas: Yes, it is.

In trying to convince his peer, Lukas decides to show that the triangle could rotate and would not change its shape. He constructs a circle with centre on the vertex of the right angle (A) and through another vertex (C) as shown in figure 5. Then, Lukas drags point C and realizes that his conjecture was wrong, because the computer visual feedback showed him that the angle size changed. When this happened, Lukas and Andre felt unable to solve the question and decided to call the teacher for help. However, soon after Lukas states:

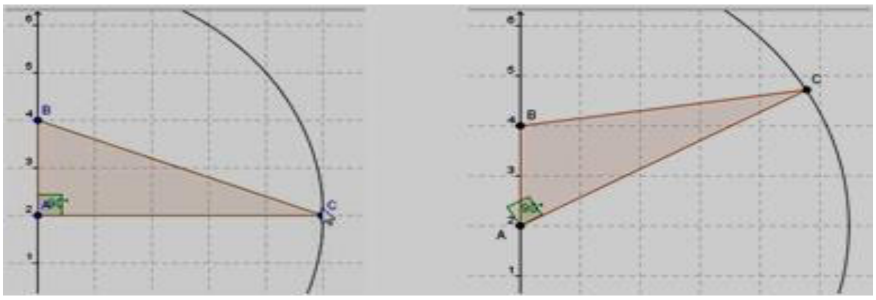

Figure 5. An attempt to rotate the triangle

Lukas: Now, I know! Andre, do you know what the challenge is? The challenge is to drag [the vertices] and keeping the angle with 90 degrees!

The teacher approached the students and asked them how the straight lines that made the right angle should be. This helped the students to think about perpendiculars and opened the way for a construction based on creating two perpendiculars to define two sides of any right-angled triangle.

According to the peer feedback model (Table 1) and to the strategies model involving technology (Table 2) proposed above we can identify a sequence of steps.

Table 1. Feedback Sequence

Performance	Andre constructs an angle with given size: 90 degrees.
Feedback	The computer shows an isosceles right-angled triangle; Lukas disagrees about the vertex of the right angle (they are trying to reproduce the figure given in the worksheet).
Reception	Andre accepts the feedback and deletes the construction.
Performance	Andre constructs a 3-sided polygon and drags vertices to display a right angle (by trial).
Feedback	The computer shows a figure similar to the given one. Lukas questions the robustness of the right angle.
Reception	Andre accepts.
Revision	Lukas suggests not to delete the figure and to use the angle with given size as before.
Performance	Andre creates a right angle using 2 vertices of the previous polygon and tries to adjust the triangle to the right angle by moving the vertices.
Feedback	The computer shows two angles with the same vertex: one right and one approximate. Lukas doubts that the triangle is right-angled and shows he is correct.
Reception	Andre tries to ignore the remark but eventually agrees.
Revision	Lukas offers a new strategy: to use the grid.
Performance	Andre protests but makes a right-angled triangle with the aid of the grid.
Feedback	The computer shows a right-angled triangle but its position isn't the wanted one. Lukas understands that the position is different but the shape is correct and they discuss.
Revision	Lukas presents a new strategy: a way of rotating the triangle by introducing a circle.
Performance	Lukas constructs a circle with centre on the right angle vertex through another triangle vertex and drags this vertex.
Feedback	The computer shows a changed triangle, which is not right-angled.
Revision	The students conclude by understanding the aim of the problem presented.

Table 2. Strategies Sequence

Interaction	Andre interacts with the software using the affordance "angle with given size" to construct an angle of 90 degrees and get a triangle ABA' right-angled at B and isosceles (figure 2).
Imitation	Andre uses a tool that he had already used in a previous task to construct a 3-sided polygon and drags vertices to display a right angle (by trial).
Association	Andre combines the two strategies used before, creates a right angle using 2 vertices of the previous polygon and tries to adjust the triangle to the right angle by moving the vertices.
Interaction	As suggested by Lukas, Andre uses a new affordance of the software and makes a right-angled triangle with the aid of the grid.
Interaction	Lukas presents a new strategy: a way of rotating the triangle by introducing a circle. Lukas creates a circle with centre on the right angle vertex through another triangle vertex and drags this vertex.
Interaction	After the teacher's question, the students think about perpendicular lines and make a new construction based on creating perpendiculars to define the sides of the right-angled triangle.

CONCLUDING REMARKS

In order to better understand the role of peer and computer feedback in student's learning, a teaching experiment using a DGS was implemented in the classroom. The focus, has shown on figure 1, was on feedback and on the strategies used to accomplish each task. The use of the two models allows us to interpret some emergent empirical evidences as a sequence of steps where performance, feedback and revisions were important for understanding the purpose of the task. In each step the pair of students learned about properties of a right-angled triangle and about the Geogebra tools.

In the episode analysed, continuous and interactive feedback fuelled by the use of the dynamic geometry environment gave rise to the so-called co-constructive dialogue between students, where knowledge is jointly constructed through dialogue (Askew & Lodge, 2000). The task involved in this episode is an adequate challenge because it allows for different ways to be solved with Geogebra. Moreover the option of students working in pairs contributed to a fruitful context to explore both the geometry underling the problem and the potentialities of the software.

The successive cycles of peer feedback indicate an obvious exploratory work of the two peers, suggesting a series of loops in the development of the problem solving activity (Askew & Lodge, 2000). The structure consistently reiterated (Fig. 6) emerges as a strong feature of the activity.

Figure 6: Feedback Cycle

As the cycles were happening the peer feedback has improved and became more consistent, promoting more thinking and more strategy search. We may notice that some of the loops emerge with blanks. This means that some phases do not appear in verbal format or in the form of actions, but they are taking place within the triad composed of the peers and the computer. Emerging strategies are mainly of *interaction* with the software, as students explore the affordances. At the same time, there arises one *imitation* strategy from the previous task, one association strategy, combining two strategies used before and some more *interaction* strategies. These strategies are products of the feedback from the peer, the computer, and in the end from the teacher. This process led students from an initial focus on reproducing a given figure to the full understanding of the aim of the construction problem even though they still required a hint from the teacher to achieve a solution. The process of learning is characterized by the emergent feedback that led to the strategies described. Nevertheless there are some limitations of the study: the analysis of strategies is just a way to get to learning, within a very specific context, with all the narrowness that it entails; the interpretations put forward on the origins of students' strategies are undoubtedly informed by our theoretical model and are a way of seeing *feedback working* on students' independent problem solving. Our future work will concentrate on further exploring our theoretical framework and analyzing other episodes, by looking at how feedback (from different sources) is actually impacting on the ways students deal with geometrical tasks with a DGS.

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ENHANCING LEARNERS' GEOMETRICAL THINKING THROUGH LESSON STUDY USING GSP

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Geometry and hence, geometric thinking continue to be one of the problematic for learners. This study employed quasi-experimental and case study research design to investigate the effects of Lesson Study incorporating Phase-Based Instruction (LS-PBI) using GSP on Thai students' geometric thinking. Three groups of mixed ability Grade 7 students (aged 12 – 13) were chosen as participants in one of the schools in Yala Province, Thailand. These groups (Group 1: N = 34, Group 2: N = 34 and Group 3: N = 35) were taught the topic of Properties of 2D and 3D shapes in turn by three different teachers. Pre-test and post-test were used to assess students' geometric thinking. Findings revealed that LS-PBI using GSP was effective in enhancing students' geometric thinking. There was a significant difference in the pre-test and post-test scores in each group and there was a significant difference in the post-test scores among the three groups of students.

Keywords: Lesson Study, Geometric Thinking, Geometer's Sketchpad

INTRODUCTION

According to Battista (2007), geometry is "a complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyse physical and imaged spatial environments" (p. 843). Geometric thinking, on the other hand, is the ability to think reasonably in geometric context (van de Walle, 2014). The ability to think geometrically will lead student to have spatial visualization – an important aspect of geometric thinking, geometric modelling and spatial reasoning that will provide ways for students to understand and explain physical environments and can be an important tool in problem solving (NCTM, 2000). At secondary level, students are expected to be able to construct and explain the compass and straight edge two-dimensional geometric figures and explain characteristics and properties of three-dimensional geometric figures such as prisms, pyramids, cylinders, cones and spheres.

Despite the importance of geometry as a school mathematics subject and despite the attempts to develop students' geometric thinking, Thai students still lag behind in mathematics and geometry in comparison to national and international averages. The examination results evaluated by the National Institute of Education Testing Service (NIETS, 2012) in the Ordinary National Educational Test of middle school students in Thailand show that the average mathematics score of secondary school students from 2008 – 2012 are 32.66%, 26.05%, 24.18%, 32.08% and 26.95%, respectively. In addition, the results from The Programme for International Student Assessment (PISA) 2009 found that the average score of Thai student is 419 which is statistically significantly below the average of The Organization for Economic Co-operation and Development and Thailand was ranked in the period of 48 – 52 in average score out of 65 participating countries (OECD, 2010). Besides this, the trend of Thai students' scores decrease continuously from PISA 2000 to PISA 2009 (OECD, 2010). In geometry achievement specifically, the Trends in International Mathematics and Science Study (TIMSS) 2007 and 2011, the average geometry achievement of Thai students are 442 and 415, respectively which were significantly lower than the international average (500) and Thailand was ranked 28th in average geometry achievement out of 49 participating countries (Mullis, Martin, Foy & Arora, 2012). Several studies (Clements & Battista, 1992; Yerushalmy & Chazan, 1993) have found that students fail in geometry because they have difficulty in visualizing geometric concepts.

The van Hiele theory of geometric thinking provides a good description of students' level of thinking (Battista, 2002). The van Hiele theory (van Hiele, 1986) which describes five levels of geometric thinking and the phase-based instruction which is a teaching strategy to move up the levels of geometric thinking through five phases of learning have been applied in many studies related to teaching and learning of geometry and show the successfulness in developing students' geometric thinking (Liu & Cummings, 2001). Besides the van Hiele theory, a much more important concern is to find ways to make students understand geometrical concepts.

During the last decade, researchers have studied and showed that using technology such as GSP was useful in developing students' understandings of geometric concepts (Connor, Moss & Grover, 2007; Myers, 2009). These studies indicate that GSP is a useful tool for enhancing children's thinking through van Hiele's hierarchy because it allows students to discover relationships among geometric concepts through investigation (Liu & Cummings, 2001). Furthermore, students should develop personally meaningful geometric concepts and ways of reasoning that enable them to carefully analyse spatial problems and situations rather than memorizing properties and definitions (Battista 2001). Hence, the integration of technology, pedagogy with the teaching content is important in developing students' understanding of a particular mathematical content.

Although ICT appears to be an important factor that makes students succeed in learning mathematics, OECD (2010) found that students who use the most ICT have the minimum score. Therefore, it follows that technology alone is not enough to improve student learning. Teacher factor is key to successful learning because effective teachers produce effective students. This suggests that professional teacher development in effective use of technology to teach mathematical topics is paramount.

Lesson study is one of the professional teacher development programs which many scholars have studied for developing teaching process and has shown success in teaching and learning because it provides opportunities for teacher to work collaboratively, have a deep understanding of the pedagogy and cultivate the skill of observation, analysis and reflection of the teacher (Becker, Ghenciu, Horak, & Schroeder, 2008; Chassels & Melville, 2009; Isoda, 2010; Knapp, Bomer, & Moore, 2008; Lewis, Perry, & Hurd, 2009; Roback, Chance, Legler, & Moore, 2006). For these reasons, this study purposes to study the effect of lesson study incorporating phase-based instruction (LS-PBI) using GSP in enhancing learners' geometric thinking.

THEORETICAL FRAMEWORK

Levels of geometric thinking

According to the van Hiele's theory of geometric thinking, students progress through five levels of thinking in relation to their understanding of geometric content:

***Level 1 Visual level*:** Students at this level recognize figures by their appearance.

Level 2 Descriptive/Analytic: Students at this level recognize/analyse figures by their properties or components.

Level 3 Abstract/Relational: Students at this level can distinguish between necessary and sufficient conditions for a concept; they can also form abstract definitions, and classify figures by elaborating on their interrelationships.

Level 4 Formal Deduction: Students establish theorems within an axiomatic system. They recognize the difference between undefined terms, definitions, axioms, and theorems. They are capable of constructing original proofs.

Level 5 Mathematical Rigor: Students understand the relationship between various systems of geometry. They are able to describe the effect of adding or deleting an axiom on a given geometric system. These students can compare, analyse, and create proofs under different geometric systems.

Phase-based instruction

Students are expected to progress from Phase 1 to Phase 5.

Phase 1 (Information): Students get acquainted with 2D and 3D geometric shapes which they encounter in real life, examine examples and non-examples of 2D and 3D geometric shapes by the pre-constructed which teacher provided on the GSP program and then they did the activities by recalled the name of 2D and 3D geometric shapes. This phase allows teachers to gauge pupils' prior knowledge.

Phase 2 (Guided Orientation): Students are guided by tasks which involving different relation of 2D and 3D geometric shapes. This phase students investigated and can understand the properties of 2D and 3D geometric shapes by dragging and unfolding the pre-constructed of 2D and 3D geometric shapes which teacher provided on GSP program.

Phase 3 (Explicitation): Students become aware of the relations and express them in words using technical language. Students discuss the properties of 2D and 3D geometric shapes from their investigation of the pre-constructed shapes provided by the teacher on GSP.

Phase 4 (Free Orientation): Students are challenged by doing more complex tasks to find their own way in the network relation. In this phase, students are given activities to answer questions such as "Which two shapes have the same number of edges, vertices, and faces"?, "Which shape has two opposite identical faces, and other faces are rectangles"?, "I have one curved face and two flat identical circular faces. Which shape I am"? and so on.

Phase 5 (Integration): Students summarize what they have learned about properties of 2D and 3D geometric shapes on the student worksheet.

RESEARCH DESIGN AND SAMPLE

A quasi-experimental research design was employed to study the effect of LS-PBI using GSP on Thai students' geometric thinking. Purposive sampling technique was used to selected three classes of mixed-ability students in grade 7 (Group 1: N = 34, Group 2: N = 34 and Group 3: N = 35) and five mathematics teachers who have more than 10 years experiences in teaching geometry from an urban secondary school in Yala Province, Thailand. The school was chosen because of its proximity to the researcher's workplace. More significantly, this school was equipped with personal computers and projectors which facilitated the use of GSP in teaching and learning.

Participant students were selected from Grade 7 because in Thailand, geometry is taught only to students in Grade 7 and 9. Therefore, it was decided that it would be more appropriate to choose Grade 7 students as participants in order to prepare them with the foundations needed in Grade 9. Besides, in order to repeat the teaching with the revised lesson plan a different group of students would be needed. Three intact classes with mixed-ability students were selected. These three groups were considered to have similar abilities as gathered from their year-end school mathematics achievement results from the previous semester.

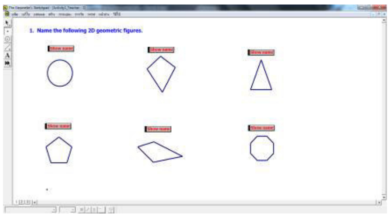
PROCEDURE

The objectives of The Basic Education Core Curriculum 2008 in mathematics for students in Grades 7 – 9 outlines that students should be able to "construct and explain stages of constructing two-dimensional geometric figures with compass and straight edge; can explain characteristics and properties of three-dimensional geometric figures, i.e., prisms, pyramids, cylinders, cones and sphere" (Thailand Ministry of Education, 2008). Hence, five lesson plans in the topic of *"Relationship between 2D geometric and 3D geometric shapes"* through phase-based instruction using GSP would be the intervention in the experiment.

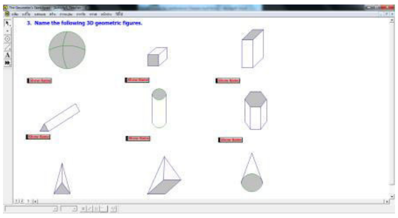
According to the van Hiele theory, progress from one level of geometric learning to the next involves five phases of learning called "phase-based instruction": Information, Guided orientation, Explicitation, Free orientation, and Integration. Therefore, phase-based instruction was used in each lesson plan. The GSP was used to facilitate teaching and learning.

INSTRUMENTS

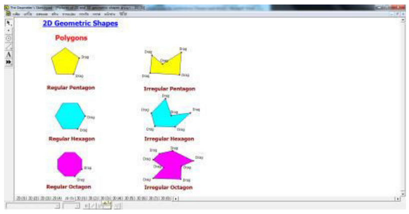
1) Lesson plan incorporating phase-based instruction using GSP (including GSP activities and students worksheets) in the topic of "Relationship between 2D and 3D geometric shapes" (Ministry of Education Thailand, 2008; van Hiele, 1986).



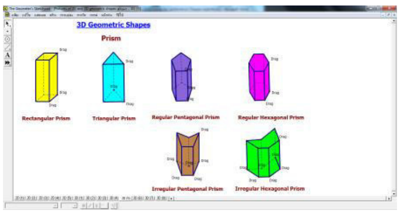
Recall the name of 2D shapes



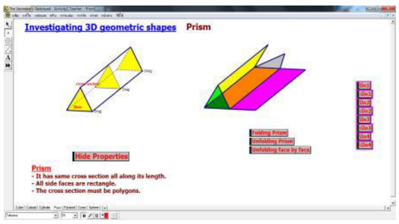
Recall the name of 3D shapes



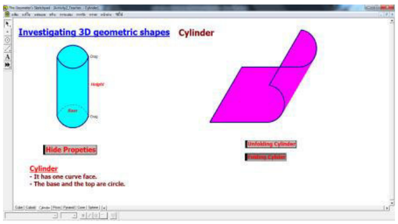
Investigating polygon



Investigating prism



Investigating prism



Investigating cylinder

2) Pretest and Posttest for assessing van Hiele level of geometric thinking of students (Mason & Johnston-Wilder, 2005; Usiskin, 1982). In both pre-test and post-test, there are 20 items of

multiple choices questions. Students are required to cross out the letter from the choices corresponding to the students' answer on the answer sheet;

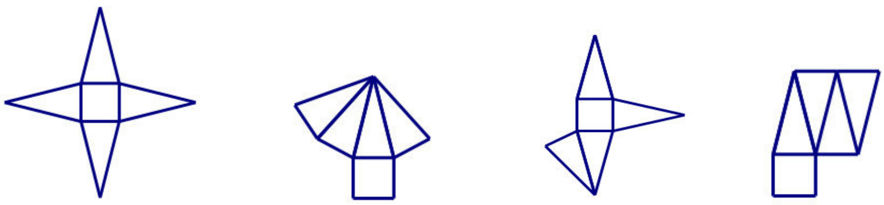
Items 1- 6 for assessing students' geometric thinking in level 1
Items 7- 12 for assessing students' geometric thinking in level 2
Items 13-20 for assessing students' geometric thinking in level 3

Sample questions from the pre-test and post-test are provided below.

Pre-test questions:
Level 2
The slicing of cone 45° declining from the base makes which cross section?

A. Isosceles triangle
B. Equilateral triangle
C. Circle
D. Ellipse

Level 3
Which of the following nets folds to make a square pyramid?



I II III IV

A. I, II and III
B. I, II and IV
C. I, III and IV
D. II, III and IV

Post-test questions:
Level 2

The slicing of which 3D geometric shape makes the triangle cross section?

A. Slicing the cylinder perpendicular to the base.
B. Slicing the cylinder 45° declining from the base.
C. Slicing the cuboid straight to the path of its diagonal.
D. Slicing the cone perpendicular to the base and pass the vertex of the cone.

Level 3

Which 3D shape is formed from the net in Figure 2?

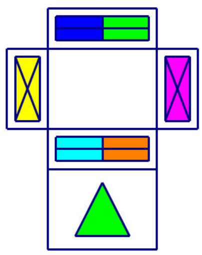
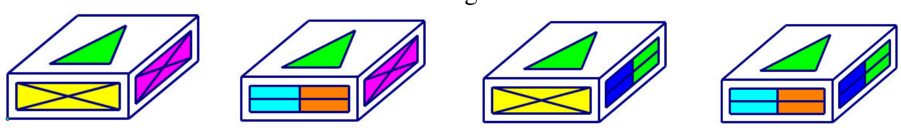


Figure 2



I II III IV

A. I B. II C. III D. IV

PROCEDURE

Prior using LS-PBI using GSP, a pre-test was administered to the students in Group 1, Group 2 and Group 3 to determine their initial level of geometric thinking. Next, the first teacher carried out the Phase-Based Instruction using GSP. During this time, the other teachers who did not teach observed the teaching and learning to see the difficulties of students in learning. Then a post-test was administered to students in Group 1. After the first lesson study session of students in Group 1, the group of teachers reflected on the teaching and learning process and revised the lesson plan by focusing on students' difficulties and the development of their geometric thinking as evidenced from their observation. Following this, the revised lesson plan was taught for a second time by the second teacher to the students in Group 2. Students' geometric thinking was observed. Then post-test was administered to the students in Group 2. The same process was repeated again to the students in Group 3.

DATA ANALYSIS

Identifying the level of geometric thinking

The criterion used in this study to identify the students' van Hiele level of geometric thinking was adapted from Usiskin (1982). If a student answered 60% or more of the items correctly at each level, he/she was considered passed in that level. Therefore, for every 6 items, a student needs to answer at least 4 items correctly to be considered satisfactory at that level.

Students are then assigned a weight sum score. A student will get 1 point if he/she meets the criterion on items 1 – 6 (Level 1), 2 points if he/she meets the criterion on items 7 – 12 (Level 2) and 4 points if he/she meets the criterion on items 13 – 20 (Level 3). Once students got the points from each level, the points in each level will be combined to become a weight sum score. Finally the classification of the van Hiele level of geometric thinking will be identified according to the Usiskin's operational definitions (Usiskin, 1982). The weight sum scores 0, 1 and 3 and 7 are assigned to the van Hiele level 0, 1, 2 and 3 respectively.

RESULTS

Students' level of geometric thinking

The initial van Hiele level of students in Group 1, Group 2 and Group 3 were predominantly at level 1 (44.12%, 47.06% and 45.71%, respectively). After the intervention, the van Hiele level of students in Group 1, Group 2 and Group 3 were predominantly at level 3 (79.41%, 82.35% and 88.57% respectively). The results show that students progressed from level 0 to level 3. Group 2 has more percentage of the students who attained level 3 than Groups 1 and 3.

Comparison of students' pre-test and post-test score

For each multiple choice questions in the pre-test and post-test, 1 point was given to the student for the correct answer and 0 point was given for incorrect answer. The total of score of each test is 12 points. Paired sample t-test was used to compare the mean scores of the pre-test and post-test of students in each group before and after the intervention. Results shows that there is a statistically significant difference between the mean scores of pre-test and post-test of students in Group 1, Group 2 and Group 3, respectively ($p < .05$). The mean scores in the post-tests were higher than the mean scores in the pre-tests in every group, which suggests the positive effect of LS-PBI using GSP.

Comparison of students' pre-test and post-test score

Analysis of covariance (ANOVA) was used to compare the post-test scores of students in Groups 1, 2 and 3 to see if students who were taught using the final revised lesson plan had the most improvement (pre-test score of the students were the covariate because it was considered as the different background knowledge of students that might affect the post-test). Results show that there is a statistically significant difference in the mean scores in post-test among the three groups of students ($p < .05$). Students in Group 3 who were taught using the last revised lesson plan showed the highest mean score, suggesting the most improvement.

DISCUSSION AND CONCLUSION

The findings show that students' initial van Hiele level of geometric thinking about the properties of 2D and 3D geometric shapes ranged from Level 1 to Level 2. After the intervention, the post-test scores show that the students' level of geometric thinking ranged from Level 1 to Level 3. However, there seems to be an increase in the number of students who showed progress in geometric thinking in Groups 1, 2 and 3. The initial van Hiele level of students in every group was predominantly Level 2 but after the intervention, the van Hiele level of students in every group was predominantly Level 3. Moreover, the results reveal that Group 3 students who were taught using the 'best' lesson plan had the highest percentage of the students who got to Level 3 (highest level). This suggests that the lesson study incorporating phase-based instruction had a positive effect on students' geometric thinking in learning the properties of 2D and 3D geometric shapes. These findings highlight the potential of this teaching and learning process as an effective strategy in teaching geometry.

The results also show that there is a statistically significant difference between the mean score of pre-test and post-test of students in Group 1, Group 2 and Group 3, respectively. Since the mean score of post-test was greater than the mean score of pre-test in every group, this suggests that each lesson plan in teaching the properties of 2D and 3D geometric shapes using phase-based instruction using GSP was effective. These findings are consistent with Choi-Koh (1999) and Chew (2009) which reported that phase-based instruction using GSP had enhanced student understanding of geometry concepts. Comparing post-test scores of students in Group 1, Group 2 and Group 3, we can conclude that there is a statistically significant difference in the mean scores of post-test among the three groups of students. Therefore, there is a difference between the post-test score of students in Group 1 who were taught using Lesson Plan 1, students in Group 2 who were taught using Lesson Plan 2 and students in Group 3 who were taught using Lesson Plan 3. Particularly, the results indicated that post-test scores after being taught using Lesson Plans 1 and 3 have statistically significant difference. However, the post-test scores after being taught using Lesson Plans 2 and 3 show no significant difference. This is probably due to students getting acquainted with the strategy of teaching early in the process. This study suggest that with a well-designed teaching and learning process using phase-based instruction, appropriate instructional tool using software which is suitable in learning geometry (e.g. GSP) and the improvement of teaching methodology and teacher competencies through lesson study are among the elements that can enhance the understanding in learning geometric concepts and geometrical thinking.

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LEARNING TO APPLY MATHEMATICS IN ENGINEERING MODELLING THROUGH CONSTRUCTING VIRTUAL SENSORY SYSTEMS IN MAZE-VIDEOGAMES

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This work is part of a research project that aims to enhance engineering students' learning of how to apply mathematics in modelling activities of real-world situations, through the construction (design and programming) of video games. This work is framed in constructionism which considers that learning is facilitated through engagement in the construction of external objects (Papert & Harel, 1991). Here, we present the work of six university engineering students who were asked to build a videogame where a virtual mobile robot has to navigate a maze; this required the modelling of combinational logic circuits, which in turn needed Boolean algebra and the simplification of logic equations. The programming of the videogame involved brainstorming and experimentation in a meaningful and motivating activity, where students were able to apply the theoretical mathematical knowledge in the design of the real world project (in this case a model of a digital systems).

Keywords: Modelling, videogames, constructionism, model-eliciting activities (MEAs), engineering

A PROJECT WHERE STUDENTS BUILD VIDEOGAMES AS A MEANS TO APPLY MATHEMATICAL MODELLING AND CREATE REAL-WORLD SIMULATIONS

A problem that engineering students often face after graduating from their studies, is that they have much mathematical knowledge unrelated to professional skills – i.e. they don't know how to apply that knowledge (which is important, in particular, in modelling activities). Therefore it is necessary to help students apply their knowledge to meet the demands of the real world.

For the past couple of years we have been involved in an ongoing project where university engineering students are asked to build videogames that require the modelling of real-world systems or situations, in order to help them learn to apply mathematical knowledge in possible applications: i.e. to develop a functional knowledge of mathematics.

The approach is based on the constructionism paradigm (Papert & Harel, 1991), which considers that learning is facilitated when the learner engages in the active construction and sharing of external objects. Thus, as described above, in order to promote learning, in our project students engage in the construction (design and programming) of videogames. This approach builds on research on the integration in education, of computers, game design and programming (e.g. Harel, 1990; Kafai, Franke, Ching & Shih, 1998; Kafai, 2006). Kafai et al. (1998) explain that game design and programming can lead to situations that naturally combine and relate issues of practice and theory, and can provide "opportunities for discussion, reflection, and collaboration within a meaningful context" (p.80). A suitable context for engaging in the activity of building a meaningful product (in our case, the programming of a videogame) is that of a learning microworld. Papert (1980) explained that a microworld is a powerful incubator of ideas (Papert, 1980). Hoyles & Noss (1987) expanded this concept of microworld as a "place" that takes into account the existing understandings and partial conceptions that the learner brings to a learning situation; the representational system (i.e. the software, programming language and other tools) for understanding a mathematical structure or a conceptual field; the pedagogical approach that includes the task design and possible interventions during the computer-based activity; and the context and social setting of in which the activities take place (e.g. whether they are carried out collaboratively). These are aspects which we consider useful for taking into account in our work.

Thus, the videogame programming activities in our project were conceived within a learning microworld with the above characteristics. On relevant aspect in this, is that the construction of videogames includes modelling processes where students construct a mathematical model (in this case, related with the design of a logic circuit) and implement it in a simulation (within the videogame).

The construction of the videogames is carried out through sequences of tasks whose design takes into consideration the six principles of Model-Eliciting Activities –MEAs (Hamilton, Lesh & Lester, 2008; Lesh & Doerr, 2003): reality, model construction, model documentation, self evaluation, model generalisation, and simple prototype.

In the next sections we provide a description (and results) of a specific activity within our project, where students were asked to build a videogame that would require the modelling of digital systems (combinational logic circuits [1]).

SAMPLE ACTIVITY: A MAZE-VIDEOGAME WITH A VIRTUAL ROBOT

This activity centres on the building of a videogame where a virtual mobile robot has to navigate a maze. A mobile robot is generally composed of a set of sensors and actuators (engines) controlled by some electronic circuit. In this case, (as explained in the following section) students have to model a combinational logic circuit that receives signals from three proximity sensors to control the two actuators (engines) of a robot. Then they have to program a simulation (using a game engine) of a virtual mobile robot navigating a virtual maze. Finally, they just have to implement its refined model in the gameplay of a game they have to design and program. Thus through this activity students apply mathematical knowledge (Boolean algebra and the simplification of logic equations), in a real world, albeit simulated, situation (the design and construction of a sensory-motor system for a mobile robot).

This activity is one of several activities of an optional course that we created for the project. Students voluntarily signed up for this optional course, because they are interested in engaging in videogame development after college. This particular activity was carried out by six engineering students (in their last year of studies: 9th semester of studies), at the National Polytechnic Institute in Mexico City, Mexico. These students had previously taken at least one course in digital system design. The activity was carried out over the course of five 4-hour sessions. During the activities, participants worked together in teams of two.

As explained above, the videogame requires building a simulation of a sensory-motor system in a mobile robot. In the next section we present the requirements involved in this task, as verbally indicated to the students.

Building a sensory-motor system for a mobile robot.

The design of a sensor system for an autonomous mobile robot corresponds to a real-life problem. In this case, students will build a simulation of this system, to be used in the videogame.

1. Students need to design a combinational logic circuit that allows the robot to navigate the maze shown in Figure 1.

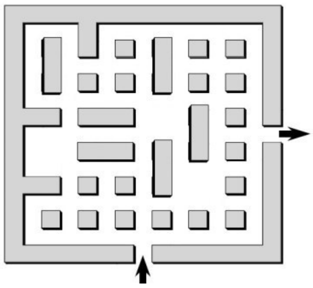

Figure 1. The maze to solve by the mobile robot

2. The design has the following requirements and restrictions: (1) between one to four sensors can be used. (2) Proximity sensors can be used: e.g. bumper or infrared (range: 1 meter). (3) The mobile robot in which the combinational logic circuit will be implemented has to have the configuration shown in Figure 2. (4) The base of the robot has a 1 m in diameter. (5) The blocks in the walls of the maze of Figure 1, can be white or black colour. (6) Each block in the wall is a square of 1.25 m.

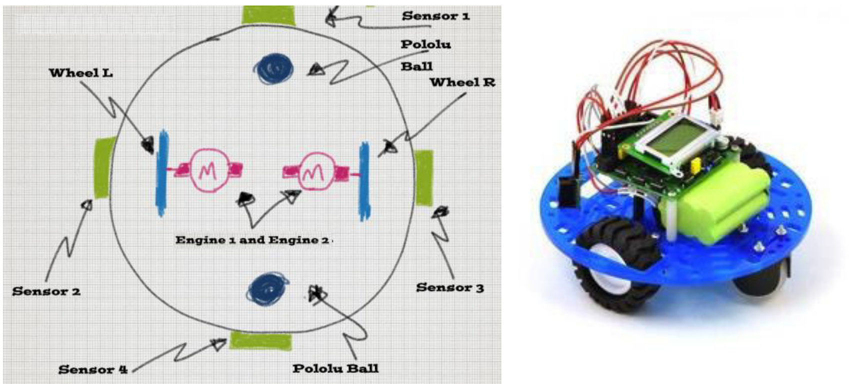

Figure 2. Configuration of mobile robot

3. Engine 1 and Engine 2, in Figure 2, are controlled as shown in Figure 3. The sensory-motor system of the mobile robot is composed by three sensors (1 to 3); these sensors send a signal (0 low or 1 high) to the Combinational Logic Circuit (CLC). The CLC connects its output to a microcontroller (μc) which in turn controls the direction and rotation of each engine through the H-Bridge [2].

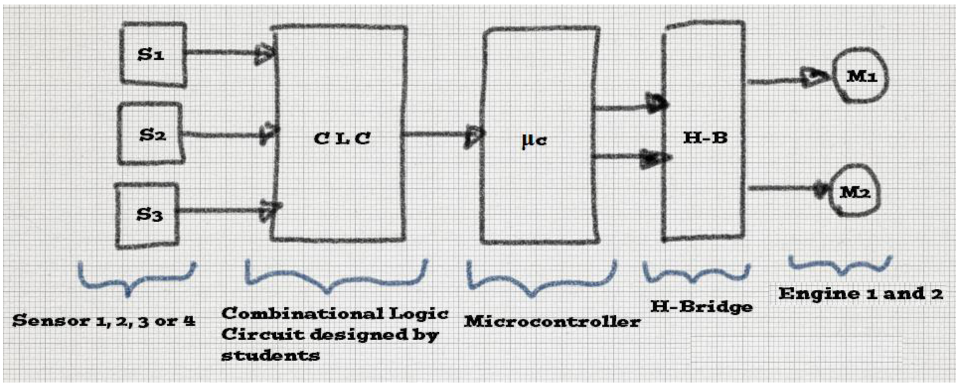

Figure 3. Block diagram of the engine controller

4. For the design of CLC, students need to define Boolean equations – which determine the operation of the logic circuit that receive signals from the sensors (1, 2, and 3) and control the engines (the direction and rotation) – and build a circuit with Logic Gates, following the truth table in Table 1. That table contains three columns: the first column gives the direction or rotation of each engine, which will allow the robot to move (forwards, backwards, left and right), and the second and third columns are the required outputs of the CLC.

Engine status	Output 1	Output 2
M1↑ M2↑ (Forwards)	0	0
M1↓ M2↑ (90° Left)	0	1
M1↑ M2↓ (90° Right)	1	0
M1↓ M2↓ (Backwards)	1	1

Table 1. Truth table for the CLC Output

As in any real-life situation, there isn't a defined path for building this system. Thus students will need to analyse different paths for constructing an adequate model; specifically, they need to analyse all possible situations for the robot to successfully navigate the maze, and propose what they consider is the best sensor array to control the actuators.

The videogame-building activity.

The videogame will The videogame-building activity, which was structured through a sequence of three tasks that take into account the six principles, mentioned above, of MEAs (Hamilton, Lesh & Lester, 2008; Lesh & Doerr, 2003), in order to encourage students to apply concepts about mathematical modelling in a real project. These tasks were:

Task 1: Modelling - Mathematisation. We asked to students solve the problem (described in detail below) that required the designing of combinational logic circuits for controlling mobile robots. Here, each student had to individually propose a mathematical model for the problem. The construction of this model required students to have, and use, previous knowledge of Boolean algebra and simplification of logic equations.

Task 2: Modelling - Programming simulations. Here, students used the videogame engine to program a simulation of the problem situation. This task required that students apply the mathematical relationships of the model within the videogame engine.

Task 3: Construction of the videogame. In this task, each team of two students had to develop, collaboratively, an idea for their videogame through a brief "story board" of up to 4 images, adding a brief description of the gameplay. The gameplay requires the use of the models built in the previous tasks, which form the core of the videogame which requires to solve puzzles and complete levels.

Between each task, students have a moment for reflection and interchange of ideas, which helped them understand how to apply modelling processes in the tasks.

RELATIONSHIP BETWEEN THE MEA DESIGN-PRINCIPLES AND THE TASKS

Having described how the activity and tasks were setup, we now explain how the six principles of MEAs were applied in the tasks carried out by students.

- *Reality*: The problem of designing and programming a model of CLC in the activities is a realistic problem. Furthermore, constructing a real videogame is something that genuinely engages students in the task of mathematical modelling. Thus, this principle appears in all tasks.
- *Model construction*: Students build a first version of the mathematical model (Task 1) and they make a refinement and simplification of this model when having to adapt it, first, for constructing a simulation in the game engine (Task 2), and then implementing it in the actual videogame (Task 3). Thus, students need to have clear understanding of the relationships necessary to program the simulation and, later, the videogame.
- *Model documentation*: In task 2, students had to describe the mathematisation of the CLC and other documentation found in the story board of videogame.
- *Self evaluation*: During the stage of refining and simplifying the model (Task 2), students need to evaluate their models and, through discussions, made decisions for finding an implementable version of the model into the game engine. Also, the reflection process took into account what elements are involved in the modelling process and how the entire process of the videogame creation, transforms and enriches their concepts from their initial conception.
- *Model generalisation*: In Task 2, students need to establish appropriate restrictions and relationships that leading to a general model that could be used again in other situations and with other technological tools.
- *Simple prototype*: When students set appropriate relationships and constraints for establishing a model with simplified equations (Task 2), they can also provide simple and structured steps to generate a set of tests, such as the simulation of the model, in order to test its effectiveness.

In the next subsections, we show how a pair of students, Javier and Felipe, carried out each task.

SAMPLE RESULTS

Task 1: Mathematical modelling

This was an individual task, where students had to build a model of CLC, that later could be used in the programming of the videogame (in Task 3). Here we present a model built by one student, Javier, that took into account the specifications of the sensory-system.

X	Y	A	B	C
Output 1	0utput 2	Sensor 1	Sensor 2	Sensor 3
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
1	0	1	0	0
0	1	1	0	1
1	0	1	1	0
1	1	1	1	1

Table 2. Truth table for the CLC designed by students

As a first step, Javier designed the truth table shown in Table 2. The first two columns show the CLC output of Table 1. The three columns A, B, and C, are the outputs signal of Sensor 1, Sensor 2 and Sensor 3. Then the student wrote Boolean equations defined as sums of products, as follows:

$$X = AB'C' + ABC' + ABC \text{ ... (Eq. 1)}$$
$$Y = AB'C + ABC \text{ ... (Eq. 2)}$$

Simplifying Eq. 1:

$$X = AB'C' + ABC' + ABC$$
$$X = C'(AB'+AB) + ABC$$
$$X = C'[(A(B'+B)] + ABC$$
$$X = AC' + ABC \text{ ... (Final expression Eq. 1)}$$

Simplifying Eq. 2:

$$Y = AB'C + ABC$$
$$Y = AC(B' + B)$$
$$Y = AC \text{ ... (Final expression Eq. 2)}$$

Thus, Javier's proposed circuit was:

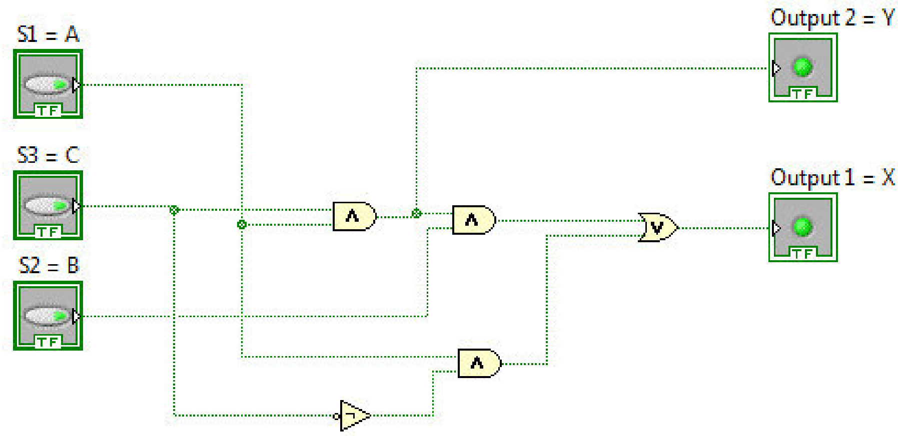

Figure 4. Logic circuit proposed by Javier

Task 2: Programming simulations.

This task requires going through a stage of refining and simplifying the model, proposing appropriate restrictions and relationships; as well as defining the characteristics of the computational objects that will simulate a mobile robot, giving it attributes (shape, colour, and size) and some Boolean properties (Morgan Theorem and other theorems for simplification of logic equations) that define the Boolean algebraic equations of the CLC.

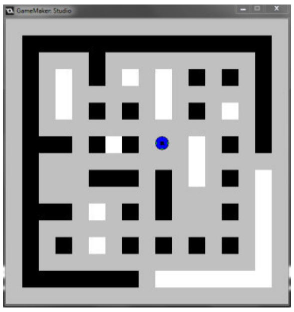

Figure 5. Sample of simulation

In the Figure 5, we show a screen-shot of the simulation of the mobile robot navigating the virtual maze created by the pair of students formed by Javier and Felipe. For programming this simulation, the students had to take into account that the game engine (Game Maker Studio) does not support "true" Boolean values, and actually accepts any real number below 1 as a false value, and any real number equal to (or greater than) 1 as being true. Table 3 provides some of the expressions in the game engine code (Game Maker Studio expressions) that correspond to the final equations arrived at by the students in Task 1 (in this case, the students used Javier's equations).

Final equations	Expressions in Game Maker Studio	Boolean operators used
X = AC + AB	*X = A* **&&** *C* || *A* **&&** *B*	&& (And), || (Or)
Y = AC	*Y = A* **&&** *C*	&& (And)

Table 3. Code for final equations in Game Maker Studio

Task 3: Construction of videogame.

In this task, Javier and Felipe created a brief "story board" (see Figure 6) to describe the game play.

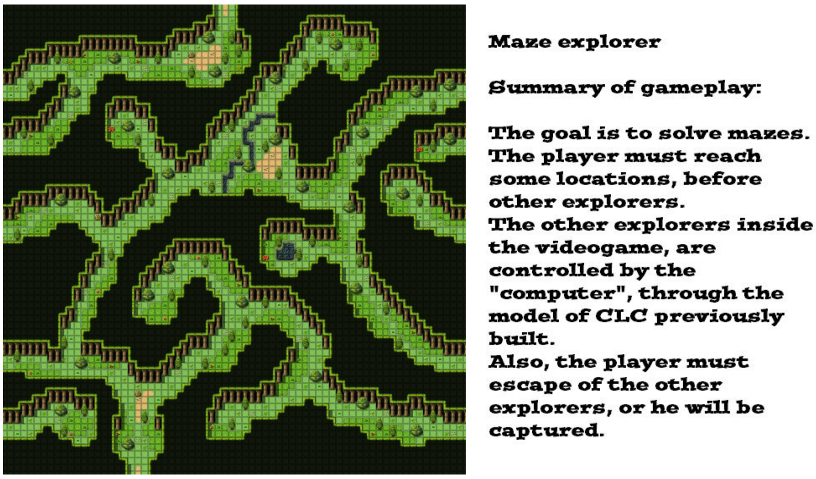

Figure 6. Videogame's storyboard

We observed that all six students showed great interest in this activity, because they want to engage in videogame design when they graduate. Also they commented that these tasks allow them to get involved in carrying out a project where they could apply their theoretical knowledge.

Figure 7. Screenshot of the videogame built by Javier and Felipe

As in the case of Javier and Felipe (see Figure 7), students were also able to apply their mathematical and technical skills and relate them to aesthetic and narrative issues, achieving a functional game with a nice design to the view and good integration of sound and music.

FINAL REMARKS

The above tasks confronted students with a real problem (the design of sensory-motor system in a mobile robot) that is significant in a professional context. Through the programming of the videogame, students engaged in producing a working model that was meaningful to them and thus gained a deeper understanding of all the elements involved in the modelling process. The mathematical modelling of CLC required students to apply their mathematical knowledge (algebra and the simplification of logic equations) in a real application (design and construction of a sensory-motor system in a mobile robot). The task sequence, in which students collaborated in constructing products and shared these, helped them expand their perspective and understanding of how apply theoretical knowledge in real-life projects and how such real-life projects can be carried out.

The problem presented here, is adequate for a student in his final year of engineering studies. Our objective with these activities, is not as a technical challenge, but a way of giving students the experience and practice of how their theoretical mathematical, scientific and engineering knowledge (which they learn in the classroom) can be applied in the real world; thus transforming that knowledge, into a functional knowledge.

In our project, we continue exploring the methodology of having students build videogames that involve modelling systems. Some other topics we are working with involve the design of a mechanical system and the control of robotic arms.

NOTES

1. Combinational Logic Circuits (CLC) consist of basic logic NAND, NOR or NOT gates that are "combined" or connected together to produce more complicated switching circuits. The outputs of CLCs are only determined by the logical function of their current input state – logic "0" or logic "1" – at any given instant in time.

2. H-Bridges are circuits used in robotics and other applications to allow DC motors to run forwards and backwards.

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WHY BUTTONS MATTER, SOMETIMES
HOW DIGITAL TOOLS AFFECT STUDENTS' DOCUMENTATIONS

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The language that students use for documenting their solutions, ideas and actions is influenced by their use of digital tools. This paper reports on a study on how students' language changes when working with digital tools. Linguistic categories were empirically developed in order to describe the language that is used, e.g. buttons, commands or expressions referring to the operating system. The qualitative analysis of examples discussed in this paper shows the role of describing varieties when students document their actions, e.g. when referring to buttons. The discussion reflects on two referential units that documentations refer to, namely the digital tool itself and mathematics. Based on the analysis of different examples, specific situations will be discussed, in which different describing varieties might be appropriate. On that basis, the normative question of adequacy of documentations can be approached from an empirical perspective.

Keywords: written documentation, language, digital tool, functional reasoning

INTRODUCTION & THEORETICAL REFLEXIONS

The effects of using digital tools in mathematics classroom have been explored in multiple ways. Digital tools affect the ways students learn (Artigue 2002) as well as the mathematical objects that are dealt with (Lagrange 1999). This paper reports on results of a study that explores the way in which students' documentations change in the light of the use of CAS in 10th grade when working on functional reasoning.

Students' documentations as a field of research in mathematics education

Students' documentations have been a subject of attention in mathematics education in recent years. Many examples show the way in which students' documentations especially change when a digital tool is used in mathematics classroom (Weigand, 2013; Ball, 2003). Words like "define", "equation", "substitute" or "solve" are standard part of the lexical repertoire for those solutions, in which digital tools were used (Ball & Stacey, 2004). CAS-solutions contain more function notations like *f(x)*, especially when comparing CAS and non-CAS solutions (Ball, 2003). Furthermore, students make use of the syntax of the digital tool (Weigand, 2013, p. 2769), especially when referring to certain commands (e.g. *propFrac((x*^{3}*-3x)/(x+1))*) or to the operating system (e.g. *Lists & Spreadsheet*).

Besides these empirical descriptions of characteristics of students' documentations when digital tools are used, there is an important second dimension in the context of the scientific discussion. It reflects the normative question of what can be counted as an adequate documentation. For example, Ball & Stacey (2005) report on a study where teachers had very diverse opinions on that question. While, for some teachers, commands were a natural part of the documentation in the students' individual exercise book, others did not accept written commands as being an acceptable part of a written documentation. Ball & Stacey concluded that students need "explicit guidance about what calculator language was acceptable in written work" (2005, p. 119).

Weigand (2013) has proposed a catalogue of criteria for adequate written documentations. Besides others, these criteria ask for understandable documentations that not only contain the expressions that can be seen on the screen, but also: "The solution describes the mathematical activities; it is not only a description in a special 'calculator language'" (Weigand, 2013, p. 2772).

Especially for doing mathematics with digital tools, the reflection on the mathematical process is as crucial as the critical reflection of one's calculations and certain solutions. Yet, it is a challenge for many students to verbalize and make explicit the actions that lead to a certain solution, to point out phenomena that seem to be obstacles within the working process and thus give insights into the process of their actions and not only into their solutions. The following example illustrates that challenge. When the student Ray is asked to document how he can determine intersection points of two given functions he documents: *Hit menu→6→4*. Ray documents the buttons he presses on his CAS. A student's documentation like this is no unusual phenomenon, yet it is irritating for a mathematician's eye on standard mathematical notation or technical language.

Taking the example above, the documentation *Hit menu→6→4* does not meet these criteria introduced by Weigand (2013). The student Ray documents the buttons being pressed, there is no semantic dimension referring to the mathematical concept or activity and the documentation describes how to handle the digital tool rather than the description of mathematics. Ray's documentation can be seen as an example of a **naturalistic description**. This term is used to refer to the idea of precisely describing natural phenomena. The philosophical idea of naturalism points out that the only way to understand our reality is to describe nature precisely with the help of natural sciences (Popper, 2002). Hence, there are no other entities like non-physical mental substances (Descartes, 1637) than those being described by natural sciences.

This idea is used to analyse documentations like the above. The student Ray uses buttons as a form of naturalistic language that carefully describes the physical and manual steps of his action. In the light of the criteria, following Weigand (2013), this documentation would not be acceptable. For Ray though, his documentation seems to be a proper way to solve the task, which asks him to document how to find intersection points. For the student Ray, there seem to be situations, in which it is appropriate to use a naturalistic language like this. Hence, for him, it seems to be an adequate expression in the light of the task. This paper draws on a study that empirically analyses the students' use of language in written documentations. On that basis the normative question of adequacy of different documentations will be raised. In this sense, the question of what can be counted as an acceptable documentation in a certain situation cannot be answered without looking at the empirical reality. The specific relation of normativity and empirical conceptual reality in this study relies on ideas brought in by Kant (1786).

The use of empirical categories to approach a normative question

The goal of the empirical study (Schacht, in press) was to develop empirically reconstructed lexical categories of the language that students use when working with digital tools. In traditional linguistics, the term lexical category is used in a grammatical notion for describing parts of speech, like nouns, adjectives or verbs. In this study, the term *lexical category* is used in a broader sense to describe empirically generated and coded categories of language that students use when working with digital tools (Dörnyei, 2007). These categories are generated within an open coding process by using the Grounded Theory (Strauss & Corbin, 1990). In this paper, the focus of the analysis is on the description of students' documentations when working with digital tools. In the outlook, the empirical analysis is then used to approach the normative question of adequacy of students' documentations. Therefore, one of the basic assumptions of the study goes back to Kant (1786). Other than Descartes, he introduced the idea that concepts cannot be seen as some kind of individual mental representation (Descartes, 1637). Rather, Kant argued, humans are the ones that apply concepts to the world. In his view, concepts have a certain normative force and humans follow a certain conceptual rule when applying the concept to the world – correct or not correct according to the conceptual rule itself. This is what Kant refers to: "thought is cognition by means of conceptions"(Kant, 1786). In this sense, conceptual acting is highly normative in the first place and whenever we use concepts, norms are already in play. Kant „developed this insight in the form of a normative theory of concepts: judging and acting are thought of as applying concepts, where the concepts determine what we have made ourselves responsible for by having a belief or performing an action, the content to which we have committed ourselves. One of the central tasks of philosophy is to understand the normativity of human belief and agency." (Brandom, 1999)

Following Kant, the question of what may be seen as an adequate documentation cannot be discussed without analysing the empirical reality of conceptual acting. Therefore, in this paper, the empirical lexical categories are used as a tool to describe the language that students use when working with digital tools. On that basis, empirical norms empirical norms in mathematics classroom will be discussed.

Lexical categories within a two-dimensional grid

The empirical analysis was carried out with a two-dimensional grid. The different reconstructed lexical categories can be found on the horizontal axis. These categories were empirically developed. *Button* is an example for one lexical category that was empirically reconstructed. That means, that expressions containing buttons are coded as *button* in the horizontal dimension. On the vertical axis, each expression was coded as *content*, *action* or *choice* (see table 1). These categories were developed in advance in order to categorize the specific function of the documentations (Neubrand, 1990). For example, if students describe a certain kind of mathematical doing, the expression was coded as *action* in the second dimension. Ray's documentation *Hit menu→6→4* was coded as *button: action*. Note that the lexical categories (horizontal axis) were empirically reconstructed and the mathematical performance (vertical axis) was not.

Table 1: The six lexical categories on the horizontal axis. Each documentation that was categorized as a certain lexical category was also coded with respect to the mathematical performance.

Lexical Categories → Mathematical performance ↓	**Command**	**Button**	**Operating System (system)**	**Mathematical symbolic expression (math_dig)**	**Expressions referring to the digital tool**	**New expression**
Content						
Action						
Choice						

Because a detailed description of the different categories can be found elsewhere (Schacht, in press), I will briefly summarize the description of those categories that are relevant to better understand the empirical data. A documentation is categorized as "*button"* when students document buttons that they pressed during their working process. Documentations that are categorized as "CAS as an *operating system"* (short: *system*) if they contain expressions that explicitly refer to the digital tool as an operating system. This category may include references to the user interface (*Open the application* ***Lists & Spreadsheet***) or to certain operations (*Open a new* ***document***). Documentations are categorized as "*Expressions referring to the digital tool"* (short: *math_dig*), if students document regular mathematical or colloquial expressions that refer to the digital tool, like *Draw the graph with* ***CAS***.

In the light of the theoretical reflexions, the study had the following underlying research questions:

- In which way do students use different lexical categories in order to express mathematical ideas and actions?
- In which way does the use of different lexical categories affect the normative question of adequacy of what counts as an acceptable written record and what are possible consequences for different situations in mathematics classroom?

METHODOLOGICAL CONSIDERATIONS AND DESCRIPTION OF THE STUDY

The study was carried out with tasks on functional reasoning with students that used TI Nspire CX CAS in 10th grade of different upper secondary schools in Germany. The paper pencil test was carried out with N=63 students in 10th grade in three different subgroups of N_1=21, N_2=23 and N_3=19 from two different schools. The students worked on tasks about functional reasoning. The tasks relevant to the data presented here will be introduced in the next section. When the data was collected, the different lexical categories were empirically generated within an open coding process.

In the next paragraph, each example will be described and analytically discussed by referring to the theoretical considerations presented above, namely by coding each example with respect to the lexical categories. On the basis of the lexical analysis, different describing varieties will be identified. Based on that analysis, situations will be discussed, where the forms of lexical usage in written documentations might be adequate. This way the normative question of adequacy will be discussed on the basis of empirical considerations of this study.

RESULTS

The following examples illustrate a variety of different lexical resources students use when working with digital tools in mathematics classroom. Different kinds of examples will exemplify this variety. Each example will show different facets of the students' use of language.

Example 1: Naturalistic language in documentations

The first example in fig. 1a shows a student working on a task to describe different ways to determine the intersection points of two given functions f and g with $f(x) = x^4 - x^3 - 5x^2 - x - 6$ and $g(x) = x^3 - 6x^2 + 6x - 5$ by using the CAS.

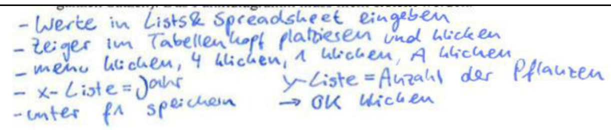
	Fig. 1a: Transcript. Being displayed in *Graph*(s, F.S.) and hit menu →6→4.
	Fig. 1b: Transcript. - Enter the values in Lists & Spreadsheet - Place the cursor in the table head and click - click menu, click 4, click 1, click A - x-List = Year y-List = number of plants - safe as f^ → click OK

Fig. 1a and 1b: Naturalistic describing varieties

In the example of fig. 1a, it is possible to reconstruct three different categories:

Table 2: Different lexical categories that can be reconstructed in the transcripts above (italic in the left column) with examples

2-dimensional code, 1st dimension: lexical category; 2nd dimension: mathematical performance	Example for lexical expression
expressions referring to the digital tool (math_dig) : choice	*being displayed*
CAS as an operating system: action	in *Graph*(s, F.S.)
buttons : action	hit *menu →6→4*

The example in fig. 1a shows a documentation that is prototypical for the use of naturalistic language in documentations of students working with digital tools. The student describes a way to determine the intersection points by using a language that refers to the digital tool rather than to mathematical concepts. First, the student refers to the operating system of his tool by choosing the *Graph(s)-Menu* in order to work on the task. Second, working in the *Graph(s)-Menu*, something (the student probably refers to the graph of the function) is *being displayed*. This passive verb form indicates that the CAS *displays* the graph in this situation and the CAS is the active part. Hence, the CAS itself, on the lexical level, is treated as an individual co-worker. Furthermore, the expression refers to the digital tool rather than to the mathematical object by using the verb *to display* rather than *to draw*. Finally, the student uses buttons in order to mark how intersection points can be determined.

This documentation shows one way in which naturalistic language can be used in order to describe steps within the working process. By using expressions like *to display* or *to hit*, the student describes his individual actions very precisely. It is a characteristic feature of such a naturalistic language that there is just very little reference to mathematical objects but rather to the digital tool and the individual manual actions done by or with it.

This can also be analysed in the second example in fig. 1b. The student was given a set of data, in which the number of plants was given over a period of time. It was the task to determine a regression function and to document the actions. In the example above, each line consists of a variety of naturalistic expressions like *enter, place the cursor in the table head, click, safe as*. Every expression not only refers to the digital tool within each step of the working process rather than to the mathematical concepts involved. Furthermore, expressions like *place the cursor and click* refer to the students' manual actions when using the CAS.

Example 2: The tension between mathematics and the tool as referential units in documentations

The second example shows a student working on a task to describe ways to determine the intersection points of two given functions f and g with $f(x) = x^4 - x^3 - 5x^2 - x - 6$ and $g(x) = x^3 - 6x^2 + 6x - 5$ by the using the CAS.

- graph zeichnen lassen und Schnittpunkte ablesen	Transcript: Let the graph be drawn and read off the intersection points

Fig. 2: Student' documentation using technical language

In fig. 2, the student offers a possible way to determine the intersection points by using the CAS, namely by taking the graphs of the functions and read off the intersection points. Two expressional units can be identified in this documentation: (i) *Let the graph be drawn* and (ii) *read off the intersection points*. The first expression (i) was categorized as *math dig: choice*. That means that the example was coded as *Mathematical expressions referring to the digital tool (math_dig)* in the first dimension as a lexical category. This code is chosen because the expression contains regular technical language like *graph* and *draw*. The student uses regular mathematical technical language within her documentation, which refers to the mathematical concepts like *graph* or *intersection points*. Yet, her expression has a specific connotation to the digital tool: The student writes that the graph is *being drawn.* This passive verb form refers to the CAS drawing the graph. This expression reveals that the CAS is seen almost as an external co-worker for some steps within the working process. Because the student chooses a geometrical way to determine the intersection points, the expression is coded as *choice* regarding the mathematical performance. The second expression (ii) was coded as *math dig: action*. Again, the expression *read off the intersection points* uses mathematical technical language (*math_dig*) to document the second step in the process of determination. Hence, regarding the lexical dimension of the whole documentation, technical language is used in order to document the solution.

This example reveals some important features in the students' language when working with digital tools. On the one hand, an expression like *the graph is being drawn* shows the strong influence of the use of a digital tool by using the passive word form, compared to the active pendant *I draw the graph*. On a linguistic level, this passive form reflects the relationship of the digital tool and the student, since the CAS is being seen as the active part when generating the graph of the function. Lexical phenomena like these can be observed quite often. On the other hand the student uses technical language that refers to mathematical concepts. While the examples in fig. 1 take into account the documentation of many detailed manual steps within the working process and – by doing so – use a naturalistic language that refers to the digital tool rather than to the mathematical concepts, the analysis of the example above shows crucial differences.

In fig. 1, the students use a naturalistic language describing which *buttons* being *hit* or that the digital tool *displays* the graph. In comparison to fig. 2, this reveals different aspects that can be outlined for naturalistic describing variants: First, the semantic dimension of the naturalistic language being used refers to the digital tool. Second, the language often reveals the digital tool itself as an autonomous co-worker. Third, the language being used mostly refers to manual actions within the working process rather than to the mathematical concepts being used. The examples show the different characteristics of such a language in a specific way. In the next section, the examples will be discussed with respect to the normative dimension in order to address the question, in which way the language can be counted as *appropriate* in mathematics classroom.

DISCUSSION AND OUTLOOK

The discussion of the examples analysed above will contain three different levels. First, some main features of the language with respect to the lexical categories in the different examples will be pointed out briefly. Second, on that basis, I will discuss characteristics of different classroom situations and point out, in which way each of the examples above might meet some of these characteristics. This will problematize the normative dimension of different normative criteria for the use of language in different classroom situations. Third, implications for mathematics classroom will be discussed.

Characteristics of language when working with digital tools

The examples described above reveal some specific characteristics of the language that students use for documentation when working with digital tools. In both examples the students document their actions within the working process. It is important to note that this is an explicit focus of the empirical study. Further research is needed regarding the question whether the results presented above vary depending on the kind of documentation (e.g. in written conversations, letters or informal reports (Manning 1996)) or depending on documentations of solutions only (without any comments on the working process) and descriptions of the students' actions.

The characteristics presented above can be identified in two dimensions, first with respect to the lexical categories being used and second regarding the referential units. The analysis of the different lexical categories reveals the type of language being used: whereas the student in the second example (fig. 2) uses the mathematical technical language in order to describe how the intersection points can be found, the students in fig. 1 use lexical terms, that precisely describe actions carried out by hand or by the digital tool like placing the cursor or clicking certain buttons. The results show that the language in the first example (fig. 1) is highly influenced by the use of the CAS itself. The specific character of this language even becomes more apparent when comparing it to situations, in which students don't have a digital tool. Suppose a student would write in a similar naturalistic manner: *I first pressed the pencil on the paper to mark the columns of the table. Then I drew a second line in order to mark the table head.* It seems hard to imagine that there is a situation in mathematics class, in which this seems to be an appropriate documentation of ones action after creating a table for documenting some given data. But, it seems to be specific for the use of digital tools, that students use naturalistic language for descriptions of actions (fig. 1) when working with digital tools. Hence, for describing the individual actions when working with digital tools, the examples reveal the distinction between a rather mathematical technical register that uses language referring to the mathematical concepts and a rather naturalistic language referring to the manual actions.

This distinction can also be observed by looking at the different referential units. The following table 2 illustrates the distinction of referential units regarding naturalistic and mathematical-technical language by referring to the examples above.

Table 2: The referential units of describing varieties when using naturalistic and technical language.

Language → Described actions and objects ↓	Naturalistic language Referential unit	Example	Technical language Referential unit	Example
Action	Manual action	*place the cursor*	Mathematical action	*to determine*
Object	Digital tool	*Menu-4-1-4*	Mathematical concept	*intersection point*

The distinction between the description of students' actions and of certain objects shows characteristics of documentations using naturalistic or technical language. Regarding the use of naturalistic language, the students refer to manual actions (*placing the cursor*) and to the digital tool itself (*table head*). In this case, both on the level of describing actions and objects, the referential unit is the digital tool. Instead, when using technical language, the description of mathematical actions (*to determine*) and mathematical concepts (*graph, intersection point*) refer to the mathematics and mathematical practices. Fig. 1a and 2 also show, that although the semantic content of generating the graph of a given function is equal in both cases, the referential units and the language being used differ in various ways (digital tool→*operating system* vs. mathematical objet→ *graph*).

Specific language in specific situations

It is one of the key features of the theoretical assumptions that the norms of what can be seen as an acceptable documentation will be a result of an empirical reconstruction rather than an a priori definition. This paper will give a brief outlook on what is meant by reconstructing the empirical reality of norms. In a first step, different situations will be identified, where the analysed language might be appropriate. Hence, on the basis of the analysis and the discussion above, there are two kinds of situations where the use of each kind of language might be appropriate.

Table 2 shows the different referential units of the varieties to document one's actions. There are two prototypic situations where each kind of language can be rather appropriate. Thinking of situations where students keep track of how to operate the digital tool, it can be useful to carefully document each step within the working process. For example, in rather complex situations, which afford multiple steps when working with the tool (e.g. generating a regression function from a given set of data) it is useful to log one's actions in order to memorize and document the actions. In a situation like this it is useful to use a rather naturalistic vocabulary that refers to the digital tool (e.g. by noting which buttons were being pressed). In some situations, it might even be necessary to note where to place the cursor because it is an essential part of the working process.

On the other hand, there are situations in class where students should master mathematical technical language that does not depend on the digital tool. In those situations, students should explicitly document their mathematical actions (by using proper mathematical technical language) and therefore refer to the mathematical concepts.

In the light of this distinction it is crucial to both give respect to the students' conceptual development as well as to their lexical development in mathematics classroom. The development of technical language needs developmental support as well as the conceptual development does. In the light of this discussion, there is a need to distinct different norms of appropriateness in different situations in mathematics classroom described above, which take into account both the learning process itself as well as those situations where students are expected to master a certain lexical body of technical language.

Why buttons matter, sometimes – Implications for mathematics classroom

Based on the analysis of two different empirical examples, two situations of using specific lexical resources were identified. This approach was based on the assumption that normative questions of which kind of documentation is appropriate in mathematics classroom cannot be clarified without looking at the empirical reality. Further research is needed to develop a normative framework for using language when working with digital tools. In the light of the theoretical considerations above, this framework should be developed on the basis of empirically reconstructed norms of students' documentations. This paper shapes out the specific language students use when documenting their actions. One of the main findings is the distinction between naturalistic and mathematical-technical lexical resources. Further research should observe similar phenomena in different mathematical areas (e.g. (dynamic) geometry, stochastic).

The different situations of lexical adequacy that have been identified in respect to the empirical data involve different norms of appropriateness for the students' documentations. For classroom actions, it is therefore necessary to make different norms of adequacy explicit when working with the students. In this view, the lexical development in mathematics classroom is a crucial part of the learning process itself. In the light of these findings, it should be one of the goals in class to problematize lexical consciousness. This means that it should be transparent to the students, which language is adequate in which situation and why.

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TECHNOLOGY IN MATHEMATICS TEACHING: NO USE AT ANY PRICE

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More than 25 years of experience in mathematics teaching lead the author to the conclusion: Technology in Mathematics Teaching: Less is More. The international assessments TIMMS and PISA had a big impact on the mathematics education at school in Germany. The paradigm changed. "Reduce the curriculum to basics. Reduce the predominance of teaching the formal skills therefore improve the understanding." (Baptist & Raab, 2007). Mathematics should be linked to real world problems. At school the focus is now on modelling using technology. Now we can observe the results of this development at university. Written exams show terrible elementary mistakes we rarely have seen before. And even overall fairly good students demonstrate these mistakes. We will give examples and little statistics of some of these problems. We also compare the level of books for mathematics used at German schools over years.

Keywords: Elementary mistakes, Underestimation of exercising

LOOK BACK TO THE NINETIES AND WHAT HAVE CHANGED?

I attended the conferences ICTMT 3 (1997) at Koblenz and ICTMT 5 (2001) at Klagenfurt. There I emphasized that technology in mathematics teaching at university enables to utilize the total new possibilities of virtual experiments in mathematics. I hoped that technology gives chances to get students more active. The aim was to enhance the mathematics understanding of the students. Indeed, students became more active, but the effect was much smaller than I expected.

What has changed since the nineties?

In the nineties and in the early 2000er years we observed a lack between the mathematical performance of the university freshmen and their expected competencies. Now the gap becomes much bigger between how the freshmen perform and how they should do perform. (Schwenk & Kalus, 2012; Schwenk & Berger, 2006). In the nineties the students had been much better trained at school in calculating without electronic tools than nowadays. Technology at school was limited to simple pocket calculator. Today computer algebra calculators are widely spread and widely used at school.

The international assessments TIMMS and PISA had a big impact on the mathematics education at school in Germany. The so called "SINUS Transfer" project was established in 2007 (Baptist & Raab, 2007). The SINUS experts suggested:

- Reduce the content to the basics,
- reduce teaching formal skills,
- avoid plantations of exercises,
- improve the understanding,
- model real world problems using technology,
- avoid drilling exercises by marching in step,
- promote individual learning,
- teach as moderator of students who follow their own responsibility.

One of the objectives was to point out that mathematics is useful and needed in everyday life. For that a frequent “modelling of real world problems” entered the schools. But here we have to object that nobody is able to model a problem without knowing anything about the field to be modelled. As a consequence instead of real world problems, exercises dressed up in a carnival costume appear. In addition to that the modelling is often already done by the authors of the exercises, which means that the students do not really need to model. An examples is given in figure 1, see below.

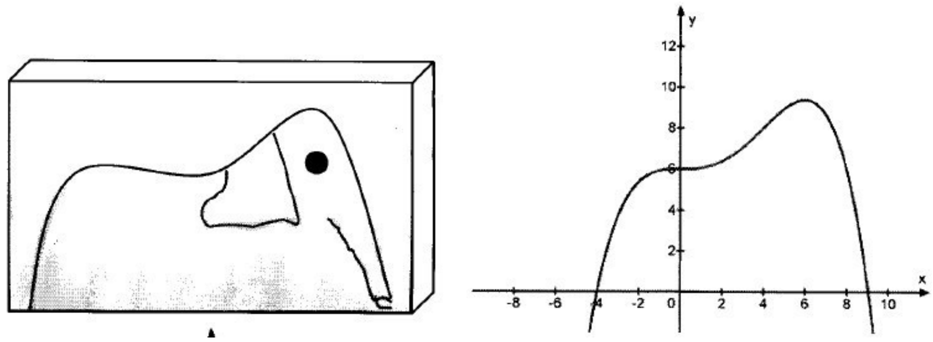

Figure 1: A wooden elephant modelled entirely by polynomial of 4th order.

The story of this problem is to get the need of wood for the elephant. The contour is determined by a given polynomial of 4th order. But no producer of toys will model an elephant with a single polynomial. A similar example is the modelling of an artificial island by a given polynomial of 3rd order (Anonymous, 2013). There are many other examples dealing with unrealistic situations like the trajectory of flying birds following a straight line. Then modelling the trajectory by parameterisations with vectors should give possible meeting points of the birds. It is not convincing that these kinds of problems far away from reality will increase the motivation of students. Modelling focuses often to the process of generating the solution, then calculations are often regarded as secondary and are delegated to the pocket calculator.

In parallel computer algebra pocket calculators (CAS calculator) arose in schools. The use of the CAS calculator was forced by the supervisory school authorities. For some years the school authorities are offering two versions of the centralized final A-level examination (Abitur) in mathematics so that the schools can choose the CAS version or the normal version. In order to prepare the students for the CAS-examination CAS calculator are intensively used at school.

CONSEQUENCES: EXAMPLES

In the following I give some examples of severe mistakes I found in written examinations of a mathematics course for electrical engineering students. Mostly these severe mistakes are known for a long time. But what is new, is the frequent occurrence of them. One of my prominent examples of basic deficits in freshmen mathematical skills was that 4 of 52 Students of mine in a written test without using a pocket calculator stopped at $\sqrt{9}$ and did not simplify it to 3. The concept of a square root seemed to be reduced to the corresponding button of the calculator. Square root was strictly linked to the use of pocket calculator. Of course, all the students knew the value of $\sqrt{9}$, but it seemed that the automatism had gone. This has been my starting point of further investigations and of statistically documenting the mistakes.

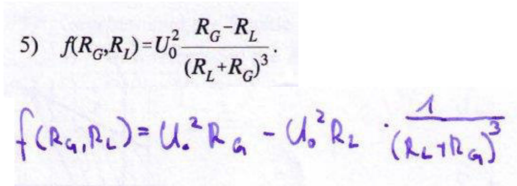

Figure 2: Transforming a fraction into a product

The example of figure 2 shows a missing ability transforming a fraction into a product. The problem was: Build the partial derivative with respect to R_G. 20% of all participants showed this mistake (see figure 2). But only a part of the participants tried to avoid the quotient rule by transforming the term into a product. Even though only this part of all participants could fall into this trap after all 20% (7 of 35) of all participants did not bear in mind that a fraction bar has an effect like brackets.

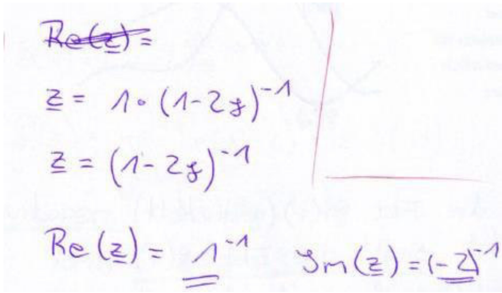

Figure 3: Inverse of a sum

The next example is taken from a quick test (see figure 3). Pocket calculators were not allowed. The problem was to identify the real and the imaginary part of the given complex number. 29% of all participants built the sum of the single inverses instead of looking for the inverse of the sum.

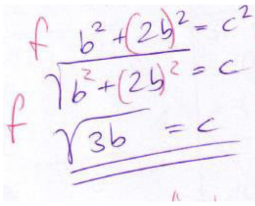

Figure 4: Square of a product

The problem in figure 4 was to calculate the length of the missing edge of a right triangle. One leg of the triangle was given by b and the hypotenuse by $2b$. 39 % that means 14 of 36 of all participants failed in squaring the product. By the way the figure 3 shows other nice mistakes, but these were of singular occurrence.

LOOKING FOR REASONS

An evaluation of the final marks points out that even overall fairly good students do these kind of simple severe mistakes. In general most students have the knowledge to do it correctly. However, if students are confronted with a single mathematical operation they will be able to solve it correctly. But when students have to split their attention to more than one point they get confused. Elsbeth Stern, an expert in cognitive psychology at the ETH Zürich, Switzerland, emphasises the role of practise: “A person, who is not experienced in reading, has to transform every letter into a phoneme and has arduously to construct a word out of it. RAM capacity is occupied that is lost for understanding the content.” (Stern, 2006, translated by Schwenk). Many of our engineering freshmen have comparable difficulties. They are not experienced in reading formulas, they spell the mathematical expressions. Thus RAM capacity is bound that goes lost to capture the meaning and the structure of the expression. Like reading, doing the basics in mathematics has to be automated. Automation releases RAM capacity, this is needed for the creative process of understanding and problem solving and saving new information.

If during mathematics education a CAS pocket calculator is used to early or/and too much this will lead to less routine of elementary mathematical skills. Competencies will not reach an automated level. Spitzer says "The brain permanently adapts itself to its use." (Spitzer, 2012) This means: If students are used to computer algebra systems, they do not really know how to deal with fractions, because they can restrict to copy fractions in an optical way to the CAS computer. Then the CAS computer will analyse the expression not the students. Finally the student's concept of fractions fades. If students use CASystems for differentiations, the computer will differentiate not the students. It is the computer that analyses the function and chooses the right differentiation rule, but not the students. The result is that the students get a lack of understanding of formulas.

Pocket calculators draw graphs of functions and students just copy it. Therefore the concept of functions is reduced to single buttons of the calculator. The difference between the multiplication button and p. e. the cos-button is disappearing. I observed mistakes where applying functions is considered as multiplications. For example the product rule was used for the differentiation of $\cos(2x)$ with one factor to be cos and the second factor to be $2x$. There are many students with a poor developed concept of functions.

An intensive use of computers obstructs students to feel responsible themselves for their knowledge. So it happens, that finally they have linked any square root to the corresponding button und do not simplify simple square roots by their own (see above).

A colleague of mine reported her students felt disadvantaged because of their poor abilities they got from school. At school they had used intensively a CAS calculator. They said they did not understand their input to the calculator but got at school the best mark. They always solved equations by the solve-button, functions were plotted immediately and the graphs were not discussed.

The same change of paradigm of teaching mathematics at German schools has started earlier at the end to the eighties in the Netherlands. Krieg, Verhulst and Walcher reported about a protest of students in the Netherlands against the low level of mathematics teaching at school. The students expressed their alarm in a public letter to the minister of education Maria van der Hoeven, signed by 10 000 students. This letter has become popular under the motto "Lieve Maria". (Krieg, Verhulst, & Walcher, 2008)

As we pointed out at a SEFI conference we also must take into account that student' life and our daily life in general have changed. There is a flood of information. Looking for solutions is now a quick and impatient online search. New devices like digital cameras, mobile phones etc. must be self-explanatory without long instructions. The young generation can be regarded as natives of our digital world, while the older generation, born in the non-digital world, are immigrants. The natives of the digital world are experts in the trial-and-error method and in looking for answers just by one click. But they are not trained for a time consuming, systematic and deductive acquisition of knowledge. They are not well trained to follow formal rules of either a natural language or a mathematical language. (Schwenk & Kalus, 2010)

CHANGE OF PARADIGM OF TEACHING: CRITICISM

Behind the modern way of teaching mathematics lies an underestimation of the importance of the formal calculus and of exercising. Spitzer says (as cited in Siebert, 2003) "Train at first simple but fundamental examples ". Exercising simply problems is a prerequisite for the following abstract level. "Often it is not clearly distinguished between the role of calculating in the context of solving real world problems and the role of learning calculating itself in the context of learning mathematics." (Schröder, 2015, translated by Schwenk).

A Comparison of German school books (7. years of school) demonstrates the dilemma.

4. a) $[(-6)+(-8)]\cdot(-7)$ e) $\frac{-24}{-3}+5\cdot(-6-11)$ i) $\frac{-3\cdot[(+18)+(-6)]}{27:(-3)}$

b) $(-8-4)\cdot(6-2)\cdot(-\frac{2}{3})$ f) $\frac{[-29-(-15)]\cdot(-5+16)}{-6-1}$ j) $\frac{(3-8)\cdot(-25+12)}{[3-(-6)]\cdot(-\frac{5}{18})}$

c) $[(-9)+6]:[-3-(-8)]$ g) $[(-11)-(+14)]:(-\frac{1}{5})$ k) $\frac{(2{,}6-4{,}6)\cdot(-\frac{1}{2})-1}{\frac{3}{7}:(-\frac{18}{35})-2\cdot(-\frac{7}{24})}$

d) $[11-17-(13-2)]\cdot(-8+2)$ h) $(-\frac{1}{2}+4{,}8+\frac{7}{5}):(-\frac{3}{4}:2)$

Figure 5: German school book used in 1992-1999

5 Berechne.
a) $\frac{2}{7}\cdot(-\frac{3}{14}):(-\frac{2}{5})$ b) $(-\frac{4}{3}:\frac{8}{9}):(-\frac{3}{4})$ c) $\frac{3}{2}:(-\frac{9}{4}):\frac{8}{15}$ d) $(-\frac{1}{4})^2:(-\frac{7}{8})$

6 Schreibe zunächst als Term und berechne.
a) Dividiere die Summe aus −5 und 7 durch $\frac{1}{2}$.
b) Wie groß ist der Quotient aus dem Quadrat von $\frac{1}{3}$ und dem Produkt aus 3 und $-\frac{2}{5}$?
c) Subtrahiere den Quotienten aus $-\frac{3}{8}$ und $\frac{1}{4}$ von $-\frac{3}{2}$.

7 Während eines Hochwassers hat Herr Schneider täglich gemessen, wie weit das Wasser unter der Kante der Türschwelle seines Hauses steht: −12 cm; −14 cm; −11 cm; −17 cm; −8 cm. Berechne, wie hoch das Wasser durchschnittlich unter der Schwelle stand.

Figure 6: A German schoolbook used in 2006-2010

The figures 5 and 6 show each the most complicated exercises dealing with fractions of rational numbers. Even though the books are written in German language, it can be easily checked that the problems of the newer book are simpler. The style of the book got nicer. We see more pictures. But I think that practising more complicated fractions of numbers is a good exercise for later algebraic terms with variables, and it is a good exercise to find the appropriate order of the single calculations. Thus the literacy of formulas will be enhanced. If these calculations are delegated to a pocket calculator or even to a computer algebra calculator students will miss exercising how to analyse the expression. They will not know how to start at which point of the fraction. Finally students will not know that a fraction bar has an effect like a bracket. Simple problems prepare for abstractions of the next higher level.

Complaining poor elementary mathematical skills is widely spread among universities in Germany and in other European countries. It was a topic on several conferences on engineering education I attended like the 17th SEFI (European Society for Engineering Education) Mathematics working Group Seminar 2014 at Dublin (for example (Breen, Carr & Prendergast, 2014)), or the 12.

Workshop Mathematik für Ingenieure 2015 at Hamburg (for examples the presentations of Susanne Bellmer, Kathrin Thiele & Gerhard Wagner or Thomas Schramm). This was also confirmed by private discussions on two further conferences in 2014.

One obstructive of the new teaching methods was to avoid instructing useless formal calculus and to force understanding. But it seems what we get is neither formal calculus nor understanding. It seems that an active discussion between teaching staff at school and university has not taken place up to now.

LIST OF FAILED EXPERIMENTS IN GERMANY

Reforms in the education field are experiments in vivo and a favourite playground of politicians. After every election the politicians can easily demonstrate their change management in reforming education. Here are some examples of the past that failed in Germany.

1974 in Germany the set theory was introduced at the primary school. The aim was to improve the structural thinking. But parents got confused; they had not been able to support their children doing the homework. Finally the set theory disappeared from primary school.

Next reform was a reform of the final years of upper schools. Students should learn based on their own responsibility. In 1972 a modular system of basic and advanced courses was established, pupils freely chose the courses. Just the minimum number of advanced courses was fixed. Learning became an exemplary learning. In the years after they went back step by step on these reforms. In 1987 the government introduced the obligatory courses German, mathematics, and a foreign language. In 1995 the responsible ministry tried to improve the preparation for university (A-Level, Abitur) within German, mathematics, and a foreign language. In 1997 after TIMMS the paradigm of teaching changed as described above, it was the starting of CAS calculator in schools. The last withdraw was in 2014. The German state Baden-Württemberg will ban CAS calculators in the final mathematics exams as from the year 2019.

CONCLUSION: NO USE AT ANY PRICE

The use of CAS calculators and computers should be carefully weighed. It is necessary that the students got definitely well trained in pencil and paper skills. Paper and pencil skills should also be practised during the phases of using the computer. If it is not continuously trained there will be a loss of skills; the less the practice is established, the more the skill vanishes.

One of the most important guiding principles should be: if it is not necessary to use a computer, then it is necessary to use no computer (Bauer, 1988).

In schools there should also be enough tests without any electronic devises in order to reinforce the elementary skills. The students must feel responsible for their own knowledge.

It is important, that the use of computers must not substitute pencil-paper skills. For example the computer could be used as a control device of solutions by hand. Afterwards the computer could be used for some big real world problems.

Our summary at the ICTMT 5 is still valid: The “phase of critical reflection is very important to working with computers. Otherwise there are two dangers using computers: 1. The teachers are too enthusiastic about the ‘nice’ facilities of the computers so that the students might not be able to follow their teaching. 2. The problems of or for the students may be covered. Before using computers, the teacher has to consider how to check afterwards (using operationalized aims) that his aims have been reached.” (Schwenk & Berger, 2002)

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DEVELOPING HIGHER ORDER THINKING IN MATHEMATICS: THREE DIFFERENT INQUIRY BASED MODELS IN A DIGITAL ENVIRONMENT

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The importance of the design and the use of tasks with the use of technology is widely acknowledged. The paper presents three different task designs that aim to develop higher order thinking in mathematics with the use of technology. These models were guided by aspects of the Integrated Thinking Model about higher order thinking, features of the inquiry based learning and also utilized pedagogical principles regarding the use of technology. The three models discussed in the paper, approach inquiry based learning in different way. The degree of pedagogical guidance and the sequence of the tasks in the three instructional models are different. The three models are the following: the "open" (open-ended tasks with minimal guidance), the "guided" (guided tasks → open ended tasks) and the "mixed" (open-ended tasks → guided tasks → open-ended tasks). Specific examples of these three environments are presented.

Keywords: Higher order thinking, Inquiry based learning, Task design, Tasks sequences

INTRODUCTION

According to a number of researchers, technology can be used to enhance students' mathematical learning and develop higher order thinking in mathematics (e.g., Jonassen, 2000; Heid & Blume, 2008). In addition to this, Drijvers (2014) concluded based on the reports of recent review studies that "there is a modest support for the claim that the use of digital technology may have a positive effect on student achievement" (p. 24). But at the same time, he underlined that "little is known about decisive factors that explain these effects, or about successful approaches in teaching that may optimize the possible benefits" (Drijvers, 2014, p. 24). In other words, there is a need to identify the best ways to teach with technology. For this, in this paper we propose three different ways to teach with technology for developing higher order thinking in mathematics by utilizing the pedagogical opportunities offered by technology.

Nowadays, a lot of attention is given to the teaching approach of inquiry based learning (e.g., Maaß & Artigue, 2013). There are research results that show that among other technological environments, inquiry based technological environment was the most successful in improving students' mathematical learning (e.g., Eysink et al., 2009). However, it is open to debate which type of inquiry based learning should be implemented in the mathematics classrooms. Setting off from this point, in this paper we propose three different inquiry based models of task design which may facilitate higher order thinking with the use of technology. To this end, we will present a definition of higher order thinking (HOT) in mathematics which we will rely on it to explain how the proposed inquiry based models of task design promoted HOT. In the following, we will provide a brief overview of the literature related to the role of technology and inquiry based learning on the development of HOT in mathematics. Based on these, we will propose three different models of task design and we will exemplify it by presenting examples suitable for elementary school students.

THEORETICAL BACKGROUND

Higher Order Thinking (HOT)

Based on the Integrated Thinking Model (Iowa Department of Education, 1989), for someone to reach HOT, a combination of content/basic knowledge, critical thinking, creative thinking and complex thinking processes are necessary. These four components are considered interrelated and dependent on each other. In the mathematics education literature, there is no model that takes in consideration all the components of the above model to define HOT in mathematics. For example, a number of researchers indicated that creative thinking is part of HOT processes in mathematics in terms of the ability to provide numerous, different and original solutions in mathematical tasks (e.g., Silver, 1997), critical thinking is part of HOT processes in mathematics in terms of logical thinking and the ability to analyse and evaluate examples (e.g., English, 2011) and complex thinking processes is part of HOT processes in mathematics in terms of solving modeling problems (e.g., English, 2011). For this, Sophocleous and Pitta-Pantazi (2015) proposed a definition for HOT in mathematics based on the Integrated Thinking Model (Iowa Department of Education, 1989). Specifically, for someone to achieve HOT in mathematics it is necessary to integrate the above components in the context of mathematics. In particular, basic mathematical knowledge is the mathematical knowledge and understanding which constitutes the basis for an individual to extend his/her mathematical thinking. Critical thinking in mathematics is the ability to analyze, connect and evaluate information based on criteria or logic in order to take a decision or solve a problem. Creative thinking in mathematics can be described as the act of "integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution" (Levav-Waynberg & Leikin, 2009, p. 778). Complex thinking processes in mathematics are those processes required by individuals in order to respond to complex problems. These include a combination of basic mathematical knowledge, critical thinking and creative thinking.

The Role of Technology on the Development of HOT in Mathematics

Using the Integrated Thinking Model by Iowa Department of Education (1989), Jonassen (2000) suggested that the use of technology can promote HOT. At the same time, research results seem to offer further support to this statement, since they suggest that the use of technology promotes aspects of the Integrated Thinking Model in mathematics. For example, a number of studies have shown that technology facilitate students: (a) conceptual understanding in mathematics (Heid & Blume, 2008), (b) critical thinking in mathematics; illustrated through abilities to compare and connect representations (e.g., Pitta-Pantazi, Sophocleous & Christou, 2013), abilities to pose conjectures and examine them (e.g., Fahlgren & Brunstrom, 2014), (c) creative thinking in mathematics; exemplified though students abilities to provide multiple and original solutions (Kordaki, 2014) and (d) complex thinking processes; demonstrated through students work on modelling problems (e.g., Mousoulides, 2013).

Inquiry Based Learning and HOT

A model that can be used to describe clearly the way that technology used to promote HOT is inquiry based learning (e.g., Artigue & Blomhøj, 2013). Inquiry based learning is grounded on the principle that the student is the centre of learning and is asked to work in a similar fashion to that of a mathematician (Maaß & Artigue, 2013). This means that students need to observe a phenomenon, pose questions, search for mathematical ways to solve problems, model situations, search sources and ideas, analyze data, interpret and evaluate solutions, hypothesize, generalize and connect situations for developing mathematical concepts (Maaß & Artigue, 2013). Moreover, teachers in this environment promote students' reasoning, support students thinking and make connections to students' experiences (Maaß & Artigue, 2013).

Different types of inquiry based learning can be found in the literature based on the degree of pedagogical guidance and the amount of information given. One type is the Open inquiry, where students must identify the problem to be solved as well as the solution method to be used. Another type is the Guided inquiry, where the teacher presents the problem and students decide the way to solve it. A third type is the structured inquiry, where the teacher gives the problem as well as the way in which the students should solve it (e.g., Bruder & Prescott, 2013). Despite the wide use of these types of inquire based learning in research, we did not come across any research studies that compared these different types of inquiry based learning in order to promote HOT in mathematics with the use of technology.

PROPOSED MODELS

Based on the lack of the task design models to develop HOT in Mathematics in a Digital Environment and their necessity, in this paper we tried to develop and describe three different models of task design that promote HOT in mathematics with the use of technology.

To construct the three task designs that promote HOT in mathematics with the use of technology we took in consideration the following three factors: (a) content, (b) the way in which technology is used and (c) the degree of pedagogical guidance. Drijvers (2012), found most of these factors to be crucial for the successful or not so successful integration of technology.

The content of the lessons was the same and consequently the lessons had the same learning goals. In particular, the three models of task design aimed at students becoming able to develop content knowledge of specific mathematical concepts and at the same time developing critical, creative and complex thinking in mathematics. The content of the examples that are presented in this paper is the interpretation and construction of distance-time graphs and velocity-time graphs.

The three models are based on the principle of learning with technology, where technological tools were used to "support meaning making by students" (Jonnassen, 2000, p. 8). The three models also emphasised learning which occurs when students know how and why to use the functionalities of the software in mathematics learning (Assude, 2007). We therefore, presented students with tasks where they were learning the functionalities of the software simultaneously with the development of mathematical concepts. The examples that are presented in this paper required from students to use dynamic representations: animation, distance-time graphs, velocity-time graphs, table, given by SimCalc environment (Kaput & Roschelle, 1998) and the functionalities of gizmos applet: Elevator Operator (http://www.explorelearning.com/index.cfm?method=cResource.dspView&ResourceID=1017) to accomplish the aims of the lessons.

Although, the three task designs evolved through the sequence of engagement, exploration, explanation and extension, suggested by the 5Es inquiry based model, the degree of guidance and the sequence of the tasks differed. More details about the structure of each task design are given below. Figure 1 presents the structure of each model.

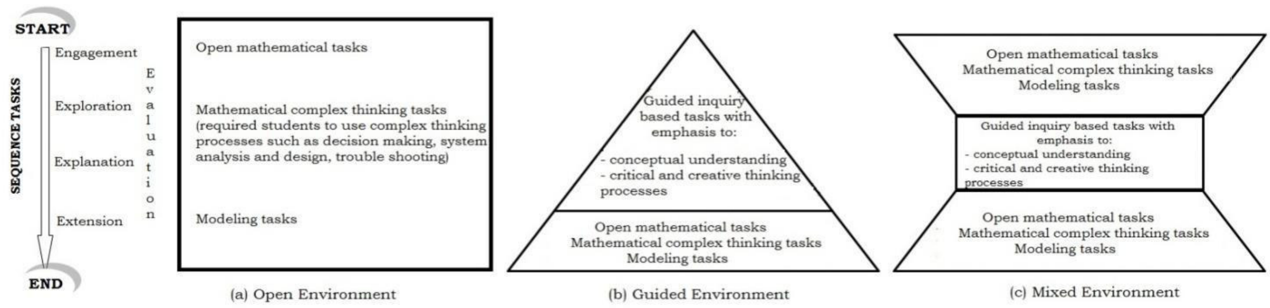
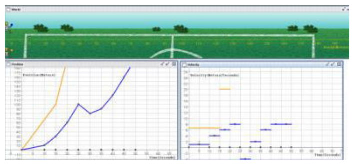


Figure 1. Structure of the three different inquiry based models of task design

Description of "Open Inquiry Based Digital Environment"

The "Open Inquiry based Digital Environment" included only open-ended tasks from the beginning, until the closure of the lesson. The open-ended tasks did not provide students with a direct solution path, but gave all the students the opportunity to work on it and solve it (Foong, 2000). These tasks were also "rich enough to challenge students to reason and think, to go beyond what they expect they can do" (Foong, 2000, p. 52). Moreover, these tasks demanded from students to use the HOT processes (conceptual understanding, critical thinking, creative thinking, complex thinking processes). Modeling tasks, may be considered a form of open-ended tasks, where students are asked to produce a model for a complex situation (Lesh & Zawojewski, 2007). Such environment is similar to Jacobson, Kim, Miao, Shen, and Chavez (2010) "Low-to-low structure" environment. During such environment, minimal guidance is offered to students. An example of this model is given in Figure 2.

Engagement - Exploration – Explanation	
TASKS	***COMMENTS***
Seven runners took part in a race of 200 m. The camera, which recorded the video of the race, has broken and there is no video showing the runner who finished first. However, these are the  comments of the runners. Find the runner who finished the race first by creating the race of each runner in the SimCalc environment, in the file "Runners.2mw" where the race of the two runners is given. Runner A: *It took me total of 65 seconds to finish the race. As I was running, my rate increased. At the 100 m of the race I reduced my velocity, because another runner appeared in front of me.* Runner B: *It took me a total of 45 seconds to finish the race. My rate had its ups and downs, it was not stable.* Runner C: *In the first 5 seconds, I covered 50 m. From the 80 m up to the 120 m my rate reduced, because my leg hurt.* *In a similar fashion Runners, D, E, F, G were described.*	Students worked with an open/complex problem. This problem included exploration of distance-time graphs and velocity-time graphs. Students had to simultaneously work with complex information to find the answer. For instance, students were expected to analyse the data (text, graphs, animation), evaluate them, find connections and differences among them and synthesize the different information in order to design the race of each runner (thus higher order thinking processes).

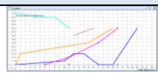
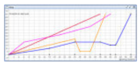
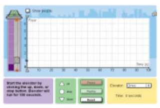
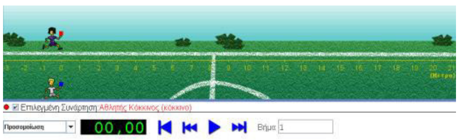
Elaboration / Extension	
A. Create a story based on the following distance-time graph.  **B.** Open the file "cars.2mw" and explain the way the blue runner can come first.  **C.** Create your own story in the SimCalc environment. Try to create a story that no one else would think of. Try to give, as many stories as possible.	These open tasks offer opportunities to students to apply the knowledge and processes that they used in the previous task and extend them.

Figure 2. An example of "Open inquiry based digital environment"

Description of "Guided Inquiry Based Digital Environment"

The "Guided Inquiry based digital environment" began with activities that were more investigatory and guided and demanded conceptual understanding and usage of critical and creative thinking processes. It then gradually proceeded to activities that demanded more complex thinking skills, from students (such as problems employed in the Open Environment). This sequence of task is similar to what Jacobson et al. (2010) call "High-to-low structure" environment. Moreover, this environment utilizes aspects of the Guided Inquiry Based Learning (Goos, 2004). An example of this task is illustrated in Figure 3.

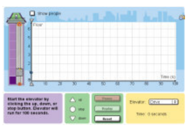
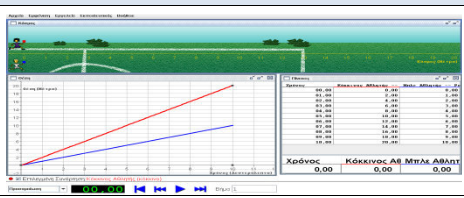
Engagement	
TASKS	***COMMENTS***
THE ELEVATOR Open the application: http://www.explorelearning.com/index.cfm?method=cResource.dspView&ResourceID=1017 and answer the questions.  (a) Press the up button  to start the elevator. After a while press the pause button . What does the graphic representation look like while the elevator moves up? (b) Press the continue button  and follow the stop button , to stop the elevator. What does the graphic representation look like when the elevator stops moving? (c) Press the continue button  and follow the down button . What does the graph look like while the elevator moves down?	Students are asked to answer specific questions regarding the functionality of the gizmos applet. Thus, the questions guided students to investigate and analyze the different forms of movement of an elevator (one variable).
Exploration – Explanation	
RACE Open the file «Runners1.2mw».  It contains a video of two runners, the Red and Blue. 1. Watch the video and answer the following questions. (a) Describe what you see in the video. (b) How long was the race and how many meters did each runner cover? (c) Which of the two runners finished first and how much time did he need? Watch the video and this time activate the option which enables each runner to leave trace behind him while running. (d) Based on the trace that you see, what are the differences between the two runners?	These tasks are part of exploration-explanation phase of this environment. In these tasks students were guided to investigate the meaning of the animation, the table and the distance-time graph given by the SimCalc environment.

<table>
<tr><td>2. In the SimCalc environment activate the table and the distance-time graph. Study them carefully and answer the following questions.
(a) What is the content of the table?
(b) In which second is the runners at the18th m;
(c) How is the table related to the video? What pattern do you see?
(d) What does the distance-time graph show?
(e) How is the distance-time graph related to the table and video?</td><td></td></tr>
<tr><td colspan="2">Elaboration / Extension</td></tr>
<tr><td>A. Using SimCalc construct the following scenario. Once you construct it find which car finishes first and how much time does each car need.
Three cars are participating in a 24 km car race.
▪ The blue car finishes in 15 minutes and is moving with a steady speed.
▪ The pink car while being first in the first 9 minutes slows down and finishes second.
▪ The green car starts from the 2nd km and until the 5th minute it reaches the 4th km. afterwards it stops for 2 minutes due to a technical problem and continues with constant speed of 6 km per minute.
B. Create your own story in the SimCalc environment. Create a story that noone else would can think of. Give, as many different stories as possible.</td><td>These open tasks offer opportunities to students to apply the knowledge and processes that they investigated in the previous tasks and extend them.</td></tr>
</table>

Figure 3. An example of "Guided inquiry based digital environment"

Description of "Mixed Inquiry Based Digital Environment"

The "Mixed inquiry based digital environment" combines characteristics of the two previously mentioned environments and this is why we named it Mixed Environment. This environment relates to Jacobson et al. (2010), "Low-to-high" structure environment. Particularly, it begins with a problem solving situation where students do not directly know the answer but students are expected to employ complex thinking skills. Students solve open-ended tasks or modeling tasks, just like they do in the in the "Open" Environment. Afterwards, students are involved in tasks that are more investigatory and guided (Guided Inquiry Based Learning). At the end students are again involved in answering open-ended tasks. An example of this model is given in Figure 4.

<table>
<tr><td colspan="2">Engagement</td></tr>
<tr><td>TASKS</td><td>COMMENTS</td></tr>
<tr><td>THE ELEVATOR

Open the application:
http://www.explorelearning.com/index.cfm?method=cResource.dspView&ResourceID=1017.
(a) Lead the elevator so that it will transfer the people to the floors they want to go. Write your observations.
(b) Lead the elevator so that it will transfer the people to the floors they want to go, in as little time as possible. Write your observations.</td><td>Students are asked to answer an open-ended task since they don't know anything regarding line graphs that they are asked to develop (by using the functionality of the gizmos applet). This task challenges students' reasoning.</td></tr>
<tr><td colspan="2">Exploration – Explanation</td></tr>
<tr><td>RACE

1. Open the file «Runners1.2mw». In it there is a video, a table and a distance-time graph. Study them and answer the following questions.</td><td>These tasks are part of exploration-explanation phase of this environment. In these tasks students are guided to investigate the</td></tr>
</table>

(a) Describe the content of the video, what you see in it. (b) How many metres was the race that each runner did? (c) Which of athletes finished first and how much time did he need? (d) What is the content of the table? (e) In which way is the table related to the video? Which pattern do you observe? (f) What does the distance-time graph illustrate? (g) In which way is the distance-time graph related to the table and video?	meaning of the animation, the table and the distance-time graph given by the SimCalc environment.
Elaboration / Extension	
A. In the SimCalc environment construct the following scenario. Once you do so, find the one who finished first and the time taken. In a street race of 24 km three cars are racing. ▪ The blue car finishes in 15 minutes and is moving with a steady speed. ▪ The pink car while being first in the first 9 minutes slows down and finishes second. ▪ The green car starts form 2 km and until the 5th minute it arrives at 4 km. afterwards it stops for 2 minutes due to a technical problem and continues with constant speed of 6 km per minute. **B.** Create your own story in the SimCalc environment. Try to create a story that no one else would think of. Try to give, as many different stories as possible.	These open tasks offer opportunities to students to apply the knowledge and processes that they investigate in the previous tasks and extend them.

Figure 4. An example of "Mixed inquiry based digital environment"

CONCLUSIONS

The paper presented three different ways that utilized the opportunities provided by technology to promote HOT in mathematics. It is important to compare different environments with technology and not only to compare use of technology and no use of technology, since the literature emphasised that little is known about the best ways to teach with technology (Drijvers, 2014). In the future, we will try to implement the above task designs, investigate their effectiveness and impact on students' mathematical learning and HOT in mathematics and re-evaluate them in the light of findings.

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POSTERS

THE POTENTIAL OF AUTHORING CREATIVE ELECTRONIC MATHEMATICS BOOKS IN THE MC-SQUARED PROJECT

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The European 'MC-squared' project is fostering several so-called 'Communities of Interest' (CoI) (Fischer, 2001) in a number of European countries. These communities work on designing and developing digital, interactive, creative, mathematics textbooks, called c-books. The c-books are made in the online digital authoring environment in which authors can construct books with various interactive 'widgets'. Here we demonstrate some of the key features of the authoring environment and suggest how c-books can function as a useful catalyst for teacher professional development.

Keywords: mathematical e-books, technology, authoring, creativity

THE MC-SQUARED PROJECT

The MC squared project (http://www.mc2-project.eu) aims to design and develop a new genre of authorable e-book, which we call 'the c-book' (c for creative), extending e-book technologies to include diverse interactive components, learning analytics and collective design. As a research lens, literature from communities of interest (COI) is used (Fischer, 2001). Below we present the key features of the platform and reflect on the first cautious steps of the English COI - offering observations about the role that the project technology is playing in teacher professional development.

KEY FEATURES OF THE C-BOOK PLATFORM

Figure 1 shows how the c-book platform accommodates the authoring of c-books (ie creative books); these are digital mathematics textbooks that consist of pages with carefully designed interactive elements called widgets. The circles at the bottom denote the pages; this particular page to the left has some text, an open answer textbox bottom left and Statistical Representation widget to the right. There are many more widgets ranging from basic ones like equation boxes and multimedia ones to full-fledged manipulatives and micro worlds. This demonstrates one key aspect of c-books: they have interactivity. In addition c-books are responsive in that they can provide feedback to students and teachers (see Figure 2 as well).

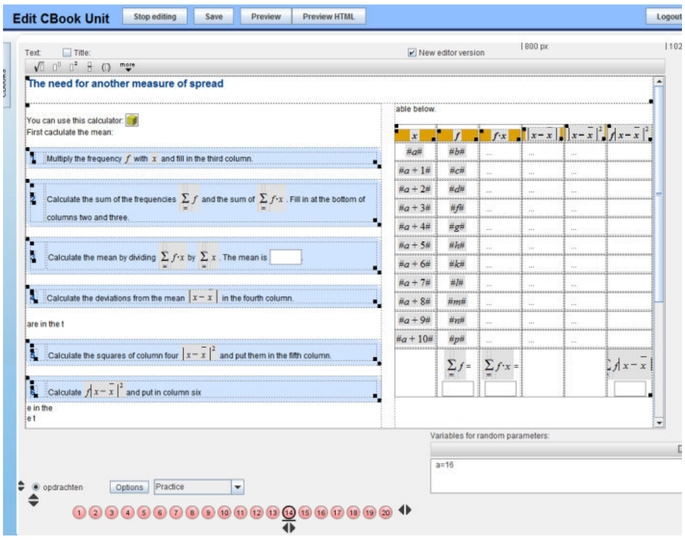

Figure 1: authoring a page of a c-book

The c-book platform is capable of storing results. This means that the complete platform does not only present c-books but also stores student data which is subsequently used for the delivery of useful and appropriate Learning Analytics. C-books can be accessed via the World Wide Web, HTML5 under development, making it possible to use and read the book anytime, anyplace, anywhere with an internet connection. The c-book platform also allows for the authorability of c-books, as already shown in Figure 1. The content of a c-book can easily be changed. The c-book platform has communication features that allow for collaborative work. As the c in a c-book stands for 'creative', the c-book platform is meant to be a catalyst for creativity (see e.g. in Figure 2).

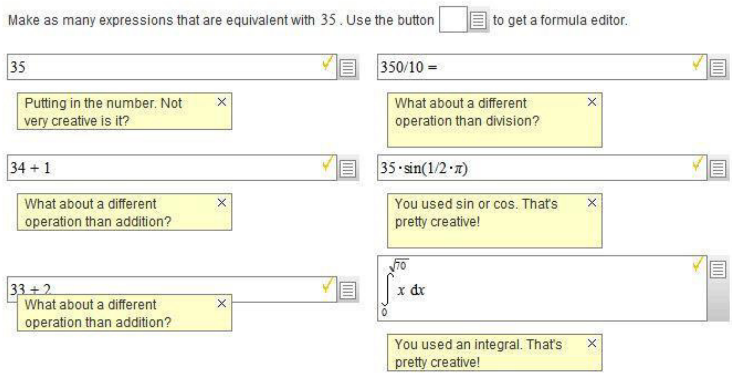

Figure 2: experimenting with creativity and feedback in the MC-squared platform

The two features of authorability and communication enable teachers to use the c-books within the c-book online platform as boundary objects (Akkerman & Bakker, 2011) where researchers, teacher educators and teachers, amongst others, co-design and use resources for teaching. The c-book platform brings together all aforementioned distinct elements in one integrated platform. Although one could arguable say that technologies exist for most of the elements mentioned, the integration of all of these in the c-book platform, to our knowledge, makes it unique.

C-BOOKS FOR PROFESSIONAL DEVELOPMENT

Within the CoI (community of interest) in England we are finding that c-books can function as catalysts to teacher professional development. Within these communities, teachers who co-design and use resources for teaching are contributing to their own professional development in ways suggested by Jaworski (2006). We are observing that c-books and the MC-squared technology might provide a useful focus of the overlap of domain, pedagogical and technical knowledge that Mishra and Koehler (2006) refer to as technological pedagogical content knowledge (TPACK). The development entails appreciating that only by engaging in design activities, teachers in the project can develop a better understanding of the relationships between technology, pedagogy, and the content being taught.

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TABLETS IN THE CLASSROOM

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This poster aims to present some instructional aspects related to teaching and learning of mathematics in the first cycle of basic education with the use of mobile technologies, such as tablets. In Basic School of Marchil (Portugal), the use of tablet's aims to provide students with an extension of the subject taught by the teacher and developing skills and competences that are required for inclusion in our modern society in constant change.

Keywords: Mathematics, instructional practices, tablet, technologies, knowledge

INTRODUCTION

Moran (2014) refers that the traditional classroom is becoming a suffocating place for all, especially for young people. Children and young people feel unsatisfied, teachers feel stressed because there are deeper issues that require new pedagogical projects. We seem to insist on an outdated model, centralized, authoritarian with underpaid teachers struggling to teach a number of subjects that students do not value. If we do not change direction quickly, the school will become uninteresting.

TEACHER'S PRACTICE IN CLASSROOM

This poster aims to report on how the lessons are developed in a classroom of third grade (8-9 years old) at Basic School of Marchil, in Faro, Portugal. In an attempt to innovate in this school, the work with the tablets began in October 2014 as many of students were buying their equipment over the past few months. Today, all of the twenty five students in the class already have tablets. Just one of them was purchased by the school.

Rules for the use of tablets and parent participation

At the beginning of the school year, at a meeting with the parents, we decided that the preferred day for the use of tablets would be Friday in the afternoon, when the students have their weekly plan, are given time for autonomous projects and activities; each student should be responsible for charging the battery of his/her own tablet previously, in order to work without interruptions; at school, the tablet could only be used in the classroom environment to reduce the risks of accidents.

Tasks suggested to the students and used apps

All the tasks suggested to the students aim to have a connection with the contents of all subjects: Portuguese language, mathematics and environmental studies. In the case of the two students with special educational needs, the applications MyScript Calculator, King of Math, Math Duel has been used for the training of calculation, since it allows to work on calculations in a very attractive and natural way, facilitating a more autonomous work and, at the same time, the support of the teacher and of other students.

Sometimes, students who finish their work sooner, use the tablets to formulate mathematical problems on other days than only Friday. In this case, students use a QR code reader to access a Google form (this form is available at http://goo.gl/KLVKXl) and thus they create their own mathematical problems. These problems were shared with their colleagues in the same class and with students from another school with whom they exchange ideas and communication. Along with the use of standard forms of creating mathematical problems, the teacher has also customized excel files for the training of mental calculation.

Productions of the students and the most relevant learning

After six months of using the tablet, students have been learning to use the tablet not only for entertainment, through games, but also they have learned to send emails, to make video conferences, researching different information: dictionaries, images, text, and so on. They learned the risks of surfing the internet and through social networks, such as photo sharing, publishing messages and through the use of tools such as Google forms, the murals of Padlet, the Excel, QR codes and other applications. They have developed writing skills and mathematical thinking and explored the capabilities of their tablets for educational purposes and for their personal development, which is increasingly linked to technologies. Students say often: I love Friday!, Friday is my preferred day!.

CONCLUDING REMARKS

At the beginning of the project there were some difficulties, particularly because of the different devices with different operating systems (Android and iOS) and different software versions. Students took their tablets to school full of installed software but essentially games. And all the students just wanted to play, ignoring all the other capabilities and features of the tablets. We had to create Google accounts for each student in order to facilitate communication. After overcoming the first difficulties, the students began to learn some ways to set their tablets: connect to the wireless network, create and add an account, create a password to initialize the tablet and organize the desktop.

The following are some parents' statements about the use of tablets in the classroom:

Parent 1: The use of tablets in the classroom is a good thing and should be encouraged. The tablet can be a learning tool, even for some contents of the curriculum, making them more attractive to students, primarily in mathematics and Portuguese. This tool should be taught to be used and not be seen only as a "game deck" or "youtube videos". I advocate that the use of tablets should be controlled and should be determined by the teacher, such as the time for their use, as the use on Friday. …

Parent 2: I believe that the use of tablets the use in the classroom, in a controlled manner, brings benefits to our children. These instruments are increasingly used as working instruments in companies and it is necessary that this generation feels comfortable with these new technologies.

Taking into account the gains: increased motivation, more learning, either in mathematics or in Portuguese language, development of knowledge about the technologies, we can consider the use of technology, in general, and tablets, in particular, as an advantage for the development of students in our constantly evolving society. In the future, we intend to continue to use the tablet in the classroom, as we are doing now, but developing some other skills, producing videos, spreadsheets, presentations using PowerPoint and Prezi application to work with colleagues from various countries: Austria, Malta, Turkey, France, Ireland and Cyprus in an Erasmus + project entitled: From blackboard to whiteboard.

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USING DIGITAL TOOLS FOR COLLABORATIVE VISUALIZATION OF INTEGRALS BY ENGINEERING STUDENTS

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This poster introduces a research which aims to answer questions on how engineering students can collaborate using a digital visualization tool, and how the tool can contribute in the learning process at university level. In an empirical investigation students will be given designed mathematical tasks in which they can use a digital tool to explore different aspects of integrals.

Keywords: Collaboration, Digital visualization tool, Engineering Education, Integrals

INTRODUCTION

All engineering students need to learn mathematics during their studies. Hence, research on the learning of mathematics is very important. The growing field of research in mathematics education for engineers reveals that many engineering students have a challenge to apply learnt content to new problems, fail their exams, or even drop out of their studies (Carpenter & Lehrer, 1999; Felder & Brent, 2005; Speer, Smith III, & Horvath, 2010).

Two important concepts within the study of engineering mathematics at university level are “Indefinite integration” and “Definite integration, applications to areas and volumes”. According to the European Society for Engineering Education (SEFI) curriculum framework (Alpers et al., 2013), it is necessary to cover these concepts during the first year of a university engineering course. Although researchers are currently investigating how students understand the basic concepts of first -year calculus, including the integral, much is still unknown regarding the cognitive resources that students hold and draw on when thinking about the integral, according to Jones (2013). We want to contribute to this field of research, but with a focus on the use of digital tools.

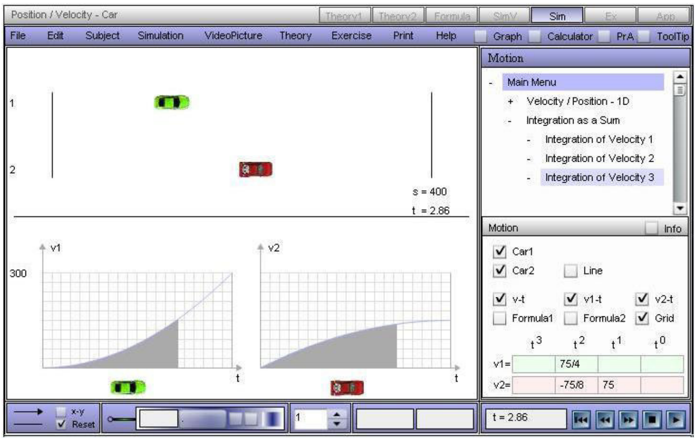

Figure 1. Interactive visualization of integrals

Digital tools have found their way into university’s classrooms and lecturers need to incorporate them in a useful and efficient way. This development has brought new research foci, such as the use of digital tools for visualizations, both in learning and in teaching (Goos, Galbraith, Renshaw, & Geiger, 2003; Presmeg, 2006; Swidan & Yerushalmy, 2014; Yerushalmy & Swidan, 2012). Studies show that students are struggling, and some of the reasons may be that they use digital tools in inappropriate ways or that the tools are ineffective for the learning of mathematics. In our study, we will look into the interplay of collaboration among students and digital visualizations of calculus concepts. We will use SimReal+, which is a digital tool developed at our university and used in engineering courses. It contains interactive visualizations of mathematical concepts, simulations of physical phenomena and instructional videos. Figure 1 shows an interactive visualization of a situated, implicit integral: given a velocity function, the position of the car is the area under the graph, which can be calculated with an integral.

METHODS

In our study we will create a controlled environment for students to work with a digital tool, investigate how students work collaboratively using the tool, how they communicate with each other and in what ways visualizations and collaborations can interact and be beneficial for their learning process. Students will get designed tasks for group work. There will be interactivity between students and the tool, and among students, while working on tasks in which they explore different aspects of integrals. The participants will be selected among engineering students in their bachelor degree program at University of Agder (campus Grimstad).

Both qualitative and quantitative methods will be included, such as observations of students' activity and a survey. At this stage, we don't have any results to present. Our poster will explain the tool and how it will be used in our research.

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FACIAL EXPRESSION ANALYSIS AS A DATA ANALYSIS METHODOLOGY

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Communication involves both verbal and nonverbal means to ensure a message is conveyed. Facial expression is acknowledged in contemporary views as one the most influential factors in non-verbal communication. This study investigates the use of the Facial Analysis Coding System (FACS) as a tool that can improve researchers' skills in reading the emotions of others to analyse data other than depending merely on the spoken language.

Key words: Facial expression, emotions, non-verbal, FACS

LITERATURE REVIEW

Facial expression analysis has been an active research topic for behavioural scientists since the work of Darwin in 1872. In recent years much progress has been made in the construction of computer systems that can aid understanding of this natural form of human communication. This study uses the Facial Action Coding System (FACS) which breaks down facial expressions into distinctively measurable units, providing a tangible approach to investigating the causes and effects of facial expressions (Ekman, 2006). FACS is, however, labour-intensive, susceptible to problems of inter-coder reliability, and limits the usage of data. Moreover, research to date neither presents a clear model of how to interpret this nonverbal communication, nor gives guidelines for researchers on how to use it to help answer their research questions (Onwuegbuzie *et al*., 2011). Saneiro *et al*., (2014) have used FACS to detect emotions in students' interactions while solving mathematics problems. Their analysis shows that participants' emotional reactions in a real learning scenario seem to be influenced by the duration of the task, its difficulty level, and the valence and arousal levels reported. All these factors have an impact on the facial expressions and body movements observable when learning tasks that involve cognitive processes, such as mathematics, are being carried out. The data collected in this research on teachers' perceptions of student-centred learning in the online environment, is qualitative. These data are based on verbal-communication with respondents via interviews and focus groups. To minimise the risk of researcher bias in this study, data were collected on the participants' non-verbal communication using audio and video recordings of the interviews and focus groups.

METHODOS

The method of this study focuses on the analysis of six facial expressions which correspond to distinct universal emotions: disgust, sadness, happiness, fear, anger, and surprise. Cues for facial expression are generalized, as suggested by Ekman (2006), using only "macro" facial expressions which usually last between half a second and four seconds. The software used for extracting and representing these cues is nVISO. Generally, the use of FACS involves four stages: face acquisition, facial data extraction, and representation, and facial expression recognition. A Faces Coding Sheet (FCS) created by Kring and Sloan (2007) was used to recognise facial expressions. This sheet is based on the following steps: (1) Expression (the result of first and second step) (2) Frequency (3) Valence (positive or negative feeling) (4) Intensity (low-Fairly low- Medium-Fairly high- High). Frequencies and durations of expressions were totalled and recorded according to Kring's and Sloan's (2007) method. An example of the applied method is explained in the next table.

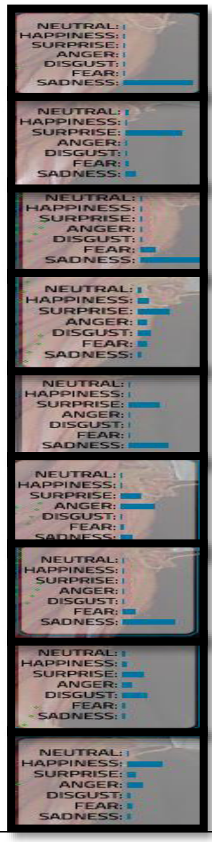
Q: What are your concerns as online tutor?			
A: Interviewee (online tutor):	Extracted Emotions	nViso result	Final result
I am always thinking of what is coming	Sadness		This script lasts for about a minute. It extracts a tutor's feelings about being asked by students a question for which he does not know the answer. Faces Coding Sheet (FCS) shows that: This is a predominant negative feeling of sadness. The highly frequent positive feeling is surprise, in short duration (<50% of bar length). Happiness is extracted once, in short duration (nearly 50% of bar length), when the tutor emailed the answer to students in due time.
that being asked by my students a question	Surprise		
that I have no answer	Sadness-Fear		
Simply, I will reply	Surprised-Disgust-Anger-Fear		
Guys, I do not know the answer	Surprise-Sadness		
Tomorrow, early morning	Anger-Surprise-Sadness		
I will email it to you	Sadness-Fear		
I do not sleep till I search for the answer	Surprise-Disgust-Anger		
and email it to them in due time.	Happiness-Anger-Surprise		

Table1. The applied methods to analyse facial expressions

CONCLUSION

FACS can assist in research to improve the reliability and trustworthiness of collected data by providing information from facial expressions that supplements the verbal message. This is particularly important in analysing affective dimensions of engagement with mathematics. However, despite efforts towards evaluation standards, there is still a need for more standardised evaluation procedures. Due to the constantly increasing interest in applications for human behaviour analysis, and technologies for human-machine communication and multimedia retrieval, this is a rapidly growing field of research to which this presentation seeks to make a contribution.

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DIGITAL RESOURCE QUALITY AND EVALUATION: A PRE-SERVICE TEACHER EXPERIENCE

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This poster briefly describes a pre-service teacher experiment concerning the development of quality assessment criteria of digital resources, as a way to get them involved with activities related to the contributions of the technology in mathematics education and its use in classroom.

Keywords: Digital resource, quality assessment, pre-service teachers.

INTRODUCTION

Although recommended in curricula and official texts of a number of countries and institutional efforts of accessibility to equipment and software, integration of digital resources in mathematics teacher practices can still be considered marginal, at all educational levels (Artigue, 2010). The large number of computational resources available on the Internet makes it difficult to identify relevant resources, quality and the needs to adapt them to a particular context of education. In addition, the present offer does not solve the problem of ownership that needs a change in teachers' skills and their representations of the role of technology in learning mathematics. In fact, for the use of computer technology in the classroom, it is necessary, concerning the teacher side, a pedagogical knowledge of such use. Based on the results of a study conducted in the European project *InterGeo* (www.i2geo.net) – which aimed offering to mathematics teachers of secondary education in different European countries, digital resources of Dynamic Geometry (DG) of "good quality", freely usable and widely adaptable to a particular context of education (Trgalová & Jahn, 2013) – an experience with future teachers of Mathematics was developed to identify their point of views on the quality of digital resources available on the web. This paper reports a part of that experience.

THE EXPERIMENT CONTEXT

It was carried out various activities with a group of 17 Mathematics undergraduates, in the context of the discipline entitled "Analysis of Teaching Texts". In particular, they were organized in groups, in order to face the following tasks: i) Identification of criteria for assessing the quality and potential of digital resources for the teaching of mathematics; ii) How to adapt a criteria grid, provided by the teacher-researcher[1], by comparing them with the ones previously and spontaneously listed; iii) Reviewing, based on this grid, a digital resource selected by the group; iv) Propositions of changes aimed at improving the quality of the resource.

SOME STUDENT COMMENTS & PROPOSALS

In the previous survey related to the analysis criteria of a digital resource, the students pointed out many different analysis aspects (instrumental, the mathematical content, didactic and pedagogic). They are relatively broad or general, and, to be mentioned, strongly influenced by previous work on the development of analysis criteria textbooks of mathematics. To name a few, it describes below partial remarks and proposals of some groups.

Group 1: It should be easily accessible and also motivate the students. Group 3: It should instigate challenges to the students, and present a clear and appropriate language to their levels. Group 4: Correct information, may not have conceptual errors or induce errors.

Table 1. Criteria appointed initially by student

After the confrontation with the questionnaire that was suggested by the class teacher, the students have commented some difficulties and perceptions of the presented criteria.

S3: In the beginning, it was not easy to think about the criteria ... Well, it remains difficult, but now, with the proposed questionnaire, it is already possible to think about some proposals and to understand how they are important. S7: Participation in the group is essential. Each one gives his contributions... I had not even thought about the suggestions or follow-up of my colleagues and about many questionnaire criteria.

Table 2. Student's perceptions of the evaluation questionnaire

With the evaluation of resources selected by the group and the related discussion, we can observe some changes on the students' understanding of criteria and aspects of the use of digital technologies in the classroom. Motivated by the questionnaire content, some didactic aspects were emphasized in the analysis of the students (see Table 3).

Group 1: But "what if the students are not able to build"? "… it should be described the possible difficulties of the students, and give further guidance to the teacher". Group 3: The activity is interesting but has a lot prerequisite. Maybe it takes several classes to prepare students.

Table 3. Identification of didactic aspects

The contributions of a particular resource, towards integration with functional improvement or task redesign, were one of the aspects emphasized by students. Many groups show have mentioned that in their analysis.

Group 2: Use of the software is peripheral, "only at the end". We propose the construction of regular polygons since the beginning and its decomposition into triangles. This construction can be helpful when the considerations about the deduction of the formula. Group 4: "It is not only the use to be used". The use makes the difference!

Table 4. Perceptions of the potential of technology

FINAL REMARKS

This experience has shown that involving students in developing criteria and quality assessment of digital resources, assumes an important role concerning the activities related to the use and contributions of technology in mathematics education, though not immediate. The training potential of this type of activity can be extended to situations of effective use of resources by students, complementing the analysis a priori with reanalysis after use in the classroom. This is an expected step in the second phase of the experiment.

NOTES

1. The analysis grid was developed by the author and his students – Ricardo Canale (scholarship of Program "Ensinar com Pesquisa", USP) and Fabian Fontoura – based on the generalization of the evaluation questionnaire criteria of *InterGeo* project and based on models found in the present literature.

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ASPECTS OF SCAFFOLDING IN A WEB-BASED LEARNING SYSTEM FOR CONGRUENCY-BASED PROOFS IN GEOMETRY

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This paper focuses on the pedagogical underpinnings of the design of a web-based learning support system for lower secondary school pupils who are just beginning to learn how to tackle deductive proving in geometry. In particular, we explain how the key features of our web-based system can scaffold learners' learning of congruency-based proofs in geometry.

Keywords: web-based learning system, scaffolding, congruency, geometry

WEB-BASED SYSTEM FOR CONGRUENCY-BASED PROOFS

In an on-going research project we are developing a web-based learning support system (currently available in English, Chinese and Japanese) that is designed for lower secondary school pupils who are just starting to tackle congruency-based proofs in geometry; see Miyazaki et al. (2011, 2013) and our project website: http://www.schoolmath.jp/flowchart_en/home.html

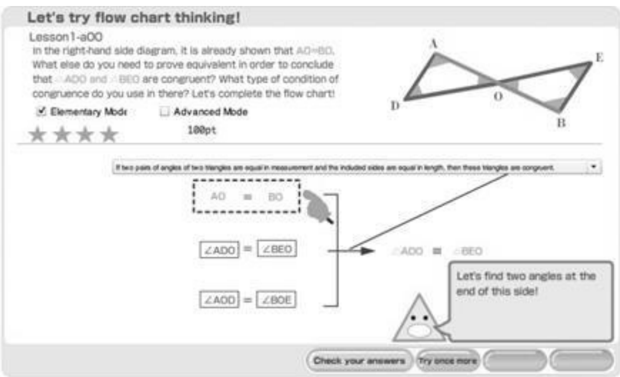

Figure 1: proof task within the web-based learning support system

To design the proof tasks in our web-based learning support system we are using two pedagogical ideas: the format of flow-chart proofs, and forms of 'open problem' proof tasks. A flow-chart proof provides a 'story line' of the proof and have been shown by McMurray (1978) and others to help students to visualize the structure of geometrical proofs. 'Open problem' situations are tasks where students can construct multiple solutions to the proof problem by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion.

When using this learning system, pupils can tackle geometric proof problems by dragging sides, angles and triangles to on-screen cells. As this happens, the web-based system automatically translates the figural elements to their symbolic form. Pupils also select from a choice of congruency conditions. From each set of actions, feedback is provided from the system. In trying to find other solutions in open problems, the pupils can review their previous solutions. Thus the system offers opportunities for students to learn geometric proofs in a way that is different from traditional textbook-based learning.

SCAFFOLDING IN TECHNOLOGY-BASED LEARNING ENVIRONMENTS

Since the introduction of the notion of instructional scaffolding by Wood, Bruner and Ross (1976), recent studies have suggested that technology-based learning environments can function as scaffolding to support learners' learning progression (eg Sherin, Reiser, & Edelson 2004). Here we follow Sharma and Hannafin (2007, p. 29) in considering scaffolding to be "a two-step process of supporting the learner in assuming control of learning and task completion". In the first step the learner is provided with "appropriate support to identify strategies for accomplishing individually-unattainable learning goals or tasks" (ibid); in the second step the assistance gradually fades as the learner becomes increasingly competent. As such, in technology-based learning environments, scaffolding can be conceptualized as "the provision of technology-mediated support to learners as they engage in a specific learning task" (ibid).

SCAFFOLDING FUNCTIONS OF THE OPEN FLOW-CHART PROVING SYSTEM

In terms of technology-mediated support our web-based support system provides the following: a) automatic translation of figural to symbol elements, b) reviewing learner's previous correct answers, c) arranging geometric proof tasks according to their complexity, and d) automatic feedback in accordance of different types of errors.

Using the framework proposed by Sherin, Reiser, & Edelson (2004) we have undertaken an analysis of the scaffolding functions of our web-based system as compared to classrooms without the system. We found that in classroom situations without our web-based learning system, pupils learnt how to construct (simple) formal proofs by following the teaching sequence suggested by an approved textbook. Compared to classrooms with our system, there was less chance to utilise flow-chart proofs and usually less use of 'open problem' tasks. In contrast, when learning with our web-based system, pupils had the opportunity to learn how to construct (simple) geometric proofs through constructing flow-chart proofs in open situations with the technology. By considering the features of the web-based learning system we identified the following 'scaffolding functions:

- Students can drag and drop graphical representations into flow-char box, and the system automatically changes them to symbolic form so that learner can concentrate on logical relationships between each element
- Students can enhance their structural understanding of geometric proofs by visualizing two kinds of deductive reasoning and their combination within the flow-chart proof format
- Students can be encouraged to think forward/backward interactively when constructing a proof using the web-based flow-chart format
- Automatic feedback and reviewing answers enriches learners' thinking by encouraging them to construct different valid proofs for the same 'open' proof problem

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LECTURERS' ATTITUDES TOWARDS INTEGRATING PEN-ENABLED TABLET PCS IN TEACHING ENGINEERING MATHEMATICS: EXPERT VS NOVICE

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The study investigates the attitudes and experiences of two lecturers involved in integrating pen-enabled Tablet PCs (penTPCs) in teaching engineering mathematics. One lecturer has 3 years of experience in using penTPCs (an 'expert'), while the other is a lecturer who is just beginning use of a penTPC in teaching (a 'novice'). Summary of the interviews with them are presented in the paper with some analysis of the similarities between the lecturers.

Keywords: pen-enabled tablet, teaching, mathematics, engineering

INTRODUCTION AND THEORETICAL FRAMEWORKS

In recent years pen-enabled tablet PCs (penTPCs) became a useful educational technology in teaching and assessment especially in engineering and mathematics. They are widely used inside the classroom as a handwriting tool (in conjunction with data projectors) providing greater visibility and flexibility than traditional black/whiteboards and outside the classroom for creating screencasts and for handwritten marking.

Lecturers from the School of Engineering have pioneered the usage of penTPCs within the Auckland University of Technology in teaching engineering and engineering mathematics courses beginning in 2010. Some lecturers have become proficient and have shared their experienced by giving demo-seminars, professional development workshops and conference presentations nationally and internationally. However the use of penTPCs is still far from widespread in teaching even in that school. Recently the mathematics department purchased several penTPCs and some lecturers are about to start using the device in their teaching for the first time. This study involved two lecturers, each representing one of two groups: the first group are experienced users of penTPCs from engineering with at least 3 year experience ('experts') and the second group are lecturers from the department of mathematics who are just about to start using penTPCs in their teaching ('novices'). Our goal was to compare the attitudes and beliefs of the two chosen lecturers towards using penTPCs in teaching engineering/mathematics by conducting and analysing interviews with them. This study records some initial insights of lecturers, within the context of the development of a broader design research approach to the implementation of the use penTPCs in a tertiary teaching environment.

THE STUDY

Both selected lecturers are very experienced lecturers. Lecturer A has a PhD in engineering and more than 25 years of experience in teaching university engineering and mathematics courses. Lecturer B has a PhD in mathematics and over 30 years of experience in teaching university mathematics courses, and engineering mathematics in particular. Both lecturers are males of a similar age. Both are Associate Professors, are very interested in the teaching and learning, and regularly attend teaching/learning forums and seminars and have been winners of several teaching/learning awards. We believe that while their discipline 'content knowledge' and 'pedagogical knowledge' are similar, their 'technology knowledge' is very different (Mishra & Koehler, 2006; Thomas & Hong, 2013). Lecturer A (expert) has extensive experience in many programs such as computer algebra tools and rapid prototyping tools such as Matlab, using them to solve complex industrial problems and in running professional development workshops nationwide. He has been using a penTPC in teaching for 3 years. In contrast, lecturer B (novice) has limited experience in using software in teaching. He has familiarity with the main features of a penTPC and is just about to start using a penTPC in his teaching.

We conducted interviews with the two lecturers to investigate their attitudes and personal beliefs towards integrating penTPCs in the teaching of mathematics based disciplines. In particular, we are interested in their reasons for adoption of penTPCs and possible influence on their teaching style. The excerpts from the interviews are below.

Interviewer: What are your reasons for adopting penTPCs in teaching and which particular aspects of your teaching are enhanced (or expected to be enhanced) the most when you use them in teaching?

Lecturer A: By using the Tablet I am actually replicating the best practice in engineering. We are not only teaching about engineering we are actually doing it. No 2 reason is that by writing you keep the students focused and awake. The 3rd point that I think is interesting is that by using a Tablet and by using handwriting there is a level of informality which is beneficial. One of the other things that I do as I teach is of course I am facing the students, I am facing them pretty much all the time. One of the things we do is record everything we are doing and I said earlier that development was key and students learn how to problem solve, and they learn the steps that you go through problem solving. They can replay this in their own time back at home.

Lecturer B: My main reason is the mobility in the classroom and the eye contact. I expect that the use of penTPC would give me an opportunity to face students all the time, regardless of me using slides or writing on the screen. The mobility in the room and facing the students most of the time that will make the learning environment more active, friendly and informal. Another reason is the simplicity of changing from PPT slides to Internet, to MATLAB, to handwriting that gives me the opportunity to use a variety of different representations of concepts; also consistent high quality visibility and zooming for different media that makes the delivery smooth and coherent.

Not unexpectedly, there are many similarities between the views of the novice and those of the expert and it is apparent that the expectations of the novice have been informed by the experiences of the expert(s) who are already using and demonstrating methods of use, and these experiences have been a driver for technology acceptance. Both lecturers see value in the affordances of penTPC that can allow for an improved learning environment that includes removing some barriers between the lecturer and students (by consistently facing their students, enhancing mobility in the room) and providing consistent high quality visibility of material. They also indicated that an improved capability to switch between media may encourage using a wider variety of different representations of concepts, resulting in a modification of their teaching approaches.

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COMPARATIVE CASE STUDIES IN COLLEGIATE MATHEMATICS: TEACHING COLLEGE ALGEBRA COURSES IN HYBRID AND ONLINE FORMS WITH ONLINE INTERACTIVE AND EDUCATION SOFTWARE EMATH

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The increasing demand in online classes and effectiveness of online teaching and learning systems became important factors for most universities including Georgia State University to start offering 100% online courses along with traditional and hybrid courses. This poster is to show an examination of hybrid and online college algebra courses and their effects on student's achievement and attitudes towards College Algebra courses by using eMath online educational system.

Keywords: Comparison Online Traditional Hybrid eMath

GSU COLLEGE ALGEBRA MATH 1111

GSU Math 1111 College Algebra regular classes are running with a modified emporium model that meets in class 1 hour and the computer lab 3 hours per week. The online class has no meeting time. All classes have the same assignments online. The online class uses the discussion board for weekly discussion assignment.

ASSESSMENT SETTINGS

All assessments are online. Assessments counted for class grade include homework, quiz, unit test, final exam, problem solving activity, and discussion (online class only):

Homework: Average 3 per week; the best of unlimited number of attempts counted; solutions viewable immediately after submission or mistakes; concepts, hints, and answer checking available for each question; system-automated personalized remedial features count towards assignment grade; 10% of final grade.

Quiz: Average 3 per week; maximum of 3 attempts; must have 90% in corresponding homework to start; best attempt recorded; 10% of final grade.

Test: 4 unit tests; pre-tests and personalized remedial practice available; 1 attempt; password protected; 40% of final grade.

Final Exam: 1 attempt; password protected; time limited; pre-test and personalized remedial practice available; 25% of final grade

PSA: Maximum of 4 attempts; best attempt recorded; solutions viewable immediately after submission; 5% of final grade.

Discussion: For online classes, students choose a question from an assignment each week and post their solutions on the discussion board; counted in class attendance; total 10% of final grade.

Attendance: For regular classes attendance counts for 2.5% of final grade; lab attendance counts for 7.5% of final grade; total 10% of final grade.

EMATH – INTELLIGENT COURSEWARE FOR ADAPTIVE LEARNING

eMath is an intelligent courseware that provides personalize adaptive learning assistance. eMath can be customized for institute specific content and program. For this course, eMath provided:

Lesson Guides: Corresponding eBook concepts relevant to each lesson. Customizable by the instructor.

Study Guides: Specific to each assignment for homework, quiz, and test practice. Question guides contain sample questions with detailed solutions, answers, and objective and prerequisite concepts. Knowledge guides contain eBook content relevant to the assignment. Automatically generated and toggled on or off by the instructor.

Remedial Options: System generated Recommended Task for each student, based on assignment due dates, assignment performances, and overall knowledge mastery status. Learn and Drill feature specific to assignments in which students can earn credit toward assignments by reviewing and practicing specific concepts; toggled on or off by the instructor. Work on Your Weakness feature allows students to rework missed questions on assignments, rather than re-taking the entire assignment; toggled on or off by the instructor.

Analysis/Reporting: Exportable, editable gradebook. Knowledge mastery reports for overall mastery of class and mastery of specific assignments. Formative assessment reports for each assignment.

SUMMARY DATA

Fall 2014 Weighted Assessment Average (WAA)				
WAA	Regular Class (688 students)		Online Class (46 students)	
90-100	353	51%	13	30%
80-89	205	30%	19	43%
70-79	73	11%	9	20%
60-69	37	5%	4	9%
<60	20	3%	1	2%
Summer 2014 Weighted Assessment Average (WAA)				
WAA	Regular Class (85 students)		Online Class (44 students)	
90-100	28	33%	15	34%
80-89	33	39%	16	36%
70-79	15	18%	7	16%
60-69	2	2%	3	7%
<60	7	8%	3	7%

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THE PROBLEM@WEB PROJECT: DIGITALLY SOLVING AND EXPRESSING PROBLEMS BEYOND THE CLASSROOM

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This poster focuses on the technology strand of the Problem@Web project. We describe part of the results of the project, by particularly considering how the youngsters tackle and solve moderately challenging mathematical problems using the digital tools of their choice in an online problem solving competition.

Keywords: Digital technologies; Mathematical problem solving; Online mathematics

THE PROBLEM@WEB PROJECT

The Problem@Web project (2011-2014) is a research project on which we studied students solving mathematical problems in the context of two online competitions: SUB12 and SUB14. In these competitions the participants tackle and solve moderately challenging mathematical problems using the digital tools of their choice, whenever they decide to use technology, either at their home environment or other places. SUB12 aims at 5th and 6th graders (10-12 year olds) and SUB14 addresses students in 7th and 8th grades (12-14 year olds). Each competition entails two distinct phases: the Qualifying and the Final. The Qualifying phase takes place entirely online, drawing on the competition website for delivering a series of ten mathematical problems, one posted online each fortnight, while the Final is a half-day on-site contest held at the campus of the University of Algarve. During the Qualifying phase the participants send their answers to the problems by e-mail or through the electronic message editor available on the website; in either case they can propose digital solutions and attach any kind of files they wish. For each problem it is explicitly requested an explanation of the participant's thinking to achieve the solution.

The research empirical field of the Problem@Web project is thus based on those two web-based inclusive mathematical competitions, which induce strong digital communicative activity and easily connect with students' homes and lives. By looking at the global context of the competitions SUB12 and SUB14 as a rich multi-faceted environment, the project explored three main strands: technology, creativity and affect in the problem solving activity of young students. In this poster we focus on the first main strand, technology usage, and we aim to study students' ways of thinking and strategies in mathematical problem solving, their forms of representation and expression of mathematical thinking and their use of technology in their problem solving approaches.

MATHEMATICAL PROBLEM SOLVING WITH DIGITAL TECHNOLOGY BEYOND THE CLASSROOM

The use of digital technologies in problem solving offers young students very different ways of learning and doing mathematics. According to Santos-Trigo (2004), the use of technology in problem solving provides an important window for students to observe and examine connections and relationships that become relevant during the solution process. The use of different tools, such as Excel or Geogebra, opens up the possibility of examining situations from perspectives that involve the use of various concepts and resources.

In this study we consider mathematics problem solving as a mathematization activity. Such mathematization processes promote the development of certain conceptual models. In turn, such underlying conceptual models often become expressed in several different representational systems, including new representational elements that are afforded by digital tools, and can be seen as instances of a new digital-mathematical discourse. The activity of solving problems includes expressing mathematical thinking at its core. Rather than separating the solving stage from the reporting stage, we propose that these are two intimately connected aspects of problem solving and that such connection is eventually deeper when the use of digital tools is available to support the expression of thinking. Therefore, descriptions, illustrations, explanations, and all the material incorporated in the final product is actually the path taken for the product to become a product, as argued by Lesh and Doerr:

> …descriptions, explanations, and constructions are not simply processes students use on the way to "producing the answer", and, they are not simply postscripts that students give after the "answer" has been produced. They ARE the most important components of the responses that are needed (Lesh & Doerr, 2003, p. 3).

Problem solving is thus conceptualized as a concurrent process of mathematization and of expressing mathematical thinking or, in a condensed manner, as a solving-and-expressing activity.

METHODS

Given the context of the SUB12 and SUB14 competitions and our research aims, we chose to adopt a qualitative and predominantly interpretative method. We collected all the emails and answers submitted throughout three editions of the competitions, we interviewed several young participants, as well as some former participants, and also mathematics teachers who followed the competitions as well as youngsters' parents and relatives. We performed a qualitative content analysis (Mayring, 2004) for the data analysis. We believe that such kind of analysis is particularly appropriate when dealing with rich and authentic empirical data, in order to understand them in the context where they originated and capture the meaning they convey in that context, namely the online mathematical competitions.

RESULTS

The analyses undertaken have uncovered the aptitudes of young competitors in taking advantage of everyday digital tools and of their representational expressiveness in the construction of a problem solving strategy (Carreira, 2012). Another emerging aspect is the perception of the existence of different degrees of robustness of the solutions submitted, mainly in terms of the strategies that competitors develop, with a particular technological tool, to solve a given problem. In the case of the use of Excel and GeoGebra we propose the identification of different levels of sophistication and robustness of technology-based solutions to the problems, based on an inspection of the characteristics of the tool use and its connection to the conceptual models underlying students' thinking on the problems.

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HANDLING 3D GRAPHIC OBJECTS DIRECTLY FOR THE LEARNING OF VECTOR EQUATIONS

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In our laboratory we build learning contents for vector equations of lines or planes in three dimensional (3D) space on the web, and students operate 3D object in this virtual space. In this presentation we show two attempts to simplify the operations of 3D objects in virtual space. The first approach is to improve the method of pointing a position; the second approach is to use a 3D sensing device as an input.

Keywords: 3D graphic object, vector equation, interactive worksheet

BACKGROUND

Linear algebra was a foundation field of mathematics for electrical engineers. Some novice learners, however, found it hard to learn because it was often taught with highly abstract approaches. They were taught from definitions and theorems to manipulations, but all symbolically, without referring to its graphical representations or real world applications. Engineering students found it especially hard to follow these instrumental explanations. In our introductory lesson, we let the students use interactive worksheets (Nishizawa et al., 2014), on which students could manipulate the position or angle of a 3D object and observe interactively changing parameters of the vector equation, before we lectured the relations of graphic and symbolic representations and explained the mechanism.

However, for such learning contents in virtual space, user interface was a bottleneck. Because computer screens and pointing devices such as mice or track-pads were 2D in nature, handling an object in the virtual 3D space required more complicated operations than in a 2D space. Because of this complexity, some students lost the connection of graphic and symbolic representations and the contents lost their advantage.

In this presentation we show two attempts to simplify the operations. The first approach is to improve the method of pointing a position, which is restricted on a plane parallel with the screen. The second approach is to use a 3D sensing device as an input. The user can locate some points in 3D space by holding her hand above the sensing device. The points follow the movement of her fingertips.

MORE DIRECT INTERFACE

The first approach is improvement of point motion direction. In the current system, handled point moves on a plane which is parallel with xy-plane. In this method points can also move toward the depth direction, and students are hard to recognize the movement. New system sets a virtual plane in 3D-virtual space which is parallel with the computer screen, and this virtual plane restricts the location of the point. Selected point does not move to the depth direction. In the new system, users feel less conflict between points the actual and their perception. If they want to move the point out of the virtual plane, they could rotate the whole virtual space to see the object from different viewpoint so the virtual plane changes its angle.

The second approach is to use a 3D positioning sensor: Leap Motion Controller (2015) as an input device. This sensor could detect the positon and state of hands and fingers, and we developed new system with this sensor. The user locates a plane in the 3D space by holding her hand above the sensor. When she moves her hand vertically, the plane in virtual space moves upward or downward. If she tilts her hand, the plane changes its angle to the xy-plane. It gives her the perception that she is controlling the object directly.

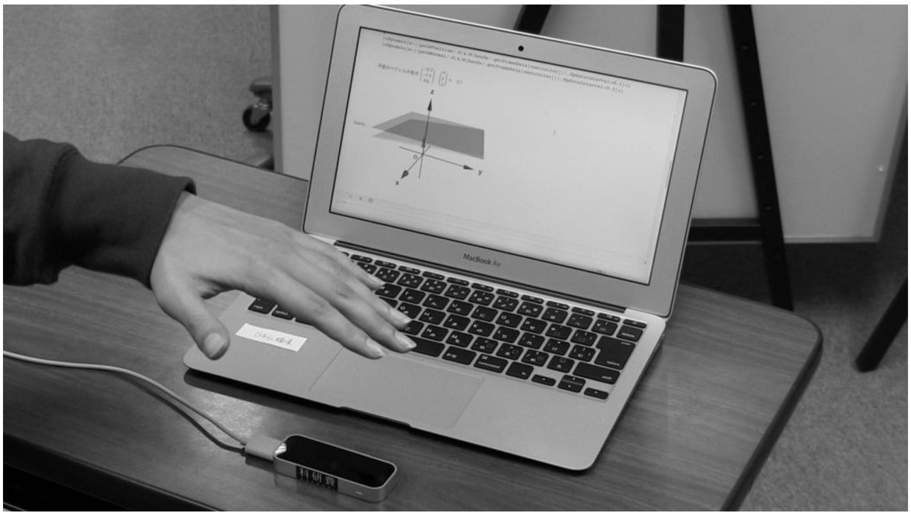

Figure 1. Controlling a 3D graphic object in virtual space with Leap Motion Controller

The first approach has an advantage when we want to use the learning contents embedded into Web pages, because it does not require any additional hardware. Second one is more direct but hard to apply to Web-based learning systems. It is to be used for stand-alone systems.

The second approach is quite useful for the introduction of studying planes in 3D space, for example in studying the relation between a sphere and the tangent plane with the sphere. The content displays an origin-centred sphere and a plane tangent with the sphere in virtual space. Students can control the plane with their hand. When they change the angle of plane, the sphere keeps the same position and distance to the origin. This experience would be linked to the graphical interpretation of inner product of two vectors, the lack of which was suggested to be a bottleneck understanding vector equations of planes for slow learners.

CONCLUSIONS

We examined two approaches to improve input interface for the interactive worksheets, on which students could manipulate the position or angle of a 3D object and observe interactively changing parameters of the vector equation. They were more direct than current interface and expected to help students to visualize the 3D object and possibly to understand the connection of graphic and symbolic representations more easily.

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SOLVING MATHEMATICAL PROBLEMS ON THE SOCIAL NETWORK FACEBOOK – A CASE STUDY

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The aim of this study is to investigate the support that Facebook can provide in the development of mathematical skills with regard to problem-solving. Following that premise, a case study was developed with students of the bachelor degree in Social Education. The results show that the participation in the group contributed to the learning process and, in particular, to problem solving.

Key words: Problem solving; Facebook; Teaching and learning; Collaboration.

INTRODUCTION

Currently the student's learning process stands out as a very fast development product, because at the time of the student's entry in the school he/she already has a set of informal knowledge of mathematics which cannot be neglected. In higher education, with the implementation of the Bologna process, practices have changed and the way teaching is perceived has changed, with among others, tutorial orientation, thus enabling the "teaching to be centred on the skills acquired by the student, and not only in the content of the subjects" (Correia, Lopes, & Nunes, 2006, p.14).

Thus, currently the core of the teaching-learning process puts aside the notion of pre- established knowledge to open the way to problem solving and research tasks. This will give the students more awareness of their mathematical skills, giving them competencies to learn new content by themselves (Henriques, 2010). Ponte, Matos & Abrantes (1998) stress that the student is increasingly the creator of his/her own knowledge.

So, we want our work to foster the collaboration and creativity with the support of social networking, since this can generate a sharing space that promotes the teacher-student and student-student communication. The possibility of the student registers their mathematical thinking and sharing it with other colleagues can make social networks add value in the context of the Mathematical Education.

OBJECTIVES

This research wants to contribute to the knowledge of the social networks' potential in the teaching and learning of Mathematics, as well as, to seek the benefits of carrying out mathematical problems through a Social Network.

METHODOLOGY

In order to apply a methodological process that responds to the objectives listed in this study, the research is of a descriptive and exploratory nature. This is a case study since the investigator will not interfere or modify the situation but try to understand it in its origin. Thus, we do not aim to meet the general characteristics of the entire population, but to understand the specific features of a situation (Ponte, 1994).

Our purpose is to unravel how social networks, especially Facebook, enhances learning strategies to solve mathematical problems. In order to achieve that, interviews were carried out and its content was analysed.

STUDY

A profile named Macs.ESocial was created on the social network Facebook, which was joined by all students enrolled in the curricular unit Mathematics Applied to Social Sciences. The aim of this profile was to promote the resolution of mathematical problems or related topics, where the majority of students had expressed greater difficulties.

The teacher/researcher started by sharing a problem on a weekly basis to be solved by students in the mentioned social network. However, the students themselves soon took the initiative of sharing problems to be solved by their classmates. Most of them were participating by commenting the proposed problems or every doubt pointed out by a classmate. Soon the idea of posting just one weekly problem faded and the need of daily checking the contributions of the students and to show more problems to be solved for those involved emerged.

RESULTS OBTAINED

In general, we know that transformations generate expectations, difficulties and denials in those who suffer them. We clearly felt an initial resistance. One student said "at the beginning, many people complained, because it was something that we had never done before". This mistrust was originated, in our opinion, by the strong ties that the educational system establishes with the traditional view of education, i.e., with a standardized learning with goals, objectives and competencies. In our perspective, the group created on Facebook made the development of creativity possible, the aptitude of written communication, the critical spirit towards a result obtained, the acquisition of mathematical concepts and the overcome of negative expectations that university student's show in relation to Mathematics.

FINAL CONSIDERATIONS

Nowadays it becomes imperative that educational institutions understand how social networks can work as an auxiliary teaching methodology which allows and generates learning.

It is our purpose to promote the training of capable, critical people who know how to mobilize knowledge aiming at good decision-making (Serrazina & Oliveira, 2005). The teacher will thus have the ability to build new, dynamic and practical strategies, in such a way that they awaken in the student the pleasure to learn in a weighted and critical way.

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TASK DESIGN WITH GEOGEBRA 3D

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The aim of our poster presentation is to present some examples of task design and analysis, in Euclidian Geometric, using the onto-semiotic approach.

Keywords: Task design, geometry, secondary school

DESIGN AND ANALYSIS OF TASKS

Mathematical tasks are central to students' learning because the tasks convey messages concerning what mathematics is, and what doing mathematics involves. In addition to this, the systematic approach to task design and analysis play critical roles in mathematics education (Artigue, 2009).

Following the ideas of Godino, Batanero, and Font (2007) in design and analysis task it is necessary to take into account six types of primary entities: *Problem situation*; *Language* (e.g., terms, expressions, notations, graphs) in its various registers (e.g., written, oral, sign language); *Concepts* (approached through definitions or descriptions); *Propositions* (statements on concepts); *Procedures* (e.g., algorithms, operations, calculation techniques); *Arguments* (statements used to validate or explain the propositions and procedures, of deductive nature or another type).

In relation to geometric reasoning, Duval (1998) refers three types of cognitive processes that accomplish specific epistemological functions - **visualization** (relating to spatial representation), **construction** (resorting to tools) and **reasoning** (in particular the discursive processes to broaden the processes of knowledge, for demonstration and interpretation). These different processes can be carried out separately and these three types of cognitive processes are intimately connected and their synergy is cognitively necessary for proficiency in geometry. The question we are to answer is: How can we use a GeoGebra3D in the teaching of mathematics productively?

An example of the design and analysis of task is given in Figure 1. A group of 26 students, in 10th grade of Secondary School, solve the task [1]:

> Consider, in referential Oxyz, the sphere defined by: $(x + 1)^2 + (y - 3)^2 + z^2 \leq 16$.
> 1. Write the equations of planes tangent to the sphere, parallel to the plan xOy.
> 2. Find the intersection of the sphere with the planes:
>   2.1. $z = 0$; 2.2. $x = 0$; 2.3. $y = 1$;

The expected solution of question 1 is $z = -4$ and $z = 4$.

For the 2nd question, in the 1st case, it is expected that the students answer: $(x+1)^2 + (y-3)^2 + 0^2 \leq 16 \Leftrightarrow (x+1)^2 + (y-3)^2 \leq 16$, in other words, the intersection of the sphere with the plane $z = 0$ is a circle with center $(-1, 3, 0)$ and radius 4, contained in the plane $z = 0$. In the 2nd case it is expected that the students answer: $(0+1)^2 + (y-3)^2 + z^2 \leq 16 \Leftrightarrow (y-3)^2 + z^2 \leq 15$, so the intersection of the sphere with the plane $x = 0$ is a circle with center $(0, 3, 0)$ and radius $\sqrt{15}$, contained in the plane, $x = 0$. In the

[1]. Adapted from Costa, B., & Rodrigues, E. (2014). Novo Espaço Parte I: Matemática A 10.º Ano (1.ª ed.). Porto: Porto Editora

last case, with plane y = 1, the solution is: $(x+1)^2 + (1-3)^2 + z^2 \leq 16 \Leftrightarrow (y-3)^2 + z^2 \leq 12$, in other words, the intersection of the sphere with the plane y = 1 is a circle with center (-1, 1, 0) and radius $\sqrt{12}$, contained in the plane y = 1.

Let us observe the mathematical objects that take part in solving the problem and their primary relationships: Language (sphere, plans, circle, center of a circle and sphere and points coordinates, radius of the circle); Concepts/Definitions (definition of circle, cartesian equation of the sphere); Procedures (appeal to the viewing situations involving the intersection of spheres with planes parallel to the coordinate planes; algebraic calculus, process visualization → reasoning).

In class, the exercises were followed by the images that make up the Figure 1:

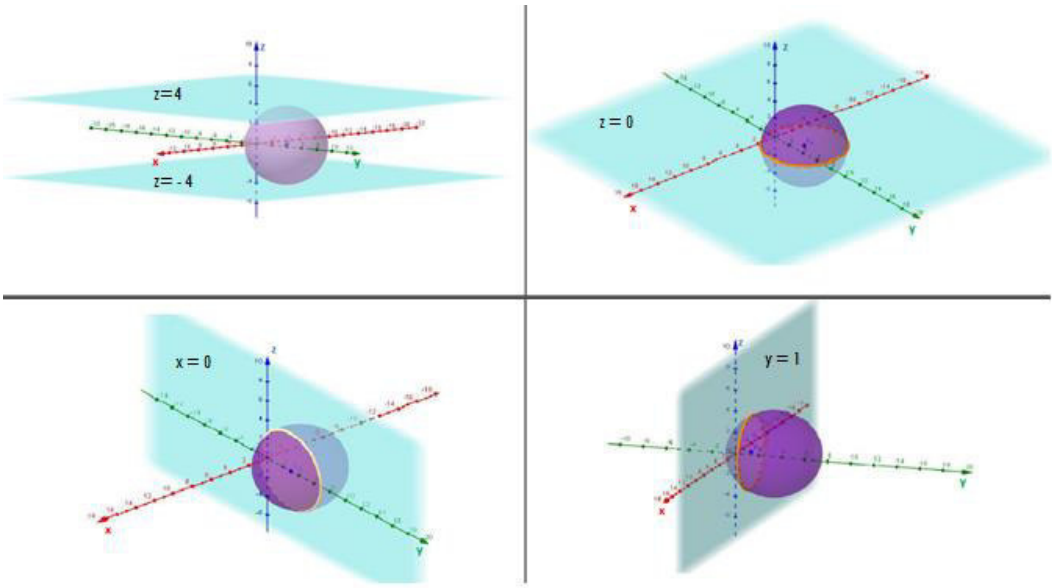

Figure 2. Construction made by resorting to GeoGebra3D

The visualisation supported students' solutions. For example, with the visualization of the intersections of different planes with the sphere, a student said that the center of the circle and the respective radius is not enough to define the circle in question, since it could be in several different planes. So, the student felt the need to state the plane encompassing the circle. This example illustrate that visualization and reasoning are intimately connected and their synergy is cognitively necessary for proficiency in geometry.

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WORKSHOPS

COMBINING REALISTIC MATHEMATICS EDUCATION AND THE BRIDGE21 MODEL FOR THE CREATION OF CONTEXTUALISED MATHEMATICS LEARNING ACTIVITIES

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Bridge21 is a learning model that is supportive of a collaborative, technology-mediated, and inquiry-based approach to education. It challenges conventional models of teaching, empowering students to learn through technology and team-work. Research in Trinity College Dublin has developed a set of design heuristics for mathematics learning activities that integrates this paradigm with that of Realistic Mathematics Education (RME). Attendees will be introduced to the methodology and will actively engage in tasks consistent with the approach.

Keywords: Contextualised Learning; Post-Primary Education; Realistic Mathematics Education

RATIONALE

It has been suggested that, within an appropriate pedagogical framework, the use of technology in the classroom can make mathematics more meaningful, practical, and engaging (Drijvers et al. 2010; Olive et al. 2010). A combination of social constructivist educational theories, which have been shown to align particularly well with the affordances of technology (Bray and Tangney 2013; Patten et al. 2006) and the process of mathematization in Realistic Mathematics Education (RME) (Gravemeijer 1994; van den Heuvel-Panhuizen 2002), has the potential to offer such a framework.

However, activities that combine a technology-mediated, social constructivist and RME approach to mathematics learning do not fit easily into the conventional classroom with its didactic teaching and short class periods (Wijers et al. 2008); so-called 21st Century (21C) learning models may be more appropriate. 21C learning emphasises a student-centred, active approach, with importance being placed on key skills such as collaboration, communication, creativity and problem-solving, as well as on content (Dede 2010; Voogt and Roblin 2012).

BRIDGE21 & RME

Bridge21 (www.bridge21.ie) is a 21C learning model that has been designed to support a collaborative, technology-mediated, and inquiry-based approach to post-primary education (ages 12-18). It challenges conventional models of teaching, empowering students to learn through technology and team-work, while the teacher adopts the roles of 'facilitator' and 'leader' (Johnston et al. 2014). The model has its foundations in socially constructivist learning theories, and draws on the patrol model of the Scouting movement. Our research has shown that implementation of the Bridge21 model in conjunction with an approach to task design consistent with RME (Tangney et al. 2015; Bray et al. in press) has the potential to have a positive impact on students' engagement with and confidence in mathematics. Some of the activities tested to date include Plinko and Probability, the Barbie Bungee and the Human Catapult (Bray et al. 2013, in press Bray and Tangney 2014; Tangney et al. 2015).

THE WORKSHOP

The workshop will follow the CPD model advocated by Bridge 21 and will adhere to the principal of 'learning by doing' – participants will engage in immersive activities, in which they will experience the Bridge21 methodology and model in action. The mathematical content that will be addressed by the activities includes: functions (linear and quadratic); data collection and analysis; line of best fit and extrapolation; and probability and bias. Participants in the workshop will be required to work in groups and should have the free software Kinovea (www.kinovea.org), GeogGebra (www.geogebra.org) and Tracker (www.cabrillo.edu/~dbrown/tracker/) on at least one device per two people. All other equipment will be supplied.

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USING GEOGEBRA TO STUDY COMPLEX FUNCTIONS

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The aim of this workshop to present some of the strategies studied to use GeoGebra in the analysis of complex functions. The proposed tasks focus on complex analysis topics target for students of the 1st year of higher education, which can be easily adapted to pre-university students. In the first part of this workshop we will illustrate how to use the two graphical windows of GeoGebra to represent complex functions of complex variable. The second part will present the use of the dynamic color Geogebra in order to obtain Coloring domains that correspond to the graphic representation of complex functions. Finally, we will use the three-dimensional graphics window in GeoGebra to study the component functions of a complex function. During the workshop will be provided scripts orientation of the different tasks proposed to be held on computers with Geogebra version 5.0 or high.

Keywords: GeoGebra, Complex Functions

INTRODUCTION

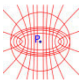
The main objective of this workshop is to lead the participants to recognize how and why the use of the proposal tasks are an added value for the understanding and learning of the mathematical content associated with them. The tasks proposed to the participants are the same as the ones designed for students of the first year of engineering and science courses can and should be used as an educational tool in collaborative learning environments. The main advantage in its use in individual terms is the promotion of the deductive reasoning (conjecture / proof).

The applications presented here can trigger exploration in the different teaching degrees. As an example, it can be used for illustrating the Fundamental Theorem of Algebra for pré- university students. These applications can be viewed as an intermediate step for visualizing and understanding Möbius Transformation and their relation with rigid movements of the Riemannian Sphere.

In addition to the time required for the tasks it is also planned to have an extra time for exchanging ideas about the impact of this type of work with students from different educational degrees and attending different courses.

GRAPHIC WINDOWS AND TWO-DIMENSIONAL REPRESENTATION OF COMPLEX FUNCTIONS

Featuring GeoGebra multiple windows, these can be used simultaneously in order to provide a computational environment that interacts with two Cartesian representations, these two GeoGebra windows are an excellent model for the domain and codomain of a complex function (Breda, A., Trocado, A., & Santos, J. ,2013). This session will be proposed various tasks involving the representation of: a) points in the complex plane, a graphical window, and the image display these points by a complex function, the second graphic window; b) points in certain subsets of the complex plane by analyzing the images of these subsets of a complex function, the second graphics window, using the Locus command, and the Trace function of a point; c) use the following command to create grids in the representation of domain points to a complex function analyzing their images.



Tool image:
Tool name:
EccentricityConcentricGird
Command name:
ECgird
n=Slider[1, 10, 1, 1,100, false, true, false, false]
P=1+i
d=Slider[0, 4, 0.1, 1,100, false, true, false, false]
EC={Circle[P,d],Sequence[(x - real(P))² / (i d / n)² + (y - imaginary(P))² / (1 - 1 / (i d / n)²) = 1, i, 0, n, 1]}

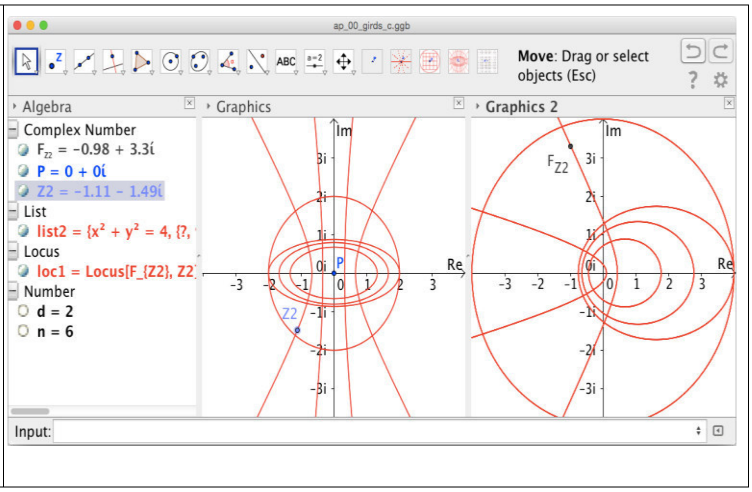
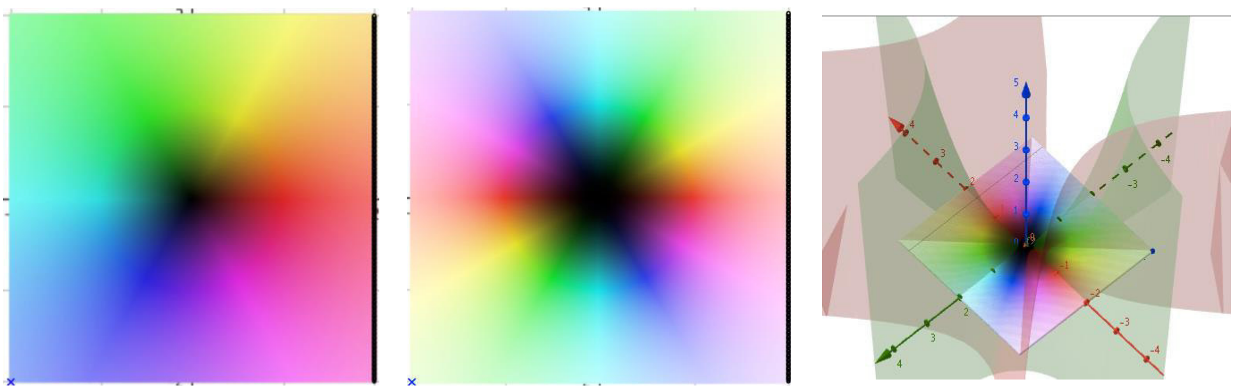


Figure 1. Subset of complex numbers and their image by function $f: C \rightarrow C, f(z)=z^2$.

Figure 1 shows the example code to use to get points on the complex plane a family of conical, represented in the graphics window, as well as the representation of the image of these points by a complex function is in the graphics 2 window of GeoGebra.

Dynamic Colors, Coloring Domains in Representation of the Graphic of a Complex Function in Geogebra

The use of dynamic colors associated with a point allowed Rafael Losada (2009) and Antonio Ribeiro obtain the first representations of fractal images involving complex numbers (Breda, et al, 2013, p. 63). Subsequently, the potential of the dynamic color GeoGebra led to Breda, and Dos Santos (2013) to apply them to obtain color domains thus obtained is the complex variable representation of complex graphic functions (Breda et al, 2013, p. 78).



$I(z) = z \wedge z \in C$ $f(z) = z^2 \wedge z \in C$.

Figure 2. Coloring Domain: identity function, f (z) =z² ∧ z∈ C and components maps of *f*.

This workshop will be presented two essential tasks for Coloring Domains of the complex plane. The first task indicates the strategy for construct a scanner of the plan with GeoGebra, the second adjusting the scanner of plan, developed in the first task, to get the Coloring Domain associated to the graph representation of a complex function, as can be observed two examples the two pictures, on the left, of Figure 2.

3d Graphics Window for the Graphic Representation of Components Maps of a Complex Function

In GeoGebra, the various properties of objects can communicate between different windows and after version 5.0 there is a three-dimensional window, thus we can get the graph representation of each component maps of a complex function. Thus, the last tasks of this workshop will be aimed at graph representation of component maps of a complex function, as well as, analyzing some of its properties. Thus we can simultaneously display various graphical representations (see the third image, the far right of Figure 2) that contribute to improving the visualization of different properties associated with a complex function. In addition, will also be presented, one sample of Coloring Domains applied on the Riemann sphere in order to study Möbius transformations (Breda, A., & Santos, J., 2015).

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THE NUMBER STORIES PROJECT: A DATABASE OF DYNAMIC REAL-WORLD ACTIVITIES

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The Number Stories Project, in which a collection of web-based dynamic mathematics materials is being developed, will be introduced and participants will have the opportunity to explore the online database and a number of activities. We are interested in potential partners to enable us to expand this database to include activities of interest in a wide diversity of international contexts.

Keywords: digital curriculum, mathematics applications, New Cabri apps, real-world, interactive, real-time formative feedback, on-line

THE NUMBER STORIES PROJECT WORKSHOP

The Number Stories Project (CEMSE, 2014) is an exploration of how to build and organize dynamic mathematics materials on the web. Number stories are real-world questions based on real-world contexts supported by factual sources. The screenshot below shows a variety of arithmetic number stories available in the database.

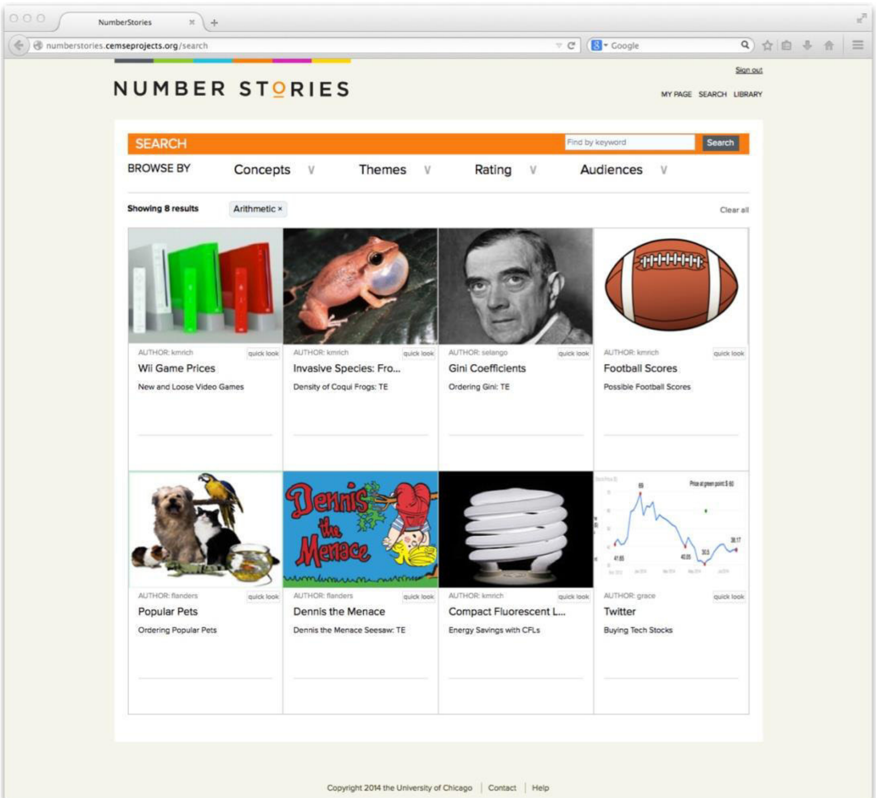

Figure 1. A selection of arithmetic number stories

Both contexts and questions are written for a wide variety of individual users such as school students at any level, teachers, teacher-educators, home-schoolers, district supervisors, curriculum developers, and others who are simply interested in how mathematics can be used to solve problems or model situations in their lives.

Unlike a traditional curriculum project where real-world problems may be used as a means to achieve specific mathematics learning goals, the main aim of the NS project is to promote understanding about how mathematics is used in daily life and to enable solvers to gain in their confidence and ability to apply mathematics in real situations through the use of engaging questions, dynamic activities and the provision of formative feedback. Further, problems within the database may be clustered to develop personalized curricula organized around any criteria such as mathematics concepts, situational themes, and/or user experience.

The *Number Stories* problems are web based. There is no expectation that they be downloaded, printed, and solved with pencil-and-paper techniques. The project is committed to making each problem take advantage of a dynamic mathematics environment, including use of dynamic geometry and algebra tools, links to the source material of contexts and questions, intelligent feedback to user's responses, and the opportunity for each user to rate and review other people's contexts, questions, or solutions. The project is currently using the *New Cabri* authoring application for activity development because we found that it exceeds the capabilities of alternative applications for efficient creation of rich, dynamically-interactive online activities with real-time feedback. However, the Number Stories platform can also host activities written with a range of other software.

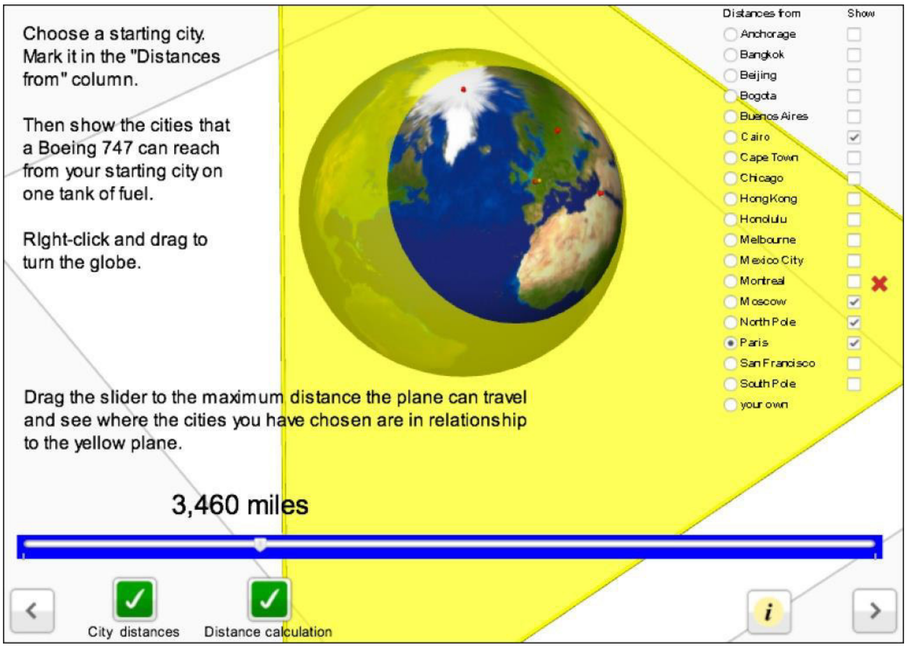

Figure 2. An example of a *Number Story* powered by the *New Cabri*

The aim of this workshop is to introduce the project and to allow participants to explore the online database and a selection of activities.

We would like to expand the database to include problems drawn from as wide a range of international contexts as possible and would welcome potential partners.

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EXPLORING MATHEMATICS THROUGH MULTIPLE REPRESENTATIONS

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Technology provides the ability to view different representations of mathematical concepts simultaneously: graphical, geometric, numeric and possibly symbolic. When dynamically linked multiple representations help students to see connections and build up knowledge step by step.

TI-Nspire technology can also improve algebraic reasoning. Its algebraic – CAS – functionality is interactive and dynamic and it links graphical and algebraic representation in both directions. This makes TI-Nspire technology a real and unique mathematical teaching and learning tool to help students to get a better understanding of abstract math concepts – focusing on the educational value of exploration, in combination of being a calculation & graphing assistant.

Being document based, it gives students the possibility to save their reasoning when e.g. trying to get out of a real world problem a mathematical model, to solve it and to reflect back to the original problem to see what solutions our meaningful. Same arguments count for teaching and demonstrating and even to build up an abstract reasoning to proof a mathematical property or theorem. The document structure, with independent problems and linked pages within a problem, gives students and educators the possibility to save their work in one document, possible aligned to the content of textbooks.

To meet different classroom needs and technology use, TI-Nspire Technology offers the same functionality on handhelds and as computer software or iPad App; with the possibility to exchange documents between these platforms:

- Teachers can use the computer software to create materials for students using handhelds and share it easily with the students in the classroom
- Students can easily transition their work between the handheld and computer and share it with teachers for assessment or to make their homework at a computer at home

TI-Nspire exists out of the following applications – all accessible is one interface:

- Graphs & Geometry
- Calculator
- Notes with interactive math boxes
- Lists & Spreadsheet
- Data & Statistics

together with an integrated Program & Script Editor and a Data Collection application.

Via concrete examples and interactive hands-on activities the participants of the workshop will become acquainted how multiple representations in a document based graphing technology can help getting students more involved in their own learning process and how to guide efficiently in their learning: creating curiosity, trial and error and reflecting their findings.

The following topics will be covered:

- Animations to help concept understanding

  How can scripts integrated in documents help students understanding and linking mathematical concept; from to Fractions from Ratios to linear functions

- Amazing systems of linear equations

  Discover and explain solution patterns when changing parameters of system of linear equations

- Cubic curiosities

  Conjecture and proof surprising properties of standard cubic functions

- Dynamic, Interactive statistical explorations

  How dynamic visualizations help understanding abstract statistical concepts and how to do to collect collaborative classroom data for statistical analyses

During the workshop, the hands-on activities will be done with TI-Nspire CX CAS handheld Technology (no prior knowledge of the technology is needed), in combination with the classroom management system TI-Nspire Navigator.

PART B

PORTUGUESE LANGUAGE CONTRIBUTIONS

PAPERS

Theme: Resources

TECNOLOGIAS DA INFORMAÇÃO E EDUCAÇÃO MATEMÁTICA/ INFORMATION TECHNOLOGY AND MATHEMATICS EDUCATION

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This work aims to discuss the process of integration of technology by the teacher of Mathematics according to the model of innovation from Rogers (2003). This author presents concepts and explanatory categories, which may help understand the elements present in the process by which the teacher goes through from his contact with the initial idea of teaching with the use of information technology, IT, until his confirmation or not of adopting this proposal in his work. Thus, it is necessary an understanding about innovation and educational innovation, which implies an incursion on technological innovation and, ultimately, on the training of the teacher in face of a new pedagogical practice. The model of Rogers (2003) reveals itself as an important contribution to an understanding of the many variables that can play a role in the inclusion of technologies in the pedagogical practice.

Keywords: Innovation, ICT, Teacher Training, Mathematics Education

A INOVAÇÃO NA EDUCAÇÃO

A ideia de inovação pode ser adequada para compreender a situação por que passam os professores ao se depararem com uma "nova" situação de trabalho, na qual a experiência acumulada no ensino é muito importante, porém, insuficiente para atender às novas demandas pedagógicas e tecnológicas.

Crenças e expectativas são elementos que podem surgir na abordagem da inovação, pois os professores, no tempo próprio de cada um, desenvolvem uma relação particular com a inovação e, ao mesmo tempo, ao trabalhar em equipa pode criar uma relação coletiva com a mesma, facilitando a superação de certos receios diante da novidade

Para desenvolver este tema podemos nos orientar pelas seguintes questões: *quais possíveis relações podem ser estabelecidas no processo do uso das tecnologias da informação na Educação Matemática pelo professor e os estágios de Inovação-Decisão do modelo de Rogers (2003)? Que elementos podem contribuir para a adoção do uso das tecnologias na prática pedagógica dos professores?*

A inovação tem sido foco de estudos de pesquisadores em diferentes áreas do conhecimento, inclusive na área educacional. Rogers (2003), da área da sociologia, contribui como referencial teórico para pesquisas na área da educação. Fullan (2009), Fullan e Hargreaves (2000), Cardoso (1997, 1999), Hernandez *et al.* (2000), são autores que discutem aspectos da inovação na área educacional. No Brasil, alguns autores como Saviani (1980), Farias (2006), Teixeira (2010), contribuem com formulações sobre inovação educacional relacionada à formação de professores, com o uso de tecnologia em educação.

No contexto educacional, a ideia de inovação recebe denominações, tais como mudança ou mesmo ruptura de uma prática para outra. No entanto consideramos que não se trata necessariamente de ruptura, pois diante da inovação considerada como o uso das TI, o professor pode reagir procurando se adequar a ela a partir de seus conhecimentos, habilidades, visão de mundo e convicções pedagógicas, estabelecendo novas relações com os elementos inovadores e agregando a eles outros conhecimentos. A inovação, portanto, se estabelece em um contínuo da ação docente.

Para Saviani (1980, p.25), a inovação pode ser entendida de diferentes pontos de vista, dependendo da concepção filosófica adotada: humanista, analítico e dialético e segundo o autor

> De acordo com a concepção "humanista" tradicional, a inovação será entendida de modo acidental, como modificações superficiais que jamais afetam a essência das finalidades e métodos preconizados em educação. Inovar é, pois, sinônimo de retocar superficialmente. De acordo com a concepção "humanista" moderna, inovar será alterar essencialmente os métodos, as formas de educar. Já do ponto de vista "analítico", inovar não será propriamente alterar nem acidental e nem essencialmente. Inovar será utilizar outras formas. [...] Quer dizer, inovação educacional traduz-se pelo uso de outros meios (ou "*media*") que se acrescentam aos meios convencionais, compõem-se com eles ou os substituem. [...] Já para a concepção "dialética", inovar, em sentido próprio, será colocar a educação a serviço de novas finalidades, vale dizer, a serviço da mudança estrutural da sociedade.

Podemos compreender a inovação no sentido "analítico" apresentado por Saviani (1980), pois a prática do professor não é alterada essencialmente, embora incorpore novos elementos e outros meios. Faria (2012, p.12) entende que "a experiência anterior do professor tem uma ressignificação em um novo contexto, ao mesmo tempo em que atribui um sentido próprio à inovação de acordo com a relação que estabelece com ela"

Hernandez *et al.* (2000, p.26) apresentam um estudo sobre inovações nas escolas e uma das principais ideias desenvolvidas é a que relaciona a inovação com as atitudes dos professores. Nesse sentido, afirmam que as múltiplas aproximações da ideia de inovação educativa se situam em um "contínuo", no qual, em um extremo, se encontra o sentido sócio pedagógico da inovação, que se define como 'uma ideia, prática ou material percebido como novo por parte da unidade de adoção pertinente'. Já no outro extremo, os autores dizem que a inovação 'implica uma mudança planejada' com a finalidade de capacitar a organização ou o professor para alcançar seus objetivos.

Sobre as dimensões das inovações tecnológicas no campo educacional, Teixeira (2010, p.1) afirma que as iniciativas de aplicação de inovações "nos sistemas educativos em diferentes países ensejaram pesquisas que viriam a constituir a inovação educacional como objeto de estudos de especialistas e políticos". A autora chama a atenção para o fato de que "inovação não é solução mágica que possa resolver todos os problemas da educação" e que "deve ser acompanhada de questionamentos como: a quem interessa? Por quem foi proposta ou implementada? E a quem poderá beneficiar?" (Teixeira, 2010, p.2).

Assim como para Hernandez *et al.* (2000) e Farias (2006), também defendemos, que a necessidade da adesão do professor ao processo inovador é fundamental para que este tenha consistência e efetividade. Para esses autores, a inovação é entendida na perspectiva de algo a ser construído ou a se construir.

Nesse sentido, tais autores constatam que "as inovações não perduram se não se conta com os docentes" (Hernandez *et al*, 2000, p.23). Ao considerar o papel do professor como importante no processo de inovação, Fullan (1982) também é enfático ao afirmar que são as ações e concepções dos professores que promovem mudanças na educação.

Pesquisas como a de Motejunas (1980), Cardoso (1997), Fullan e Hargreaves (2000), Hernandez *et al.* (2000), Farias (2006) e Fullan (2009), entre outros, fazem referências a mudanças que acontecem na escola por meio de reestruturação curricular, pela inserção de novos métodos de ensino ou pela implantação de projetos. Tais estudos resultaram em importantes aspectos da relação dos professores com o processo de mudança e dizem respeito à forma de inclusão, à maneira como é feita a sensibilização, à predominância da imposição dos agentes externos ou da adesão do professor ao processo. Muitas dessas iniciativas de inovação têm origem em instituições governamentais ou particulares; outras são formuladas pela escola e, ainda, existem aquelas que emergem de anseios dos próprios professores.

A inserção das TI na escola ou a adoção de novos métodos de ensino são situações que promovem discussões acerca da entrada e da permanência da inovação no meio educacional. Pesquisas como as de Cardoso (1999), Hernandez *et al*. (2000) e Farias (2006) evidenciam que, quando impostas de cima para baixo, essas mudanças possuem baixo grau de adesão da comunidade escolar, em particular dos professores.

No que se refere à inclusão das TI no sistema escolar e à sua relação com professores e alunos, destacam-se, entre outros, os estudos de educadores como Masetto (1998, 2003), Moran (2000); Belloni (2003, 2005) e Behrens (2007), que veem na tecnologia potencial para a instauração de novas concepções nos processos de ensino e de aprendizagem.

De acordo com Teixeira (2010, p. 3), "ao fazerem uso de instrumentos tecnológicos, os sujeitos, de alguma forma, podem modificar seu uso e por eles serem modificados". Para a autora, no que se refere à educação, "intensifica-se o discurso da necessidade do uso dos 'novos' instrumentos tecnológicos, bem como, na formação de professores".

Ao trazer para o foco desta discussão a inovação no campo educacional ligada às práticas dos professores, fazemos referência aos aspectos pedagógicos, tecnológicos e de gestão nesse contexto.

INOVAÇÃO SOB O OLHAR DE ROGERS

Rogers (2003) apresenta um modelo *de inovação-decisão* em que esquematiza o processo pelo qual o indivíduo passa do conhecimento mais geral sobre a inovação, aprofundando esse conhecimento, formando uma opinião ou uma atitude a seu respeito, até chegar à decisão de adotá-la ou rejeitá-la para, enfim, no caso de adoção, trabalhar com a implementação e confirmar esta decisão.

Ao definir o seu entendimento sobre inovação, Rogers (2003, p.12) afirma que esta é "uma ideia, prática ou objeto que é percebido como novo pelo indivíduo". Para ele, não é importante se tal ideia é nova no sentido de se tratar de uma descoberta recente, mas o que interessa é como a pessoa percebe esta nova ideia. A pessoa que acaba de entrar em contato com uma nova ideia pode até ter recebido, anteriormente, algum conhecimento a seu respeito, no entanto, o que importa, é a relação que, a partir de então, será estabelecida entre esta pessoa e a nova ideia.

Desse modo, um professor pode saber sobre o uso das TI na educação e já ter uma opinião formada a respeito ou o uso das TI na educação pode não representar novidade alguma. Quando este professor, porém, se depara com a possibilidade ou necessidade de se relacionar com esta modalidade de ensino, seja como gestor ou como professor, desempenhando, enfim, algum papel neste contexto, ele passa a encarar o uso das TI na educação como uma nova ideia, como uma novidade, pois a sua relação, também, é ressignificada.

Esta relação pressupõe etapas pelas quais o, motivações e certezas; formulando suas opiniões e tomando suas decisões quanto à possibilidade de aceitar ou não a inovação, em especial diante das exigências próprias da linguagem da Matemática.

O modelo do processo de *inovação-decisão* de Rogers envolve cinco estágios: conhecimento, persuasão, decisão, implementação e confirmação, os quais são representados na Figura 1.

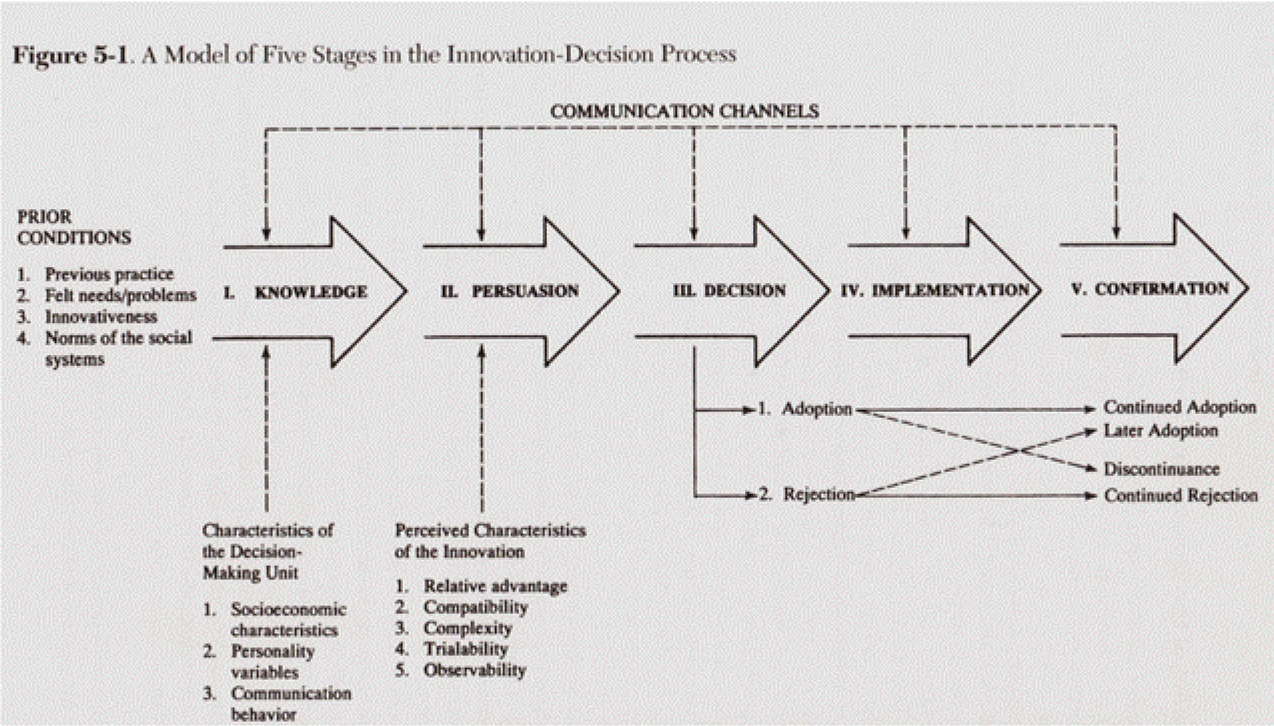

Figura 1: Modelo do processo de inovação-decisão (Rogers, 2003, p.170)professor percorrerá buscando informações

Podemos verificar que no processo de *inovação-decisão* a pessoa percorre etapas e, embora representadas linearmente na Figura 1, a relação que vier a ser estabelecida entre a pessoa e a nova ideia é que determinará o seu percurso. Neste contexto, os meios de comunicação estão presentes constantemente, denotando um alto grau de interferência nesta relação.

A partir das condições prévias do indivíduo, a motivação inicial, ou seja, seus conhecimentos a respeito da nova ideia, também interferem, assim como a necessidade de buscar informações e conhecimentos mais aprofundados sobre a mesma.

Ao levantar os componentes que surgem nesta nova prática do professor de Matemática que está iniciando esta experiência, é importante buscar identificar elementos que possam facilitar o estabelecimento desta relação.

Ao longo do processo, alguns elementos serão decisivos para determinar uma atitude positiva ou não diante da inovação: a percepção das vantagens e desvantagens com relação à inovação; seu nível de complexidade; e em que medida atende ou não às suas necessidades e anseios. Estes elementos estão representados na Figura 1, pelas características identificadas da inovação.

Na etapa da persuasão pode-se incluir, ainda, a observação da inovação e a sua manipulação como elementos fundamentais para que a pessoa forme uma opinião própria a respeito da ideia que lhe está sendo apresentada como solução, ou possibilidade de solução, para suas necessidades.

Relacionando isso com a atividade do professor, nos reportamos a esta fase como aquela em que o professor recebe formação específica para o desenvolvimento de suas atividades; discute com seus pares a respeito do que irão desenvolver; as atividades planejadas; observa o andamento das atividades e desempenha as funções de autoria. A partir daí, reúne condições para avaliar a experiência que está vivendo e formular sua opinião com relação ao trabalho com o uso das TI.

No caso de ensino de conteúdos matemáticos, a inserção de entes abstratos necessita de recursos que facilitem a visualização das ideias a serem transmitidas. Torna-se, assim, um diferencial importante, visto que o professor dispenderá de um esforço a mais para implementar tais recursos e para criar novas formas que facilitem a compreensão dos esquemas gráficos, das fórmulas e dos exemplos que o professor costuma oferecer em suas explicações no dia-a-dia.

Enfim, situações em que professores se colocam diante do desafio de desenvolverem cursos com o uso das TI são muito complexas, ainda mais quando as especificidades pedagógicas, metodológicas, tecnológicas e de gestão do ensino exigem deles mais do que trazem da experiência adquirida.

Ao entrar em contato com a inovação, o professor procura obter novos conhecimentos, estabelecendo associações com a sua bagagem anterior e efetuando trocas de informações com seus pares, a fim de afirmar as suas opiniões, esclarecer suas dúvidas e efetivar suas atividades de trabalho. Desse modo, o professor vai, pouco a pouco, adequando seu desempenho e aperfeiçoando suas condições de trabalho ao longo do caminho, a partir das estratégias que estabelece pessoal e coletivamente com a inovação, ou seja, com o uso das TI.

As características da inovação, tal como elencadas por Rogers (2003), por fazerem parte do processo mental, são percebidas pelos professores de duas maneiras: individualmente, pois cada membro formula as suas próprias opiniões a respeito da inovação; e coletivamente, pois a equipe negocia as suas ideias para que o trabalho tenha convergência, tomando um caminho comum, ou seja, aquele em que um membro, mesmo que não esteja fortemente persuadido a respeito do que será feito, sabe que conta com o apoio dos demais membros.

A etapa da *Persuasão* pode, ainda, se apresentar em vários temas e subtemas: nas interpretações referentes aos conhecimentos prévios, quando os professores procuram estabelecer relações entre os seus conhecimentos e aqueles requisitados para o trabalho com o uso das TI; na formação para o trabalho e durante o trabalho e no suporte da instituição. A percepção e o desenvolvimento de opinião por meio de uma aproximação subjetiva da inovação são muito importantes, principalmente para que os professores tomem uma atitude positiva diante da inovação, uma vez que a mesma não sobrevive se não tiver a adesão dos professores e sem que se tenham apropriado do seu significado.

Com relação à etapa de *Decisão*, como o próprio Rogers adverte, ela não ocorre somente no final do processo de *inovação-decisão*, mas continuamente, pois, a cada momento, a adoção da inovação é questionada por alguma situação ou acontecimento.

Pode-se identificar momentos relacionados à etapa de decisão quando os professores expressam o desejo de buscar mais formação específica para as questões relacionadas ao uso das TI, ao mencionarem planos para o preparo de sua disciplina e ao sentirem a necessidade de avaliar o que haviam feito até então**.**

A etapa de *Implementação* do modelo do processo de *inovação-decisão* está relacionada diretamente com o fato de colocar a inovação em uso. A partir do momento em que os professores deram início ao uso das TI, passa-se para esta fase do processo descrito por Rogers.

Ao se confrontarem com a prática percebem que esta etapa representa, concretamente, uma formação continuada, que complementa a formação inicial e chama a atenção para especificidades do uso das TI no contexto da Matemática, ainda não percebidas.

A etapa de *Confirmação* é a última a ser apresentada no modelo do processo de *inovação-decisão* de Rogers, como mostrado na Figura 1, porém, não é, necessariamente, a última a ocorrer no processo, já que este é considerado contínuo ao longo do tempo. E mais, além da decisão de aceitar ou rejeitar uma inovação se estabelecer em um contínuo, à medida que o indivíduo recebe uma informação, pode modificar ou reforçar a sua decisão anteriormente tomada.

PROFESSOR DE MATEMÁTICA: NOVOS PAPÉIS E COMPETÊNCIAS

A formação do professor é um trabalho contínuo que se transforma conforme as necessidades da atuação docente em sala de aula, na produção de materiais, na administração das atividades acadêmicas, na atuação política e nas questões profissionais que lhe dizem respeito.

Ponte, Oliveira e Varandas (2003, p.160), ao falarem sobre as contribuições das TI para o desenvolvimento do conhecimento do professor e sobre a sua identidade profissional, consideram que aprender a trabalhar com as TI pode auxiliar "o desenvolvimento de uma identidade profissional" do professor, "estimulando a adoção do ponto de vista e de valores próprios de um professor de Matemática".

De acordo com Ponte, Oliveira e Varandas (2003, p.162), as TI, ao se tornarem parte do ambiente de trabalho do professor, o modificam, alterando também o modo como o professor se relaciona com os outros professores. Isso proporciona um "impacto importante na natureza do trabalho do professor e, desse modo, na sua identidade profissional", conforme observam os autores.

Instigar o professor ao desenvolvimento para alcançar as finalidades mencionadas por Ponte, Oliveira e Varandas (2003) requer, do professor, um esforço em direção às mudanças que assume com relação a novas possibilidades de trabalho docente, à mobilização de conhecimentos, crenças e concepções.

Independentemente de o trabalho estar focado no ensino e do tipo de ferramenta tecnológica que está sendo usada, a ideia apresentada por Penteado (2000) estabelece uma relação entre uma situação em que o professor possui pleno domínio, ou seja, a *zona de conforto*, para se lançar em uma "nova" situação, demarcada pelas incertezas e descobertas que se encontram na *zona de risco*.

O que a autora denomina de "zona de risco" corresponde ao "encontro" com a inovação, no sentido de que é neste novo espaço que as relações se configuram com as incertezas, crenças e expectativas diante deste trabalho.

Os discursos sobre as necessidades de formação convergem para o desenvolvimento e aprimoramento da prática do professor de Matemática em função de vários fatores, como as mídias, a pesquisa como incentivo à produção de conhecimento e a habilidade de trabalhar cooperativamente. Esse discurso se amplia com as possibilidades de uso das tecnologias no ensino.

Ao descrever sobre o trabalho do professor na educação, Belloni (2003, p.79) observa que o papel a ser desempenhado é composto por múltiplas funções, devido ao uso dos meios tecnológicos e afirma que "o uso mais intenso dos meios tecnológicos de comunicação e informação torna o ensino mais complexo e exige a segmentação do ato de ensinar em múltiplas tarefas".

Em sua definição sobre o papel do professor, Belloni (2003, p.81) aponta a "transformação do professor de uma entidade individual em uma entidade coletiva" como a característica principal.

Ao mencionar os desafios do redimensionamento do papel do professor diante do trabalho de transpor do ensino tradicional para um ensino cada vez mais *mediatizado*, (Belloni, 2003) considera a necessidade de que esse professor "aprenda a trabalhar em equipa e a transitar com facilidade em muitas áreas disciplinares. Será imprescindível quebrar o isolamento da sala de aula convencional e assumir funções novas e diferenciadas". (Belloni, 2003, p.29).

Existe uma forte tendência de formar estudantes para exercer funções que ainda são desconhecidas ou indefinidas, possibilitando a eles a aquisição de autonomia para aprender o suficiente que lhes permita continuar a sua própria formação ao longo da vida pessoal e profissional. Essa tendência deve acompanhar a formação de professores seguindo essa mesma lógica, ou seja, recebendo a formação adequada, os professores não fogem à regra geral das necessidades de desenvolvimento social imposta a outras profissões, aceitando a inovação e, consequentemente, evoluindo em sua profissão. A respeito das competências necessárias ao professor, Belloni (2003, p.87) nos apresenta algumas pistas para os profissionais da educação. A primeira delas está relacionada à aquisição de um domínio de técnicas indispensáveis em situações educativas cada vez mais *mediatizadas*. Em seguida, estão as competências de comunicação, *mediatizadas* ou não, que possibilitem ao professor trabalhar em equipa. A terceira pista está ligada à capacidade de sistematizar e formalizar procedimentos e métodos para desenvolver a prática pedagógica e a quarta pista se refere à capacidade de transmitir seus saberes e experiências de maneira que outros possam utilizá-los em suas próprias necessidades.

Com este raciocínio, entende-se que a formação do professor de Matemática para atuar no ensino com o uso das TI é um campo a ser explorado, principalmente por buscar compreender os fenômenos que ocorrem com relação à prática docente. Atualmente, os professores em geral, e os professores de Matemática, em particular, encontram-se desafiados pelo processo de inovação das práticas educativas, representado pela mediação tecnológica e pelas especificidades pedagógicas e de gestão próprias desta modalidade de ensino.

Essas necessidades propostas de formação acrescentam elementos novos ao desenvolvimento profissional e, consequentemente, contribuem para o fortalecimento da identidade profissional do professor.

CONCLUSÃO

As etapas do modelo de inovação de Rogers (2003) podem permitir a identificação dos elementos necessários sobre o que pode auxiliar o professor a se relacionar com o uso das TI de modo a aceitar esta novidade e desenvolver competências para as novas temporalidades.

A atuação docente com o uso da tecnologia exige o repensar do fazer pedagógico. Novas competências, atitudes e valores são requeridos dos professores, observando-se aspectos pedagógicos, tecnológicos e de gestão específicos deste novo modo de ensinar e aprender.

Diante da inovação considerada como o uso das TI, o professor reage procurando se adequar a ela a partir de seus conhecimentos, habilidades, visão de mundo e convicções pedagógicas, estabelecendo novas relações com os elementos inovadores e agregando a eles outros conhecimentos. A inovação, portanto, se estabelece em um contínuo da ação docente.

Com este raciocínio, pode-se concluir que a prática do professor com o uso das TI é resultado de um processo entre seus conhecimentos e experiências profissionais e o que adquire ao se relacionar com esta nova situação e sua decisão de aceitação.

Em suma, as competências dos professores que vão trabalhar com as TI necessitam de uma reconfiguração em direção à incorporação da inovação. Tais competências, porém, não são adquiridas em tempos e espaços determinados: é resultado de um trabalho pessoal de construção de identidade e de persuasão, característico do processo de *inovação-decisão de Rogers (2003)*, tal como já descrito.

Assim, como discutida por Hernandez *et al.* (2000), a inovação é interpretada, de modo distinto, por cada pessoa a ela exposta. Caberá à pessoa estabelecer uma relação própria com a nova ideia, o que, consequentemente, poderá determinar formas diferentes de lidar com suas particularidades.

Pesquisas sobre a formação do professor de Matemática para o ensino com o uso das TI, podem contribuir para discussões acerca desta realidade, qual seja, a de ampliar as reflexões sobre o ensino da Matemática com o uso das TI, principalmente buscando explicitar elementos que contribuam para a busca da superação das dificuldades e especificidades desta área de conhecimento, assim como a linguagem e a abstração, próprios da Matemática em um ambiente tecnológico.

Esperamos que esse trabalho possa representar o início de uma discussão frutífera diante da problemática de compreensão sobre as necessidades formativas do trabalho pedagógico a serem apreendidas pelo professor de Matemática no contexto do uso das TI, seus desafios e suas negociações.

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O USO DOS TABLETS NO ENSINO DA GEOMETRIA NOS ANOS INICIAIS DO ENSINO FUNDAMENTAL: UMA EXPERIÊNCIA COM O APLICATIVO "SIMPLY GEOMETRY" / THE USE OF TABLETS IN TEACHING GEOMETRY IN THE EARLY GRADES OF ELEMENTARY EDUCATION: AN EXPERIENCE WITH THE APPLET "SIMPLY GEOMETRY"

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In this paper we report an experience on the use of tablets in teaching geometry to a third grade class of elementary school. The teacher is a participant in a continuing education program and the teaching sequence that she has put into practice in the classroom allowed students to build knowledge through practical activities using the tablet for exploring the applet "Simply Geometry". The present study focuses on the use of the tablet as a teaching tool in mathematics classes, which can contribute as much to the teaching as to the learning of geometry topics.

Keywords: Tablet; Teaching; Learning; Geometry; Early grades.

INTRODUÇÃO

Inúmeras pesquisas têm mostrado as contribuições do uso de recursos tecnológicos para a aprendizagem da Matemática. A utilização apropriada destes recursos pode auxiliar e contribuir de modo importante para a aquisição do conhecimento pelos alunos. Entretanto, a sua utilização em sala de aula ainda é um grande desafio aos professores, já que muitos deles não se sentem confortáveis neste contexto e precisam se adaptar a novas formas de pensar o ensino. Para envolver o professor nesse ambiente tecnológico é fundamental que ele seja preparado pedagogicamente e tecnicamente, para poder se apropriar dos conhecimentos necessários e contribuir para a aprendizagem dos seus alunos. "Acreditamos que a verdadeira integração da tecnologia somente acontecerá quando o professor vivenciar o processo e quando a tecnologia representar um meio importante para a aprendizagem" (Bittar, Guimarães & Vasconcellos, 2008, p. 86).

Nessa perspectiva, pretendemos discutir neste trabalho algumas possibilidades do uso do *tablet* como uma ferramenta pedagógica, tanto em relação ao modo como o professor integra esta tecnologia nas suas aulas quando ensina Matemática, quanto em relação à aprendizagem dos alunos. Os dados foram coletados junto a uma professora e sua turma de alunos do 3º Ano dos Anos Iniciais do Ensino Fundamental, na faixa etária dos oito anos. A professora é participante do curso de formação continuada, intitulado "O uso de *tablets* nas aulas de Matemática nos Anos Iniciais do Ensino Fundamental". Acompanhamos a mesma nas sessões do curso, como formadoras, e também na escola, auxiliando no planejamento das aulas e no desenvolvimento das mesmas, quando a professora integra o *tablet* nas suas aulas.

Essas ações fazem parte de uma pesquisa de mestrado (conduzido pela primeira autora), mas neste artigo, trazemos o recorte de uma das aulas em que a professora utilizou o aplicativo "*Simply Geometry*", com o objetivo de trabalhar com os seus alunos algumas noções de geometria plana e geometria no espaço. Buscamos no decorrer deste relato, por meio da sequência didática elaborada pela professora, analisar e discutir algumas contribuições e/ou limitações deste aplicativo como auxiliar no ensino da geometria nos Anos Iniciais.

O *TABLET* COMO FERRAMENTA PEDAGÓGICA

O desenvolvimento das tecnologias digitais tem sido comandado por solicitações e necessidades da sociedade atual, mas para que o seu avanço ocorra de modo significativo no sistema educacional, Barcelos & Batista (2013) afirmam que um longo caminho ainda precisa ser percorrido. De fato, tais tecnologias parecem encontrar um impasse no contexto educacional, que apresenta dificuldade para acompanhar tal desenvolvimento, enquanto os alunos são cada vez mais seus usuários fora da escola. Reconhecemos os *smartphones* e os *tablets* como algumas dessas tecnologias, também caracterizadas como dispositivos móveis. Saboia, Vargas & Viva (2013) contemporizam que essas tecnologias vêm sendo difundidas nas mais diferentes áreas e sua utilização tem-se expandido entre as gerações anteriores e as novas gerações. Os mais jovens, em particular, "já incorporam tais dispositivos como uma extensão do lar ou de seu próprio corpo" (p. 3-4). Os autores constatam:

> A existência e o uso destas tecnologias não se evidenciam somente no momento em que vemos um dispositivo em uso, mas culturalmente nossas ações, nossas relações e nosso vocabulário denunciam que estamos fortemente influenciados por esta era digital. Os assuntos nas rodas de amigos, os textos escolares, científicos, os namoros entre outras relações sociais não necessitam mais da presença física para que ocorram (Saboia, Vargas & Viva, 2013, p. 4).

Em torno do uso pedagógico desses dispositivos, está crescendo um campo de pesquisa denominado *Mobile Learning* (*m-learning*), que estuda como as tecnologias móveis podem contribuir para a educação. Para Batista, Behar & Passerino (2010) o desenvolvimento de recursos pedagógicos para os dispositivos móveis é essencial para a efetiva aplicação de *m-learning*.

> A popularização dos dispositivos móveis é um aspecto positivo em termos educacionais, uma vez que favorece o alcance de um grande número de pessoas, sem requerer deslocamentos físicos. Porém, além deste aspecto, existem diversas características, tais como mobilidade, interatividade, aprendizagens em contextos reais, e práticas colaborativas, que têm motivado pesquisas em m-learning (Batista, Behar & Passerino, 2010, s/p).

Contrário ao computador de mesa que se encontra em laboratórios ou salas de informática, e que exige a locomoção dos alunos até esses locais para a sua utilização, os dispositivos móveis podem ser transportados de uma sala para outra, colocados em cima da carteira, movidos para vários locais da sala de aula, ou fora dela, tornando-se uma ferramenta integrante da aula (Batista & Freitas, 2010; Fister & McCarthy, 2008). Além disso, Saboia, Vargas & Viva (2013) preconizam:

> As tecnologias móveis têm possibilitado que o processo de comunicação e a difusão da informação ocorram em diferentes espaços e tempos, sendo duas de suas características a portabilidade e a instantaneidade. Características que permitem a uma grande parcela da população o acesso a informação em qualquer lugar e a qualquer tempo, seja em tempo real ou não. Outra característica a destacar é a larga produção destas tecnologias, resultando em um custo mais acessível e uma massificação tecnológica [...] (p. 8).

Neste trabalho destacamos o uso do *tablet* como uma ferramenta pedagógica. Segundo Barcelos, Batista, Moreira & Behar (2013, s/p) "os *tablets* são dispositivos que oferecem diversos recursos que podem facilitar a visualização de conteúdos, estimular atividades cooperativas e o desenvolvimento de projetos e, assim, contribuir para a realização de diversas atividades pedagógicas".

De acordo com Barcelos & Batista (2013), existem alguns indicativos de que esses dispositivos promovem a colaboração e a interação entre alunos em sala de aula, graças às características da portabilidade e da conectividade. No entanto, as autoras destacam que para determinar as potencialidades e eventuais limitações no uso pedagógico do *tablet*, é preciso uma análise mais profunda. E fazem notar que, embora exista uma variedade de aplicativos educacionais para os *tablets*, muitos foram criados para situações que não necessitam a intervenção do professor. Portanto, "a utilização dos mesmos, em sala de aula, pode requerer estratégias adequadas para que esses aplicativos possam colaborar para os objetivos pedagógicos pretendidos" (Barcelos & Batista, 2013, p. 169). Também segundo Fister & McCarthy (2008), a conetividade do *tablet* tem a vantagem de permitir que os alunos possam discutir questões matemáticas com outros ao redor deles, mostrar seu trabalho e, em seguida, explicar os seus resultados. Incentiva, portanto, o esforço de partilha, bem como as competências individuais de pensamento criativo que são igualmente importantes. O *tablet* tem ainda a seu favor o fato de ser muito atraente para a geração mais nova que lida de forma desinibida com a tecnologia.

Batista, Behar & Passerino (2010) consideram adequados para a aprendizagem os recursos que não buscam reproduzir os atuais cenários de aprendizagem mais tradicional, mas sim criar novas oportunidades que não seriam possíveis sem a tecnologia móvel. Coaduna-se com essas afirmações, Moran (2013, s/p), quando afirma:

> Educar é, simultaneamente, fácil e difícil, simples e complexo. Os princípios fundamentais são sempre os mesmos: Saber acolher, motivar, mostrar valores, colocar limites, gerenciar atividades desafiadoras de aprendizagem. Só que as tecnologias móveis, que chegam às mãos dos alunos e professores, trazem desafios imensos de como organizar esses processos de forma interessante, atraente e eficiente dentro e fora da sala de aula, aproveitando o melhor de cada ambiente, presencial e o digital.

O APLICATIVO "*SIMPLY GEOMETRY*" E A SEQUÊNCIA DIDÁTICA PROPOSTA

Está hoje bem documentado pela pesquisa sobre o ensino e aprendizagem da Matemática que os alunos apresentam dificuldades em diversas áreas desta disciplina. Neste trabalho, visamos ilustrar possíveis contribuições da tecnologia, em especial do *tablet*, para a aprendizagem da Matemática nos Anos Iniciais do Ensino Fundamental; iremos aqui considerar a geometria, que segundo Bittar, Guimarães & Vasconcellos (2008) é um assunto que tem sido relegado a um segundo plano no ensino da Matemática. Na sequência didática proposta e desenvolvida pela professora participante em nosso estudo, o conteúdo trabalhado se refere a formas geométricas planas e tridimensionais. A professora iniciou a aula com a exploração do aplicativo "*Simply Geometry*" (disponível em https://play.google.com/store/apps/details?id=com.nery&hl=pt_BR), para que por meio dele os alunos pudessem reconhecer e diferenciar as formas, bem como perceber algumas características especiais de cada grupo. Na sequência apresentamos um breve roteiro do aplicativo, cujo objetivo é identificar formas geométricas planas e espaciais. Este aplicativo pode ser utilizado de forma a levar os próprios alunos a classificarem os polígonos como figuras planas e reconhecerem que nos sólidos geométricos as faces são constituídas por polígonos e que os mesmos apresentam volume. O seu idioma é o Inglês. Quando o aplicativo é iniciado, surge uma tela com as opções iniciais (Figura 1).

Figura 1. Opções iniciais

O aplicativo possui quatro tipos de atividades livres, sendo elas: *Lineup* (alinhar); *Sort* (selecionar); *Patterns* (padrões) e *Build Matrix* (construir matriz). As demais opções só podem ser acessadas se forem compradas. Para começar a jogar deve-se selecionar uma das opções tocando sobre a mesma.

Selecionando a opção *Lineup* (Figura 2), a atividade proposta é ligar a forma geométrica com seu respectivo nome. O movimento do dedo sobre a tela permite criar as linhas de ligação. Mesmo o aplicativo estando em inglês, isto não impede a sua compreensão. Nesta fase do jogo, aparecem tanto figuras geométricas planas, quanto as espaciais.

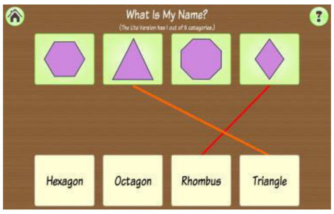

Figura 2. Lineup

Figura 3. Sort

Para retornar ao *menu* inicial e escolher outra atividade, basta tocar a figura da casa no canto superior esquerdo da tela. Selecionando o botão "*Sort*" aparecem várias peças com formas geométricas (Figura 3). A tarefa é identificar quatro delas que possuem a característica solicitada. Para selecionar, se deve segurar a peça e arrastá-la até à área branca no centro da tela. Quando o quadro central estiver completo de modo correto, o jogo avança automaticamente para a próxima fase, que consiste em situações similares à primeira. Nesta fase do jogo o aluno percebe algumas diferenças entre as formas 2D e as 3D, e também é levado a identificar os vértices das formas, chamados no aplicativo por "cantos".

A próxima atividade é *Patterns* (Figura 4), a qual envolve conceitos de sequência e geometria. O objetivo é continuar completando a sequência que o aplicativo propõe. Para isso se seleciona a figura geométrica desejada, levando-a até o ponto de interrogação. Ao completar corretamente, passa-se para outra fase em que o grau de dificuldade vai aumentando. O objetivo principal é a concentração do aluno, que brincando, consegue identificar figuras geométricas que completam a sequência.

A última atividade gratuita, proposta pelo aplicativo é a "*Build Matrix*" (Figura 5), em que se deve completar uma tabela com figuras geométricas fornecidas pelo jogo. Essa sempre é a soma de duas figuras geométricas e deve ser colocada no cruzamento da linha com a coluna dessas formas. O aluno, por meio dessa tarefa, consegue-se localizar na tabela, identificar linha e coluna e separar as duas formas que constituem a figura geométrica fornecida.

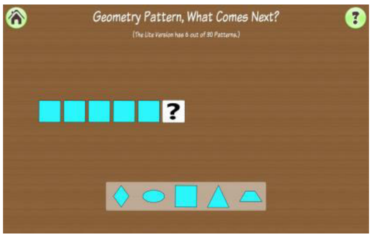

Figura 4. Patterns

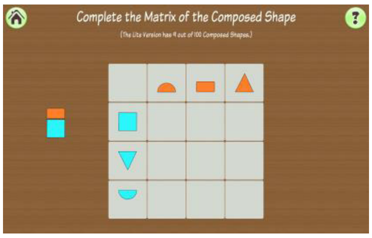

Figura 5. Build matrix

Durante a exploração do aplicativo a professora circulou entre os alunos para observar o modo como interagiam com o jogo. Percebeu que muitos deles, iam por "tentativa e erro", pois o aluno arrasta as opções e o jogo rejeita, ele tenta novamente até que esteja correto e seja aceita. A professora apontou algumas palavras-chave, traduzindo-as junto com os alunos, e ao mesmo tempo ia explicando algumas diferenças entre as formas geométricas. Ressaltamos que os alunos têm a disciplina de Inglês, portanto, a professora não considerou a questão do idioma uma barreira para a utilização do aplicativo.

Terminada a exploração do aplicativo, a professora colocou perguntas sobre o jogo e o que eles haviam percebido, enquanto jogavam. A turma é bastante participativa, todos quiseram falar ao mesmo tempo, e disseram que "gostaram muito" e que tinha "várias formas geométricas"; em função disso a professora conduziu a discussão, sempre questionando e os alunos respondendo, mas eles também faziam perguntas acerca do que estava sendo discutido. A professora explicou o que seriam as figuras planas, explorou a questão do número dos "cantos" e que se tratava de formas geométricas bidimensionais. Também discutiu com a turma as formas tridimensionais dizendo aos alunos que a diferença entre as duas é que além da altura e da largura, estas apresentam profundidade, então segundo a professora "podemos guardar coisas dentro delas".

Após a explanação dividiu a turma em grupos e pediu que colocassem sobre a mesa o material (caixinhas, embalagens e objetos que pudessem "guardar algo dentro"), que havia pedido na aula anterior. Então solicitou que relacionassem os objetos com algum sólido geométrico. Ela circulou pelos grupos e os alunos falavam, caracterizando os objetos, pelas faces, por sua forma redonda, ou se possuía pontas, enfim descreviam os objetos por meio das suas características geométricas. A professora então instruiu que cada grupo deveria escolher um dos objetos e apresentar para a turma, identificando as suas faces e o seu nome. O primeiro grupo trouxe um cubo mágico, os alunos descreveram que suas faces eram quadradas e alguns alunos já o identificaram. A professora aproveitou e falou do paralelepípedo, mostrando a diferença entre os objetos e destacando as faces. Os outros grupos trouxeram latinhas com forma de cilindro, bolsinha com forma de esfera, caixinha de chocolate, entre outros objetos. Destacamos o que disseram os alunos do quinto grupo:

Professora: É uma caixinha de pasta de dente, uma embalagem, que figura é essa?
Aluno 3: A figura é com retângulos.
Professora: Sim, ela é formada por faces retangulares, mas que nome se dá a ela?
Aluno 4: Paralelepípedo.

Um aluno, que não pertencia ao grupo, interrompeu a professora e comentou:

Aluno 1: Profe! Oh, profe! Mas ela tem uma face quadrada...
Professora: Sim, mas ela não tem todas as faces quadradas, se fossem todas iguais seria o quê?
Turma: Cubo!
Professora: Ela possui faces quadradas e faces retangulares. As faces são diferentes. Então ela é um?...
Turma: Paralelepípedo!

Para o Aluno 1 ainda havia uma confusão em relação às diferenças entre o cubo e o paralelepípedo, para ele o paralelepípedo, necessariamente, deveria ter faces retangulares, e se o sólido tiver faces quadradas, mesmo que sejam apenas duas, não se configurava um paralelepípedo. Por meio do seu questionamento, ele pode compreender que o cubo só pode ter faces quadradas, enquanto o paralelepípedo pode possuir tanto faces quadradas quanto retangulares, ou apenas retangulares, e, portanto, é justamente as faces que diferenciam o cubo do paralelepípedo. Ficou claro que alguns alunos já conheciam alguns sólidos, como por exemplo, o cubo e as pirâmides, bem como as formas geométricas planas básicas, como triângulo, retângulo e círculo, entre outras. A professora complementou que as figuras espaciais são classificadas como sólidos geométricos e destacou que as figuras planas compõem as suas faces. A partir desta definição entregou a cada grupo um sólido de papel, e solicitou que cada grupo abrisse estes sólidos para planificá-los. Circulando entre os grupos perguntava que figuras geométricas apareciam em cada sólido planificado. Chamamos a atenção para o grupo que planificou um cilindro.

A professora pegou o cilindro planificado, o fechou e perguntou:

Professora: Que forma geométrica é essa?
Aluna 8: Assim é um retângulo.
Professora: Assim, não é um retângulo. Assim, é um cilindro.
Aluna 6: O que é um cilindro?
Professora: Cilindro pode ser um copo, uma latinha de "refri", isso é cilindro. Quando eu abro tirando essas duas... Que figura é essa?
Aluna 8: Retângulo.
Professora: E essas aqui o que são?
Alunas 6, 8,10: Círculos!
Professora: Um retângulo mais quantos círculos?
Alunas 6, 8, 10: Dois.
Professora: Um retângulo e dois círculos formam o quê?
Aluna 6: Eu vou lembrar, profe...
Aluna 8: Cilindro!
Professora: Claro que tem os que são abertos, tirando um círculo ele fica igual a um copo. Vocês agora vão pintar um desses aqui (círculo) e esse aqui (retângulo).

Nesse grupo foi possível perceber que as alunas ainda não reconheciam o cilindro, e que só ouvir ou ver alguém falando sobre algo, pode não fazer sentido, porque um grupo havia apresentado anteriormente um objeto da forma de um cilindro, no aplicativo também apareceu cilindro em quase todas as fases do jogo. Com a experiência da planificação elas tiveram a oportunidade de manusear o sólido, tanto na forma fechada, quanto aberta, deste modo compreender para identificar. Quando a professora solicitou que pintassem um círculo e o retângulo pretendia que, ao destacar as figuras geométricas, os alunos percebessem que as faces dos sólidos são figuras planas. Isso foi solicitado a todos os grupos que depois fizeram uma socialização em que cada grupo apresentou o seu sólido e indicou as figuras planas que constituíam as suas faces.

Na atividade seguinte a professora buscou explorar outras características das formas tridimensionais, além das faces trabalhou as arestas e os vértices. Primeiramente, ela justificou que eles não deveriam brincar com comida, abriu um pote com gomas coloridas e passou por toda a sala distribuindo uma para cada aluno degustar. Depois começou a construir um cubo utilizando as gominhas coloridas e palitos. Construído o cubo, ela explicou que as gomas eram os vértices, os "cantos" que foram apresentados no aplicativo, e os palitos eram as arestas. Depois contou junto com os alunos, quantas arestas e quantos vértices continha o cubo. Na sequência, disse que ia abrir uma exceção, que eles poderiam "brincar" com as balinhas, "porque isso iria ajudá-los a aprender". Então distribuiu várias gomas e palitos de tamanhos diferentes para cada grupo e instruiu que construíssem os sólidos que haviam planificado. Todos fizeram os sólidos pedidos e também outras construções com a sobra do material fornecido pela professora.

A REFLEXÃO SOBRE A AULA

Conversando com a professora sobre as atividades propostas a partir do uso do *tablet*, ela considerou que a aula foi muito produtiva. Ela já havia trabalhado com os *tablets* com a turma, na semana anterior, utilizando um aplicativo que explorava as figuras geométricas planas. Falou em relação ao planejamento e a sua expectativa de como seria a sua postura no desenvolvimento das atividades propostas.

> Professora: O que eu faço, como que eu vou intervir, que intervenções eu vou fazer? Eu percebo que nessa aula eu consegui fazer mais intervenções com eles, mais pontuais, também. Gostei muito...

Em relação à geometria relatou:

> Professora: [...] a gente acaba não trabalhando tanto, fica muito aquela questão da Matemática de cálculos, e mais, e menos, as operações, e o quanto a gente acaba perdendo de outras áreas da Matemática que não são tão exploradas. E que fazem toda a diferença depois.

Queremos recordar que a professora é integrante do curso de formação e que nós, os formadores, levamos os *tablets* para a escola e auxiliamos no planejamento e no desenvolvimento das atividades. Neste sentido a professora declarou:

> Professora: Mas para mim assim, hoje, claro com a estrutura da Universidade, vocês trazem. Está sendo muito mais qualificado estar na sala com os *tablets* do que no laboratório de informática. Porque aqui nós temos, eu acho, que em torno de dezoito computadores, mas dos dezoito, apenas seis funcionam, então são três alunos por computador... até pela questão da disposição das classes. Que é importante eles poderem estar observando os colegas, né, trocando ideias, assim.

A professora também destacou que na utilização dos *tablets* os alunos se comunicam, trocando experiências, orientando o colega de como eles fizeram e que isso é facilitado em função da distribuição das classes em forma de círculo, em que os alunos ficam um ao lado do outro e não um atrás do outro, como nas aulas tradicionais. Quanto à utilização do *tablet* nas aulas, ela enfatiza que para trazer bons resultados é importante que seja trabalhado com objetivos bem claros e definidos.

> Professora: É porque a questão é assim, tu pode dar uma aula "super boa" sem tecnologia nenhuma, a tecnologia não vai dizer: "ah a professora está dando aula naquela sala com *tablets* e está sendo mil maravilhas". Então essa ideia tem que se tirar, também. Ah, está dando aula com *tablet* é professor "top de linha", não, não é. É a ferramenta

> e o que tu faz com ela. Tem a ferramenta, tudo bem, mas o que tu está fazendo com ela? Eu acho que é isso.

CONSIDERAÇÕES FINAIS

Consideramos que é possível, e cada vez mais imprescindível, que alunos e professores se apropriem de recursos tecnológicos úteis para a aula. Os *tablets* estão se tornando mais acessíveis e compreender suas potencialidades pedagógicas é fundamental para a formação dos professores. A experiência relatada permitiu observar que o fato do *tablet* não fazer parte do cotidiano de vários alunos, não ocasionou dificuldades no seu manuseio. Os alunos participaram ativamente no desenvolvimento de todas as atividades propostas e foi possível perceber que a sequência didática adotada pela professora contribuiu na compreensão dos conteúdos de geometria. Também a mescla entre o aplicativo e as atividades práticas e de contato com objetos contribuiu na construção do conhecimento dos alunos, como afirma Vasconcellos (2008, p. 81): "É preciso proporcionar às crianças diferentes oportunidades para que desenvolvam habilidades que lhes permitam gradativamente trabalhar com o conhecimento geométrico mais elaborado". Os *tablets* podem se constituir em uma importante ferramenta auxiliar no ensino da geometria e de outros conteúdos, ser um facilitador do trabalho pedagógico, aprimorando a forma de ministrar as aulas, tornando-as mais dinâmicas. O seu uso também pode "apoiar situações de aprendizagens colaborativas, nas quais o aluno tem um papel ativo no processo, construindo o seu conhecimento por meio da interação com o grupo" (Batista, Bahar & Passerino, 2010, s/p). A comunicação matemática e a discussão de diversas questões surgidas no interior dos grupos foi uma das dimensões muito relevantes da proposta didática. A socialização destas observações pode ser largamente facilitada pelo recurso a aplicativos, de forma intencional, planejada e integrada, isto é, enquanto ferramentas pedagógicas. Tal como concluem Fister e McCarthy (2008), no seu estudo, um dos aspetos de grande importância no uso do *tablet* é a facilidade que estes oferecem em partilhar informação de forma expedita, aumentando a interação entre os alunos e entre eles e o professor. Verificamos que um resultado semelhante foi assinalado pela professora nos seus comentários sobre a aula que desenvolveu.

Certamente, entende-se que utilizar recursos tecnológicos não significa resolver todos os problemas educacionais. Porém, não tirar partido do potencial das tecnologias digitais, em especial dos *tablets*, é correr o risco de manter a escola fora do contexto atual.

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O JOGO *ONLINE* COMO FERRAMENTA PARA AUXILIAR NA RESOLUÇÃO DE PROBLEMAS MATEMÁTICOS / ONLINE GAMES AS TOOLS TO SUPPORT MATHEMATICAL PROBLEM SOLVING

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This work aims at promoting the integration of problem solving and pedagogical use of technological resources in mathematics learning. The research involves 6th graders from three state schools in Vale do Taquari/RS – Brazil. For the study, we carried out a teaching experiment based on the use of the Moodle platform. The research follows a case study methodology, with the purpose of finding the advantages of the use of technology in the problem solving activity. We intend to see how the previous use of digital applets was helpful to the students in the development of a strategy to solve the problems.

Palavras-Chave: Mathematical problem solving; Online games; Technologies; Elementary School.

INTRODUÇÃO

No Brasil, tal como em outros países, os estudantes revelam muitas dificuldades na aprendizagem da matemática. Os Parâmetros Curriculares Nacionais brasileiros (PCNs) (Brasil, 1997) destacam a necessidade de promover mudanças no ensino da matemática de modo a que este não se reduza somente a procedimentos mecânicos, sem qualquer significado para os alunos. Neste sentido, os PCNs defendem a necessidade de "reformular objetivos, rever conteúdos e buscar metodologias compatíveis com a formação que hoje a sociedade reclama".

O presente estudo, tem como objetivo promover a integração da resolução de problemas, articulando-a com a utilização pedagógica de recursos tecnológicos na aprendizagem da matemática. Este trabalho faz parte de duas pesquisas: uma pertencente ao Programa Observatório da Educação, aprovado pelo edital 038/2010/CAPES/INEP, que tem por objetivo desenvolver estudos e pesquisas para melhorar a Educação Básica no Brasil, que está sendo desenvolvido no Centro Universitário Univates, em Lajeado/RS; outra, aprovada pelo Programa de Internacionalização da Pós-Graduação no RS, com apoio da FAPERGS (Fundação de Amparo à Pesquisa do estado do Rio Grande do Sul) que está sendo desenvolvida por pesquisadores brasileiros e portugueses. Cada um dos projetos tem um foco ligado a esta pesquisa: o Observatório, vinculado à resolução de problemas e o Internacionalização à utilização das tecnologias na aprendizagem da matemática.

Este estudo procura integrar as recomendações nacionais e internacionais no âmbito da resolução de problemas no ensino da matemática e da utilização das tecnologias (PCNs, 1997, NCTM, 2000). Por outro lado, nos apoiaremos ainda nos resultados da investigação desenvolvida no âmbito do projeto português Problem@Web, que inspirou este estudo.

Para tanto, foi formulado o seguinte problema de pesquisa: "De que forma o uso de aplicativos tecnológicos facilita a resolução de problemas de matemática?"

Os recursos tecnológicos e a resolução de problemas de matemática

Os recursos tecnológicos estão, cada vez mais, presentes no dia-a-dia dos alunos. Atualmente, os jovens dispõem de uma variedade de tecnologias que lhes permite comunicar com os outros, partilhar vídeos, músicas e outros recursos, com grande familiaridade e agilidade. No entanto, a Escola e, em particular, a sala de aula está longe de tirar partido desta relação de proximidade. Parece-nos importante que a Escola promova oportunidades para que os jovens utilizem estes recursos nos processos de ensino e de aprendizagem da Matemática.

Acresce ainda o facto de os jovens atualmente utilizarem com frequência diversos jogos computacionais como forma de entretenimento. Importa pois tornar e transformar esses jogos numa ferramenta para a aprendizagem, como defendem Mendes e Grando (2006, p.4):

> Ao jogar, o aluno desenvolve outras habilidades como aprender a conviver e cooperar com os outros, observar regras, cumprir acordos, comunicar ideias, desejos e emoções. Assim, é possível verificar as potencialidades pedagógicas que os jogos computacionais apontam para o processo de ensino-aprendizagem de Matemática.

De acordo com Carvalho e Ivanoff (2010), presentemente o desafio não está apenas em ensinar ou aprender, mas em ensinar e aprender com as tecnologias. Estes autores ressaltam que devemos ampliar o campo da educação sem alterar os procedimentos formais, sendo necessário para tanto que professores e alunos trabalhem em conjunto. Em consonância com estas ideias, Mendes e Grando (2006, p.4), acrescentam que

> Ao observarmos o comportamento de uma criança brincando e/ou jogando, percebe-se o quanto ela desenvolve sua capacidade de fazer perguntas, buscar diferentes soluções, repensar situações, avaliar suas atitudes, encontrar e reestruturar novas relações, ou seja, resolver problemas. É a partir desta perspetiva que entendemos ser importante a utilização do jogo computacional na educação.

Reconhecem ainda a importância do jogo juntamente com a resolução de problemas, afirmando que:

> O jogo pode ser utilizado nas aulas de Matemática na perspetiva de resolução de problemas como um gerador de situação-problema e desencadeador da aprendizagem do aluno, ou seja, um instrumento pelo qual os problemas podem ser propostos durante e após o jogo, levando os alunos a refletir sobre o movimento do pensamento de resolver o problema. Na ação de jogar várias situações-problema são propiciadas: pelo contexto do jogo, pela ação dos adversários, pela intervenção pedagógica do professor e/ou pelos problemas escritos. Tais situações podem ou não vir a ser um problema para o sujeito, dependendo da maneira como ele se sinta desafiado a resolvê-lo (Mendes e Grando, 2006, pp.4-5)

A Matemática tem papel decisivo, tanto na vida escolar quanto no dia-a-dia do aluno, pois permite resolver problemas do cotidiano e no mundo do trabalho. Valente (1999, p. 2) destaca:

> Quando o aluno usa o computador para construir o seu conhecimento, o computador passa a ser uma máquina para ser ensinada, propiciando condições para o aluno descrever a resolução de problemas, usando linguagens de programação, refletir sobre os resultados obtidos e depurar suas ideias por intermédio da busca de novos conteúdos e novas estratégias. (...) o aluno usa o computador para resolver problemas ou realizar tarefas como desenhar, escrever, calcular, etc.. A construção do conhecimento advém do fato de o aluno ter que buscar novos conteúdos e estratégias para incrementar o nível de conhecimento que já dispõe sobre o assunto que está sendo tratado via computador.

De acordo com Carvalho e Ivanoff (2010, p. 3), “três práticas estão sempre presentes no processo de ensinar e aprender com tecnologias de informação e comunicação: utilização de bases de dados e informações, comunicação e interação e construção de conteúdo”.

Valente (1999, p. 36) descreve o papel do aluno na utilização das tecnologias, destacando as principais habilidades que um estudante deve desenvolver durante suas resoluções das situações problemas.

> (...) deve ser ativo: sair da passividade de quem só recebe, para se tornar ativo caçador da informação, de problemas para resolver e de assuntos para pesquisar. Isso implica ser capaz de assumir responsabilidades, tomar decisões e buscar soluções para problemas complexos que não foram pensados anteriormente e que não podem ser atacados de forma fragmentada. Finalmente, ele deve desenvolver habilidades, como ter autonomia, saber pensar, criar, aprender a aprender, de modo que possa continuar o aprimoramento de suas idéias e ações, sem estar vinculado a um sistema educacional. Ele deve ter claro que aprender é fundamental para sobreviver na sociedade do conhecimento.

Nesse contexto, a Matemática abordada numa ótica de resolução de problemas, permite desenvolver o conhecimento dos alunos auxiliando na formação de cidadãos, uma vez que, conseguindo ter sua própria forma de resolução, criando e desenvolvendo suas estratégias, desenvolvendo autonomia de sua aprendizagem. Entretanto, como ressaltam os PCNs (Brasil, 1997, p. 22):

> (…) as orientações sobre a abordagem de conceitos, ideias e métodos sob a perspetiva de resolução de problemas ainda são bastante desconhecidas; outras vezes a resolução de problemas tem sido incorporada como um item isolado, desenvolvido paralelamente como aplicação da aprendizagem, a partir de listagens de problemas cuja resolução depende basicamente da escolha de técnicas ou formas de resolução conhecidas pelos alunos.

Deste modo, a resolução de problemas surge como a estratégia mais adequada para responder às necessidades da sociedade atual. De acordo com as palavras de Dante (2002, p. 15):

> Mais do que nunca precisamos de pessoas ativas e participantes, que deverão tomar decisões rápidas e, tanto quanto possível, precisas. Assim, é necessário formar cidadãos matematicamente alfabetizados, que saibam como resolver, de modo inteligente, seus problemas (...) a resolução de problemas como parte substancial, para que desenvolva desde cedo sua capacidade de enfrentar situações-problema.

A atividade de resolução de problemas permite ao aluno alcançar a solução mais adequada à situação colocada, de forma criativa, libertando-o de resposta única, possível pela simples aplicação de uma regra ou algoritmo. Neste sentido, na intervenção a resolução de problemas seguiu as quatro fases propostas por Polya (1995) para resolver problemas:

- Compreensão do problema: deve-se entender o enunciado e retirar os dados do problema;
- Estabelecimento de um plano: estabelecer os passos para encontrar a solução para o problema;
- Execução do plano: nesta etapa, desenvolvimento do plano, com paciência e flexibilidade, podendo acrescentar detalhes, verificando cada passo, para não ocorrera erros;

- Retrospecto: é a análise da execução, "reconsiderando e reexaminando o resultado final e o caminho que levou até este". Conseguindo solucionar o problema por outras formas e verificar qual é o caminho mais fácil.

Em consonância com essas ideias, Smole e Diniz (2001, p. 11) descrevem:

> A competência da resolução de problemas envolve a compreensão de uma situação que exige resolução, a identificação de seus dados, a mobilização de conhecimentos, a construção de uma estratégia ou um conjunto de procedimentos, a organização e a perseverança na busca da resolução, a análise constante do processo de resolução e da validade da resposta e, se for o caso, a formulação de outras situações-problema.

Também os PCNs (1998) destacam a necessidade de formar os jovens de hoje mostrando-lhe que a "Matemática pode dar uma grande contribuição à medida que explora a resolução de problemas e a construção de estratégias". Os referidos documentos frisam ainda a necessidade de desenvolver nos estudantes a "capacidade de investigar, argumentar, comprovar, justificar e, em simultâneo, estimular a sua criatividade", tanto no seu trabalho individual quanto coletivo.

PROCEDIMENTOS METODOLÓGICOS E A INTERVENÇÃO PEDAGÓGICA

O principal objetivo deste estudo é o de compreender "como" os alunos resolvem os problemas apresentados após uma experiência com um aplicativo *online* relacionado. Tendo em conta este propósito optou-se por uma metodologia qualitativa, em concreto por um estudo de caso. Cada caso é constituído pela turma e seu professor de matemática. O fato de se optar por três turmas, em três escolas distintas, com três professores diferentes, permite afirmar que se trata de um estudo de caso múltiplo (Yin, 2005).

Lüdke e André (1986, p.18) explicam que um estudo de caso qualitativo "é rico em dados descritivos, tem um plano aberto e flexível e focaliza a realidade de forma complexa e contextualizada", além de ser uma pesquisa com preocupação de "compreensão de uma instância singular", ou seja, algo que possui importância própria. Deve-se ressaltar que o objeto de estudo é único, sendo uma "representação singular da realidade", e a pesquisa possui por objetivo "retratar uma unidade em ação". De acordo com os Lüdke e André (1986) o estudo de caso qualitativo encerra um grande potencial para conhecer e compreender melhor os problemas da escola.

A razão que nos leva a considerar a existência de três estudos de caso, deve-se às características únicas existentes em cada uma das salas de aulas observadas. Um dos aspectos a realçar é o fato da investigadora, em cada caso, ter assumido papel diferente. Embora a proposta de intervenção tenha sido planejada, pensando a pesquisadora como observadora da sala de aula. Este posicionamento foi diferente em cada caso. Na primeira escola a pesquisadora assumiu o papel de observadora, tendo a professora da turma admitido que pretendia realizar as propostas com os seus alunos sem qualquer apoio da pesquisadora. Na segunda escola a professora da turma colocou a aplicação da intervenção nas mãos da pesquisadora e, por fim, na terceira escola existiu uma parceria entre a professora titular da turma e a pesquisadora, onde ambas auxiliaram os alunos no desenvolvimento da intervenção.

Esta intervenção foi desenvolvida através de uma plataforma *Moodle* onde se colocaram oito problemas. Cada problema foi acompanhado de um jogo *online* e de um questionamento. O aplicativo digital e o problema selecionado estão em sintonia; o objetivo de associar a cada problema, um aplicativo foi o de proporcionar aos alunos uma experiência inicial que funcionasse como ajuda para a resolução do problema proposto e não como um tutorial. A opção por proporcionar aos alunos um estímulo e ajuda inicial na resolução de cada problema está relacionada com resultados do projeto de investigação Problem@Web que mostram que grande parte dos alunos envolvidos necessitou de ajuda para resolver o problema proposto. Desta forma procurou-se antecipadamente prevenir dificuldades, propondo este passo inicial.

Participaram nesta pesquisa 72 alunos do 6.º ano do Ensino Fundamental, cujas idades variam entre os 10 e 15 anos, de três escolas estaduais do Vale do Taquari/RS - Brasil. Os alunos realizaram a atividade durante o período de aula, trabalhando em duplas. A recolha de dados foi feita pela investigadora e primeira autora desta comunicação. Cada sessão foi gravada e, posteriormente, transcrita para posterior análise. Para além da observação foram ainda recolhidas todas as produções dos alunos e aplicado um pequeno questionário em cada problema que pretendia conhecer o contributo do trabalho com o jogo na resolução do problema. Todas as produções dos alunos foram colocadas na plataforma *Moodle*.

Breve descrição da intervenção

Para esta comunicação selecionamos um problema de geometria, pois no Brasil, normalmente, este conteúdo é deixado para ser trabalhado no final do ano e muitas vezes o mesmo não é abordado. Assim esse problema foi escolhido para verificar se os estudantes conseguem resolver um problema de geometria relacionado ao seu cotidiano sem preocupação com o tipo de cálculo que será necessário para a sua resolução. Cabe ressaltar, que nesse trabalho não será divido as resoluções em casos específicos, pois ficaria muito extenso.

Como aditivo para este problema foi apresentado o jogo *online* gratuito *Construindo com lego* (Fig. 1) (disponível em http://www.jogosfas.com/jogar-lego-construcccao-id8105.html). Este jogo de lego tem como objetivo a construção de uma casa ou um carro e, para tal, são dadas peças de lego de diversas cores e tamanhos e ainda janelas, portas e outros elementos como mostra a figura 1. Os alunos têm de ir selecionando as peças de acordo com forma da construção que pretendem fazer. O jogo *online* tem muitas semelhanças com os tradicionais legos.

Figura 1. Construindo com lego

Após a exploração do jogo pelos alunos foi apresentado o problema (Fig. 2) e foi-lhes dito que podiam usar qualquer recurso tecnológico, no computador, no celular, ou o papel e lápis. No final teriam sempre que postar a resolução no Moodle.

Problema 6: **Uma torre de Lego**

A Maria tem muitas peças de lego vermelhas e verdes, todas com a mesma forma.

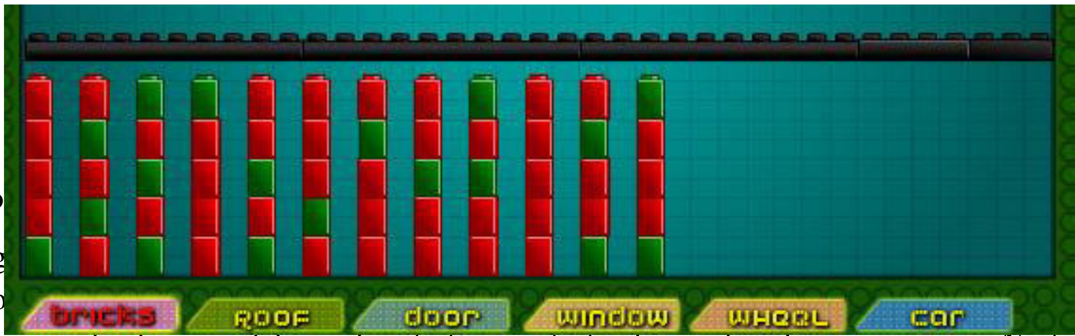
Ela começa a fazer uma torre vertical, encaixando as peças uma sobre as outras. A torre pode ser toda da mesma cor mas não pode ter duas peças verdes seguidas. Quantas torres diferentes com 5 peças poderão ela construir desta forma? Fonte: http://fctec.ualg.pt/matematica/5estrelas/07-08/x.12.6.pdf.

Figura 2 - Problema 6

O problema apresentado tem como objetivo a construção de uma torre de Lego, apresentando assim várias semelhanças com o jogo inicialmente proposto aos alunos. No entanto, para resolver o problema os alunos não dispõem das peças como no jogo, necessitando assim de elaborar uma estratégia, verificando as condições do problema. Há claramente a necessidade de uma abstração e de um raciocínio mais elaborado do que o requerido no jogo.

As resoluções apresentadas pelos alunos foram agrupadas de acordo com as estratégias utilizadas em três grandes grupos.

A maioria dos alunos recorreu ao jogo para encontrar uma resolução para o problema na medida em que o aplicativo possibilitava a construção de uma torre. Poucos alunos encontraram a resposta correta, sendo que cerca de 80% dos participantes apresentaram 12 como solução do problema (Fig. 3).



A opção

> O jog … ploração do co … da, pelo aluno, quando ele joga, elaborando estratégias e testando-as a fim de vencer o jogo. Neste sentido, defende-se a inserção dos jogos no contexto educacional numa perspetiva de resolução de problemas, garantindo ao processo educativo os aspetos que envolvem a exploração, aplicação e explicitação do conceito vivenciado.

Outros alunos optaram por uma estratégia diferente, abandonando o jogo e recorrendo à planilha do Excel para apresentar a resposta ao problema. A planilha foi utilizada para pintar as células (Fig. 4) procurando assim seguir uma estratégia semelhante à que o jogo *online* permitia. Esta estratégia seguida pelos alunos é bastante semelhante à relatada por Carreira et al. (2015). Também na pesquisa do Problem@Web muitos alunos recorreram ao Excel para apresentar as suas resoluções mas para construir tabelas ou pintar as células.

Outros alunos optaram por recorrer ao *Word* para apresentar a resolução do problema, justificando a resposta encontrada. Neste estudo encontramos diversos resultados alinhados com o projeto Problem@Web no que se refere à utilização do *Excel* e do *Word*.

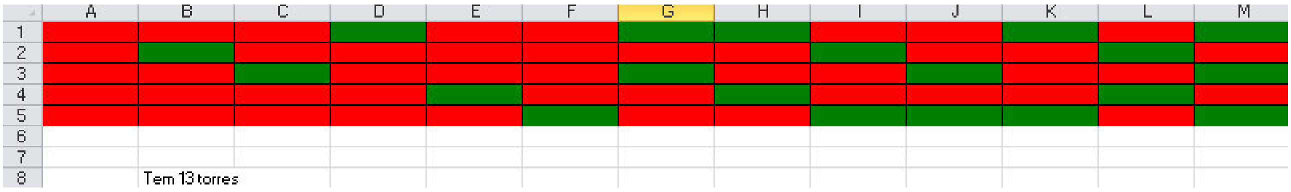


Figura 4. Resolução do problema 6, por meio da planilha de Excel

Tal como referido, após a resolução de cada problema os alunos responderam a uma pequena questão: "o jogo ajudou você a resolver o problema?" Os estudantes dispunham de três possibilidades: muito, pouco ou nada. O gráfico 1 sintetiza as respostas dos alunos neste problema.

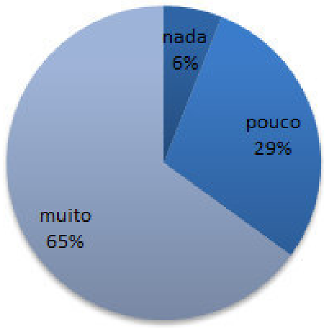


Gráfico 1. Respostas para o questionário do problema 6

De acordo com as respostas dos alunos podemos afirmar que a estratégia de apresentar, inicialmente, um jogo que permite a experimentação, potemcia a resolução do problema resultando em benefícios para a obtenção da solução, na medida em que lhes proporcionou uma "ajuda" para a construção da Torre de Lego.

CONSIDERAÇÕES FINAIS

Esta intervenção proporcionou aos alunos de três turmas uma experiência que aliou a resolução de problemas à utilização das tecnologias. Alguns dos alunos envolvidos nunca tinham tido oportunidade de utilizar o computador em sala de aula. Também a atividade de resolução de problemas é pouco usual nas salas de aula em que foi feita esta proposta. A resolução de problemas é uma atividade altamente recomendada no PCNs no Brasil, tal como cada vez mais se recorre à utilização das tecnologias. No entanto, a implementação em sala de aula desta atividade e deste recurso apresenta inúmeras dificuldades aos professores; talvez por esta razão uma das professoras optou por colocar a intervenção nas mãos da pesquisadora, outra professora optou por solicitar a colaboração da pesquisadora e apenas uma das professoras assumiu esta intervenção sem o seu auxílio. Este trabalho mostra-nos que a resolução de problemas e a utilização das tecnologias, em sala de aula, não é ainda algo em que os professores se sintam confiantes. Desta forma também é natural que os alunos possuam dificuldades na resolução de problemas e na utilização do computador. Este estudo também nos mostrou que o fato de aliarmos a resolução de problemas às tecnologias encoraja e motiva os jovens, mas é igualmente importante reconhecer que a possibilidade de um "jogo" inicial que funciona como ajuda para resolver o problema se mostra uma estratégia eficaz permitindo que os alunos não desanimem na resolução do problema.

Este estudo, embora ainda se encontre em desenvolvimento já nos permite afirmar que é fundamental associar a resolução dos problemas com jogos *online* e com ferramentas no computador que ajudem os alunos. A resolução de problemas é uma atividade muito importante na educação matemática que terá de ser implementada com apoio das tecnologias.

Com essa proposta pretende-se ainda fortalecer o diálogo entre a comunidade acadêmica e os envolvidos no processo educacional. Almejamos que o resultado desta experiência fomente a utilização de recursos digitais em sala de aula e contribua para uma melhoria da implementação da resolução de problemas, primeiramente, pelos professores da Educação Básica do Vale do Taquari e, futuramente, por outros professores do Brasil na medida em que as atividades desenvolvidas estarão disponíveis para que outro professores as possam implementar.

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A CALCULADORA GRÁFICA NA PROMOÇÃO DA ESCRITA MATEMÁTICA / THE GRAPHING CALCULATOR IN THE PROMOTION OF MATHEMATICAL WRITING

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Through writing, students express many of their processes and ways of thinking. Since at high school level some of the activities are carried out with the graphing calculator, we intend to investigate the contribution of this resource to promote the mathematical writing in the learning of continuous nonlinear models at 11th grade. Adopting a qualitative methodology, we collected and analyzed the students' writing productions. What they write when using the calculator gives evidence about the information valued (when they sketch graphics without any justification); about the strategies used (when they define the viewing window and relate different menus on the graphing calculator); and about the reasoning developed (when they justify the information given by the calculator and the formulation of generalizations and conjectures validation).

Keywords: Mathematics Writing; Graphing Calculator; Mathematics Teaching.

INTRODUÇÃO

No acompanhamento da prática pedagógica de futuros professores de matemática, como formadores de professores, constatamos que a calculadora gráfica é mais utilizada como um auxiliar de cálculos do que na formalização de conceitos matemáticos em estudo. Como a calculadora é um material de uso obrigatório na aula de matemática do ensino secundário português e atendendo às dificuldades que os alunos revelam ter na escrita matemática, consideramos pertinente tirar partido deste material na promoção da aprendizagem. A nossa intenção é corroborada pelas recomendações metodológicas do programa do ensino secundário, onde se defende que as estratégias de ensino devem assegurar atividades em que os alunos "descrevam os raciocínios utilizados e interpretem aquilo que se lhes apresenta de modo que não se limitem a 'copiar' o que veem" (Ministério da Educação, 2001, p. 16). A capacidade de comunicar por escrito e aplicar os conteúdos matemáticos por parte dos alunos levou-nos a averiguar o contributo da calculadora gráfica na promoção da escrita matemática na aprendizagem de modelos contínuos não lineares de alunos do 11.º ano.

A atividade de escrever permite estabelecer relações entre conceitos 'antigos' e conceitos 'novos', desenvolvendo o raciocínio e ajudando também na organização do discurso escrito (Morgan, 1998). A escrita é uma boa ferramenta para exercitar a memória, uma vez que muitas discussões orais poderiam ficar perdidas sem o registo em forma de texto. Para Santos (2005), "um estudante que compreende e domina um determinado conceito deve ser capaz de escrever sobre ele, ressaltando suas certezas e possíveis dúvidas" (p. 128). Segundo este autor, a razão para o uso da escrita na aula de Matemática está em procurar organizar o raciocínio, elaborar definições com as próprias palavras, construir exemplos, questionar sobre possíveis dúvidas, interpretar uma determinada ideia, estabelecer conexões e atribuir novos significados a conceitos familiares. Ball (2003) considera que a comunicação escrita tem dois objetivos: registar o método de resolução usado e comunicá-lo a outrem. No desenvolvimento dessas atividades, Lee (2010) defende que a escrita matemática promove o conhecimento e a compreensão de tópicos matemáticos. Colocar uma ideia no papel requer uma reflexão cuidadosa e atenta, ajudando a aprender e a reter os conceitos explorados.

Desde que foi introduzida nos programas de Matemática, em 1997, a calculadora é um dos recursos tecnológicos mais utilizado nas aulas. Com a evolução da tecnologia, os alunos possuem calculadoras cada vez mais poderosas e com mais funcionalidades. Segundo Anderson et al. (1999), a calculadora veio mudar a dinâmica da aula de matemática e a forma de resolver problemas, visto que "tradicionalmente, tanto a formulação do problema matemático como a interpretação da solução eram vistos pelos professores com menor importância em comparação com o processo de resolução do problema" (p. 498). Para estes autores, com a ajuda da calculadora na resolução de problemas, os alunos têm mais possibilidades de desenvolver capacidades cognitivas elevadas. Significa isso, tal como defendem Fernandes e Vaz (1998), que "a simplificação do cálculo permite mais tempo para explorar atividades matemáticas mais profundas e significativas" (p. 44).

Ball e Stacey (2003) defendem que usar a calculadora na sala de aula exige uma atenção cuidada sobre o que constitui um bom registo escrito, fornecendo novas oportunidades na resolução de problemas. Para auxiliar alunos e professores a mudar a natureza dos registos escritos, estas autoras desenvolveram um conjunto de diretrizes para usar na aula de matemática, propondo que uma boa comunicação matemática pode ser alcançada se o aluno: (i) anotar todo o seu raciocínio; (ii) anotar toda a informação envolvida, incluindo a sintaxe da calculadora; (iii) certificar-se de que o plano a seguir é claro; (iv) selecionar a informação, pois nem todos os passos intermédios são necessários. O registo do raciocínio (R), da informação (I), do plano (P) e de algumas respostas (A) – RIPA – promove a comunicação escrita com a calculadora gráfica na aula de matemática.

Neste trabalho, entende-se por escrita matemática o registo que o aluno faz quer da informação que retira da calculadora gráfica, através de várias representações (gráfica, simbólica e tabular), quer do conteúdo matemático presente nessa informação.

METODOLOGIA

Com este estudo pretendemos averiguar o contributo da calculadora gráfica na promoção da escrita matemática na aprendizagem de modelos contínuos não lineares por alunos do 11.º ano de Matemática B. Esta temática integrou a prática pedagógica de uma futura professora, que é um dos autores do texto. Nas suas estratégias de ensino, os tópicos dos modelos não lineares (traduzidos pelas funções exponencial, logarítmica e logística) foram estudados a partir de tarefas que implicavam, na maior parte delas, a utilização da calculadora gráfica. Esta experiência decorreu durante 15 aulas. Atendendo à natureza do objetivo delineado, adotámos uma abordagem qualitativa e interpretativa na procura de compreender as atividades dos alunos na resolução de tarefas em contexto de sala de aula (Bogdan & Biklen, 1994). Com esta finalidade, os dados foram recolhidos através dos registos escritos que os alunos produziram, em pares, na resolução das tarefas propostas com recurso à calculadora gráfica.

Para evidenciar o que os alunos escrevem a partir da utilização da calculadora gráfica, a informação recolhida é apresentada e analisada segundo as seguintes categorias, adaptadas de Ball e Stacey (2003): (i) Informação; (ii) Estratégias de utilização da calculadora gráfica; e (iii) Raciocínio.

RESULTADOS

Seguidamente apresentam-se os resultados organizados segundo as categorias antes referidas.

Informação. Consiste em verificar se o aluno ilustra a informação que retira da calculadora sem apresentar qualquer justificação. Por exemplo, no estudo da concentração dum fármaco no sangue, os alunos transcrevem apenas a representação gráfica elaborada pela calculadora:

Admite que a concentração do fármaco "Saratex", em miligramas por litro de sangue, t horas após a administração a um doente, é dada pela expressão $C(t) = t \times 1{,}05^{-2t}$. Durante quanto tempo a concentração do fármaco no sangue é superior ou igual a 2,5 miligramas por litro?

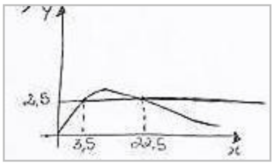

Figura 1. Representação gráfica, sem resposta escrita (P2).

Nesta resolução o par P2 reproduziu o gráfico na folha, indicando os pontos da interseção dos gráficos das funções $y_1 = 2{,}5$ e $y_2 = x \times 1{,}05^{-2x}$, mas não apresentou qualquer resposta. Este tipo de situações, em que apenas é apresentada a representação gráfica sem dar a resposta à tarefa, aconteceu mais nas primeiras aulas. Com o hábito de utilizarem a calculadora gráfica e, sobretudo, de registar e interpretar o máximo de informação que retiravam desta, os alunos aperceberam-se da importância da estratégia que delineavam no ato de recorrer à calculadora.

Estratégias de utilização da calculadora gráfica. Para além da mera informação que retira da calculadora, ao nível da representação gráfica, o aluno tem que ser capaz de definir corretamente a janela de visualização e relacionar diferentes menus da calculadora gráfica.

Definição da janela de visualização. À medida que os alunos adquirem habilidade em trabalhar com a calculadora, apercebem-se da importância de considerar a janela de visualização que se adequa ao contexto do problema, como exemplifica a resolução dos alunos à seguinte tarefa:

A quantidade Q de cafeína num indivíduo, t horas após a ingestão da mesma, é dada pela expressão $Q = Q_0 \times a^{-t}$. Um indivíduo tomou uma chávena de café que contém 80mg de cafeína. Sabe-se que o tempo de semivida da cafeína no organismo é de, aproximadamente, 4 horas.

Informação suplementar: A semivida de uma substância cuja quantidade decresce é o tempo necessário para que essa quantidade passe a metade. Admite que para valores inferiores a 15mg de cafeína no organismo a mesma deixa de exercer efeitos estimulantes. Determina graficamente, recorrendo à calculadora, o período de tempo em que a cafeína funcionou como estimulante. Apresenta o resultado em horas e minutos (os minutos arredondados às unidades).

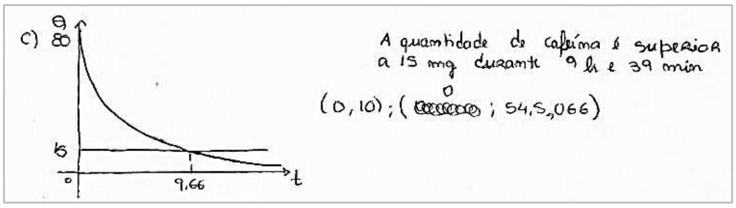

Figura 2. Esboço gráfico que traduz o momento em que a cafeína deixa de exercer efeitos (P5).

Nesta resolução, os alunos tiveram em conta o domínio da função segundo o contexto do problema: a variável dependente é o tempo (t), logo não consideraram o semieixo negativo Ox, e a variável independente representa a quantidade de cafeína presente no organismo de um indivíduo (Q(t)), não fazendo sentido considerar o semieixo negativo Oy. Atendendo a estas características, os alunos indicaram a janela de visualização que utilizaram, embora num formato que não é usual (trocaram os parênteses retos pelos curvos), o que permite ao professor ou aos seus colegas obter o mesmo esboço gráfico. Como era pedido o período de tempo em que a cafeína funciona como estimulante, sabendo que esta deixa de exercer esse efeito para valores inferiores a 15mg de cafeína, representaram graficamente as funções $y_1 = 15$ e $y_2 = 80 \times 2^{-\frac{t}{4}}$ para determinar o ponto de interseção dos respetivos gráficos. Nesta representação, a maior parte dos alunos definiu uma janela de visualização que lhes permitiu interpretar e responder à questão colocada.

Relacionar os diferentes menus. Com a calculadora gráfica, os alunos têm acesso a diferentes menus, tais como o de Estatística, Cálculo, Gráfico, Tabela, Dinâmico. Na resolução de algumas tarefas propostas, os alunos poderiam recorrer ao menu gráfico e ao menu tabela. Nem sempre foi preciso recorrer simultaneamente a estes dois menus, o que só aconteceu em função da natureza da tarefa proposta. Por exemplo, no estudo do número de Neper importava que os alunos estabelecessem conexões entre a informação expressa quer pelo menu Gráfico quer pelo menu Tabela para compreenderem como se obtém este número através da resolução da seguinte tarefa:

A Maria depositou 6000€ num banco à taxa anual de 10% de juros. O banco pratica capitalizações (número de vezes que os juros são acrescentados ao capital, ao longo do ano) variáveis: anual, semestral, trimestral, mensal e diária. Analisa o que acontece se a Maria pudesse optar por uma capitalização a todo o instante.

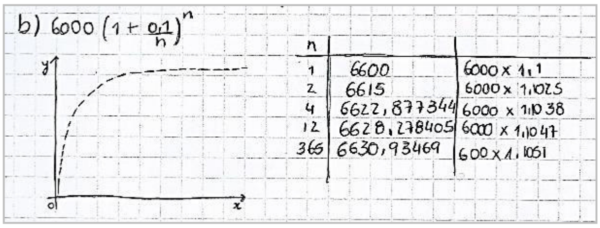

Figura 3. Relação entre diferentes menus da calculadora gráfica (P4).

Os alunos puderam assim determinar, através do confronto entre os valores da tabela e a visualização da representação gráfica da situação, o fator de capitalização do dinheiro depositado no banco a todo o instante. Neste registo, os alunos indicam as variáveis nos eixos, novamente x e y, mas não traçam a assíntota horizontal porque a calculadora não lhes fornece essa informação. Por outro lado, ao traçar o esboço do gráfico a partir da origem não consideram que o zero não faz parte do domínio da função. Quando se atribui à variável independente valores na vizinhança de zero, à sua direita, a função tende para valores próximo de 6000 e não de zero. Este lapso registado pelos alunos deveu-se ao facto de na janela de visualização que consideraram não ser possível 'ver' o eixo das abcissas. Ao transporem a informação da calculadora para o papel não revelaram capacidade crítica para confrontar o esboço obtido com o comportamento dos valores expressos na tabela.

A discussão da tarefa serviu para introduzir o número de Neper através da visualização do comportamento do gráfico da sucessão $\left(1+\frac{1}{n}\right)^n$, dos valores expressos na tabela e do enquadramento dos termos da sucessão, como mostra o seguinte registo efetuado por um par de alunos:

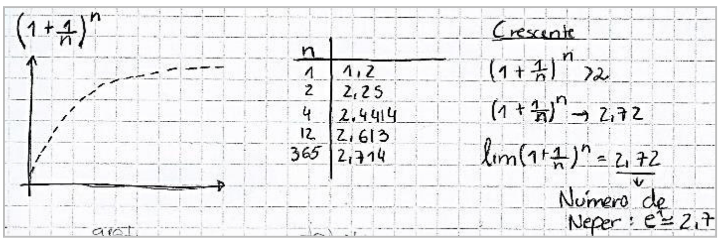

Figura 4. Introdução do número de Neper (P4).

Nesta resolução, os alunos usaram três tipos de representação (gráfica, tabular e analítica). Na tabela, os alunos aperceberam-se de que à medida que o valor de *n* aumenta, os termos da sucessão aproximam-se de 2,7. No gráfico, não apresentaram as variáveis dos eixos nem a assíntota horizontal. A associação da informação presente nestas duas representações levou os alunos a registarem que a sucessão é estritamente crescente, logo o conjunto dos seus termos é minorado pelo primeiro, determinado erradamente na tabela, e majorado pelo limite dos termos da sucessão.

Raciocínio. A utilização da calculadora gráfica impele o aluno a verbalizar, apresentar e registar os seus raciocínios, formular generalizações e validar conjeturas.

Verbalizar, apresentar e registar raciocínios. Relativamente à forma como o aluno dá a conhecer o seu raciocínio no que regista no papel, destacam-se dois procedimentos: (i) recorrer somente à verbalização para explicar o pensamento da interpretação da informação que retira do gráfico representado na calculadora; e (ii) registar simultaneamente a representação gráfica da calculadora e a verbalização de como interpreta a informação. Estes procedimentos verificaram-se, por exemplo, na resolução de pares de alunos distintos à seguinte tarefa:

2. No início de 1972, havia quatrocentos lobos num determinado parque natural. As medidas de proteção fizeram com que o referido número aumentasse continuamente. Os rescursos do parque permitem que o número de lobos cresça até bastante perto de um milhar, não permitindo que este valor seja ultrapassado.
Nestas condições, apenas uma das expressões seguintes pode definir a função P que dá o número aproximado de lobos existentes no parque natural, *t* anos após o início das medidas de proteção:

(A) $\frac{1000}{1+e^{-0,5t}}$ (B) $\frac{1000}{1+1,5e^{-0,5t}}$ (C) $\frac{1200}{1+2e^{-t}}$ (D) $1000-\frac{600(t^3-1)}{e^t}$

Qual é a expressão correta? Numa composição, com cerca de dez linhas, explica as razões que te levam a rejeitar as outras três expressões **(apresenta três razões diferentes, uma por cada expressão rejeitada)**.

Na resolução desta tarefa, houve alunos que recorreram à representação gráfica de cada uma das funções que não 'saiu' do ecrã da calculadora, aliada à informação que é fornecida no enunciado, para justificar as razões que os levaram a rejeitar cada uma das expressões.

2
(A) = 500
(B) = 400
(C) = 400
(D) = 1600

Para determinar a expressão correta, substituimos nas diferentes expressões o t por 0 e observamos as que para t=0 davam 400. A primeira não podia ser a expressão correta pois para t=0, y=500. A terceira também não pois, embora para t=0, y=400, a função ultrapassa y=1000. O último também não porque para t=0, y=1600. Logo, a resposta é a (B).

Figura 5. Respostas dos alunos sem gráfico (P5).

No registo escrito do par P5, os alunos analisaram o número de lobos no instante inicial, rejeitando assim as opções A e D. Para decidir entre as opções C e B determinaram a assíntota horizontal, o que lhes permitiu verificar que o gráfico da expressão C ultrapassa o milhar de lobos estipulado. Neste caso, recorreram à calculadora, mas sem transcrever o esboço do gráfico, não tendo também apresentado justificações diferentes para cada expressão como era pedido. Outros alunos recorreram simultaneamente à representação gráfica e à verbalização das suas justificações para dar a conhecer os seus raciocínios, como exemplificam os registos elaborados pelo par de alunos P7 (Figura 6).

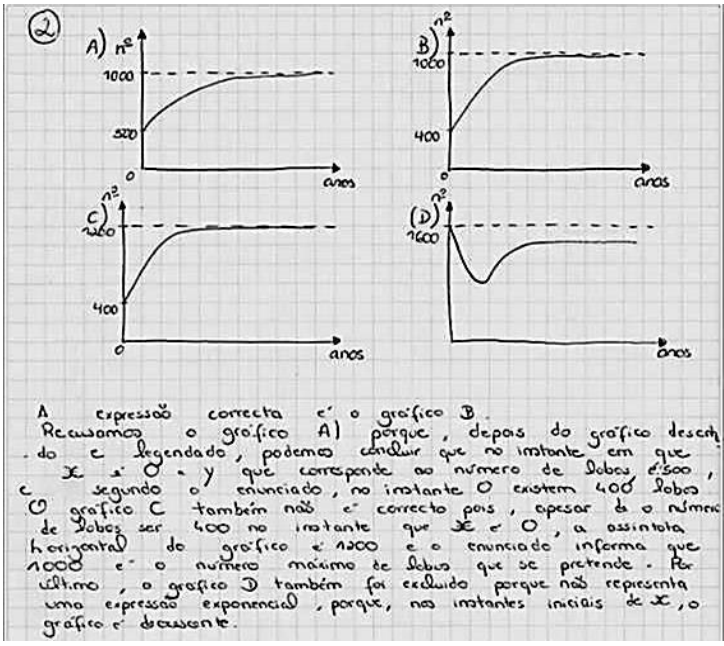

Figura 6. Representação gráfica com justificação dada pelo par de alunos P7.

Nesta resolução, o par representou graficamente as quatro expressões e só depois tirou conclusões, indicando a opção correta e um motivo para rejeitar as restantes. Relativamente aos gráficos, denota-se algum rigor dos alunos na representação, tendo em conta o domínio da função, a assíntota dos gráficos de cada uma das funções e a indicação das variáveis nos eixos segundo a contextualização do problema. Na justificação que os alunos escreveram, apresentam um motivo para rejeitar cada um das opções incorretas.

Formular generalizações e validar conjeturas. Quando os alunos encontram um modelo para uma dada situação, têm de ver se é o que melhor se adequa, analisando, por exemplo, o coeficiente de correlação, como se verificou na resolução da seguinte tarefa:

Num estudo realizado em Portugal sobre o número de infetados pelo vírus da SIDA, efetuou-se a primeira recolha de dados no ano de 1988. A tabela seguinte apresenta os dados relativos ao número de infetados pelo vírus da SIDA em Portugal entre 1988 e 1996.

Anos	Nº de infetados
1988	522
1989	891
1990	1389
1991	2032
1992	2934
1993	3940
1994	5189
1995	6764
1996	8789

Representa graficamente os dados e analisa a evolução do número de pessoas infetadas ao longo deste período de tempo. Que modelo melhor se ajusta aos dados registados?

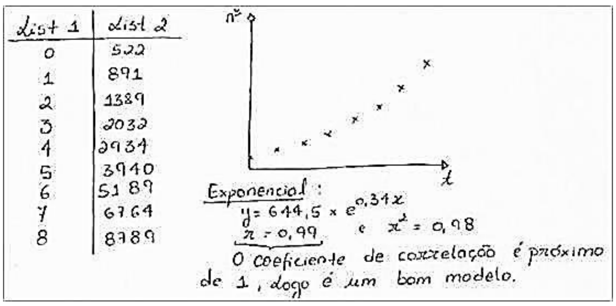

Figura 7. Modelo exponencial como sendo o que melhor se ajusta aos dados (P1).

Os alunos teriam de recorrer ao menu de Estatística, inserir os dados nas listas, considerando o ano 1988 como sendo o ano 0, o ano 1989 como sendo o ano 1 e assim sucessivamente. Analisando a nuvem de pontos, já conseguiriam ter uma ideia de qual seria o melhor modelo.

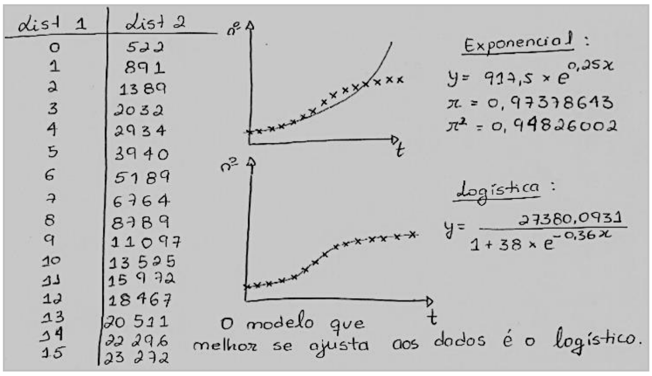

Figura 8. Modelo logístico como sendo o melhor que se ajusta aos dados (P1).

Os alunos introduziram os dados nas listas estatísticas da calculadora gráfica, recorreram ao modelo exponencial, afirmando que o coeficiente de correlação está muito próximo de um, sendo um bom modelo para os dados apresentados. Experimentaram outros modelos, recorreram novamente ao valor do coeficiente de correlação ou mesmo ao facto de passar por todos os pontos marcados, validando assim o modelo exponencial como sendo o que melhor se ajusta aos dados. Com base nestes resultados, os alunos acrescentaram mais dados à lista e procuraram ver se o mesmo modelo se adequava aos novos dados. Ao acrescentarem os dados na lista da calculadora gráfica, a nuvem de pontos foi alterada, como se observa no registo do par P1. Apesar de o coeficiente de correlação continuar próximo de um, o esboço do gráfico não passa em todos os pontos. Tentaram, então, o modelo logístico, traçaram o gráfico e verificaram que passava em todos os pontos, validando o modelo e considerando como sendo o que melhor se ajusta a estes novos dados.

CONCLUSÕES

Da análise e interpretação dos dados recolhidos, constata-se que os alunos recorrem à calculadora gráfica para transcrever para o papel informação, estratégias de utilização da calculadora e expressar o seu raciocínio. Relativamente à informação que retiram da calculadora, os alunos registam sobretudo os esboços gráficos de modelos contínuos não lineares estudados sem qualquer verbalização escrita. Trata-se de um primeiro nível de utilização deste recurso, meramente instrumental, sobretudo quando as expressões que definem as funções não são familiares aos alunos. Como referem Waits e Demana (1994) e Dallazen e Scheffer (2003), a calculadora tem uma função utilitária na exploração de novos conceitos e procedimentos, bem como na resolução de tarefas que seria impraticável por outros meios.

Ao adquirirem uma maior habilidade na utilização da calculadora, os alunos manifestam cada vez mais à vontade em trabalhar com este recurso, o que se traduz no registo das estratégias a que recorrem para tirar partido das potencialidades gráficas e numéricas da calculadora. Nessas estratégias destacam-se a definição da janela de visualização e a articulação entre os diferentes menus da calculadora. A definição da janela de visualização da calculadora é uma das estratégias mais importantes a desenvolver nos alunos (Consciência, 2013). Inicialmente, os alunos tendiam a representar graficamente uma função sem atender à definição dos intervalos que lhes permitia perceber o comportamento dessa função. Posteriormente começaram a atender à janela de visualização, mas nem sempre registavam os intervalos que consideravam, o que indicia que esses intervalos resultavam de tentativas. A perceção acerca do aspeto gráfico de determinada expressão, da ordem de grandeza dos valores correspondentes às variáveis que contextualizam uma dada situação e das características da própria função são, segundo Consciência (2013), determinantes no registo dos intervalos da janela de visualização. No que diz respeito à articulação entre diferentes menus da calculadora, os alunos inicialmente utilizavam sobretudo o menu gráfico, e posteriormente estabeleceram conexões entre o menu gráfico e o menu tabela e entre o menu gráfico e o menu estatístico. Para Rocha (2002), esta capacidade desenvolve-se com a forma como os alunos integram a calculadora nas suas atividades de estudo sobre funções.

Relativamente à forma como organizam as suas respostas e dão a conhecer o seu raciocínio, os alunos, numa primeira fase, recorreram à calculadora para elaborar justificações sem apresentar o respetivo esboço gráfico. Numa segunda fase, tendem a registar simultaneamente a representação gráfica da calculadora e a verbalizar a forma como interpretaram a informação. Os alunos apercebem-se da importância que tem o confronto entre o que se pensa e regista com o que se vê. Para Ball e Stacey (2005), uma boa utilização da calculadora gráfica não dispensa o aluno de ter que pensar sobre a informação nela obtida, importando confrontar o que se obtém com o que se conhece.

Potenciar a capacidade de raciocínio dos alunos decorre também da formulação de generalizações e validação de conjeturas. Na generalização de um dado modelo, os alunos tiveram a oportunidade de discutir o que melhor se ajustava aos dados apresentados. Por vezes, o modelo que conjeturavam como sendo o ideal para o problema em estudo era refutado através da comparação do coeficiente de correlação de outros modelos. Para além desta comparação, validaram as suas conjeturas através da análise dos esboços gráficos dos modelos encontrados. O recurso a argumentos visuais na escrita matemática faz com que os alunos, como recomenda Lee (2010), sintam a necessidade de ter a certeza do que interpretam e do que validam.

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ABORDAGEM DA CONVERGÊNCIA DE SEQUÊNCIAS INFINITAS EM AMBIENTES INFORMATIZADOS VISANDO À CORPORIFICAÇÃO DO CONCEITO / APPROACH TO INFINITE SEQUENCE CONVERGENCE IN COMPUTER-BASED ENVIRONMENTS AIMING AT THE EMBODIMENT OF THE CONCEPT

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This paper presents part of a master's research, which used a qualitative methodology and aimed to verify whether the development of activities based on the embodiment of concepts – with the use of software – favors the understanding of sequences convergence and numerical series and assists the transition from elementary to advanced mathematical thinking. However, only the analysis of the sequences will be presented here. Exploration activities were implemented and built based on the theoretical frameworks of Advanced Mathematical Thinking and the Three Worlds of Mathematics, which sought the embodiment of the convergence concept through the software's manipulation and visualization. Data analyses lead us to believe that the activities promoted the embodiment of the convergence concept and offered opportunities for the development of the elementary to the advanced mathematical thinking.

Keywords: Tree Worlds of Mathematics; Advanced Mathematical Thinking; Convergence; Sequence; GeoGebra.

INTRODUÇÃO

É notória a reconhecida importância atribuída à disciplina Cálculo Diferencial e Integral nos cursos da área de exatas. Autores como Baruffi (1999) consideram o curso de Cálculo como básico, amplo e integrador e outros como Franchi (1993) destacam sua importância como linguagem para representação de fenômenos da realidade e como ferramenta para resolução de problemas. Porém, o seu ensino também tem sido amplamente discutido, devido ao alto índice de reprovação e evasão. Segundo Catapani (2001) estudos apontam que as causas do problema vão desde a forma tradicional de ministrar a disciplina até a falta de motivação de professores e alunos. A falta de conceitos básicos de Matemática do estudante também é apontada por Rezende (2003) como dificuldade.

Entre os temas tratados no Cálculo estão: as sequências e as séries infinitas. Para Bagni (2005) a abordagem destes conteúdos em sala de aula não é simples, uma vez que os alunos têm dificuldades ao passar das operações finitas para o infinito. Percebemos que o entendimento do comportamento de sequências e séries infinitas requer um pensamento matemático qualitativamente diferente do utilizado antes do aluno ingressar em um curso superior.

Diante do cenário descrito surgiu o interesse de se fazer uma pesquisa de mestrado (realizada pela primeira autora deste artigo e orientada pela segunda autora), cujo objetivo principal era verificar se o desenvolvimento de atividades baseadas na corporificação dos conceitos, com a utilização de *software* de geometria dinâmica, favorece a compreensão da convergência de sequências e séries numéricas infinitas e auxilia a transição do pensamento matemático elementar para o avançado. Para a concepção e análise das atividades desenvolvidas nos apoiamos no quadro teórico *Pensamento*

Matemático Avançado (*Advanced Mathematical Thinking*) e, em especial, no denominado *Três Mundos da Matemática* (*Three Worlds of Mathematics*) que descreveremos na próxima seção.

Para este artigo trazemos um recorte da pesquisa que foi realizada no segundo semestre do ano de 2011, com um grupo de alunos do curso de Engenharia de Produção que cursava a disciplina de Cálculo II em um Instituto Federal de Ensino, numa cidade do interior do estado de Minas Gerais, Brasil, sob a responsabilidade da pesquisadora e primeira autora deste artigo. A metodologia adotada teve cunho qualitativo. Para a coleta de dados foram utilizados: registros dos alunos das resoluções das atividades, gravações de áudio, gravações das telas dos computadores e notas de campo da pesquisadora. Foram realizadas aulas de laboratório (utilizando o *software* GeoGebra), com a aplicação de atividades exploratórias, para os alunos trabalharem a corporificação dos conceitos de convergência, por meio da experimentação e formulação de conjecturas que foram discutidas, refutadas e/ou confirmadas em aulas teóricas.

O PENSAMENTO MATEMÁTICO AVANÇADO E OS TRÊS MUNDOS DA MATEMÁTICA

Dreyfus (1991) coloca o pensamento matemático avançado como um processo complexo que consiste em um conjunto de processos de aprendizagem que interagem entre si, tais como: representar, visualizar, generalizar, classificar, conjecturar, induzir, analisar, sistematizar, abstrair ou formalizar. Para ele a distinção entre o pensamento matemático elementar e o avançado está na complexidade e na forma como o conceito matemático é tratado. Além disso, para passar de um nível para o outro são necessárias a abstração e a representação, pois com esses dois processos é possível gerenciar a complexidade da Matemática. A abstração pode ser considerada como o processo mais importante, pois se um aluno desenvolve a capacidade de conscientemente abstrair a partir de situações matemáticas, terá alcançado um nível avançado do pensamento matemático.

Dreyfus (1991, p. 30) considera a possibilidade de utilização de tecnologias. Para o autor ao utilizar um ambiente de aprendizagem computadorizado podem surgir novas ideias e pode ocorrer o reconhecimento de relações até então desconhecidas. Neste tipo de ambiente muitos dos relacionamentos que em geral estão implícitos, como diferentes representações para um mesmo conceito, podem ser explicitados, contribuindo para o aluno reconhecer tais relações e relacionar novas ideias, levando à formação de conceitos.

Entre os pesquisadores sobre o pensamento matemático avançado está Tall (2002) com o quadro teórico dos Três Mundos da Matemática. No desenvolvimento cognitivo da Matemática, distingue três mundos que interagem entre si e que possuem diferentes formas de demonstração, são eles: o *mundo conceitual-corporificado* (ou somente *mundo corporificado*) que está na base do pensamento matemático e fundamenta-se na percepção e na ação; o *mundo proceitual-simbólico* (ou somente *mundo simbólico*) que é o mundo dos símbolos que utilizamos para o cálculo e manipulações na álgebra, geometria, cálculo, entre outros; e o *mundo formal-axiomático* (ou somente *mundo formal*) que é baseado e é expresso em termos de definições formais.

Segundo Tall (2004), o mundo corporificado "se desenvolve a partir de nossas percepções do mundo e é composto de nosso pensamento sobre as coisas que percebemos e sentimos, não só no mundo físico, mas em nosso próprio mundo de significados mentais" (p. 2). Desta forma, este mundo está baseado nas percepções e reflexões feitas sobre as propriedades de um objeto, que inicialmente é visto e percebido no mundo real, mas depois é imaginado na mente (Tall, 2008). A prova no mundo corporificado pode ser realizada por meio de experimentos onde é possível *ver* o resultado acontecer.

O mundo simbólico inicia-se por ações que são encapsuladas em conceitos matemáticos, usando símbolos que tornam possível mudar de processos de fazer matemática para conceitos de pensar sobre o que fazemos (Tall, 2004). Neste mundo a prova é feita por meio de manipulações de símbolos algébricos e de cálculos numéricos para generalizar ideias.

O mundo formal pressupõe a construção de um sistema axiomático (Tall, 2004). As demonstrações são feitas por meio de prova formal, utilizando-se de axiomas e definições para que as deduções sejam feitas.

Para o autor os três mundos interagem entre si e apresentam interseções. Para o desenvolvimento cognitivo da argumentação podemos utilizar no mundo corporificado linguagens cada vez mais sofisticadas para deduzir e definir as propriedades de objetos. Ao tentarmos simbolizar o que foi corporificado ou corporificar o que está descrito de maneira simbólica, estamos na intersecção entre os mundos corporificado e simbólico. O desenvolvimento dentro do mundo simbólico envolve uma compreensão cada vez mais sofisticada das ações sobre a manipulação dos símbolos.

Utilizando-se de linguagens cada vez mais sofisticadas, são desenvolvidos níveis mais avançados de corporificação e simbolismo, sendo que essas experiências podem levar a definição e a dedução de propriedades. É possível passar dos argumentos baseados na experiência para os que compõem um sistema axiomático, presentes no mundo formal.

Concordamos com Tall (1991, 2002) que no curso de Cálculo deve haver uma combinação entre os mundos corporificado e simbólico, não tendo como objetivo unicamente o mundo formal, ficando essa abordagem para um curso de Análise. Defendemos ainda que nos cursos de Cálculo devemos transitar entre os mundos corporificado e simbólico, por meio de atividades. Essas atividades precisam permitir trabalhos que, iniciando no mundo corporificado, possam lançar bases para tratamentos formais posteriores atingindo uma interseção dos Três Mundos da Matemática, que foi chamado por Fonseca (2012) de *base do mundo formal*.

Segundo Tall (2002), ambientes informatizados, com o uso de *softwares* manipuláveis como os de geometria dinâmica, possibilitam abordagens corporificadas, dando fundamento significativo para as mais refinadas ideias da Análise. Além disso, entendemos que um *software* de Matemática Dinâmica, como o GeoGebra, permite que os objetos matemáticos sejam explorados pelos estudantes de maneira diferente das imagens estáticas contidas nos livros didáticos de Cálculo. Essa dinamicidade possibilita organizar atividades diversificadas que podem levar o aluno a realizar experiências de pensamento, favorecendo a corporificação de conceitos de Cálculo (Tall, 2002). Segundo a interpretação de Fonseca (2012), utilizando atividades que permitem a visualização e a manipulação da Matemática, é possível, não apenas a corporificação, como também a proceitualização a partir da reflexão sobre a ação realizada e do uso da simbologia adequada.

Softwares de Matemática Dinâmica como o GeoGebra possibilitam o trabalho com diferentes representações de um mesmo objeto. Dessa forma é possível a constituição de ambientes nos quais o aluno pode representar, visualizar, manipular objetos matemáticos, conjecturar, induzir, generalizar, analisar, sistematizar e até mesmo abstrair sobre algum conceito matemático. Desta forma, este tipo de ambiente, também pode favorecer a passagem do pensamento matemático elementar para o avançado.

É importante explicitar como interpretamos as diferentes formas de prova da convergência de sequências, peculiares de cada um dos Três Mundos da Matemática. Para isso exemplificamos com a sequência $a_n = \frac{2n}{n+1}$. No mundo corporificado é possível provar a convergência por meio da visualização dos termos da sequência em duas formas de representação: como pontos em uma reta e como gráfico de uma função com domínio discreto. A figura 1 exemplifica tais representações, sendo que os pontos do eixo vertical (Q, pontos em uma reta) são as projeções ortogonais dos pontos do gráfico da função (P) sobre o referido eixo:

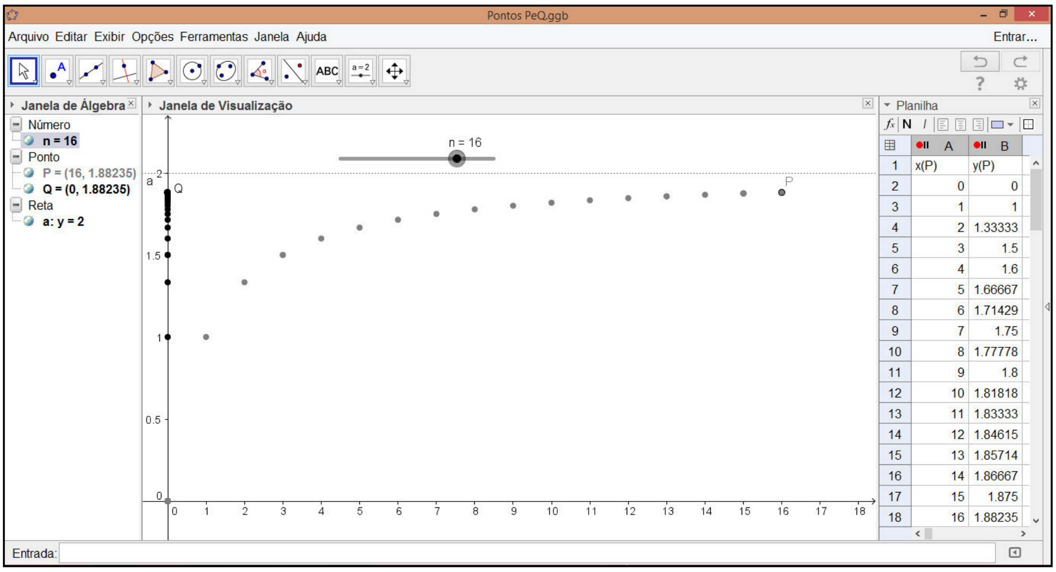

Figura 1. Corporificação da convergência da sequência

A figura anterior pode ser aceita como uma prova corporificada, uma vez que os pontos da sequência se aproximam cada vez mais de dois, sem dar a entender que irão ultrapassá-lo. Também é possível observar por meio dos pontos sobre uma reta, que as distâncias entre os termos da sequência e o possível valor de convergência se tornam cada vez menores, tendendo a zero.

Já a convergência da sequência, no mundo simbólico, pode ser comprovada ao calcularmos o limite de a_n quando n tende ao infinito e analisarmos o resultado final, ou seja,

$$\lim_{n\to\infty}\frac{2n}{n+1} = \lim_{n\to\infty}\frac{2n}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)} = 2 .$$

Para finalizar, a verdade no mundo formal é verificada ao mostrarmos que, para cada $\varepsilon > 0$, existe um inteiro correspondente N tal que, se $n > N$, então $\left|\frac{2n}{n+1} - 2\right| < \varepsilon$.

ATIVIDADES VISANDO À CORPORIFICAÇÃO DA CONVERGÊNCIA DE SEQUÊNCIAS

A pesquisa teve como proposta desenvolver um conjunto de atividades que possibilitasse ao aluno, com base na corporificação do conceito de convergência, construir os conceitos de convergência de sequências e séries infinitas. Como ambiente foi utilizado o *software* de geometria dinâmica GeoGebra. O GeoGebra possibilita representar um mesmo objeto matemático nas formas algébrica, gráfica e numérica, sendo possível manipular os objetos e acompanhar todas as modificações que são feitas quando alguns valores são alterados.

Construímos atividades visando à exploração de expressões, imagens e valores numéricos dos termos de sequências convergentes e divergentes, no contexto dos Três Mundos da Matemática, iniciando pelo mundo corporificado, passando pelo simbólico e buscando alcançar a interseção dos Três Mundos da Matemática, que chamamos de base para o mundo formal. Instigamos os alunos a buscarem a validação de suas conjecturas manipulando as sequências e verificando as respostas apresentadas pelo GeoGebra. Tivemos por objetivo fazer com que os alunos desenvolvessem o conhecimento matemático, inicialmente por meio das percepções e sentimentos vindos da visualização e da manipulação com o auxílio do *software*. Ao solicitar aos alunos que escrevessem as conjecturas por eles elaboradas, justificando-as, buscamos fazer com que a passagem do mundo corporificado para o proceitual ocorresse de forma natural. As definições formais e as provas das conjecturas seriam feitas nas aulas teóricas.

As aulas teóricas foram conduzidas de maneira a fazer com que os conceitos, trabalhados nos mundos corporificado e proceitual, fossem evocados para serem uma introdução ao mundo axiomático. Visamos a todo o momento passar pelos Três Mundos da Matemática, conforme exposto anteriormente, mas também nos preocupamos em desenvolver o pensamento matemático avançado na medida em que as provas características de cada mundo ocorressem.

No GeoGebra os pontos das sequências estudadas foram visualizados como gráfico de uma função, como pontos sobre uma reta e também pelos seus valores numéricos representados em uma planilha.

Tivemos a preocupação de elaborar as atividades exploratórias de forma que o conhecimento do aluno fosse sendo construído aos poucos. Buscamos, a partir da visualização dos termos da sequência, como pontos do gráfico de uma função representados no plano cartesiano, levá-los a observar que ela pode ser representada como uma função discreta com domínio nos inteiros positivos e imagem nos reais. Incentivamos a observação do comportamento de seus termos iniciais, para então, depois de conjecturar uma possível convergência, observar o que acontece com a sequência quando a quantidade de termos aumenta. Só então inserimos uma nova visualização, como pontos de uma reta, que teve por objetivo observar se os valores da sequência estão se aproximando ou se afastando uns dos outros. Acreditamos que, baseados nas visualizações gráficas e nos valores numéricos obtidos na planilha, os participantes poderiam elaborar conjecturas e testar hipóteses para obter respostas às perguntas a eles apresentadas nos roteiros de atividades.

Iniciamos as atividades exploratórias com a sequência $a_n = \frac{5}{n}$, uma vez que ao observarmos os dez primeiro termos, ainda não é possível chegar a uma conclusão sobre sua convergência. Sendo assim, ao pedirmos aos alunos que construíssem apenas os dez primeiros termos e analisassem, em qualquer uma das representações, o comportamento dessa sequência, apareceram muitas respostas do tipo “está diminuindo". Em seguida foi pedido que os alunos aumentassem a quantidade de termos, sendo essa quantidade definida por eles, e novamente analisassem o comportamento da sequência, observando o ponto *P* e os valores numéricos apresentados na tabela. Na gravação do áudio da dupla de alunos A05 e A12, é possível perceber como a utilização do *software* auxiliou na corporificação da convergência desta sequência. Eles perceberam que a sequência estava se aproximando de zero e acreditaram inicialmente que o valor poderia ser igual a zero. Em seguida aumentaram a quantidade de termos para 10000 e desconfiaram que a sequência apenas se aproxima de zero. Para verificar se a conclusão a que chegaram estava correta, aumentaram a quantidade de termos para 90000 e depois para 1000000. Posteriormente a isso, eles concluíram que a sequência apenas tende a zero, como está simbolizado na figura 2.

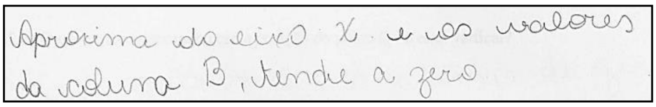

Figura 2. Resposta da dupla A05 e A12 em relação ao comportamento da sequência $a_n = 5/n$

Ainda na atividade anterior, ao serem perguntados como a sequência se comportaria quando os valores de *n* se tornassem cada vez maiores, a maioria dos alunos colocou os termos "tende", "aproxima" e "é igual a" zero, dando uma ideia de que os alunos conseguiram visualizar a convergência da sequência. Importante destacar que os termos "converge" e "diverge" não foram trabalhados anteriormente à realização das atividades.

Além da sequência $a_n = \frac{5}{n}$ foram exploradas outras sequências, escolhidas de forma a contemplar diferentes situações. Exploramos as sequências: $a_n = n^2$, que diverge rapidamente; $a_n = \frac{n+1}{n}$, que converge para um valor diferente de zero; $a_n = (-1)^n \frac{n^2}{2^n}$, sequência alternada convergente; e $a_n = (-1)^n \frac{n}{n+1}$, sequência alternada divergente.

DESTACANDO ASPECTOS DA ANÁLISE DOS RESULTADOS

Para análise dos resultados buscamos estabelecer relações entre a corporificação do conceito de convergência e os processos de proceitualização e axiomatização caracterizados no quadro teórico dos Três Mundos da Matemática. Para isso, buscamos nos nossos dados indícios de apropriação do conceito de convergência em cada um dos três mundos, assim como nas intersecções entre eles. Também analisamos como se deu a transição do pensamento matemático elementar para o pensamento matemático avançado no conjunto das atividades realizadas. Pela impossibilidade de espaço nesse artigo não discorreremos sobre cada um dos aspectos acima mencionados. Traremos, a título de exemplo, alguns dos resultados evidenciados e expressados pelos alunos participantes.

Consideramos para efeito de análise que os alunos que conseguiram *ver* e *perceber*, por meio da manipulação do GeoGebra, que os termos da sequência se aproximam de um determinado valor, corporificaram o conceito de convergência. Entendemos que visualizar é mais do enxergar os termos que estão representados, é realizar experiências de pensamento tentando interpretar o que enxergaram para além do número finito de termos apresentados. Aqueles que utilizaram as palavras "tende", "aproxima" e "vai para", consideramos que chegaram à intersecção entre os mundos corporificado e

simbólico. Isso foi manifestado pelos alunos em muitas situações. Uma delas é a que analisa o que acontece com os valores da sequência $a_n = \frac{5}{n}$ pela observação das representações como função e como pontos de uma reta.

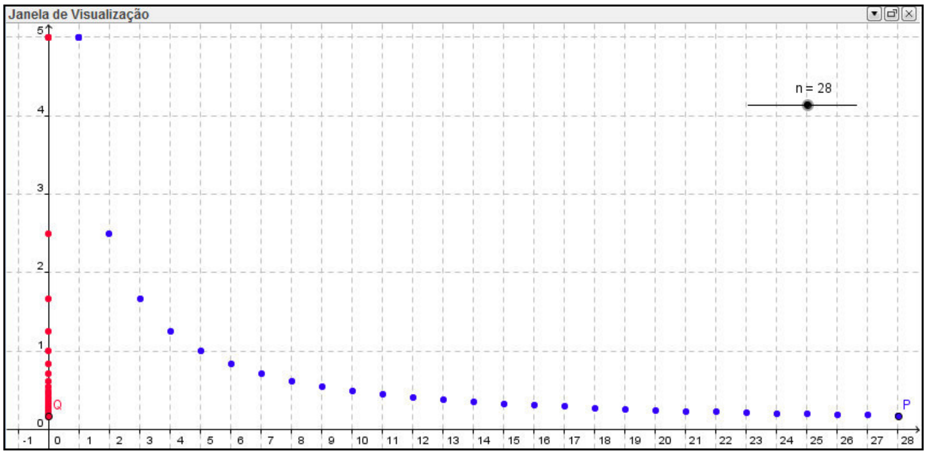

Figura 3: Duas representações gráficas da sequência $a_n = 5/n$

Dois alunos, trabalhando em dupla, variaram o valor de n e dialogaram comparando as imagens desses pontos nas duas representações. Concluíram que havia uma relação entre os pontos P e Q. Que ambos tendiam a zero e que a coordenada y de P e Q era a mesma em cada ponto. Concluíram que os valores diminuem a medida que n cresce, tendendo a zero.

Outra forma de corporificação da convergência seria perceber que os termos da sequência se aproximam cada vez mais uns dos outros. Trazemos uma manifestação de alunos que evidencia essa percepção:

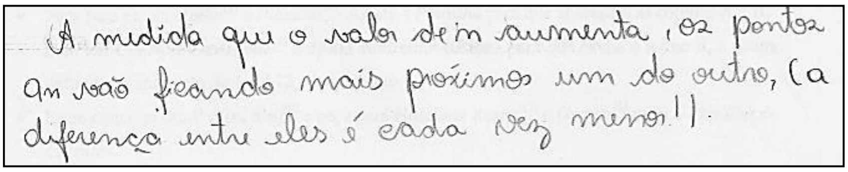

Figura 4. Resposta dos alunos A02, A08 e A17 sobre o comportamento da sequência $a_n = 5/n$

Entendemos essa manifestação de corporificação como uma base para expansão teórica futura do critério de convergência de Cauchy. Tall (2002) denomina isso como raiz cognitiva, ou seja, como algo que sendo significativo para os alunos, contém as sementes da expansão cognitiva para definições formais e posterior desenvolvimento teórico.

Entendemos que os recursos do *software* GeoGebra utilizados nas atividades tiveram influência decisiva no processo de corporificação do conceito de convergência e nos processos de visualização que contribuíram para a formação desse conceito. O GeoGebra possibilitou trabalhar com diferentes representações de maneira integrada e explorar os conceitos do ponto de vista dinâmico, da sequência em construção. Os recursos do *software* possibilitaram aos alunos manipular, experimentar e formular conjecturas sobre o comportamento das sequências. Essas conjecturas eram verificadas pelo aluno ao aumentar a quantidade de termos da sequência. Quando essa prova não ficava clara para algum dos alunos ou quando havia dúvidas sobre as conjecturas, iniciava-se uma discussão entre os alunos que formavam a dupla, de forma que cada um desenvolvia melhor sua argumentação para convencer o outro a respeito do que havia observado. Para responder as perguntas sobre as atividades, os alunos analisavam as diferentes representações, discutiam com os colegas e expressavam suas conclusões. Entendemos que esses alunos alcançaram fases avançados dos processos de aprendizagem indicados por Dreyfus (1991) como característicos do pensamento matemático avançado, a saber: eles representaram, visualizaram, analisaram, deduziram e passaram a ter forte ligação e troca flexível entre as diferentes representações.

CONSIDERAÇÕES FINAIS

As atividades, construídas buscando a corporificação dos conceitos, exigiram uma postura ativa dos alunos, contribuindo para que se tornassem agentes do processo de desenvolvimento cognitivo da Matemática, estimulando a observação e ação sobre o observado, possibilitando a visualização e a construção de imagens mentais dos conceitos, permitindo a formação do conceito de convergência. Os dados nos levam a crer que, grande parte dos alunos, com suas experiências de pensamento, formou a imagem mental de que a convergência de uma sequência ocorre quando os termos da sequência se tornam cada vez mais próximos de um valor, tendo assim corporificado o conceito de convergência e estabelecido uma raiz cognitiva para definição formal da convergência pelo limite e posterior desenvolvimento teórico. Os dados nos levam a inferir que a proposta também contribuiu para uma transição mais tranquila entre os mundos corporificado e simbólico, pois, ao serem instigados a responder questões cada vez mais elaboradas, os alunos tiveram que expressar sua compreensão dos aspectos trabalhados utilizando algum tipo de linguagem e também desenvolveram a argumentação. Entendemos que a transição foi facilitada, pois a corporificação possível por meio da ação sobre o observado facilitou a compreensão e fez com que os alunos tivessem mais convicção de suas conclusões. Além disso, após a realização das atividades, foi possível utilizar as respostas dos alunos para chegar à sistematização dos conteúdos, mostrando a ligação entre o que foi corporificado e sua escrita formal. A análise também nos leva a crer que as atividades facilitaram a transição entre o pensamento matemático elementar e o avançado. Em Fonseca (2012) é possível conhecer mais detalhes sobre a análise realizada.

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FORMAÇÃO CONTINUADA PARA PROFESSORES E ACADÊMICOS: O ESTUDO DA GEOMETRIA EUCLIDIANA POR MEIO DO SOFTWARE GEOGEBRA / TEACHERS' AND UNDERGRADUATES' CONTINUING FORMATION: THE STUDY OF EUCLIDEAN GEOMETRY WITH GEOGEBRA

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Current discussion underscores the results provided in a university extension project on the teaching of Geometry by software GeoGebra. Eleven undergraduates from the Math course and 9 teachers from the high school participated in the project. The project´s final suggestion was the implementation of activities involving Geometry and GeoGebra. It should be emphasized that no teacher took part in this stage due to wariness in the use of the computer and software. In most cases the teachers did not remember the concepts of the matter under analysis, whereas the undergraduates had fewer difficulties in this matter. It should be underlined that the development of the project was basic to show the need for more in-depth knowledge in Geometry and the use of other technologies. The project made possible favorable experiences for the undergraduates´ teaching formation.

Keywords: Euclidean Geometry. Software GeoGebra. Continuing Formation. Basic Education.

INTRODUÇÃO

No atual modelo de educação do Brasil, o processo de escolarização da Matemática, em seu aspecto formal e sistematizado, inicia-se, principalmente nos primeiros anos do Ensino Fundamental (com crianças de 6 a 10 anos), nos quais, por hipótese, devem ser construídas as bases para a formação Matemática dos futuros cidadãos. Diante disto, por meio do projeto de extensão denominado "Formação continuada de professores de Matemática", buscou-se oferecer formação complementar para os acadêmicos do curso de Matemática do Instituto Federal Catarinense – IFC – Câmpus Concórdia, e formação continuada para os professores que atuam nas séries finais do Ensino Fundamental (11-14 anos), no que diz respeito aos conteúdos de geometria e o uso de GeoGebra.

Neste contexto, acredita-se que a formação continuada pode sanar dúvidas quanto aos conteúdos de geometria e contribui para a inserção de recursos tecnológicos na Educação Básica. Em geral, os professores de Matemática desconhecem os softwares de Matemática que podem ser utilizados em sala de aula. Sendo assim, há necessidade de formação continuada para esses professores, que dê conta dos avanços e novas perspectivas para a educação. Benincá (2004, p. 100) destaca que "o professor que não se transforma, atualizando-se, não tem como acompanhar os processos de mudança que ocorrem no mundo".

Com relação aos conteúdos abordados, destaca-se que eles são de extrema importância tanto para a formação do professor como para o acadêmico. No que se refere ao ensino de geometria, Lorenzato (1993) destaca que a geometria está em toda parte, porém é preciso conseguir percebe-lá:

> Mesmo não querendo, lidamos em nosso cotidiano com ideias de paralelismo, perpendicularismo, congruências, semelhanças, proporcionalidade [...] seja no visual (formas), seja pelo uso no lazer,

> na profissão, na comunicação oral, cotidianamente estamos envolvidos com a Geometria (Lorenzato, 1993, p. 5).

Os Parâmetros Curriculares Nacionais (PCN) para o terceiro e quarto ciclos do Ensino Fundamental, abordam que os conceitos geométricos "consistem parte importante do currículo de Matemática no ensino fundamental, porque, por meio deles, o aluno desenvolve um tipo especial de pensamento que lhes permite compreender, descrever e representar, de forma organizada, o mundo em que vive" (Brasil, 1998, p. 51).

FUNDAMENTOS TEÓRICOS, DISCUSSÃO E ANÁLISE DOS RESULTADOS

Este trabalho relata as experiências vivenciadas durante o projeto de extensão cujo objetivo foi estudar e discutir conceitos de geometria euclidiana plana por meio do software GeoGebra. O curso teve duração de 32 horas, sendo seis encontros presenciais de 4 horas e 8 horas destinadas à construção e implementação de uma atividade. Para finalizar o curso, os participantes teriam que produzir atividades e aplicar com alunos do Ensino Fundamental ou Médio, para observar as potencialidades ou não do ensino de geometria por meio de um software geométrico. Iniciaram o curso 11 acadêmicos do curso de matemática e 9 professores. Todos os acadêmicos realizaram todas as etapas do curso; dos nove professores somente dois terminaram a etapa teórica, mas não realizaram a implementação, os demais foram desistindo no decorrer do seu desenvolvimento.

Nos cinco primeiros encontros foram abordados conteúdos da geometria euclidiana plana, tais como: ponto, reta, plano, segmento de reta, ângulos, triângulos, ponto médio, mediana, mediatriz, quadriláteros, polígonos, simetria, construções geométricos, entre outros.

No primeiro encontro também discutiu-se a importância da geometria e da sua relevância para o ensino, bem como para o dia a dia dos alunos. Como expõe Fainguelernt (1999, p. 15) a geometria euclidiana é considerada uma ferramenta para "compreender, descrever e interagir com o espaço em que vivemos; é talvez, a parte da Matemática mais intuitiva, concreta e real". Assim, ela exige do aprendiz uma maneira específica de raciocinar, de explorar e descobrir. Fainguelernt (1999, p. 53) acrescenta que o estudo da geometria:

> É de fundamental importância para se desenvolver o pensamento espacial e o raciocínio ativado pela visualização, necessitando recorrer à intuição, à percepção e à representação, que são habilidades essências para leitura no mundo e para que a visão da Matemática não fique distorcida.

Segundo a autora, a visualização geralmente se refere à habilidade de perceber, representar, transformar, descobrir, gerar, comunicar, documentar e refletir sobre as informações visuais. As noções geométricas podem ser desenvolvidas por meio de experiências vivenciadas pelos alunos, ou seja, é preciso construir, desmontar, montar, mexer, observar objetos para que os alunos aprendam este conhecimento. Porém, mesmo que seja perceptível a importância do ensino deste conteúdo, ainda assim ela é, em geral, deixada de lado pelos professores da escola Básica, sendo um dos argumentos a falta de conhecimentos sobre o assunto. Portanto, pensando em sanar e aprofundar o estudo da geometria elaborou-se este projeto de extensão.

Destaca-se que nos encontros presenciais estudava-se alguns teoremas, definições e posteriormente utilizava-se o software para construir os entes geométricos. Afinal, de acordo com Gerônimo, Barros e Franco (2010, p. 11) "o software Geogebra pode substituir satisfatoriamente o caderno de desenho geométrico". Ressalta-se ainda, que a utilização do GeoGebra torna as aulas de geometria mais dinâmicas e ainda acredita-se que o aprendizado do conteúdo ocorre com mais facilidade. No decorrer do curso, na realização das atividades propostas, ocorreu uma aceitação positiva de todos os envolvidos, porém uma facilidade maior por parte dos acadêmicos da graduação, uma vez que os professores possuíam algumas dificuldades acerca do próprio uso do computador.

No que se refere a inserção da informática na educação destaca-se que no início das discussões sobre o uso de tecnologias da comunicação e informação na educação, muitos professores temiam ser substituídos pelas máquinas, porém, o medo cedeu lugar ao desconforto gerado pela presença das tecnologias (Borba & Penteado, 2001, p. 53-4).

Ponte (2000) expõe que, encontramos entre os professores, atitudes diversas em relação as novas tecnologias. Alguns, desconfiados e inseguros, adiam o máximo possível o uso das novas tecnologias. Outros usam na sua vida diária, mas não sabem muito bem como utilizá-las no contexto escolar. Uma minoria entusiasmada utiliza explorando novos caminhos e ideias. Os professores participantes do curso, ao serem questionados sobre estes aspectos, apontaram que usam muito pouco o computador e outras tecnologias em sala de aula, como justificativas destacaram o pouco conhecimento referente aos softwares disponíveis e também quanto ao uso do próprio computador.

Borba e Penteado (2001), destacam que o professor prefere permanecer em uma zona de conforto, ou seja, o professor procura conduzir a sua prática por um caminho conhecido. Os professores:

> Não se movimentam em direção a um território desconhecido. Muitos reconhecem que a forma como estão atuando não favorece a aprendizagem dos alunos a possuem um discurso que indica que gostariam que fosse diferente. Porém, no nível da sua prática, não conseguem se movimentar para mudar aquilo que não os agrada (Borba & Penteado, 2001, p. 53-4).

Para o professor a zona de conforto é sempre previsível e ele tem dificuldade em avançar "para o que chamamos de zona de risco na qual é preciso avaliar constantemente as consequências das ações propostas" (Borba & Penteado, 2001, p. 54-5). Os professores começaram a desistir quando perceberam a dimensão da zona de risco, ou seja, aprender algo nova para quem sabe futuramente mudar sua prática em sala de aula.

Contudo, a simples presença dos recursos tecnológicos na escola não é garantia de maior qualidade de ensino e os recursos não mudam diretamente o ensino ou a aprendizagem. Eles devem servir para enriquecer o ambiente educacional e propiciar a construção do conhecimento de forma crítica e criativa. A verdadeira função do computador não deve ser a de ensinar, mas sim a de criar condições de aprendizagem e para isso acontecer, necessita-se de professores capacitados e interessados em trabalhar com os recursos disponíveis. De acordo com Lovis e Franco (2013),

> Para que o professor possa utilizar os recursos tecnológicos presentes nas escolas é preciso que ele conheça as possibilidades educacionais destes recursos, uma vez que a sua disponibilidade não garante que ele será utilizado em benefício da educação. Esse fato aponta para uma necessidade de investir na formação e aperfeiçoamento do professor de forma continuada. A formação continuada parece ser um dos suportes mais importantes para o desenvolvimento das competências e saberes relacionados às novas tecnologias e ao seu uso na prática pedagógica. (Lovis & Franco, 2013, p. 152)

Assim, uma das necessidades é investir na formação e na preparação do professor para que ocorra uma utilização coerente das novas tecnologias. O professor precisa conhecer as possibilidades de uso dos recursos disponíveis para utilizá-los como instrumento para a aprendizagem, e ainda, é preciso que o professor saiba utilizar as ferramentas do computador (abrir/salvar arquivos, fazer buscar, etc). A formação acadêmica e continuada do professor é um alicerce fundamental para que isso possa ocorrer.

No entanto, estudar um conteúdo matemático com o auxílio de um software depende da existência de um software que atenda às necessidades do professor na hora de elaborar atividades sobre um assunto específico. Um dos softwares que tem se destacado no ensino de geometria é o GeoGebra.

E para salientar o desempenho do software no contexto das salas de aula, ocorreu o último momento do curso, no qual solicitou-se que os participantes construíssem e implementassem uma atividade com o GeoGebra. Destaca-se novamente que nenhum professor realizou esta etapa do curso. Em conversas informais com os dois professores que terminaram a parte teórica do curso, eles destacaram que não se sentiam seguros para utilizar o software e por isto não fizeram a implementação. As ministrantes responsáveis pelo curso se propuseram a auxiliar os dois professores, inclusive indo até as suas escolas para a realização da implementação, mesmo assim os professores não aceitaram, o que as deixou um pouco chateadas. Uma das justificativas dos professores é o pouco conhecimento acerca do próprio uso do computador, o que dificulta a utilização do software, alegando o medo de imprevistos na realização das atividades, tais como uma falha no computador, alguma página não disponível, a instalação do próprio programa, enfim, como discutiu-se anteriormente os professores temiam adentrar em uma zona de risco a qual eles não conheciam.

Quanto aos acadêmicos, eles procuram as escolas próximas ao IFC e solicitaram, junto à direção e professores, a autorização para a realização da atividade. No último dia do curso eles socializaram os resultados obtidos com o desenvolvimento da atividade.

Destaca-se a seguir as atividades realizadas pelos acadêmicos. Destes, três grupos trabalharam com os conceitos de plano, ponto, retas, segmentos, triângulos equiláteros e ponto médio. Outro grupo abordou a questão de simetria utilizando-se da bandeira do Brasil, aproveitando a Copa do Mundo estava sendo realizada no Brasil. Houve um grupo que não trabalhou com geometria, mas com conceitos de funções, mostrando assim a questão dos gráficos com o auxílio do programa.

Todos os acadêmicos comentaram que obtiveram resultados positivos. Assim, destaca-se que eles conseguiram observar que ao utilizar-se de uma tecnologia é possível dispor um ensino de matemática que fuja da abordagem tradicional. Como evidenciam Burak, Pacheco e Kluber (2010, p. 211):

> Há de se considerarem as formas de utilização do computador e da internet, pois estes são instrumentos que podem criar condições para a superação do modelo tradicional de ensino, uma vez que podem provocar uma nova forma de atuação, independente e diversificada, do professor e do estudante.

Afinal, foi exposto pelos acadêmicos que durante as atividades, os alunos interagiam, perguntavam e sugeriram outras opções de atividades que poderiam ser trabalhadas com o software. Ainda, destaca-se o comprometimento dos alunos frente às atividades propostas.

Alguns acadêmicos aplicaram um questionário durante a realização da implementação para observar qual a opinião dos alunos quanto à utilização do GeoGebra e o ensino de geometria. Quando questionados sobre a utilização do software, todos os alunos destacaram que gostaram de conhecer o programa e acharam mais fácil para compreender o conteúdo. Por fim, destaca-se os comentários de três alunos:

> Aluno A: A oportunidade foi muito interessante, é uma maneira prática de verificar resultados e realizar tarefas mais rapidamente com o programa qualquer cálculo fica mais dinâmico. Acho interessante a adoção desse sistema como complemento de estudo e até mesmo como alternativa de aula, uma vez que é dinâmica e fácil de aprender e realizar cálculos que podem acelerar o resultado ou efetuar uma eventual prova dos nove para qualquer exercício. Gostei muito de conhecer o programa e achei bem fácil de entender e de usar.

> Aluno B: Foi muito boa a experiência para aperfeiçoar os conhecimentos. O programa é muito útil e bem legal de trabalhar, deveríamos ter maior contato com ele e usar na escola para transpor e entender alguns resultados de cálculos do caderno, foi importante o conhecimento do programa para saber que existem formas mais fáceis de se chegar a algum resultado e até tirar as duvidas que surgirem.

> Aluno C: O Geogebra não só é um ótimo programa, como também motiva os alunos a se interessarem na matemática, que é muitas vezes considerada uma das matérias mais difícil. A ideia de utilizar em sala de aula e na escola deveria ser adotada por todos os professores da matéria, muito pelo contrário do que pensam, não seria uma "matação" de aula e sim uma forma dinâmica, divertida e interessante.

Portanto, notou-se que, ao utilizar o software GeoGebra no ensino dos conceitos de geometria existe uma aceitação por parte do aluno e o aprendizado pode ser efetivado. Porém, vale ressaltar, que a utilização do GeoGebra acarretará benfeitorias para a educação, se ele for utilizado com objetivos pré-definidos. E, neste caso, cabe ao professor ou futuro professor, sair da zona de conforto e se arriscar na utilização de metodologias e alternativas para o ensino.

Ao desenvolver atividades com o auxílio do GeoGebra, o aluno tem a possibilidade de construir figuras e analisar se suas propriedades de fato são verificadas, formular argumentos válidos para descrever essas propriedades, fazer conjecturas e justificar os seus raciocínios. As figuras podem ser arrastadas na tela do computador sem perder os vínculos estabelecidos na construção. Além disso, é possível realizar construções que com lápis, papel, régua e compasso seriam difíceis, ou no mínimo gerariam imprecisões.

CONSIDERAÇÕES FINAIS

Nas últimas décadas, várias tecnologias foram criadas para facilitar o dia a dia do ser humano. No contexto escolar, as novas tecnologias foram e estão sendo implementadas com o intuito de complementar e aperfeiçoar o processo de ensino e aprendizagem. Porém, o conhecimento de tais recursos tecnológicos e a maneira de como utilizá-los é fator primordial neste processo de inovação. Pensando nisso o projeto de extensão apresentou um dos softwares de grande importância no que se refere ao ensino de geometria: o GeoGebra.

Neste sentido percebeu-se uma aceitação positiva por parte dos participantes. No entanto, os professores ainda não estão preparados e seguros para o uso deste recurso tecnológico. Quanto aos acadêmicos do curso de matemática, eles não tiveram dificuldades com o uso do GeoGebra e ainda conseguiram construir atividades e realizar sua implementação. Este aspecto foi bastante positivo, porque acrescentou experiências favoráveis para a formação docente. Dos onze acadêmicos, pelo menos seis nunca haviam preparado e ministrado uma aula.

As observações realizadas com os alunos da educação básica nos dão indícios de que o uso do GeoGebra pode aprimorar o ensino da matemática. Os alunos demonstraram interesse pelo software e o consideraram uma ferramenta importante para o ensino de geometria bem como para o ensino de outros conteúdos de matemática.

Com o projeto também buscou-se discutir aspectos do ensino de geometria, além em sanar e aprofundar o estudo de alguns conceitos geométricos que muitas vezes não são abordados em sala de aula. No que se refere aos conteúdos abordados observou-se que os professores, na maioria das vezes, não lembravam os conceitos abordados, os acadêmicos, neste aspecto, tiveram menos dificuldades.

No que diz respeito ao desenvolvimento do projeto, pode-se afirmar que esta primeira etapa foi fundamental para observar a necessidade por parte dos acadêmicos e dos professores no que diz respeito ao aprofundamento do conteúdo geometria e do uso de tecnologias. Notou-se a importância de cursos de formação para ambos os públicos, uma vez que com isto é possível a complementação acadêmica e a troca de experiências entre professores da educação básica, do ensino superior e dos acadêmicos do curso de matemática.

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A INFLUÊNCIA DO REGISTRO FIGURAL SOFTWARE GEOGEBRA NA APREENSÃO OPERATÓRIA E NA PESQUISA HEURÍSTICA DE FIGURAS / THE INFLUENCE OF THE FIGURAL RECORD GEOGEBRA ON OPERATIVE APPREHENSION AND HEURISTIC EXPLORATION OF FIGURES

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This article discusses the influence of GeoGebra software in the operative apprehension and heuristic exploration of geometric figures through the solving of geometry problems. GeoGebra can be considered as a figural record according to the Theory of Registers of Semiotic Representations by Raymond Duval. In this article, we present a theoretical discussion about possible changes in figural records and show the results of a survey performed with some mathematics teachers of Basic Education, from southern Brazil, on the influence of this software on the reconfiguration operation of geometric figures. We concluded that, while solving math problems that come with the visual support of illustrations, GeoGebra software offers best contribution regarding operative seizures and heuristic potential of figures compared to other types of figural records.

Keywords: Geometry. GeoGebra software. Operative Apprehension. Heuristic.

INTRODUÇÃO

Em problemas de matemática, principalmente os que envolvem conceitos de geometria, o uso de imagens pode auxiliar em sua interpretação e resolução. A imagem ou figura pode modificar o significado do texto, oferecendo uma perspectiva específica sobre aspectos a serem considerados para chegar à conclusão necessária.

A figura auxilia a resolver problemas matemáticos por desempenhar um importante papel do ponto de vista cognitivo e na maneira de "ver" e interpretar o problema. Conforme Duval (2012b, p. 286), "as atividades de construção de figura são atividades que privilegiam a formação de representação de um objeto matemático ou de uma situação matemática no registro figurativo".

No que diz respeito aos registros das figuras, Duval (2011) apresenta três características que lhes conferem um poder cognitivo particular. Em primeiro lugar, o seu valor intuitivo, que permite interpretações com um simples olhar sobre a figura; em seguida, proporcionam o reconhecimento de objetos como imagens desenhadas; e, por fim, podem ser:

> Construídas instrumentalmente seja com régua, com o compasso ou com um *software*, pois com um desenho à mão livre não poderíamos nem distinguir uma reta de uma curva, nem verdadeiramente considerar as relações entre grandezas! (Duval, 2011, p. 84).

Diversas pesquisas em Educação Matemática têm mostrado a utilização de recursos tecnológicos para o ensino e a aprendizagem da matemática. Pensando nisso, buscou-se, neste artigo, enfatizar a importância deste recurso, mostrando algumas de suas peculiaridades e funções diante da representação e do estudo de figuras geométricas. Duval (2011, p. 84), acrescenta que, "a construção instrumental das figuras, sobretudo utilizando *software*, confere às figuras uma confiabilidade e uma objetividade que permitem efetuar verificações e observações”.

O *software* GeoGebra é um *software* gratuito que abrange conceitos de Geometria, Álgebra, Cálculo e Estatística. Ele foi desenvolvido inicialmente pelo austríaco Markus Hohenwarter, no ano de 2002. Com esse *software*, é possível construir os elementos básicos de Geometria, além de figuras, gráficos de funções, cálculo de áreas, seções cônicas que podem ser modificadas dinamicamente.

> Uma das vantagens do uso do GeoGebra é que suas construções são dinâmicas [...]. Isso permite que o sujeito faça grande quantidade de experimentações que lhe possibilite construir proposições geométricas (Gerônimo, Barros, & Franco, 2010, p. 11).

A opção pela pesquisa por meio do *software* GeoGebra se deve ao fato de que atualmente, em uma região ao Sul do Brasil, implantou-se um programa governamental de inclusão digital das escolas públicas chamado Paraná Digital. Este programa está fundamentado na disponibilidade de recursos educacionais por meio de computadores e da internet, incluindo o *software* GeoGebra instalado em todos os laboratórios de informática das escolas públicas e universidades.

Para destacar a importância do *software* GeoGebra na resolução de problemas, durante a realização da pesquisa foram utilizados também, Materiais Manipuláveis e Expressões Gráficas, em busca de tentar solucionar os problemas propostos, por meio das figuras, que auxiliassem as deduções matemáticas.

Assim, esta pesquisa busca responder qual a influência do registro figural *software* GeoGebra (SG) quando comparado a outros registros, no caso os Materiais Manipuláveis (MM) e as Expressões Gráficas (EG), ao resolver problemas de equivalência de áreas e partição geométrica.

Neste texto, entende-se por MM, tudo que pode ser manipulado pelo sujeito, permitindo modificações e operações, no concreto, para resolução dos problemas, incluindo o uso de tesouras, colas, cartolinas, réguas, papéis em geral e objetos físicos que representam objetos matemáticos. Da mesma forma, as figuras realizadas por meio das EG fizeram parte da pesquisa. Aqui, entende-se por EG o uso de materiais que auxiliam a construção de desenhos e, principalmente figuras em geral, como, por exemplo, papel, régua, transferidor, esquadro, compasso, lápis etc. As figuras assim construídas formam imagens passíveis de comunicar uma ideia, um conceito ou um pensamento.

A apreensão operatória e a exploração heurística de uma figura

Em se tratando de problemas em geometria, é possível perceber a necessidade da visualização e do reconhecimento de elementos figurais que auxiliem em suas resoluções, já que nem sempre esses problemas são triviais do ponto de vista cognitivo ou matemático. Pensando nisso, chega-se à conclusão que as figuras formam um importante suporte intuitivo para as atividades em geometria, já que permitem visualizar mais do que os seus enunciados e também possibilitam modificações de seus elementos (Duval, 1999).

Essas possíveis modificações de uma figura inicial e as reorganizações dessas modificações compõem a apreensão operatória, e remetem ao papel heurístico das figuras. De acordo com Duval (2012a), a produtividade heurística de uma figura consiste em realizar tratamentos matemáticos específicos ao registro figural. Tais tratamentos estão vinculados com possibilidades de operações e/ou modificações, como por exemplo, modificações mereológicas, óticas ou posicionais, que podem ser realizadas mentalmente e materialmente. Neste último caso, contam com o auxílio de ferramentas que constituirão os registros figurais.

A operação de modificação mereológica faz-se por meio da relação parte e todo, podendo dividir uma figura em várias subfiguras sem alterar suas dimensões e tamanhos. Com o intuito de resolver problemas e como parte da modificação mereológica, a operação de reconfiguração se apresenta como um modo de explorar heuristicamente uma figura geométrica. Fazer a operação de reconfiguração em uma figura implica na reorganização de uma ou várias subfiguras diferentes em outra figura. Desse modo, uma subfigura de dimensão 2 é resultado de reagrupamentos de unidades figurais elementares também de dimensão 2: "A reconfiguração é um tratamento que consiste na divisão de uma figura em subfiguras, em sua comparação e em seu reagrupamento eventual em uma figura de um contorno global diferente" [1] (Duval, 1999, p. 156).

A seguir, tem-se um exemplo de um tratamento puramente figural de reconfiguração que constitui uma representação autossuficiente para o conhecido Teorema de Pitágoras (Duval, 2005, p. 31).

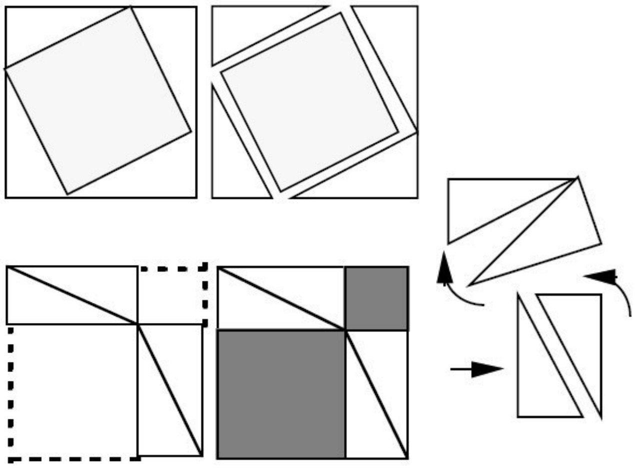

Figura 1. Teorema de Pitágoras

Inicialmente são realizadas operações de separação mereológica na unidade figural inicial (o quadrado) transformando-as em outras unidades figurais (um quadrado e quatro triângulos) – primeira reconfiguração. Em seguida, justapõem-se os quatro triângulos formando dois retângulos – segunda reconfiguração, de modo a obter a mesma região da figura inicial (o quadrado).

Além da operação de reconfiguração, há também o mergulhamento. Este também está ligado às operações mereológicas na apreensão operatória. Duval (2012a) explica que o mergulhamento é inverso à reconfiguração, pois se trata de um prolongamento da figura. Por exemplo, um triângulo mergulhado e dobrado no plano torna-se um pedaço de um paralelogramo.

Por outro lado, no que diz respeito a aumentar, diminuir ou deformar uma figura, refere-se a ações que consistem na produção de sua imagem por meio de uma modificação ótica, permitindo explorar informações por homotetia, por exemplo. E, por fim, se uma figura pode ser deslocada ou rotacionada de acordo com a necessidade, configura-se uma modificação posicional.

METODOLOGIA

Na busca por investigar a influência do *software* GeoGebra nas apreensões operatórias e na exploração heurística de figuras geométricas, foi feito uma pesquisa qualitativa, de caráter interpretativo, na modalidade estudo de caso, que contou com o auxílio de sete professores de

Matemática da Educação Básica de uma região ao Sul do Brasil, que foram submetidos individualmente a uma intervenção com tarefas sobre geometria. O objetivo foi investigar a influência do *software* GeoGebra como um registro figural, quando comparado a outros registros, como os MM e as EG.

Os professores investigados nesta pesquisa participaram antes de sua realização, de um Curso de Extensão intitulado "As Geometrias por meio de Diferentes Representações" cujo objetivo foi proporcionar o conhecimento dos registros figurais na forma de MM, SG e EG dando a oportunidade aos professores que o fizeram de, além de conhecer, trabalhar com esses registros, pois durante todo seu desenvolvimento foram propostas tarefas em sala, de tal forma que os registros fossem utilizados.

TAREFAS

As duas aplicações a seguir usam a operação de reconfiguração na equivalência de áreas e partições geométricas. Tais problemas foram elaborados por Raymond Duval (2012b) e aplicados pelo pesquisador com alunos do *cinquième* [2], porém sem o uso de ferramentas tecnológicas.

Neste trabalho, os problemas foram aplicados individualmente de modo que os professores construíam as figuras e as exploravam contando com o auxílio de três tipos de registros figurais, alternadamente: os Materiais Manipuláveis (MM), o *software* GeoGebra (SG) e as Expressões Gráficas (EG).

Tarefa 1: O problema de Euclides - mostrar a equivalência das partes 1 e 2, qualquer que seja a posição do segmento AB (Duval, 1999, p. 157).

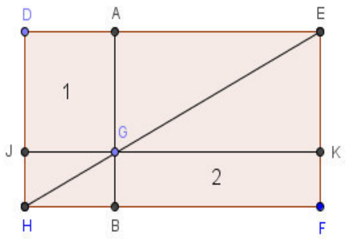

Figura 2. Retângulo

Este problema pode ser resolvido por uma modificação figural do tipo mereológica, fazendo-se uma operação de reconfiguração que consiste no fracionamento da figura inicial em subfiguras. Neste caso, por congruência entre os triângulos $G\hat{A}E \equiv E\hat{K}G$ e entre $H\hat{J}G \equiv G\hat{B}H$, conclui-se que há igualdade entre as áreas dos quadriláteros 1 e 2.

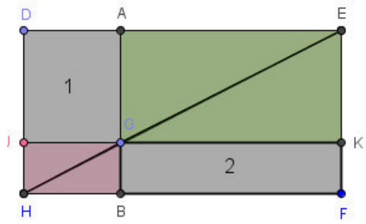

Figura 3. Solução da tarefa 1

Para resolver esta tarefa, disponibilizou-se para os professores participantes, em um primeiro momento, o retângulo construído no *software* GeoGebra de tal forma que o segmento de reta AB pudesse ser movimentado percorrendo a diagonal HE por meio do ponto de intersecção G entre JK e AB.

Como o GeoGebra é um *software* dinâmico, a figura pode ser modificada de infinitas maneiras, a figura 3, mostra três possibilidades:

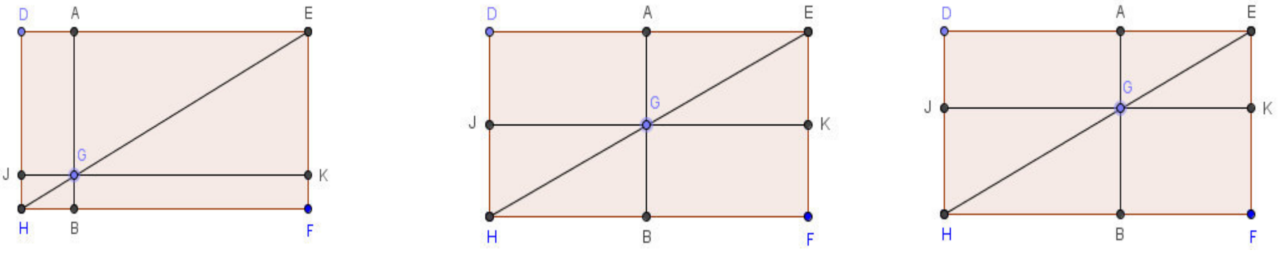

Figura 4. Modificações posicionais no retângulo

Como é possível observar, tal registro é um facilitador para estas operações, tendo em vista que o GeoGebra proporciona essa dinamicidade, não encontrada em outros registros, e, consequentemente, a identificação clara das unidades figurais presentes (ponto de interseção, diagonal, segmentos de reta) e das subfiguras (triângulos e quadriláteros). Além disso, o *software* proporciona uma forte congruência entre o registro da língua natural (o enunciado do problema) e o registro figural (representação da figura no *software*), por permitir visualizar a equivalência das partes 1 e 2, qualquer que seja a posição do segmento AB, conforme o enunciado do problema. Do total de professores, três conseguiram resolver a tarefa utilizando este registro que auxiliou a obtenção de deduções matemáticas.

Aos professores que não conseguiram resolver a tarefa com o *software*, foi disponibilizada, em seguida, uma folha branca com o desenho do retângulo incidindo no registro da Expressão Gráfica. O único professor que resolveu a tarefa neste registro apresentou-se mais suscetível a fazer marcações e tentativas de provar a equivalência das áreas por símbolos matemáticos.

Finalmente, para os três professores que não resolveram a tarefa com o uso do *software* GeoGebra ou com a Expressão Gráfica, foi disponibilizado o recurso em forma de Material Manipulável. Ou seja, um papel cartão no formato do retângulo em questão, de tal forma que o professor pudesse usar materiais adicionais do tipo tesoura, cola, régua, entre outros. Nesta etapa, os três professores resolveram a tarefa, porém de modos diferentes e empiricamente. Um professor, especificamente, recortou a área 1 e a dividiu em subfiguras (também com recorte), encaixando e colando as partes recortadas sobre a área 2, conseguindo "mostrar" a equivalência das áreas. Os outros dois professores utilizaram uma régua graduada e mediram os lados dos quadriláteros 1 e 2, calculando suas áreas e comparando-as.

Com base em todas as operações efetuadas com os registros figurais disponíveis, é possível destacar que o SG proporcionou maior proximidade entre as hipóteses (enunciado) e os tratamentos figurais do problema por dois motivos: a mobilidade do segmento AB e a possibilidade de posicionar este segmento no ponto médio da diagonal do retângulo oferecendo uma visualização de 4 (quatro) subfiguras de mesma área, sendo duas delas as solicitadas no enunciado. Neste sentido, Duval (2012b) explica que existem fatores internos à figura que disparam ou inibem a visibilidade de operações e um deles é a possibilidade da partição da figura em diversas subfiguras. Sendo assim, com o SG é possível visualizar subfiguras em tamanhos maiores ou menores dependendo da localização do segmento AB. Como visto, tal fato auxiliou na identificação da igualdade das áreas 1 e 2. O SG também se destacou por possibilitar uma operação de reconfiguração importante: a modificação posicional de translação do segmento AB e consequentemente proporcionar a pesquisa heurística da figura.

Tarefa2: Fazer a partição deste quadrado em três partes iguais, a partir do ponto médio do lado AB (Duval, 2012a, p. 128).

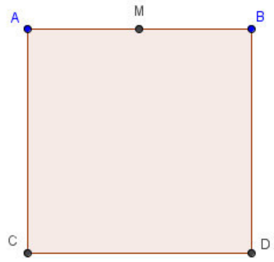

Figura 5. Quadrado

Duval (2012a) relata, em sua pesquisa, que um aluno do *cinquième* efetuou a partição do quadrado em seis colunas iguais, conforme a figura a seguir:

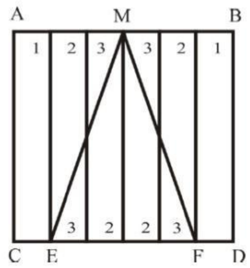

Figura 6. Reconfigurações intermediárias

Conforme a numeração nas unidades figurais da Figura 5, o aluno explicou a igualdade das reconfigurações intermediárias entre AMEC, MFE e MBDF. Observa-se que este é um problema que não oferece uma congruência entre as unidades figurais diretamente visíveis e as unidades figurais necessárias para sua resolução. Isto significa que este problema é complexo tanto do ponto de vista cognitivo quanto do ponto de vista matemático.

Nenhum professor conseguiu resolver a tarefa do mesmo modo que o aluno do *cinquième*, e somente dois professores a resolveram, ambos utilizando o *software* GeoGebra.

Em um primeiro momento, foi entregue a cada professor um papel cartão recortado (MM), com a representação de um quadrado, a marcação de seus vértices e ponto médio M, conforme a Figura 4. Porém, este registro figural dificultou a pesquisa heurística e, consequentemente, as operações na figura, visto que os professores não sentiam segurança para rabiscar e recortar o material, pois não encontravam recursos matemáticos e de medida, para comprovar a igualdade entre as três áreas. Por fim, não houve soluções para a tarefa proposta com este registro figural material.

Logo após, disponibilizou-se aos professores outro tipo de registro figural: o *software* GeoGebra. O mesmo quadrado da Figura 4 foi exposto aos professores de modo que eles pudessem efetuar as operações que achassem necessárias. Dentre os sete professores, quatro afirmaram encontrar facilidade no reconhecimento e na pesquisa junto aos elementos figurais que representavam o problema no *software*. Desses quatro professores, dois chegaram à solução correta. Segue a imagem das duas soluções:

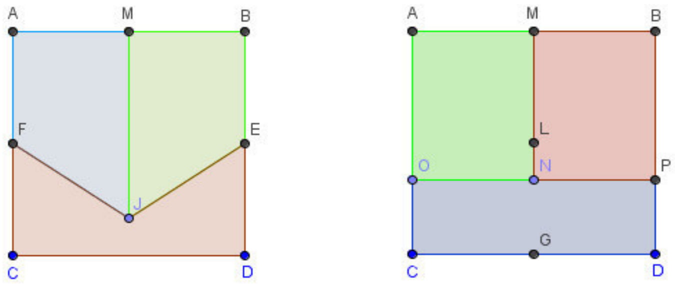

Figura 7. Soluções 1 e 2

Ambas as soluções se apoiaram em conceitos matemáticos, e também em uma ferramenta do *software* GeoGebra que é capaz de calcular com precisão áreas de polígonos, contribuindo para a conclusão correta da tarefa. O *software* possibilitou a liberdade de movimentação e modificação das subfiguras sem perda de vínculos, facilitando, assim, chegar à solução do problema. As unidades figurais também foram facilmente reconhecidas pelos professores.

Por fim, aos professores que não resolveram a tarefa por meio dos Materiais Manipuláveis ou do *software* GeoGebra, indicou-se o uso da Expressão Gráfica. Nesta etapa, dois professores não conseguiram resolver, e um professor chegou ao que ele acreditava ser uma solução, mas incorreta.

Quanto às apreensões operatórias e à exploração heurística da figura neste registro, conclui-se que diversas tentativas de modificações foram feitas, utilizando-se de lápis e régua, porém, sem sucesso.

CONCLUSÕES

Com esta pesquisa, foi possível notar que dentre os registros figurais apresentados - MM, EG e SG, o SG proporcionou tanto para a Tarefa 1 quanto para a Tarefa 2, maior exploração heurística da figura, por oferecer congruência entre os tratamentos figurais e o raciocínio dedutivo.

Além disso, o SG possibilitou operações de reconfiguração em ambas as figuras. Isto é, em habilidades, tais como, modificar uma figura em diferentes posições, visualizá-la, dividi-la em várias subfiguras, calcular suas áreas e, de posse dessas informações, raciocinar matematicamente, o uso do SG obteve destaque. Tais operações puderam ser desenvolvidas mediante este registro figural computacional.

Todas as operações e tratamentos citados têm influência direta na resolução de problemas, logo o registro figural na forma de *software* pode auxiliar na tomada de decisões e conclusões de problemas de geometria.

NOTES

1. "*La reconfiguración e su tratamiento que consiste en la división de una figura en sub-figuras, en su comparación y en su reagrupamiento eventual en una figura de un contorno global diferente*".

2. "A série *cinquième* (7º ano), alunos com idade com 12 ou 13 anos, corresponde ao segundo ano das séries finais do ensino fundamental no Brasil" (Duval, 2012a, p. 125).

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O SOFTWARE MATHEMATICA COMO APOIO AO ENSINO DE CÁLCULO I EM CURSOS DE ENGENHARIA / THE MATHEMATICA SOFTWARE AS A SUPPORTING TOOL FOR TEACHING CALCULUS I IN ENGINEERING COURSES

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The main objective of this work is to present the results of an experiment related to the use of new technologies, experienced at the Calculus I discipline of the Engineering Course at ITA (Technological Institute of Aeronautics). Initially, key issues in the current learning process were identified and it was found that such difficulties are not exclusive to the Brazilian reality. Since the use of new technologies can contribute significantly to the learning process in higher education, key aspects of such approach were identified and the methodology was applied a group of students. We conclude that the new technologies used in a conscious way can contributes to the development of the foundation of mathematical knowledge, engaging the students and stimulating their commitment to the learning process.

Keywords: Teaching Calculus, Mathematica Software, Engineering Courses

INTRODUÇÃO

O ensino de Cálculo na maior parte das universidades brasileiras tem sido objeto de análise em diversos congressos em função das dificuldades de aprendizagem apresentadas, bem como pelos altos índices de reprovação e evasão nos primeiros períodos dos alunos matriculados nestas disciplinas (Wrobel, Zeferino e Carneiro, 2013) que de acordo com pesquisas realizadas chegam até a 77,5% (Garzella, 2013).

De fato, tais índices evidenciam o que Rezende (2003) aponta como o fracasso no ensino da disciplina de Cálculo. Rezende (2003) também destaca que o problema com a disciplina de Cálculo não é uma exclusividade da realidade brasileira, visto que trabalhos sobre esse tema têm sido publicados e destacados por parte da literatura especializada internacional. Para tanto cita dois exemplos que ilustram essa situação. O primeiro foi o movimento em prol da reforma do ensino de Cálculo, iniciado na década de 80, que ficou conhecido por Calculus Reform (ou Cálculo Reformado). Um importante resultado dessa reforma foi o Calculus Consortium at Harvard (CCH) que se tornou um dos currículos mais utilizados para cursos de Cálculo nos Estados Unidos. Alguns princípios desse novo currículo, de acordo com Knill (2009), foram: - combinar uma abordagem gráfica, numérica e algébrica; - incentivar os alunos a partir de problemas práticos; - escolher temas que se relacionem com outras disciplinas do curso; - formular problemas abertos; - desencorajar as técnicas de imitação; - usar a tecnologia para visualizar conceitos estudados;- incentivar o trabalho em equipe.

O outro exemplo são os trabalhos de David Tall, que vão ao encontro à metodologia proposta pela Reforma do Cálculo. De acordo com Tall (2009) o ponto mais importante a ser observado quanto ao uso de novas tecnologias no ensino de Cálculo é a maneira como ela será utilizada e que, "claramente as habilidades de uso de software, tais como Mathematica ou Maple são recursos valiosos em seu próprio direito. No entanto, a forma com que são introduzidos na disciplina é extremamente importante" (Tall, Smith e Piez, 2009).

Desse modo entende-se, como Barufi (1999), que as mudanças ocorridas na maneira de tratar a informação apontam a necessidade urgente de tornar a interação com as novas tecnologias na sala de aula algo tão natural como, em décadas passadas, foi a manipulação do lápis ou da caneta. A realidade do aluno atualmente obriga o professor a olhar o mundo com novos olhos. Tudo agora é mais rápido, mais acessível e a sala de aula não pode ficar alheia a tudo que está presente na atualidade (Barufi, 2009). Desse modo os computadores se tornaram indispensáveis para o desenvolvimento do trabalho em sala de aula, em geral, e especificamente na área de Engenharia.

Desse modo, a experiência descrita neste trabalho utilizou o software Mathematica para ensino de Cálculo I em um curso de Engenharia do ITA teve como orientação as seguintes ideias:

a) combinar uma abordagem gráfica à aula tradicional; b)usar uma ferramenta tecnológica para visualizar conceitos chave como limite, derivada, integral, sequência e série; c) incentivar os alunos a trabalhar em pequenos grupos;

Preocupando-se também em:

a) escolher as ferramentas adequadas ao curso e à instituição; b)que a tecnologia escolhida auxiliasse os alunos na visualização de conceitos de cálculo e ao mesmo tempo contribuísse com a forma algébrica; c) utilizar a tecnologia de forma adequada; d) viabilizar a aprendizagem cooperativa.

Tais pontos podem ser encontrados claramente nas indicações dos trabalhos que ser referem ao Calculus Reform, de acordo com Murphy (2006) e os de David Tall (Tall, Smith e Piez, 2009) que se tornaram norte do desenvolvimento de tal experiência.

UMA EXPERIÊNCIA UTILIZANDO O SOFTWARE MATHEMATICA PARA ENSINO DE CÁLCULO I EM UM CURSO DE ENGENHARIA

No ano de 2010 foi proposto a um dos professores do ITA (Instituto Tecnológico de Aeronáutica) a realização de mudanças substanciais na metodologia desenvolvida na disciplina de Cálculo Diferencial e Integral I. Desse modo, a proposta de mudança não alteraria o cronograma nem a ementa do curso ministrado tradicionalmente na instituição.

Os alunos dessa turma eram do primeiro ano do Curso Fundamental de Engenharia[1] do ITA. O primeiro ponto que caracterizou uma inovação no desenvolvimento da disciplina foi a criação de uma página na Web, onde continha todas as informações referentes à disciplina, tais como: Informações gerais – Ementa do curso, plano do curso, cronograma semanal, horário de aulas, professores; Material didático a ser utilizado nas aulas; Tutoriais para uso do software Mathematica, Links para revistas sobre Ensino de Matemática [2].

O desenvolvimento da página foi importante, pois todo o material teórico, como notas de aula, exercícios resolvidos e listas complementares, estava disponibilizado no site, facilitando o acesso do aluno e maximizando o tempo de sala de aula para discussões de exemplos importantes e aulas laboratoriais.

Outro ponto importante foi o desenvolvimento das aulas, que adotou o software Mathematica como ferramenta de ensino.

Este software foi escolhido por possuir grande aplicabilidade, tanto em disciplinas de graduação, quanto em pós-graduação seja para a resolução de problemas propostos em sala de aula, como também em projetos de pesquisas.

Alguns dos objetivos de se utilizar esse software no ensino são:

a) Despertar talentos; b) Aguçar o interesse dos alunos; c) Estimular a inovação nas disciplinas e projetos de pesquisa; d) Descobrir novas alternativas para resolução de velhos problemas; e) Iniciar pequenos projetos em sala de aula;

Entende-se que, as ferramentas oferecidas pelo Mathematica podem contribuir, consideravelmente, para o ensino de Cálculo nos cursos de Engenharia adequando-se, portando, aos interesses da Instituição.

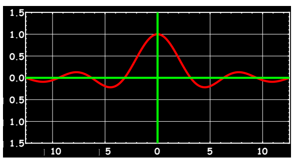
A dinâmica das aulas ocorrida da seguinte maneira: os alunos tinham 5 aulas por semana de Cálculo Diferencial e Integral I, 4 aulas teóricas em sala de aula e uma aula no laboratório de informática, ou como é chamada no ITA, na Sala Inteligente.

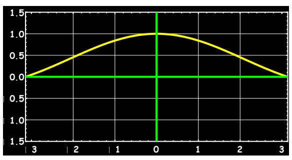
Na Sala Inteligente [3] os alunos desenvolviam exemplos que elucidavam os conceitos desenvolvidos nas aulas teóricas. Tais exemplos eram realizados no Mathematica, assim, ao mesmo tempo em que visualizavam exemplos da teoria estudada, os alunos aprendiam a utilizar os recursos do software.

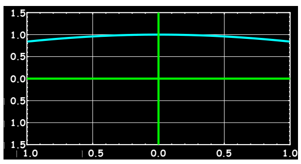
Figura 1: imagem da Sala Inteligente do ITA

A interação entre as aulas teóricas e a prática na sala inteligente, pode ser ilustrada na descrição, por exemplo, de como foi realizada semana de aula onde se introduziu o conceito de Limite. Durante essa semana foram estudados os seguintes pontos: Operações com limites; Caracterização de limite por sequências; Limites laterais; Limites fundamentais; Limites da composta; Limites no infinito e Limites infinitos.

Os conteúdos teóricos sobre esses pontos foram disponibilizados na página do curso, para que os alunos pudessem ter acesso antes mesmo de irem para as aulas teóricas. Nessas aulas o professor discutiu os principais conceitos envolvidos, resolveu exercícios, etc. A última aula da semana foi realizada na sala inteligente, onde foi proposta, com ajuda do software Mathematica, a resolução do limite da função f(x) = Seno[θ]/θ. Para isso foi solicitado que os alunos plotassem no software o gráfico dessa função com o domínio de θ variando entre -4π e 4π, na sequência com o domínio entre -π e π, entre -1 e 1e entre -0,1 e 0,1; para que os alunos analisassem o que estava ocorrendo com essa função conforme se diminuía o intervalo do domínio. Segue a sequência se gráficos obtidos com os respectivos domínios.







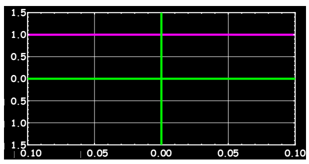


Figura 2: A função plotada com diferentes domínios

Um gráfico mostrando todas as curvas, ao mesmo tempo, também foi construído no software.

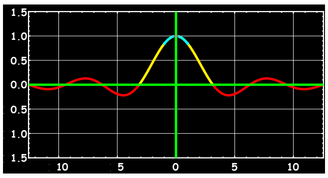


Figura 3: gráfico da função mostrando todas as curvas obtidas

Em seguida, os alunos criaram uma tabela de valores numéricos de θ vindo da direita, para a esquerda, com o valor de θ ‘tendendo’ a $1/10^n$ e também uma tabela com os valores de θ vindo da esquerda para a direita.

```
TableForm[
  Table[{θ, NumberForm[Sin[θ]/θ, 15]} /. θ → 1./10^n,
      {n, 0, 7}],
    TableHeadings → {None, {"θ", "Sin[θ]/θ"}},
    TableAlignments → Center]
```

θ	Sin[θ]/θ
1.	0.841470984807897
0.1	0.998334166468282
0.01	0.999983333416667
0.001	0.999999833333342
0.0001	0.999999998333333
0.00001	0.999999999983333
$1.\times10^{-6}$	0.999999999999833
$1.\times10^{-7}$	0.999999999999998

```
TableForm[Table[{θ, NumberForm[Sin[θ]/θ, 15]} /. θ → -1./10^n,
      {n, 0, 7}],
    TableHeadings → {None, {"θ", "Sin[θ]/θ"}},
    TableAlignments → Center]
```

θ	Sin[θ]/θ
-1.	0.841470984807897
-0.1	0.998334166468282
-0.01	0.999983333416667
-0.001	0.999999833333342
-0.0001	0.999999998333333
-0.00001	0.999999999983333
$-1.\times10^{-6}$	0.999999999999833
$-1.\times10^{-7}$	0.999999999999998

Figura 4: tabelas de valores numéricos

Chegando, finalmente à conclusão do valor do limite dessa função quando θ ‘tende’ a zero:

```
Limit[Sin[θ]/θ, θ → 0] // N

1.
```

Figura 5: resultado numérico dado pelo software Mathematica ao limite da função

O exemplo ilustrado foi apenas uma das atividades realizadas na sala inteligente para ilustrar o conceito de limite de uma função. Desse modo, as atividades realizadas no Mathematica auxiliaram na compreensão do conceito que era estudado anteriormente nas aulas teóricas.

O trabalho bimestral da disciplina de Cálculo I

Nesta etapa da disciplina os alunos foram orientados a realizar um trabalho bimestral, utilizando o software Mathematica e os conceitos estudados durante a disciplina de Cálculo I. O trabalho foi desenvolvido em grupos de 3 ou 4 alunos e o tema do trabalho foi de escolha de cada grupo, ao todo foram entregues 15 trabalhos.

Os trabalhos foram orientados para que contemplassem os seguintes pontos:

a) todos deveriam ser desenvolvidos no software Mathematica; b) teriam que envolver, no mínimo, conteúdos de Matemática desenvolvidos na disciplina de Cálculo I; c) seguir um roteiro básico contendo: Apresentação do problema, desenvolvimento algébrico dos conteúdos matemáticos envolvidos, utilização do software para ilustração gráfica do problema e da sua solução, discussão e análise da solução encontrada.

Podemos destacar inicialmente a diversidade, e até mesmo a complexidade, dos temas escolhidos pelos alunos:

1) A série infinita de Swinwahead; 2) Aplicação de limites envolvendo números notáveis; 3) Artilheiro no ponto A que tenta acertar um algo em movimento; 4) Calcular a área abaixo de uma curva; 5) Cálculo do valor de π; 6) Difração; 7) Equação do Planck, Lei de Rayleigh – Jeans, Lei de Wi; 8) Espirais; 9) Infinitude dos números primos com o auxilio de Séries; 10) Limite do movimento oscilatório amortecido; 11) Limites para resolução de problemas de Cálculo; 12) Números Irracionais notáveis; 13) Permutação caótica utilizando Séries; 14) Série de potência e Série de Maclaurin; 15) Visualização gráfica de testes comparativos entre limites.

Analisando os temas escolhidos pode-se perceber que os trabalhos envolveram os conceitos de limite, séries, sequências, integrais e derivadas. Ou seja, praticamente todos os conceitos estudados na disciplina aparecem, pelo menos em dos trabalhos.

Outro ponto importante é que os trabalhos foram desenvolvidos partindo de problemas aplicados – temas 3, 4, 6 e 10 – como também trataram de problemas puramente matemáticos com certo grau de complexidade – temas 1, 7, 13, 14. Alguns desses trabalhos envolveram também aplicações dos conteúdos de Cálculo em outras disciplinas como Física e Análise – trabalhos 3, 6 e 7. Outros ainda buscaram inspiração na história da Matemática, um conteúdo que não é amplamente difundido no curso de Engenharia, como ocorreu nos trabalhos 11, 12 e 15.

Todos os trabalhos utilizaram o software para ilustrar graficamente os desenvolvimentos algébricos e demonstrações realizadas, realizar comparações entre dois resultados diferentes e para animar as ilustrações gráficas. Segue abaixo trechos de alguns trabalhos que evidenciam o apoio dessa ferramenta.

A figura a seguir mostra os gráficos plotados valores de $n\ sen\ \frac{180°}{n}$, variando o valor de n e, também para $n\ tg\ \frac{180°}{n}$ e, em seguida, realização uma comparação entre dois gráficos mostrando os dois gráficos juntos.

Figura 6: Comparações gráficas – Tema: 5

Já a figura seguinte mostra como os alunos plotaram o gráfico no software e utilizaram o recurso de animação para elucidar os conceitos teóricos envolvidos no tema de Difração.

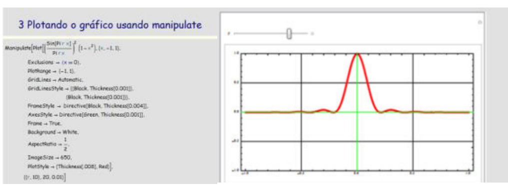


Figura 7: Utilização do recurso de animação – Tema: 6

Esses trabalhos foram utilizados pelo professor da disciplina como parte da nota final dos alunos. Ao se observar a complexidade dos temas escolhidos, as preocupações dos alunos em não só mostrar graficamente os conceitos, propriedades ou teoremas utilizados, mas também demonstrá-los algebricamente, apontam para um importante vínculo entre o uso do software nas aulas práticas e a importância do desenvolvimento da teoria. Os alunos não utilizavam o software como uma verdade única, mas como uma maneira de ilustrar os resultados obtidos algebricamente.

Os parágrafos que se seguem são trechos da conclusão de alguns trabalhos que mostram, de acordo com os alunos, como tal experiência contribui para o seu entendimento dos conteúdos desenvolvidos na disciplina de Cálculo I.

Considerações dos alunos - Tema 5:

"Concluímos, ao final de todo esse trabalho, que, dispondo de uma ferramenta poderosa como o Mathematica 7, ou até mesmo outras melhores, podemos calcular coisas que dificilmente conseguiríamos manualmente, como se mostrou o cálculo com várias casas decimais do π. Além disso, foi muito útil para nosso entendimento o uso de gráficos, tanto estáticos quanto animados, pois eles facilitam a visualização, tornando o aprendizado mais fácil".

Considerações dos alunos - Tema 12

"Concluímos, então, nossa breve explanação sobre os números irracionais π, e, φ mostrando os problemas que motivaram o estabelecimento de cada um deles e usando recursos gráficos para uma melhor visualização da convergência das sequências e séries geradoras dos mesmos".

Considerações dos alunos - Tema 8

"Foi feita uma análise detalhada sobre as principais espirais presentes na matemática e, muitas vezes, na natureza. Puderam-se utilizar recursos gráficos que em muito auxiliaram em evidenciar as diferenças entre cada tipo de espiral, além de permitirem a análise de vários valores plotados. O conceito de limite foi empregado diversas vezes, sendo, em algumas delas, muito avançados, fugindo assim dos objetivos do trabalho e, por isso, não demonstrados, mas apenas analisados graficamente. Aqueles limites que diziam respeito à teoria estudada foram demonstrados e plotados em gráficos de maneira a auxiliar sua compreensão. A visualização clara foi um dos principais preceitos deste trabalho, juntamente com o despertar do interesse pelo assunto tratado, tão presente no meio natural. Por fim, as espirais mostraram-se um interessante ramo de aplicação dos fundamentos do Cálculo e verdadeiros conjuntos de propriedades notáveis".

DISCUSSÃO

Neste trabalho foram apontados importantes exemplos sobre a relevância das discussões em relação à disciplina de Cálculo, com autores como Wrobel, Zeferino e Carneiro (2013); Rezende (2003); Barufi (1999) e Mello (2001). E ainda o fato de que tais discussões não são exclusividade da realidade brasileira, como foi destacado nos trabalhos de Oliver Knill, na Universidade de Harvard, baseado na Reforma do Cálculo e de David Tall.

Observa-se nesses trabalhos como ponto comum a sugestão do uso de computadores e software de matemática para as aulas de Cálculo, como possibilidade facilitar o entendimento gráfico de conceitos trabalhados nessa disciplina.

O Mathematica foi a ferramenta escolhida, pois além de ser disponibilizada para toda instituição, é utilizada em outras disciplinas como Física I, por exemplo, que também utiliza a sala inteligente, adequando-se perfeitamente.

Pode-se notar uma diversidade, e até mesmo a complexidade, dos temas apresentados nos trabalhos dos alunos, a associação feita por eles a problemas práticos ou temas vinculados a outras disciplinas, e até mesmo abordagens relacionadas à história da matemática. Tais particularidades no desenvolvimento desses trabalhos indicam um fator de amadurecimento desses alunos no que se refere às ideias discutidas na disciplina de Cálculo I.

Os dados mostram também a utilização do software para comparações e visualizações gráficas, cálculo de grandes proporções, dentre outras. Destaca-se também o reconhecimento dos próprios alunos em relação às facilidades advindas do uso do software, como nas seguintes falas: “(...) podemos calcular coisas que dificilmente conseguiríamos manualmente (...)”; “(...)Pôde-se utilizar recursos gráficos que em muito auxiliaram em evidenciar as diferenças entre cada tipo de espiral, além de permitirem a análise de vários valores plotados (...)”.

Desse modo os recursos utilizados se traduzem em uma primeira experiência da instituição rumo ao desenvolvimento de um curso de Cálculo I. Tal iniciativa vislumbra propiciar ao estudante um ambiente e recursos tecnológicos adequados que possa auxilia-lo na exploração dos conceitos matemáticos envolvidos no Cálculo I, e até mesmo em outras disciplinas do curso de Engenharia, por meio de uma abordagem metodológica diferenciada.

CONCLUSÃO

Esse trabalho apontou que a aprendizagem do Cálculo no Ensino Superior é objeto de análise em diversos congressos nacionais. Além disso, o tema configura-se também em uma preocupação internacional desde a década de 80, com o movimento “*Calculus Reform*”.

O uso de novas tecnologias é entendido, então, como peça fundamental ao se pensar na modernização, ou reorganização, da disciplina de Cálculo e que a sua utilização é uma preocupação central.

Desse modo, o ensino de Cálculo I, especificamente, em cursos de Engenharia, tornou-se muitas vezes um ‘mito’, devido ao excesso de reprovações e dificuldades diversas apresentadas pelos alunos e os mais variados pontos de vista dos professores no que se refere à solução do problema.

Assim, a utilização adequada, tanto ao ambiente da Universidade quanto à metodologia, das novas tecnologias é indicada como possibilidade de melhora dos problemas encontrados no processo de aprendizagem dos alunos.

Na sequência apresentou-se uma experiência realizada no ITA em que foram utilizados, além da Web, o software Mathematica e o ambiente da Sala Inteligente, especialmente desenvolvida para dar suporte ao uso de novas tecnologias. O desenvolvimento das aulas, utilizando a Sala Inteligente e os resultados apresentados pelos alunos nessa experiência, configura-se na maior contribuição do trabalho.

A criação dessa sala foi um grande passo da instituição rumo à implementação sólida das novas tecnologias nos cursos de Engenharia, não só em apoio ao processo de ensino e aprendizagem à disciplina de Cálculo, mas também de outras do Curso Fundamental. A utilização desses recursos durante as aulas influenciou positivamente os alunos da turma no juízo de valor que eles fizeram dos conteúdos de Cálculo, cabendo a eles participação ativa e envolvimento nas atividades propostas. Acredita-se que, nesse processo, os alunos construíram o conhecimento matemático em Cálculo de maneira significativa, pois tiveram a oportunidade de escolher os temas dos trabalhos apresentados livremente, apresentando o seu desenvolvimento o conhecimento adquirido durante a disciplina.

NOTAS

[1] Os Cursos de Engenharia do Instituto Tecnológico de Aeronáutica são ministrados em 5 anos. Os dois primeiros anos constituem o Curso Fundamental, comum a todas as especialidades. Os três anos seguintes constituem o Curso Profissional, que atualmente abre-se em seis especializações: Engenharia Aeronáutica, Aeroespacial, Eletrônica, Mecânica-Aeronáutica, Civil-Aeronáutica e Computação.

[2] Link para a página do curso: http://www.mat.ita.br/mat12/index.html

[3] Esta sala consiste em uma sala de aula cooperativa, que pode ser utilizada como: laboratório de ciências (em menor escala), sala de informática (otimizada por avançados sistemas de informação, que já não mais utilizam "PCs", por exemplo), a sala de vídeo e multimeios (com a inclusão de um quadro digital), sala de arte e biblioteca. Contem os seguintes equipamentos: 1 servidor interligado a rede internet do ITA, 10 computadores instalados em bancadas de 3 lugares interligados a rede internet do ITA, lousa touch screeen de 120 polegadas, 2 câmeras para filmagem e transmissão de vídeo pela WEB, 1 projetor LCD e sistema de som, com amplificação direta do servidor e possibilita a integração de importantes ambientes e recursos no mesmo local. Esse projeto foi financiado pela FINEP dentro do edital PROMOVE, que teve como objetivo a aproximação das escolas de engenharia com as escolas de ensino médio da rede pública (http://www.ita.br/grad/infraestrutura) .

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EXPLORANDO SUPERFÍCIES ATRAVÉS DE UM APLICATIVO / EXPLORING SURFACES THROUGH AN APPLET

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In this article we present an applet developed by the author to analyse and explore the main quadric surfaces. It is also reported the analysis of an experience conducted over the teaching period to several courses related with the use of such tools in the classroom. The work showed that the use of these tools in conjunction with the traditional teaching methods significantly improve the knowledge and skills of students allowing that they acquire stronger and deep knowledge of the subjects, increasing in this way their confidence and self-esteem.

Keywords: ICT; b-Learning; Mathematical User Interfaces; Higher Education

INTRODUÇÃO

Atualmente, nas Universidades e Institutos Universitários somos confrontados com a situação de que muitos dos estudantes desistem das suas instituições sem terem adquirido qualquer grau académico. Em vários países europeus a taxa de desistência está a aumentar e existem várias razões que poderão explicar porque é que isto acontece (Araque, Roldán, Salguero, 2009). A primeira que nos vem à cabeça é inevitavelmente a presente crise económica e, apesar da falta de recursos financeiros ser uma das principais razões que contribuem para os estudantes desistirem das Universidades, alguns deles têm insucesso ou desistem por causa da sua impreparação para a vida académica e o ambiente que é proporcionado no mundo universitário (Breier, 2010; Werblow, 2009). Uma das razões disto é que, em geral, o nível científico e de conhecimentos matemáticos que é ensinado nas escolas secundárias é relativamente fraco e a diferença entre o nível secundário e o universitário é elevado, tão elevado que algumas instituições em Portugal criaram o chamado "ano zero" com o intuito de fazer aumentar o nível de instrução em matemática. Outra razão está relacionada com o facto de os estudantes ficarem facilmente aborrecidos em sala de aula e não verem nenhuma ligação entre a vida académica e o mundo exterior. Quando agregamos isto tudo, criam-se imensos desafios para os professores na preparação dos conteúdos programáticos que são incorporados nas unidades curriculares. Uma maneira de tentar resolver estes problemas é construir e desenvolver aplicativos (applets) de modo a que estes apresentem os conteúdos das matérias de uma forma mais pragmática e intuitiva, num certo sentido, adaptarem-se às necessidades dos estudantes que temos à nossa frente, representando o enquadramento pedagógico mais apropriado nalgum tópico de matemática.

NOÇÕES RELACIONADAS COM SUPERFÍCIES QUÁDRICAS

As superfícies quádricas têm uma considerável importância prática, na medida em que são as curvas e superfícies mais simples depois das retas e planos, e são encontradas por toda a parte, em dimensão 2 e 3, na matemática, mas também na física e na astronomia. Em matemática, elas são usadas em várias áreas, nomeadamente em Geometria, Análise Multivariada, Análise Vectorial e são a extensão natural em três dimensões das tão conhecidas secções cónicas (elipses, hipérboles e parábolas). Noutras áreas, temos por exemplo, as órbitas dos planetas e cometas que são secções cónicas; o pico de uma torre, ou o de qualquer outra estrutura alta, move-se debaixo da ação do vento numa órbita elíptica, e o mais simples polinómio aproximador que pode ser usado para juntar duas rectas é uma parábola. Então, o que é que significa uma quádrica?

Uma quádrica afim (real) é uma superfície algébrica de ordem 2 num espaço afim euclidiano de dimensão 3 que pode ser representada pelo conjunto de pontos (x,y,z) que satisfazem a seguinte equação polinomial

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0 \qquad (1)$$

onde a_{ik} são constantes reais (i,k=1,…,4) e pelo menos um dos coeficientes a_{ik} (i,k=1,…,3) é não nulo.

Usando as matrizes

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} (\text{com } a_{ik} = a_{ki} \text{ e } i,k=1,\ldots,3), \quad B := \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \text{ e } \; X := \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

é possível representar a quádrica (1) na seguinte forma matricial

$$X^t AX + 2B^t X + a_{44} = 0, \qquad (1)$$

A equação polinomial (1) pode ser também escrita como uma forma quadrática

$$X^t AX = 0, \qquad (2)$$

onde agora as matrizes são dadas por

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} (\text{com } a_{ik} = a_{ki} \text{ e } i,k=1,\ldots,4) \text{ e } \; X := \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

Dada uma qualquer quádrica na forma matricial (1), é possível mudar as coordenadas e transformar a sua equação para uma forma mais simples, onde não existem termos cruzados (i.e., os termos *xy*, *xz* e *yz*). De forma análoga, podemos transformar as coordenadas de um ponto para outro e com esta translação eliminamos, se possível, os termos lineares (i.e., os termos *x*, *y* e *z*). Portanto, o processo de eliminar estes dois tipos de termos será um processo feito em dois passos. No primeiro passo, muda-se para outra base eliminando os termos cruzados e, no segundo passo, faz-se uma translação para a origem do referencial, removendo-se assim os termos lineares. Para além disso, o primeiro passo pode ser expresso através de encontrar uma matriz ortogonal (visto ser dado por uma rotação) P tal que $P^t AP = P^{-1}(G^{-1}A)P$ é uma matriz diagonal, onde G é a matriz da métrica. As entradas diagonais desta matriz são os valores próprios da matriz característica $G^{-1}A - \lambda I$. Claramente, se a base inicial do espaço vectorial é a base canónica, então $G = I$ e a matriz característica reveste o aspecto $A - \lambda I$.

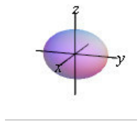
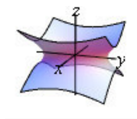
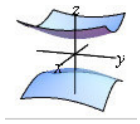
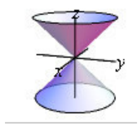
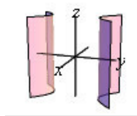
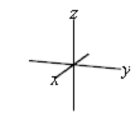
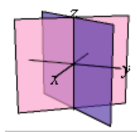
Portanto, depois deste processo de dois passos, ficamos com uma equação com a seguinte forma

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + d = 0, \qquad (3)$$

onde x', y' e z' são as novas coordenadas e *d* é uma constante. A quádrica da equação (1) diz-se que está na forma canónica reduzida.

Exemplos de superfícies e sua classificação

Para classificar as superfícies quádricas podem ser usados vários invariantes (Bromwich, 1905, Burington, 1932), nomeadamente: a característica, determinantes e o sinal do discriminante. Na Tabela (1) é mostrado a classificação das quádricas, baseada na característica das matrizes A e da matriz completa [A|B]:

Nome	Equação (forma reduzida)	Eixo	Gráfico
r(A)=r([A|B])=3 e 3 valores próprios diferentes de zero			
Elipsóide	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	n.a.	
Hiperbolóide de uma folha	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	eixo-z	
Hiperbolóide de duas folhas	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	eixo-z	
Duplo cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	eixo-z	
r(A)=r([A|B])=2 e um valor próprio nulo			
Cilindro elíptico	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	eixo-z	
Cilindro Hiperbólico	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	eixo-z	
Uma recta	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$	eixo-z	
Planos intersectados	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	eixo-z	
r(A)=r([A|B])=1 e dois valores próprios nulos			

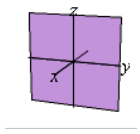
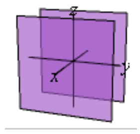
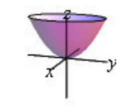
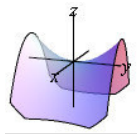
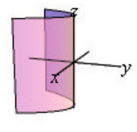
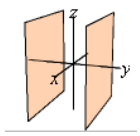
Planos coincidentes	$x^2 = 0$	eixo-x	
Planos paralelos	$x^2 - d^2 = 0$	eixo-x	
r(A)=2<3=r([A|B])			
Parabolóide elíptico	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2dz$	eixo-z	
Parabolóide hiperbólico	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2dz$	eixo-z	
r(A)=1<3=r([A|B])			
Parabolóide Cilíndrico	$x^2 = 2dy$	eixo-z	
Planos paralelos	$y^2 - d^2 = 0$	eixo-y	

Tabela 1 As principais quádricas

É também importante mencionar que qualquer plano de corte intersecta a superfície quádrica ou numa cónica própria ou numa cónica degenerada ou é o conjunto vazio.

O APLICATIVO DAS QUÁDRICAS

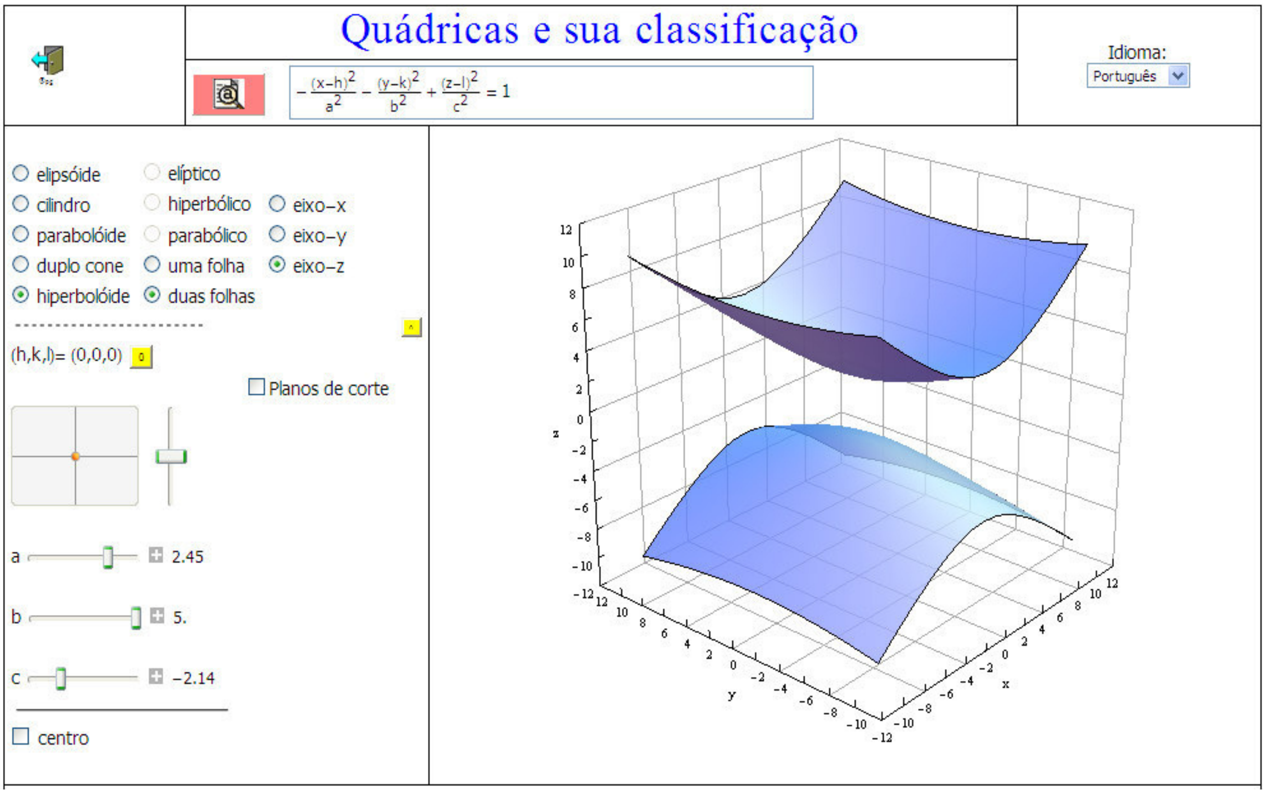
Apesar de existirem no mercado várias ferramentas de *software* (Oldknow, 2005), aplicativos e também sites na Internet, como por exemplo, o Wolfram|Alpha (Dimiceli et al., 2010) que podem ser usados para preparar os conteúdos programáticos de uma unidade curricular, uma das razões que nos conduziu à criação desta ferramenta foi a não existência no mercado de um aplicativo com as características que esta possui, senão vejamos, a título de exemplo, existem várias ferramentas de Geometria (e.g., Sketchpad, Cabri e GeoGebra, etc.) que são bastante úteis, mas em nenhuma delas é possível introduzir uma expressão algébrica ou uma equação polinomial, de modo a que se obtenha a respetiva classificação dessa superfície algébrica.

Por outro lado, poder-se-ia ingenuamente pensar que trabalhar com este tipo de ferramentas seria uma tarefa fácil. Infelizmente, e tendo por base a nossa experiência, este não é o caso, dado que a dificuldade em encontrar objetos de aprendizagem que sejam adequados às nossas unidades curriculares, o tempo despendido na sua elaboração e a dependência do *software* utilizado são alguns dos desafios com que somos confrontados. Para uma unidade curricular como a matemática, algumas destas ferramentas não são as mais apropriadas e têm sérias desvantagens que iremos mencionar na próxima secção.

Ao tentar-se delinear uma interface para conteúdos de matemática, devemos ter o cuidado com práticas que possam porventura distrair o utilizador e desviá-lo dos objetivos traçados para uma determinada unidade curricular e fazer todos os esforços para incentivar a compreensão e o prazer de aprender. No ensino da matemática tem sido reconhecido pela maior parte dos professores de matemática que é bastante importante a utilização da representação de objetos geométricos, bem como, a respetiva visualização desses nas mais diversas formas e posições (Arzarello, Ferrara, Robutti, 2012). O aplicativo fornece os meios para se alcançar os objetivos anteriormente mencionados, permitindo aos estudantes focarem-se em qualquer tipo de representação que possa germinar nas suas mentes, quando estes estão a resolver problemas, e foi desenvolvido pelo autor através da utilização do ambiente de programação do *software Mathematica*.

A interface é apresentada na Fig. 1 e está dividida em cinco painéis.

Figura 1: A interface principal



No primeiro painel, existe um botão, que permite alternar entre dois tipos de modos de utilização diferentes: o modo de visualização e o de inserção. No modo de visualização é possível escolher o tipo de quádrica que pretendemos, o seu eixo e também podemos alterar os valores dos parâmetros (o centro e os parâmetros a, b e c). As capacidades de rodar, mover e redimensionar os objetos em 3D permite aos professores/estudantes verem as representações gráficas desses objetos de determinados ângulos. Estas rotações podem ser executadas em todas as direções através da pressão de uma tecla e movendo o cursor à volta do objeto. É possível fazer redimensionamentos proporcionais em todas as direções do objeto ou apenas numa única direção. É também possível inserir planos de corte e controlar o seu deslocamento e declive para melhor visualizar possíveis interseções com as quádricas. No modo de inserção, uma das características mais importantes recai não só na capacidade de gerar aleatoriamente superfícies quádricas, mas também na possibilidade de inserir qualquer tipo de expressão e o programa por detrás irá nos dizer se a expressão que foi inserida representa ou não uma quádrica. Se for uma quádrica, então o programa irá mostrar a correspondente visualização gráfica.

A EXPERIÊNCIA EM SALA DE AULA

A utilização do aplicativo em sala de aula

É bem sabido que o sucesso na aprendizagem da matemática requer, para além de uma boa capacidade de entendimento e compreensão, a resolução de uma quantidade considerável de exercícios e problemas, tendo como objetivo um profundo conhecimento das técnicas de cálculo e da aplicação da teoria. Num curso de Geometria e mais especificamente para problemas em 3D, os estudantes devem adquirir e melhorar um conjunto de capacidades de visualização por forma a ganharem um profundo entendimento dos objetos geométricos com os quais estão trabalhando. Baseado na nossa própria experiência, acreditamos que um tipo de aprendizagem mista (b-learning) que represente uma mistura equilibrada entre os métodos tradicionais e os assistidos por computador, poderá contribuir para uma melhor forma de apresentar este tópico de geometria na sala de aula. Portanto, uma alternância entre explicações no quadro e demonstrações com computador, eventualmente, completados com a criação de vários tipos de exercícios que realcem os aspetos mais importantes das quádricas, deverá ser o caminho a seguir. Por outro lado, isto deverá ser analisado por cálculos analíticos e confirmados com a ajuda do aplicativo. Deste modo, a ferramenta educacional permite aos estudantes interagirem e verem cada quádrica em várias posições no espaço 3D e, consequentemente, ganharem uma rica experiência que lhes permite de uma maneira profunda, entenderem os conceitos que estão por detrás, indo para além dos livros de texto e de outros recursos estáticos. Finalmente, também se deverá mencionar que o aplicativo não é um substituto do processo de ensinar e deverá ser visto como uma ferramenta para ajudar e complementar o trabalho desenvolvido em sala de aula deixando todas as decisões importantes nas mãos do professor.

Dados da experiência

Durante os anos letivos de 2009/10 e 2010/11 foi feito um registo de dados provenientes de vários cursos lecionados na Universidade do Algarve, essencialmente aos cursos de Engenharia, Biologia e Biotecnologia, sobre a utilização em sala de aula de aplicativos, destinados à compreensão e ao entendimento de determinados conteúdos programáticos que são lecionados a esses cursos. Foram analisadas as classificações de vários alunos obtidas em diversos momentos de avaliação, tendo como objetivo compreender a diferenciação entre a lecionação de um conteúdo programático no qual foi utilizado um aplicativo e o mesmo sem a sua utilização. Dos dados observados (n=270) constatou-se, para ambos os grupos de dados, que a amplitude observada foi de 5 no intervalo empírico [0,5]. Para o primeiro (resp., segundo) grupo de dados, os principais parâmetros são os seguintes: a média é de $\bar{x}$=2.95 (resp., $\bar{x}$=2.36), o desvio padrão é s=1.32 (resp., s=1.10) e, para o primeiro, segundo e terceiro quartis obteve-se: $q_{1/4}$= 2.5 (resp., $q_{1/4}$=2.0), $q_{1/2}$=2.8 (resp., $q_{1/2}$=2.25) e $q_{3/4}$=3.5 (resp., $q_{3/4}$=3.15). Ao calcularmos a variação Δ, entre os dois grupos de dados, i.e., a diferença entre as classificações dos elementos dos dois grupos observados, é imediata a discrepância entre o número total de variações positivas (>0) e o número total de variações negativas (<0), cujos valores atingem, respetivamente, $\Delta > 0$=73% e $\Delta < 0$=6%, revelando portanto uma melhoria dos valores de um grupo relativamente a outro.

CONCLUSÕES

Da nossa experiência em sala de aula o aplicativo suporta uma grande variedade de métodos de ensino e a sua intuitiva interface, capacidade interativa e características dinâmicas, facilitam a aprendizagem da matemática, além de que o seu poder computacional deixa-nos espaço para melhor nos focarmos nos cenários pedagógicos e nos conceitos fundamentais que estão por detrás de cada assunto de matemática. Comparativamente com outros que são apenas usados como "quadros de giz eletrónicos", ela é uma excelente ferramenta de ensino e aprendizagem que pode efetivamente promover a aprendizagem matemática e, para além disso, pode também ser combinada com os tradicionais métodos de ensino, permitindo desta forma visões mais apropriadas e particulares bem como diferenças que possam existir entre estudantes. A análise dos resultados obtidos pelos alunos, ao longo dos anos, evidencia que, em média, os seus resultados melhoram nas matérias para as quais este tipo de aplicativos foi utilizado.

Esperamos que, com este tipo de ferramentas e outros aplicativos educacionais criados por Semião (2012) e Semião & Rodrigues (2012), isto seja um importante contributo para a Educação Matemática, fornecendo novas maneiras de envolver os estudantes na disciplina de matemática e ajudá-los a desenvolver o raciocínio e o espírito crítico. Deste modo, pensamos que num ambiente pedagógico deste género, eles ganham mais auto-confiança, mais auto-estima e menos ansiedade, que tão negativamente os afeta nas atuais salas de aula.

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Theme: Teachers

CONHECIMENTOS REVELADOS POR TUTORES EM FÓRUNS DE DISCUSSÃO COM PROFESSORES DE MATEMÁTICA / KNOWLEDGE REVEALED BY TUTORS IN DISCUSSION FORUMS WITH MATH TEACHERS

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We aimed to investigate the knowledge revealed by tutors of a continuing education course for mathematics teachers, offered in the distance. Initially, we follow the work of 32 tutors over a year, in order to typify its interventions in discussion forums with the course participants. From the results, we offer training to a new group of tutors in order to promote improvement in the actions that we consider below expectations. Between August 2012 and July 2013, we follow six tutors, the subjects of this research. For your training, TPACK theoretical framework was adopted. The research, qualitative, made use of the observation of the work of the tutors, which was analyzed by means of discursive types found in tutors' interventions in the discussion forums. Our analysis indicated that affective and attitudinal components play a key role in the exercise of mentoring in this context.

Keywords: Tutoring in Distance Education, Tutor Training, Continuing Education for Mathematics Teachers in Online Environments, TPACK, TPACK-OTE.

INTRODUÇÃO

Este texto traz um recorte da tese de doutoramento do primeiro autor, sob orientação da segunda, no Programa de Estudos Pós-Graduados em Educação Matemática da Pontifícia Universidade Católica de São Paulo. Foram acompanhados dois grupos de tutores, em um curso de formação continuada para professores de Matemática oferecido na modalidade a distância, com objetivo de investigarmos, em dois momentos distintos, os conhecimentos que emergiam de suas práticas no ambiente virtual de aprendizagem.

No primeiro momento, 32 tutores tiveram seu trabalho acompanhado ao longo de um ano, com o intuito de verificarmos quais eram os tipos de intervenções mais frequentes, realizados nos fóruns de discussão. A partir desses resultados, foi possível identificar de que forma os tutores estavam contribuindo para a formação continuada dos professores de Matemática inscritos no curso.

No ano seguinte, formamos um novo grupo de tutores para acompanhar outra turma de formação continuada para professores de Matemática. Seis dentre os novos tutores se voluntariaram para ter o trabalho acompanhado de perto, por um ano, a fim de verificarmos quais conhecimentos emergiam de suas práticas. A partir desta formação a pesquisa utilizou o quadro teórico proposto por Mishra e Koehler (2006), o TPACK, que propõe e defende os conhecimentos tecnológicos, pedagógicos e do conteúdo curricular, além de suas interseções e interações, como tipos de conhecimentos necessários à atividade profissional docente, e que entendemos que se aplica, também, à prática da tutoria em ambientes virtuais de aprendizagem.

Apresentamos, a seguir, a metodologia utilizada na pesquisa, de natureza qualitativa, assim como descrevemos o uso dos instrumentos para coleta dos dados, e os referenciais utilizados para sua interpretação, dando ênfase a análise temática de conteúdo e as tipologias discursivas para análise de intervenções em fóruns de discussão. Como principal resultado da pesquisa, destacamos a proposta de ampliação do quadro teórico TPACK, no contexto da formação continuada de professores de Matemática na modalidade a distância, incorporando os conhecimentos afetivo-atitudinais, caracterizando o modelo TPACK-OTE.

FUNDAMENTAÇÃO TEÓRICA

Nóvoa (2009) e Tardif (2002) discutem a formação de professores e sua profissionalidade em um contexto geral, que aqui particularizamos para a formação do tutor de cursos de formação continuada na modalidade a distância. Tardif (2002, p. 5) apresenta algumas questões como centrais na profissionalização do ensino e da formação de professores, dentre as quais destacamos a seguinte "Quais são os saberes profissionais dos professores, isto é, quais são os saberes (conhecimentos, competências, habilidades etc.) que eles utilizam efetivamente em seu trabalho diário para desempenhar suas tarefas e atingir seus objetivos?". Buscamos, então, responder a esse questionamento, a partir do quadro teórico proposto por Mishra e Koehler (2006) e de nossas investigações.

Shulman (1986) elencou os conhecimentos pedagógico, do conteúdo, e pedagógico do conteúdo como fundamentais para a prática docente. Esse modelo ficou conhecido como PCK – *Pedagogical Content Knowledge*, e é apontado por Godino (2009) como um divisor de águas nas pesquisas e propostas sobre formação docente, tendo dado origem a uma série de outros modelos que são pesquisados e desenvolvidos até hoje.

Mishra e Koehler (2006) estenderam o modelo proposto por Shulman (1986), incluindo o conhecimento tecnológico, e sua interação e interseção com o conhecimento do conteúdo: o conhecimento tecnológico do conteúdo; com o conhecimento curricular: agora chamado de conhecimento tecnológico pedagógico; com o conhecimento pedagógico do conteúdo: o conhecimento tecnológico pedagógico do conteúdo, como na Figura 1.

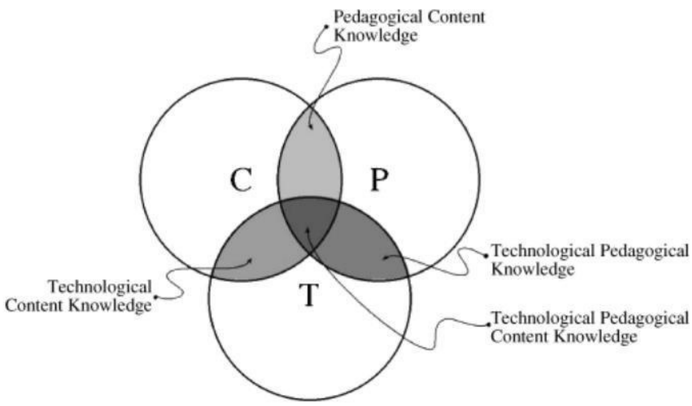

Figura 1: Diagrama TPACK.

Fonte: Mishra e Koehler, 2006, p. 1025.

Os autores apresentam o conhecimento tecnológico como o conhecimento sobre o uso de qualquer tecnologia, do livro impresso tradicional aos recursos digitais mais avançados. Envolve as habilidades necessárias para operar as tecnologias, incluindo, por exemplo, como instalar e remover um *software* ou dispositivos periféricos em se tratando de tecnologias digitais. Mishra e Koehler (2006) comentam que, como a tecnologia está em constante mudança, a natureza do conhecimento tecnológico tem também essa característica, que exige do educador constante atualização, independente da modalidade em que atue, seja ela presencial, a distância ou *blended*.

Recentemente Saad, Barbar e Abourjeili (2012) fizeram uma releitura e apresentaram uma ampliação do quadro teórico TPACK, para a formação inicial de professores. Os autores tentaram responder às seguintes questões: qual é a natureza da base de conhecimentos dos futuros professores que os habilitaria a ensinar com tecnologia? Como os programas de formação de futuros professores deveriam ser estruturados para construir essa base de conhecimentos?

Os autores propuseram, então, um modelo com cinco construtos, dois além do proposto por Mishra e Koehler (2006), a saber: pedagogia, tecnologias de informação e comunicação, conteúdo, contexto e aprendentes. Esse quadro teórico é chamado de TPACK-XL, em que o X vem de *conteXt* e o L de *Learners*. A partir das interseções desses cinco construtos independentes, os autores chegaram a 26 combinações, alcançando 31 tipos de conhecimento.

No nosso caso, a partir do TPACK, listamos as ações esperadas do trabalho dos tutores junto aos professores de Matemática em formação, e as relacionamos com os sete tipos de conhecimento elencados por Mishra e Koehler (2006). A partir daí, procuramos desenvolver, por meio da formação dos tutores, as habilidades relacionadas a esses conhecimentos. Posteriormente, a partir da observação das intervenções e da identificação das tipologias discursivas dos tutores, percebemos que o modelo TPACK fundamenta a prática da tutoria, mas não totalmente, e por isso recorremos ao trabalho de Saad, Barbar e Abourjeili (2012).

ABORDAGEM METODOLÓGICA E PROCEDIMENTOS DA PESQUISA

A abordagem escolhida para este trabalho é a de caráter qualitativo, comumente utilizada em pesquisas nas áreas das Ciências Sociais e Humanas, em particular, em Educação Matemática. A escolha dessa abordagem se deu por concordarmos que a pesquisa qualitativa "lida e dá atenção às pessoas e às suas ideias, procura fazer sentido de discursos e narrativas que estariam silenciosas" (D'Ambrosio, 2010 apud Borba e Araújo, 2010, p. 19).

A análise dos dados se dá de maneira indutiva, as abstrações a respeito do problema são construídas de baixo para cima, a partir das análises e da categorização dos dados. Não há uma preocupação em provar hipóteses previamente definidas, pois entende-se que o processo de pesquisa qualitativa é emergente, ou seja, as questões podem mudar a medida em que os dados são coletados e analisados ao longo da pesquisa (Creswell, 2010).

Fundamentados em Laville e Dione (2008), e Richardson (1999), fizemos uso da observação participante, da aplicação de questionários e da realização de grupos focais, gravados em vídeo. Já o projeto interpretativo dos dados coletados se dá por meio da análise temática de conteúdo (Richardson, 1999), em que categorias emergem dos dados e permitem uma interpretação mais fiel da realidade quando trianguladas com as tipologias discursivas (Bairral, 2004) e identificadas em recortes de intervenções dos tutores em fóruns de discussão.

Identificação dos tipos de mediação

No segundo semestre de 2012 foram selecionados tutores para atuarem em um curso de formação continuada para professores de Matemática da rede pública do Rio de Janeiro, Brasil. Esses tutores responderam a um questionário inicial, que objetivava levantar o perfil do grupo, e tiveram suas práticas no ambiente virtual observadas ao longo de um ano.

Acompanhamos, então, o trabalho de 32 tutores, com o intuito de identificar os tipos de mediação utilizados por cada um dos tutores, e categorizá-los a partir da frequência em cada fórum de discussão. Foram observados 12 fóruns com duração de 15 dias cada, e consideramos que o tutor exerceu um determinado tipo de mediação quando o tipo por nós identificado aparecia mais de duas vezes em pelo menos 9 dos 12 fóruns realizados, ou seja, pela recorrência de posturas e discursos utilizados pelos tutores nesses fóruns. Escolhemos acompanhar, estrategicamente, os fóruns entre o quarto e o décimo mês do curso, para que os tutores tivessem tempo de se familiarizar com suas atribuições e os resultados fossem mais confiáveis.

Mediação que gerencia o fórum: intervenções que iniciam os fóruns, propondo temas e formas de discussão; que organizam as discussões, a fim de facilitar a comunicação entre os cursistas; que encerram os fóruns, sintetizando as discussões.

Mediação que convida à reflexão: intervenções que buscam aprofundar as discussões a partir das falas dos cursistas, que são tratadas pelos tutores, que as transformam em novas perguntas de resposta não imediata.

Mediação que mostra domínio do conteúdo: intervenções em que fica evidente o conhecimento do material didático utilizado no curso, das releituras de conteúdos matemáticos e estratégias didático-pedagógicas propostas.

Mediação que incentiva a interação entre cursistas: os tutores vão além de convidar o grupo a opinar a respeito da postagem de um colega e relacionam postagens de diferentes cursistas, estimulando-os a se posicionarem a respeito.

Mediação que incentiva o aprofundamento das discussões: intervenções nas quais os tutores enriquecem as discussões, indo além do proposto no fórum e no material didático, sugerindo outras fontes de estudo e pesquisa, e relações com outros conteúdos matemáticos e áreas do conhecimento.

Postos os tipos de mediação que caracterizam cada categoria, os resultados da observação são apresentados na Figura 2, e mostram que a maior parte dos tutores se preocupou em gerenciar os fóruns de discussão, zelando pela organização das discussões e não permitindo conversas que não tratassem dos temas propostos. Percebemos também que, de maneira geral, os tutores se preocuparam em promover a reflexão dos cursistas, postando mensagens que os levassem a repensar ideias e posturas relacionadas aos conteúdos matemáticos e à própria atividade docente.

Por outro lado, apenas seis tutores explicitaram seu conhecimento sobre os conteúdos matemáticos discutidos no material didático. Talvez isso se justifique porque postagens mais “conteudistas” não se fizeram necessárias, mas, ainda assim, consideramos esse o principal ponto a ser melhorado em suas práticas, visto que apenas 14 tutores se preocuparam em fomentar o aprofundamento das discussões, o principal objetivo dos fóruns.

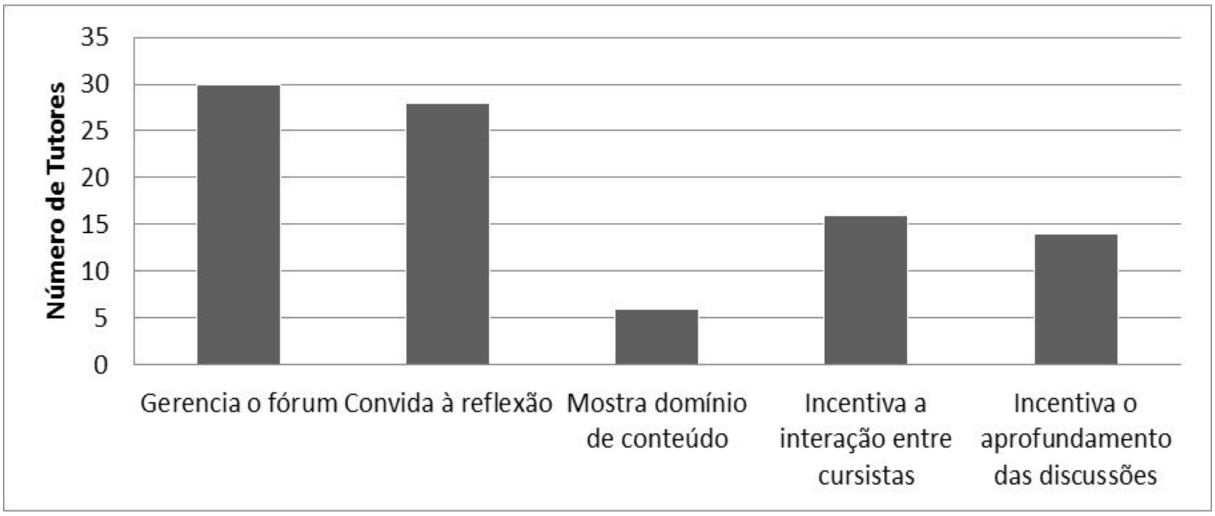

Figura 2: Frequência dos tipos de mediação encontrados nos fóruns de discussão.

Fonte: Esquincalha e Abar (2012).

Como observamos na Figura 2, poucos tutores mostraram domínio do conteúdo, um conhecimento fundamental para o bom desempenho da função, segundo o aporte teórico TPACK. Porém, as outras categorias indicam que os tutores manifestaram de maneira significativa o conhecimento pedagógico intrínseco a cada uma delas. Ressaltamos que o fato dos conhecimentos de conteúdo não terem sido manifestados nos fóruns de discussão não significa que os tutores não os têm desenvolvidos; apenas indica que precisam buscar estratégias para explorá-los na condução das discussões realizadas.

Com base nestes resultados, que ofereceram subsídios para a investigação com os seis tutores, caracterizados como os sujeitos da pesquisa, realizada entre agosto de 2012 e julho de 2013, pudemos oferecer formação para sanar o que consideramos aquém dos objetivos do curso, e os resultados advindos da observação de seu trabalho são apresentadas e discutidas a seguir.

ANÁLISE DE DADOS POR MEIO DA TIPOLOGIA DISCURSIVA

Aqui não discutimos os dados obtidos por meio do questionário e do grupo focal, que foram devidamente tratados por meio da análise temática de conteúdo. Restringimo-nos à identificação das tipologias discursivas observadas nos fóruns de discussão mediados pelos seis sujeitos desta pesquisa.

Para estudar as intervenções dos tutores, apoiamo-nos em Bairral (2007, 2004) que apresenta pesquisas sobre análise qualitativa de debates a distância, com foco nas interações entre os interlocutores e no desenvolvimento profissional dos cursistas. Em Bairral (2004) há um interesse em descobrir que tipologias discursivas estão presentes nas intervenções realizadas em um fórum de discussão entre docentes.

Ressaltamos que a técnica utilizada por Bairral (2004) é mais complexa, envolvendo a identificação de nós cognitivos que "permitem organizar uma base de informação em blocos diretos de conteúdo" (p. 2) e podem ser conectados por diferentes tipos de *links,* a partir dos quais são elaborados esquemas que conduzem a análise dos resultados. Nesse sentido, a partir de Bairral (2007, 2004) elaboramos um modelo mais simplificado, que apresentamos a seguir.

1. escolhemos, propositalmente, fóruns de discussão em que o tema fosse o desenvolvimento de algum conteúdo matemático;

2. registramos todas as intervenções ocorridas nesses fóruns em arquivos de texto distintos;

3. fizemos uma leitura exaustiva de cada um dos arquivos com o intuito de reduzir o conjunto de mensagens a subconjuntos menores, a fim de evidenciar a postura do tutor diante da discussão sobre o conteúdo matemático em tela;

4. categorizamos as intervenções, tipificando o discurso dos tutores;

5. Analisamos os dados.

As tipologias discursivas identificadas foram as seguintes: Iaa – intervenção afetivo-atitudinal que acolhe ou incentiva os cursistas; Icd – intervenção que sugere caminhos diferentes ou corrige o cursista; Idt – intervenção que discute o uso de recursos tecnológicos; Imc – intervenção que discute matemática com equívocos; Ims – intervenção que discute matemática sem equívocos; Iqp – intervenção que discute questões pedagógicas. Exemplos de intervenções dos tutores que se enquadram em cada tipologia podem ser encontradas em Esquincalha (2015).

O que podemos perceber, confrontando as tipologias discursivas identificadas com o modelo TPACK, utilizado na formação dos sujeitos da pesquisa, e com os resultados obtidos por meio da identificação dos tipos de mediação realizada inicialmente pelos 32 tutores, é que o segundo grupo de tutores já não se omite tanto em relação às discussões matemática mais explícitas, tanto que percebemos um número razoável de intervenções a respeito, inclusive com alguns equívocos conceituais, felizmente com baixa frequência, mas o suficiente para que fosse identificada como uma tipologia.

Percebemos, ainda, intervenções que estimulavam o uso das tecnologias em sala de aula, com sugestões de aplicativos e objetos educacionais digitais, mas de maneira tímida, o que nos leva a conjecturar alguma insegurança a respeito, possivelmente pela falta de experiência e algum receio de ter que falar mais a respeito, caso fossem questionados pelos professores em formação. Questões pedagógicas foram muito recorrentes, quase sempre discutindo abordagens para o ensino de determinados conteúdos, o que podemos associar com o conhecimento pedagógico do conteúdo matemático. Não identificamos nenhuma intervenção que discutisse, com profundidade, como o ensino de algum conteúdo matemático pode ser otimizado pelo uso de tecnologias, caracterizando o desenvolvimento do conhecimento tecnológico pedagógico do conteúdo.

Por fim, destacamos a tipologia discursiva Iaa, caracterizada pelas intervenções afetivo-atitudinais, que apareceram com mais frequência do que todas as outras nas postagens dos tutores. Observamos que mensagens de estímulo, elogios, conforto e mesmo as de correção, feitas de forma educada e acolhedora, respeitando os conhecimentos do professor em formação e sua experiência profissional, foram fundamentais para a baixa evasão no curso. Este tipo de conhecimento não é explorado no modelo TPACK e, por isso, inspirados no trabalho de Saad, Barbar e Abourjeili (2012), propomos o modelo TPAC-OTE, em que OTE é a sigla em inglês para Formação de Professores a Distância (*Online Teacher Education*). Em Esquincalha (2015) apresentamos as interseções deste quarto tipo de conhecimento, afetivo-atitudinal para formação de professores a distância, com os conhecimentos tecnológico, pedagógico e do conteúdo matemático, além de elencarmos algumas ações que indicam o desenvolvimento de cada tipo de conhecimento.

CONSIDERAÇÕES FINAIS

Os conhecimentos revelados pelos tutores, apresentados nas análises, se caracterizam como necessários para sua prática. Além dos conhecimentos relacionados às questões pedagógicas e ao conteúdo matemático, que eram esperados pela natureza do curso de formação continuada, percebemos uma presença ainda tímida de conhecimentos relacionados a incorporação das tecnologias para além do uso do ambiente virtual.

Em relação aos conhecimentos matemáticos revelados, nem sempre se mostraram satisfatórios. Percebemos casos em que tutores se omitiram diante de imprecisões matemáticas dos cursistas, da mesma forma que percebemos intervenções dos próprios tutores que deixaram a desejar nesse quesito. Entendemos que isso pode ocorrer em qualquer modalidade, presencial ou a distância, mas esperávamos que isso acontecesse em menor escala, diante da formação mínima de especialização, e experiência do grupo, e da possibilidade da resposta aos cursistas não precisar ser dada em tempo real, uma vez que os fóruns de discussão são ferramentas de comunicação assíncronas e os tutores tinham tempo para buscar fundamentação antes de realizarem suas intervenções.

Como o aporte teórico TPACK fundamentou esta pesquisa sobre a formação dos tutores, era natural e esperado que conhecimentos pertinentes a este modelo aparecessem em nossas análises. Já os conhecimentos que chamamos de afetivo-atitudinais, discutidos por Bairral (2004), como cordialidade, empatia, flexibilidade, capacidade de motivar, entre outros, caracterizando o estabelecimento de vínculos afetivos com os cursistas, além da capacidade de desenvolver uma escuta/leitura inteligentes, se revelaram de importância fundamental ao trabalho dos tutores.

Cabe destacar que esses foram os conhecimentos revelados pelos sujeitos da pesquisa durante o tempo em que foram acompanhados. Como apontado por Tardif (2002), esses conhecimentos são temporais, personalizados e situados, mas nos permitiram propor o modelo TPACK-OTE, que não tem o intuito de ser hermético. Ao contrário, nosso objetivo é o de ressaltar a importância dos componentes afetivo-atitudinais para a formação de professores na modalidade a distância aliados aos conhecimentos tecnológicos, pedagógicos e do conteúdo disciplinar. Possivelmente outros tipos de conhecimento sejam revelados em outros contextos, mas acreditamos que estes quatro são relevantes e merecem a atenção dos gestores de programas de formação de professores na modalidade a distância.

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FLUÊNCIA NO USO DE TECNOLOGIAS DIGITAIS: UMA INVESTIGAÇÃO COM PROFESSORES DE MATEMÁTICA DO ENSINO BÁSICO / FLUENCY IN DIGITAL TECHNOLOGIES: AN INVESTIGATION WITH MATHEMATICS TEACHERS OF BASIC EDUCATION

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This paper reports a qualitative research that has as subjects a group of public basic education teachers in a process of continuing education in the state of São Paulo (Brazil). The research proposed the development of constructions that would be answers to mathematical problems using SuperLogo and GeoGebra software. Those constructions were related to themes like "construction of regular polygons" and "trigonometric relations in right triangle." The theoretical framework of this initiative includes elements of the theory of didactic situations and theory of technology use cycle in mathematics education. The analysis of the proposals submitted by the participants allowed us to establish an important link between mathematical knowledge of the teacher and fluency in the technologies employed from the exploration of the dynamic nature of the interfaces.

Keywords: technological fluency; mathematics education; teachers' education.

INTRODUÇÃO

O relato contido neste artigo se refere a interações investigativas ocorridas no âmbito do projeto "Educação Matemática e Tecnologias: uma abordagem por meio de oficinas didáticas", realizado como parte das atividades do grupo de pesquisa "Processos de Ensino e Aprendizagem em Matemática – PEAMAT", do Programa de Estudos Pós-Graduados em Educação Matemática da Pontifícia Universidade Católica de São Paulo (PUC/SP) no ano de 2013, com apoio de agências de fomento no Brasil, como a FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) e o CNPq (Conselho Nacional de Pesquisa Científica). Na investigação mencionada, professores de Matemática do ensino básico de escolas públicas de São Paulo frequentaram oficinas gratuitas nas instalações da universidade, cujos temas estiveram relacionados ao uso de tecnologias como componentes de estratégias didáticas para o ensino de Matemática. Especificamente, neste artigo, são tratadas as interações relativas à oficina "Estratégias Didáticas com o SuperLogo em Aulas de Matemática".

A questão que se procurou responder com a investigação aqui relatada é "*de que forma a articulação entre o conhecimento matemático e estratégias investigativas pode ser realizada por professores de Matemática do Ensino Básico a partir do domínio e emprego fluente do software SuperLogo?*". Deste modo, o objetivo principal desta pesquisa consiste em analisar as interações e as produções de um grupo de professores que utiliza o SuperLogo em uma sequência didática cujas atividades foram planejadas para que os sujeitos, no intuito de resolvê-las, revisitem os conhecimentos matemáticos subjacentes.

Especificamente, o *software* SuperLogo utiliza uma linguagem desenvolvida por Seymour Papert na década de 1960 no Massachusetts Institute of Technology (MIT), de Cambridge, MA, Estados Unidos, e que foi adaptada para uso acadêmico no Brasil (em português) a partir de 1982, na Universidade Estadual de Campinas (UNICAMP/Brasil), pelo Núcleo de Informática Aplicada à

Educação (NIED). O desenvolvimento do SuperLogo faz parte da teorização, também proposta por Papert, denominada construcionismo. A teoria propõe que computador seja o ferramental para que o aluno construa seu próprio conhecimento, isto é, para que o aluno se torne sujeito ativo de sua aprendizagem.

Nas próximas seções, são descritos os procedimentos e teorias que serviram de base à pesquisa relatada neste texto.

METODOLOGIA

Quanto aos sujeitos, participaram da pesquisa aqui relatada oito professores de Matemática do ensino público, dos níveis fundamental e médio, que militam na docência entre cinco e dez anos. Os dados foram coletados durante a realização de duas sessões, com a duração de três horas cada, e trabalhados sob a perspectiva metodológica da análise de conteúdo, abordagem qualitativa considerada adequada em investigações que têm seu foco no processo em si, mais do que em eventuais resultados isolados (Bogdan & Biklen, 1994).

As sessões foram realizadas em um dos laboratórios de informática da Universidade, disponibilizado para este fim. Cada um dos professores participantes da investigação utilizava um computador específico, o que não os impedia de debater e trocar impressões sobre as atividades componentes da sequência didática. Os sujeitos tomavam ciência de suas tarefas no diálogo com o pesquisador: as propostas foram feitas como convite à investigação, o que permitiu ocultar eventuais intencionalidades didáticas, movimento típico do processo de devolução (Brousseau, 1986). De forma geral, as atividades solicitavam a criação de polígonos a partir de diferentes propostas, mais adiante descritas.

Os sujeitos foram orientados a gravar seus protocolos de resolução no ambiente SuperLogo, de modo que os mesmos pudessem ser recuperados nas análises. Suas falas também foram consideradas neste processo, uma vez que, com autorização de todos, os diálogos foram gravados. Com base em tal material, tornou-se possível explorar as seguintes categorias de análise: relação entre o conhecimento matemático necessário para consecução das atividades e a fluência em relação à interface empregada; estratégias de integração entre a lógica do SuperLogo e o conhecimento matemático e, estratégias de construção de fluência no *software* e de proposição de soluções para problemas em cenários investigativos.

SITUAÇÕES ADIDÁTICAS E ORGANIZAÇÃO DO TRABALHO

Os pressupostos teóricos que guiaram a pesquisa indicavam a necessidade de desenvolver, em conjunto com os sujeitos, um trabalho relacionado com a resolução de problemas no âmbito de situações adidáticas. Brousseau (1986) propõe o engajamento dos estudantes em situações desta natureza, assim vistas como aquelas nas quais não se pode perceber, da parte do professor, a intencionalidade didática, e que implicam no processo de devolução, pelo qual o professor propõe e os alunos aceitam dado problema como de sua responsabilidade, quanto à resolução. Deste ponto de vista, a propositura do problema prevê um contexto material, didático e teórico de caráter antagônico (o *milieu*), no âmbito do qual o processo investigativo do estudante segue por três dialéticas distintas: de ação, de formulação e de validação. O professor retoma o caráter didático da proposta quando se propõe a discutir e esclarecer sobre o estatuto do conhecimento matemático válido, o que se dá pela dialética de institucionalização. Assim, estes elementos da teoria foram adaptados, no âmbito da pesquisa, em relação aos professores participantes, uma vez que se pretendia que os mesmos percorressem, ao trabalharem com as tarefas, um percurso investigativo, não direcionado pelo pesquisador, e mediado pelas tecnologias.

Estabelecida a organização didática mencionada, a proposta desta iniciativa era a de promover o engajamento dos professores em um processo de compreensão do uso de tecnologias digitais em suas aulas de Matemática, assim entendido como uma trajetória que envolve adquirir fluência nas tecnologias empregadas (neste caso, softwares Matemáticos), pensar com as tecnologias, elaborar e desenvolver temas com as tecnologias e elaborar estratégias didáticas com as tecnologias (Oliveira, 2015). Neste artigo, as análises repousam sobre as duas primeiras etapas do ciclo, cujos elementos principais são expostos a seguir.

SOBRE A TEORIA DO CICLO DE FORMAÇÃO EM EDUCAÇÃO MATEMÁTICA

A teoria do ciclo de formação de pessoas para o uso de tecnologias na Educação Matemática, apresentada por Oliveira (2015) e utilizada em diversas pesquisas na área (por exemplo, Gonçalves, 2014; Marcelino, 2014) propõe uma reflexão a respeito das fases que envolvem a implantação das tecnologias em sala de aula. Embora o foco desse processo de construção do conhecimento mediado por tecnologias seja, evidentemente, os seres humanos envolvidos, a dimensão secundária das tecnologias também deve ser considerada. Segundo Oliveira (2015), a primeira condição para que os seres humanos utilizem a tecnologia e possam torná-la extensão da memória e do pensamento é dominar as ferramentas inerentes à interface. No contexto educacional, é importante que professores e alunos saibam utilizar a tecnologia de maneira fluente, de modo a evitar futuros obstáculos na resolução de problemas.

De acordo com o autor, essa fase compreende duas etapas distintas e integradas: (i) a exploração dos elementos da interface e (ii) a apropriação da lógica da interface em uso. A exploração dos elementos da interface está associada à desenvoltura na utilização da tecnologia, isto é, compreender o funcionamento da interface e utilizá-la por meio de seus comandos principais. Já a apropriação da lógica da interface consiste em compreender a maneira pela qual a tecnologia considera a perspectiva matemática, isto é integrar a matemática envolvida na resolução de determinado problema à forma de operação da interface.

> Desenvolver fluência equivale a saber usar com desenvoltura, de modo que este aspecto seja aliado de uma outra fluência, de caráter mental, que permite, por sua vez, ao sujeito, avançar na reorganização dos conhecimentos no âmbito do próprio processo que o leva a tomar o problema proposto como seu e investigar, em dialéticas e movimentos cada vez mais refinados, até formar uma proposição sua, que tenha o status de solução, resposta. Se a tecnologia usada é entrave, se dificulta ou intimida, se permanece oculta em seus recursos, então é provável que muito mais tolha o desenvolvimento das conjecturas necessárias à investigação de um problema matemático do que as facilite (Oliveira, 2015, p.15).

Uma vez adquirida a fluência na interface em questão, a tecnologia passa a fazer parte do cotidiano dos seres humanos, de modo a viabilizar novas possibilidades, bem como a reorganização do pensamento, como levantado por Tikhomirov (1981). Na segunda fase, portanto, os seres humanos passam a pensar com a tecnologia. Nesse momento, o sujeito passa a conjecturar e experimentar utilizando a interface, na tentativa de formular respostas às questões propostas e discuti-las com seus colegas e com o professor. Cabe lembrar a importância de conceitos como o da colaboração e da cooperação nessa fase do processo.

Posteriormente, na terceira fase, inclui-se a possibilidade de ampliar a exploração dos conteúdos matemáticos, por meio da experimentação e da resolução de problemas. Nessa etapa, os aprendizes são capazes de visualizar as conjecturas propostas e refletir sobre elas, de modo a elaborar conclusões válidas a respeito de um objeto matemático. Desta forma,

> Ao manipular uma construção geométrica a partir de um ponto ou de distintos valores numéricos, professores e estudantes podem alicerçar argumentações sobre condições de existência, generalizações, demonstrações e provas, por exemplo. Evidentemente, será este pensar integrado aqui referido, sob sua responsabilidade, que promoverá este processo, que, por sua vez, culminará em uma demonstração, por exemplo. O que se quer dizer é que pessoas demonstram, usam o conhecimento matemático, expressam seu pensamento com as tecnologias disponíveis. Desta maneira, pode ser possível desenvolver, em relação à Matemática, outras formas de pensar e conjecturar. (Oliveira, 2015, p.16).

Por fim, a quarta fase dessa teoria implica na elaboração de estratégias com a tecnologia, isto é, permitir que as pessoas envolvidas avancem a partir da investigação e da resolução de problemas, de modo a aplicar os conhecimentos adquiridos em outros contextos e outras situações e estimular o percurso investigativo autônomo. Portanto, cabe ao professor desenvolver estratégias didáticas coerentes, para que a utilização da tecnologia seja adequada ao conteúdo matemático que se pretende estudar.

De acordo com Oliveira (2009), as etapas descritas anteriormente, de alguma forma também apontadas em outros trabalhos, como os de Goos, Galbraith, Renshaw e Geiger (2005) e de Frota e Borges (2004), fazem parte de um ciclo (figura 1) que se repete conforme novos problemas são explorados:

> [...] As etapas da trajetória aqui descrita são complementares e compõem um ciclo. A partir do momento em que se julga (ou se aposta, no mínimo) que uma dada tecnologia pode ser adequada para o trabalho didático com certo conteúdo matemático, aprende-se sobre ela, a pensar com ela, a explorar e desenvolver a partir dela e a elaborar estratégias didáticas das quais ela faça parte. Esta trajetória se repete, em níveis mais elevados de uso e compreensão, sempre que temas ou problemas mais complexos são explorados. (Oliveira, 2015, p.22).

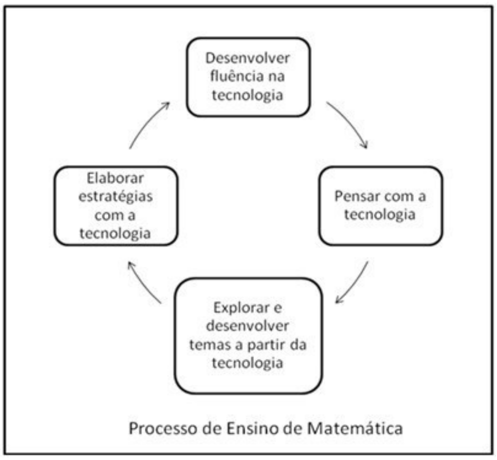

Figura 1. Ciclo do uso de tecnologias (Oliveira, 2015, p. 12)

Para que as tecnologias sejam efetivamente mediadoras nos processos de construção de conhecimento, possam reorganizar o pensamento humano e permitir a evolução de suas funções cognitivas, não se pode ignorar um planejamento consistente e voltado ao objeto matemático que se deseja estudar. As atividades analisadas a seguir indicam, no âmbito da pesquisa descrita, esta intenção.

ANÁLISES DAS ATIVIDADES

Em relação ao *software* SuperLogo, os professores que participavam como sujeitos da pesquisa jamais haviam tido qualquer experiência anterior. Assim, de acordo com a construção teórica que norteou esta investigação, seria necessário, no movimento de conjugar o emprego da interface informática com o conhecimento matemático subjacente, iniciar por ações que incentivassem a construção de fluência em relação ao sistema computacional. O que se observou, em relação à maioria dos sujeitos, é que a fluência em outros programas destinados ao trabalho didático com matemática – entre os quais o próprio GeoGebra, utilizado nesta iniciativa, e o Cabri Gèomètre, por exemplo – facilitou as primeiras etapas relativas à fluência digital demandada, menos pela similitude entre as ferramentas em questão e mais pela percepção, por parte do usuários, da necessidade de assimilar a lógica subjacente à interface empregada, ainda que cada uma delas sustente sua própria necessidade de apreensão. Em relação ao SuperLogo, as primeiras atividades, destinadas a promover a exploração da interface, procuraram trabalhar com a lógica de funcionamento dos comandos destinados a movimentar a tartaruga pela tela, em relação à medida dos segmentos traçados (comandos parafrente ou pf e paratras ou pt) e aos ângulos por meio dos quais a mesma muda de direção (comandos paradireita ou pd e paraesquerda ou pe). Assim, foi proposto aos sujeitos, após breve explanação, que inclui a função da janela de comandos do software, que os utilizassem livremente e registrassem sua percepção sobre o funcionamento dos mesmos. Um dos professores participantes fez as experimentações e comentários indicados a seguir.

> A tartaruga pode se movimentar para frente e para trás usando os comandos pf e pt. Quando escrevo pt 80, a tartaruga dá 80 passos para frente, por exemplo. Já os comandos pd e pe fazem a tartaruga virar em um ângulo desejado para a direita ou para a esquerda. Colocando o comando na linha de baixo da tela e apertando o botão executar da tela ou o enter do teclado, a tartaruga faz o que mandamos (fala de Professor Um).

Percebe-se, pelo depoimento supramencionado, que a exploração da interface permitiu iniciar o processo de apropriação da lógica da mesma, pelo menos em relação aos comandos fundamentais do *software*, e o entendimento de que o cursor (tartaruga) se movimenta como resultado dos comandos incluídos, e não de maneira aleatória. Os depoimentos e resultados obtidos pelos demais sujeitos nesta atividade inicial foram bastante semelhantes a este. Em seguida, na mesma sessão, foram encaminhadas propostas de atividades com intuito de prosseguir na exploração da interface para alcançar a fluência na mesma, mas já procurando alicerçar de forma mais consistente a apropriação sobre a lógica da ferramenta utilizada. Aqui, o objetivo era fazer com que os professores refletissem sobre a necessidade de dominar os elementos presentes no sistema computacional utilizado, ao mesmo tempo em que percebiam a correlação entre este domínio e o conhecimento matemático inerente às propostas. Desta forma, sugeriu-se que os sujeitos construíssem um quadrado de lado 100 passos de tartaruga. Nenhuma instrução adicional foi fornecida, de modo que eles próprios conjecturassem sobre eventuais respostas para a atividade propostas, em movimentos de ação, formulação e validação (Brousseau, 1986). Inicialmente, alguns professores se mostraram inseguros em relação à sequência de comandos capaz de prover uma resposta adequada. Algumas perguntas foram dirigidas ao pesquisador, que as devolvia, procurando, neste movimento, orientar em relação à responsabilidade e foco da investigação que cada um desenvolvia ao procurar a resposta. Por exemplo, à pergunta feita por Professor Dois, "– Como se desenha um quadrado com o SuperLogo?", o pesquisador respondeu "– É importante pensar na lógica pela qual funciona a interface. Afinal, o que é um quadrado?". Depois de trocarem algumas ideias, os professores Um, Três e Seis apresentaram, como resposta, os elementos constantes na figura a seguir, bem como as justificativas correspondentes.

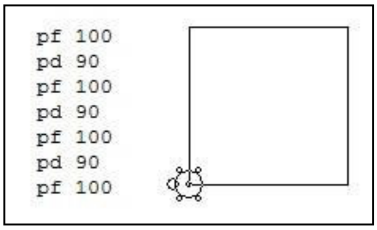

Figura 2. Resolução da atividade "construir um quadrado com 100 passos de tartaruga"

> Para construir o quadrado, pensamos em uma definição matematicamente aceitável de quadrado, e vimos que se trata de um polígono regular de quatro lados e ângulos internos de 90 graus. Isto facilitou nosso trabalho, depois foi só fazer a tartaruga dar os passos de acordo com esta ideia (fala de Professor Dois).

Os outros professores, em tempos que variaram de 5 a 12 minutos após a conclusão dos sujeitos supramencionados, também concluíram a atividade de forma satisfatória. Na consecução desta parte das atividades, foi notória a busca pela integração entre a lógica de funcionamento da interface, que já se ampliava em alguns professores, e o conhecimento matemático necessário, sem o qual pouco sentido teria a manipulação do *software*.

As atividades seguintes, que tinham um tempo mais rígido estipulado – deveriam ser feitas em 25 minutos – propunham a construção de outros polígonos regulares: triângulo, pentágono e hexágono. Os professores Um, Dois, Três e Seis terminaram a tarefa completamente no tempo dado. Os demais só o fizeram de forma parcial com o tempo de que dispunham. Entretanto, é importante destacar que, após a socialização das construções por parte dos professores que obtiveram êxito, os demais puderam usar estes resultados para concluir suas atividades, a partir de suas próprias produções (Figura 3).

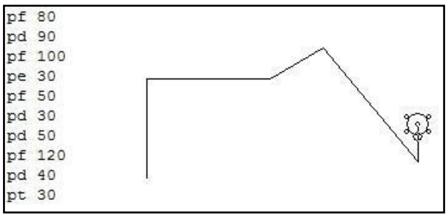

Figura 3. Comandos utilizados na experimentação de professor um e seus efeitos

Um erro comum no início desta atividade estava relacionado à compreensão do ângulo a ser usado – os professores se surpreenderam, por exemplo, com o fato de o ângulo de 60 graus não representar a resposta correta para a construção do triângulo equilátero. Diante da dúvida apresentada pelos sujeitos, o pesquisador, atento às questões de devolução e dos efeitos do contrato didático, indicados por Brousseau (1986), recomendou que os professores recorressem a outros recursos que lhes permitissem entender o que estava acontecendo. Diante disto, Professor Dois sugeriu aos demais que tentassem "refazer os passos da tartaruga" para entender a razão matemática da dificuldade que estavam tendo. O GeoGebra, já utilizado pelos professores com certo grau de fluência, surgiu como recurso adicional para o trabalho. Por meio dele, os professores compreenderam que a tartaruga deveria se voltar de acordo com os ângulos externos dos polígonos

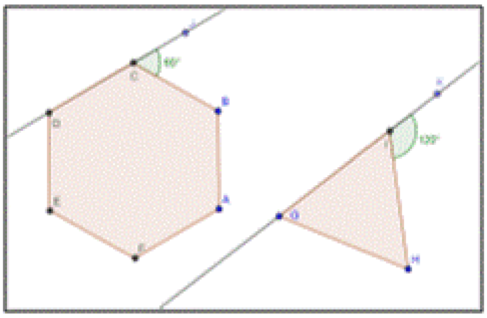

Figura 4. Utilização do GeoGebra para compreensão dos ângulos nas construções (Professor Dois)

Após a conclusão desta etapa, foi possível notar uma ampliação dos aspectos relacionados à compreensão das distintas lógicas das interfaces empregadas, já agregadas à percepção, por parte dos professores, da necessidade de mobilização do conhecimento matemático subjacente às propostas a eles submetidas. Os comentários feitos pelos sujeitos parecem indicar esta direção:

> Tem um jeito certo de usar o programa e que faz a tartaruga andar da forma que a gente quer. O que eu percebo é que a chave de tudo é a matemática. Esta coisa do polígono regular tem uma lógica. É a matemática que desvenda esta lógica. Sem matemática, o superlogo vira um joguinho... (fala de Professor Sete).

Do ponto de vista teórico, as asserções do professor Sete corroboram os elementos contidos no arcabouço teórico desta pesquisa: pensar na lógica dos *softwares* é importante, bem como compreender exatamente as diferenças entre elas, mas não sem agregar os elementos matemáticos indispensáveis para a compreensão e realização das atividades. Fala-se do desenvolvimento de um percurso que indica a necessidade de ampliar a fluência em relação às interfaces em ação no processo (Oliveira, 2015; Oliveira, 2005). De forma geral, os resultados alcançados pelos participantes ao final do tempo estipulado são aqueles que se encontram indicados na Figura 5.

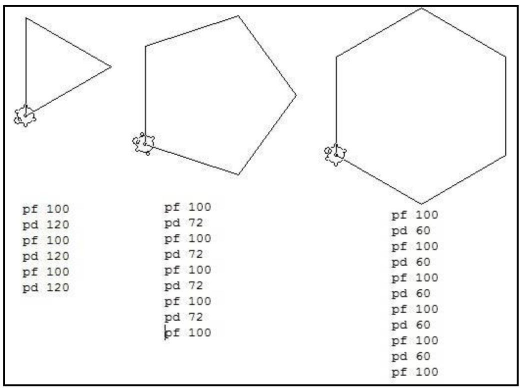

Figura 5. Resolução da atividade "construir triângulo, pentágono e hexágono regulares"

A atividade seguinte consistia em ampliar as construções pensadas até aquele momento, e instruía os participantes da pesquisa a elaborar um procedimento que permitisse construir um polígono regular qualquer, tendo sido dados o número de lados do mesmo e a medida dos lados. Para isto, o pesquisador precisou retomar o aspecto de exploração da interface ao trabalhar com as ideias de variável, de procedimento e do comando “repita” no SuperLogo, rapidamente apropriadas pelos professores. Também aqui o tempo de 25 minutos foi dado para consecução da proposta.

A integração entre as ideias matemáticas e a fluência crescente na interface pareceu decisiva para o êxito por parte da maioria dos professores – apenas dois deles tiveram dificuldades para concluir a atividade no tempo concedido. Da mesma maneira, porém, após a discussão coletiva dos resultados alcançados, os professores remanescentes concluíram suas respostas, entendendo que o ângulo externo a ser utilizado correspondia ao quociente de 360 dividido pelo número de lados de determinado polígono (Figura 6).

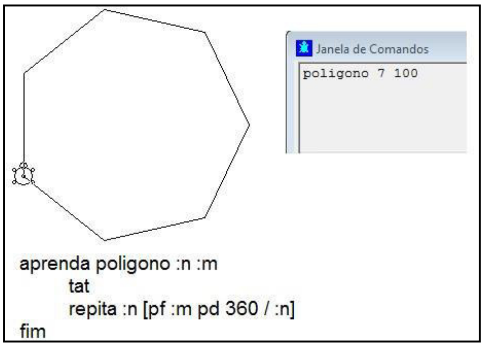

Figura 6. Resolução da atividade “construir um polígono de n lados de medida m” (Professor Quatro)

A discussão sobre o trabalho com polígonos regulares permitiu avançar para outras atividades, relacionadas à construção de triângulos retângulos com base nas razões/relações trigonométricas. Assim, solicitou-se, inicialmente, que os sujeitos construíssem um triângulo retângulo isósceles, cujos lados congruentes medissem *m*. Os professores relataram terem usado o teorema de Pitágoras para chegar à resposta esperada. Além disso, os próprios professores, utilizando as opções de ajuda do SuperLogo, identificaram o comando “raizq”, utilizado para calcular a raiz quadrada de 2 (Figura 7).

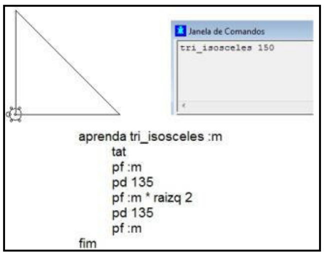

Figura 7. Resolução da atividade “construir triângulo retângulo isósceles de lado m” (Professor Cinco)

Neste momento, os professores já haviam avançado consideravelmente em relação à fluência no SuperLogo, tanto no aspecto de exploração quanto de compreensão da lógica da interface. A atividade seguinte precisaria de novos comandos e de segurança em relação à fluência. Consistia em utilizar conhecimentos relativos às relações trigonométricas para a construção de um triângulo retângulo, dadas a medida do cateto adjacente e a medida do ângulo entre ele e a hipotenusa. Imediatamente, os professores começaram a levantar o conhecimento matemático necessário para a consecução da atividade usando o SuperLogo. Tendo percebido que o pesquisador não facilitava o percurso investigativo, revelando como a resposta poderia ser alcançada, os sujeitos passaram a debater sobre os elementos matemáticos necessários e como relacionar os mesmos com os recursos da interface. Para esta tarefa, 50 minutos foram concedidos (Figura 8).

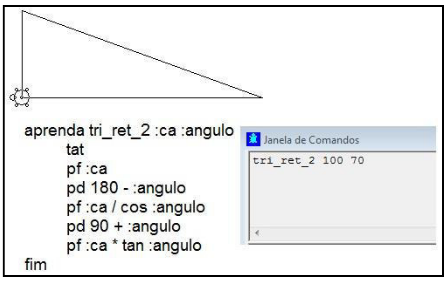

Figura 8. Resolução da atividade "construir um triângulo retângulo dados o cateto adjacente e o ângulo entre ele e a hipotenusa" (Professor Quatro)

Ao final desta atividade, foram recolhidos os depoimentos dos sujeitos, destacando os aspectos que acharam mais importantes para que concluíssem a construção. Apenas dois professores terminaram a tarefa após o tempo previsto, mas sem necessidade de quaisquer intervenções do pesquisador e mesmo antes do momento da institucionalização.

> Esta atividade exigiu mais de todos nós. Tivemos que mobilizar alguns conhecimentos de razões trigonométricas e adaptar ao SuperLogo, mas eu percebi que já ficou mais fácil, mais direto, do que no começo. Comigo foi assim também com o GeoGebra, quando tivemos a oficina. Depois que a gente entende como é a lógica do programa, é mais traduzir o pensamento matemático para ele. O bom é que dá para fazer experiências e mudar os parâmetros, avançar nas construções. Já começo, agora, pensando em que tecnologia vou usar, que combina mais com o problema. Tem coisas que não tem mais sentido fazer de outro jeito (fala de Professor Três).

CONSIDERAÇÕES FINAIS

Os depoimentos e os resultados apresentados pelos professores no percurso investigativo que percorreram indicam a relevância da primeira etapa do ciclo de uso das tecnologias, descrito por Oliveira (2009), relativo à fluência. De fato, à medida que o domínio sobre a lógica do SuperLogo avançava, crescia, também, o desempenho dos sujeitos, a velocidade de execução, a compreensão de um erro cometido (e a correção do mesmo), por exemplo. Esta fluência encaminhou a integração do pensamento matemático e as considerações lógicas relativas ao problema com a tecnologia empregada, indicando que as produções resultaram de um pensar conjunto, integrado, envolvendo as pessoas e suas extensões, no caso, o SuperLogo. Resta indicar, em relação aos mesmos sujeitos, de que forma esta trajetória encaminhou a compreensão dos mesmos para as etapas seguintes do ciclo de uso de tecnologias por professores de Matemática, quais sejam explorar e desenvolver temas e elaborar estratégias didáticas com a tecnologia empregada. A compreensão dos professores, aliás, se estendeu para o entendimento sobre a pertinência de determinadas tecnologias em relação ao trabalho didático a ser realizado, o que foi possível levantar quando os mesmos indicavam proximidades e diferenças, por exemplo, entre o SuperLogo e o GeoGebra. Outra importante percepção que merece registro está relacionada à relevância do conhecimento matemático em relação à fluência tecnológica – por diversas vezes, os professores mencionaram que o uso isolado das interfaces, desvinculado do saber matemático de referência, nada produziria em termos de avanço na compreensão dos problemas correlatos. Isto deve ser bastante destacado, já que açoda um dos pressupostos teóricos mais relevantes indicados por Oliveira (2009), bem como asserções semelhantes apontadas por Goos et al. (2003) e Frota e Borges (2004). Em relação a estes elementos, já existem dados, recolhidos em outras oficinas do projeto, que permitirão proceder outras análises, no sentido de esclarecer, em caráter de continuidade, estas outras etapas do ciclo.

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EXPERIÊNCIA DE FORMAÇÃO CONTINUADA DE PROFESSORES: USO EDUCACIONAL DE TABLETS PARA ENSINAR MATEMÁTICA NOS ANOS INICIAIS / EXPERIENCE OF CONTINUING TEACHER EDUCATION: TABLETS FOR EDUCATIONAL USE TO TEACH MATHEMATICS IN THE INITIAL YEARS OF PRIMARY SCHOOL

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This work presents partial results of an action research in progress, which aims to investigate the integration of technological resources, especially tablets, in the practice of teachers who teach Mathematics in the Initial Years of Primary School. To achieve this goal is being offered a continuing education course, whose didactic sequence is based on the exploration and problematization of educational activities using computational applications in tablets, and in the experience of teachers. In this report are socialized some educational activities that are being developed with the use of the application "Dessiner les formes", which allows the teaching of geometric shapes for the Initial Years of Primary School. The main results are that the participants are feeling safer to integrate tablets into their pedagogical practice, and that they believe in the productivity of this tool in the teaching of Mathematics.

Keywords: Tablet; Mathematics; Continuing Teacher Education; Initial Years of Primary School.

INTRODUÇÃO

O cotidiano das pessoas, cada vez mais, é influenciado com a variedade e a difusão das tecnologias que também atinge a escola, tanto no dia-a-dia dos estudantes quanto dos professores. Os dispositivos móveis são algumas dessas tecnologias e a sua utilização tem se expandido, talvez em função da sua mobilidade que permite a imediata atualização da informação, bem como a comunicação instantânea. De acordo com Artigue (2013, p. 5) "Mais tablets do que computadores são vendidos hoje e eles oferecem um número crescente de aplicativos de matemática". Nota-se que existem motivos relevantes para a afirmação da autora, pois os tablets são portáteis, de baixo custo e facilmente manipuláveis. Estes aspectos, além do financiamento destes aparelhos pelos governos estaduais brasileiros, são informações importantes para a integração dos tablets na prática dos professores da Educação Básica.

Em vista disso, esse estudo socializa resultados de um grupo de investigadores cujo objetivo é integrar as tecnologias, principalmente os tablets, nos processos de ensino e de aprendizagem da Matemática. Deve-se ressaltar que esse trabalho faz parte de duas pesquisas, uma que pertence ao Programa de Internacionalização da Pós-Graduação no RS, com apoio da FAPERGS (Fundação de Amparo à Pesquisa do estado do Rio Grande do Sul) e está sendo desenvolvida por pesquisadores brasileiros e portugueses. A outra foi aprovada pelo edital Universal 14/2013 do CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).

De acordo com os estudos teóricos discutidos nos encontros desenvolvidos pelos integrantes das pesquisas, pôde-se destacar a importância de possibilitar a integração das tecnologias nas práticas dos professores por meio de formação continuada. Neste contexto, o grupo de pesquisa propôs o curso de formação continuada intitulado "Uso de tablets para o ensino de Matemática nos Anos Iniciais do Ensino Fundamental", que está sendo desenvolvido no Centro Universitário Univates, Lajeado/RS. Neste relato, apresenta-se um recorte do curso, que socializa como os formadores desenvolvem e planejam as atividades com os tablets, bem como as percepções dos professores participantes, em relação à referida formação continuada. A pretensão foi responder a seguinte questão de pesquisa: quais as contribuições de um curso de formação continuada, com foco no uso de tablets para o ensino de Matemática, nos Anos Iniciais, na prática pedagógica dos participantes?

Para alcançar o objetivo previsto foi adotada a pesquisa-ação, que permite aos participantes refletir sobre sua prática, de forma coletiva e contextualizada. Assim, pesquisadores e professores se tornam parceiros, ou seja, uma pesquisa realizada com professores e não sobre eles.

REFERENCIAL TEÓRICO

Os jovens de hoje convivem com as tecnologias e as utilizam das mais variadas formas. Estes instrumentos de comunicação revolucionaram a maneira como eles pensam e demonstram as suas ideias. A escola não pode preterir esta realidade, pois a utilização de tecnologias digitais já faz parte do cotidiano dos alunos. Em pleno século XXI, torna-se urgente colocar a serviço do ensino e da aprendizagem da Matemática, as ferramentas tecnológicas existentes e que fazem parte do nosso dia-a-dia. Em efeito:

> Na virada do século, não se trata mais de nos perguntarmos se devemos ou não introduzir as novas tecnologias da informação e da comunicação no processo educativo. [...] Atualmente, professores de várias áreas reagem de maneira mais radical, reconhecendo que, se a educação e a escola não abrirem espaço para essas novas linguagens, elas poderão ter seus espaços definitivamente comprometidos (Rezende, 2002, p. 1).

Desta forma, torna-se possível utilizar metodologias variadas juntamente com as preferências dos estudantes, propondo ações e atividades que proporcionam o uso das tecnologias. Nesse sentido, Carreira et al. (2013, p. 56) destacam, " o acesso fácil e rápido a qualquer ferramenta tecnológica permite que os jovens desenvolvam um elevado número de competências que lhes conferem certa sofisticação e destreza na procura de conhecimentos que vão além da escola". É preciso compreender que os nativos digitais apresentam proximidade e familiaridade com as tecnologias, e que por esse motivo, possuem forma diferenciada para construir seu conhecimento (Carreira, 2009).

É importante destacar que a utilização de recursos tecnológicos deve ter objetivos claros e exequíveis. Vários autores alertam para a necessidade de cuidar da forma como se utiliza as tecnologias na sala de aula (Amado, 2007). Além disso, aludem que os professores são os principais responsáveis por essa integração, acarretando a eles certas exigências. Moran (2013, p. 1) destaca que se deve ter cuidado na inserção dos recursos tecnológicos e ressalta a necessidade de formação do professor para tornar eficaz essa utilização.

As tecnologias trazem muitas possibilidades, mas, sem ações de formação sólidas, constantes e significativas, boa parte dos professores tende, após a empolgação inicial, a um uso mais básico, conservador – repositório de informações, publicação de materiais – enquanto, os alunos podem seguir utilizando-as para inúmeras formas e redes de entretenimento, como jogos, vídeos e conversas online.

Amado (2007) e Moran (2013) destacam que para conciliar o currículo escolar com as tecnologias, sem dúvida, precisam-se investimentos nos recursos e na formação do professor. Em consonância, Gandin (2013, s/p) argumenta:

> Está muito claro, a todos os pesquisadores e os formadores que trabalham sério em educação, que não é possível acontecer a utilização de tablets na realidade das escolas brasileiras sem uma formação adequada dos docentes. O que vai facilitar a aproximação, a perda do 'medo' e a familiaridade do professor com a tecnologia é exatamente a formação. Não somente a formação acadêmica, mas a formação continuada, em serviço, preocupada com o trabalho pedagógico diário e atento à realidade, ultrapassando os muros da escola.

Em concordância com Gandin e Moran, este grupo de pesquisadores também acredita que a formação continuada é uma maneira de unir os recursos tecnológicos com a prática pedagógica dos professores. E, como afirmam Espinosa e Fiorentini (2005, p. 156), a prática docente pode ser considerada o ponto de partida e o ponto de chegada à formação dos professores, baseado no fato dos docentes "possuírem saberes específicos que são mobilizados, utilizados e produzidos no âmbito de suas tarefas cotidianas e de, com tais saberes, desempenharem seu trabalho". Os mesmos autores ratificam: "Os pesquisadores da área de formação de professores e da educação matemática, atualmente, começam a se preocupar não apenas a investigar os saberes docentes mobilizados e produzidos na prática, mas também em valorizá-los, incorporando-os à literatura relativa" (Espinosa e Fiorentini, 2005, p. 153). Portanto, as modificações que ocorrem na prática do professor com a introdução dos recursos tecnológicos e a realidade, na qual ele insere a tecnologia nessa prática, representam um desafio aos professores e aos formadores.

DETALHAMENTO DAS ATIVIDADES

O curso de formação continuada ofertado para professores dos anos Iniciais do Ensino Fundamental tem por objetivo auxiliar os professores deste nível de ensino no uso de tablets em seu fazer pedagógico, bem como explorar aplicativos para o ensino da Matemática, além de discutir a integração de aplicativos na prática pedagógica. Quinze professores participam do referido curso.

Os encontros, que acontecem desde junho de 2014, ocorrem uma vez por mês, em quintas à noite, totalizando quarenta horas de formação, num total de dez encontros, dos quais oito são presenciais e dois à distância. Durante os encontros presenciais os formadores disponibilizam atividades, utilizando aplicativos disponíveis nos tablets, relacionadas aos conteúdos de Matemática: sequência numérica, operações matemáticas, frações, números decimais e geometria. Inicialmente as atividades são desenvolvidas pelos professores participantes e, posteriormente ocorre a problematização de tais atividades, com o objetivo de incentivar a exploração das mesmas na prática pedagógica dos professores participantes.

Na sequência serão apresentadas algumas atividades exploradas referentes ao conteúdo de geometria. O aplicativo utilizado foi "Dessiner les formes", que é encontrado em https://play.google.com/store/apps/details?id=com.sinyee.babybus.shapeII.fr&hl=fr. O aplicativo é gratuito e pode ser trabalhado sem a necessidade da internet, pois o mesmo pode ser instalado no tablet, e o seu acesso pode ser feito em qualquer lugar e momento. O objetivo desse aplicativo é desenhar formas geométricas básicas e relacioná-las com alguns objetos do cotidiano. O jogo apresenta três contextos diferentes, nos quais se devem desenhar o contorno de círculos, retângulos ou triângulos, para que esses se transformem em algum objeto que apresenta seu formato de acordo com a respectiva figura geométrica.

Ao abrir o aplicativo, aparecerá a tela da Figura 1. Após selecionar o ícone do jogo, aparecerão três figuras, conforme a Figura 2. Basta selecionar uma delas para jogar.

Figura 1 – Seleção inicial

Fonte: https://play.google.com/store/apps/details?id=com.sinyee.babybus.shapeII.fr&hl=fr

Figura 2 – *Menu* inicial

Fonte:https://play.google.com/store/apps/details?id=com.sinyee.babybus.shapeII.fr&hl=fr

Quadro 1. Tela Inicial e *Menu*

Selecionando a imagem da *pizza* (forma triangular), a missão será alimentar um urso panda no canto da tela (Figura 3). Desenhar o contorno de alguma forma geométrica (Figura 4). Se o desenho estiver correto, esse se transformará em determinado tipo de alimento que deve ser levado até o urso (Figura 5).

Figura 3 – Campo inicial da pizza

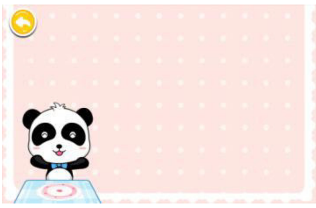


Figura 4 – Desenhando um círculo

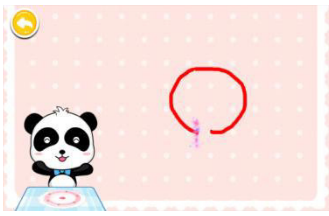
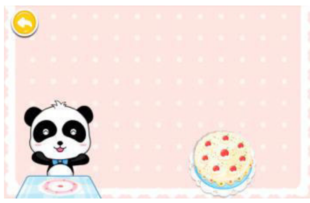


Figura 5 – O bolo



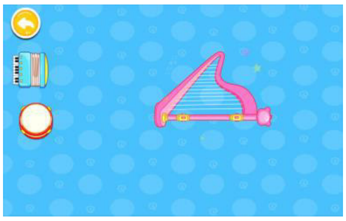
Fonte:https://play.google.com/store/apps/details?id=com.sinyee.babybus.shapeII.fr&hl=fr

Quadro 2. Desenhar e alimentar o urso panda

O tipo e o formato do alimento dependerão da forma geométrica desenhada, como por exemplo, o círculo que se transforma em um bolo ou bolacha, dependendo do seu tamanho.

De volta para o menu, selecionar a figura do peixe que levará a uma nova tela de atividades. Dessa vez, as formas geométricas se transformarão em peixes para nadar no fundo do mar (Figura 6). Para ir a uma terceira situação, no início selecionar a figura do instrumento musical. A dinâmica continua a mesma, porém agora surgirão instrumentos musicais (Figura 7). Quando desenhadas as formas geométricas, elas reproduzem sons característicos de instrumentos musicais que apresentam a forma semelhante ao desenho realizado.

Após a exploração do jogo no tablet, foram desenvolvidas atividades envolvendo geometria com a utilização do tablet e outras a partir dele. Estas últimas têm a finalidade de verificar se ocorreu a compreensão do conteúdo que foi trabalhado, na qual sugere ao professor, possibilidades de atividades práticas, com dinâmicas que podem ser desenvolvidas na sala de aula, tanto individualmente quanto em grupos.

Figura 6 – Fundo do mar	**Figura 7 – Instrumentos musicais**
	
Fonte:https://play.google.com/store/apps/details?id=com.sinyee.babybus.shapeII.fr&hl=fr	

Quadro 3. Fundo do mar e Instrumentos musicais

A seguir apresentamos algumas das atividades propostas, na ocasião em que foi trabalhado o aplicativo "*Dessiner les formes*".

1) Completar o quadro, após ter selecionado a imagem da *pizza* a qual tem por objetivo alimentar o urso panda:

Forma desenhada	**Alimento que surgiu**	**Outros alimentos que possuem esse formato**

Para o desenvolvimento desta primeira atividade proposta foi necessária a consulta ao aplicativo. Mas a atividade também solicita informações que permitem ao aluno relacionar as figuras geométricas com outros alimentos que não surgiram no decorrer do jogo.

2) Responder as questões que seguem após ter selecionado a imagem do instrumento musical:

a) Que instrumentos surgiram quando foi desenhado um triângulo?

b) Que instrumentos surgiram quando foi desenhado um retângulo?

c) Que instrumentos surgiram quando foi desenhado o contorno de um círculo?

d) Escrever outros instrumentos que também possuem as formas dos três formatos do jogo:

e) Escrever outros instrumentos que possuem formatos diferentes.

Assim, como na primeira atividade, na resolução da segunda, utilizou-se o tablet para responder aos itens "a", "b" e "c", enquanto, nos itens "d" e "e", foi necessário extrapolar o aplicativo, relacionando com os conhecimentos prévios e situações do cotidiano do aluno.

3) Após os exercícios realizados, responder:

a) Quantos lados tem um retângulo?

b) Quantos lados tem um triângulo?

c) Quantos lados tem um círculo?

Nesta atividade, objetiva-se verificar o desenvolvimento do conhecimento do aluno acerca do número de lados das figuras geométricas apresentadas e, além disso, estabelecer discussão em torno do número de lados. O aplicativo poderá ser consultado apenas como forma de observação e verificação.

4) Desenhar em folhas coloridas, retângulos triângulos e círculos (de tamanhos diferentes), recortar e montar figuras/gravuras que apresentam: somente retângulos; somente triângulos; somente círculos; retângulos e triângulos; triângulos e círculos; retângulos e círculos; retângulos, triângulos e círculos.

Esta situação foi trabalhada em grupo e o seu desenvolvimento permitiu a construção e a identificação das figuras geométricas. Também se pretendeu incentivar a criatividade, pois há necessidade de montar figuras ou gravuras, utilizando algumas ou apenas uma figura geométrica. Esta proposta de atividade não utilizou o tablet, pois um dos focos do curso é também mostrar que após o uso de aplicativos há possibilidade de usar outras atividades que decorrem do que foi explorado no recurso.

COLETA DE DADOS E DISCUSSÃO DOS RESULTADOS

O desenvolvimento do estudo é baseado na pesquisa-ação que, segundo Moreira (2011) tem por objetivo fundamental melhorar a prática, considerando ao mesmo tempo os resultados e os processos. O autor define a pesquisa-ação como uma forma de pesquisa coletiva, autorreflexiva, realizada por participantes de situações sociais, que busca ampliar a produtividade e o entendimento das suas próprias práticas sociais ou educativas, e compreenderem a relação entre tais práticas e as situações em que acontecem (Ibidem).

A coleta de dados se deu por meio das respostas dos professores aos questionamentos colocados no desenvolvimento das aulas, bem como as percepções dos mesmos, em relação ao que estava sendo proposto, e a associação que fazem entre as suas práticas e seus conhecimentos construídos nessa prática. Nos encontros presenciais, busca-se constantemente envolver os docentes, por meio de atividades e tarefas que permitem reflexão, debate e sugestões. A cada encontro são realizadas discussões, que são gravadas e posteriormente transcritas, para analisar a viabilidade das atividades propostas e os avanços que os professores estão obtendo em relação ao trabalho com os tablets e a utilização dos aplicativos na sua prática pedagógica.

Salienta-se que no primeiro encontro, procurou-se identificar o que motivou os professores a buscarem a formação e se já utilizavam tecnologias nas suas aulas. A seguir, a fala de uma professora, em relação à busca pelo curso e sobre a utilização dos tablets.

> Professora 3: Quanto ao uso dos tablets, assim, eu tenho os pequenos, eu vejo lá na escola que muitos têm tablet, e a gente acaba trazendo pelo "dia do brinquedo", usando ele como ferramenta do "dia do brinquedo". E talvez possa usar isso com outro objetivo também, porque isso acaba sendo um brinquedo, mas tem várias utilidades. Eu não sou tão velha, mas eu não sou do tempo de usar tablet. Então assim, é diferente para nós professores também, é uma discussão nova que a gente precisa estar se atualizando. E lá na escola a gente tem uma sala interativa, que tem esse uso assim, que já está se adequando, mas a gente acaba, às vezes, deixando de lado até pela nossa falta de conhecimento.

De acordo com a professora, é evidente a facilidade de acesso aos tablets. Muitas crianças já o possuem, mas para que ele se torne um aliado do professor, nos processos de ensino e de aprendizagem, é fundamental que exista um suporte pedagógico que o oriente em como lidar com essa tecnologia. Gandin (2013, s/p) considera que a formação continuada garante este apoio, ao argumentar:

> Isto significa munir o profissional da educação com as ferramentas da tecnologia, aproveitando o seu conhecimento e planejando projetos que contemplem um melhor aproveitamento das

experiências que os alunos têm em suas vidas, com a internet, redes sociais e jogos, coisas pelas quais se interessam.

Nessa circunstância, é fundamental ao professor, perceber que esses recursos podem ser seus aliados e ferramentas de apoio, tanto para ele quanto para o aluno. Na sequência destaca-se a fala de uma professora que expõe a realidade de algumas escolas, em relação à estrutura.

> Professora 2: Lá na escola a gente tem o período da informática, mas eu não acompanho para saber certo, o que é desenvolvido. Mas eu, assim, não uso muito a tecnologia em sala de aula, ainda talvez não consegui englobar bem. Algumas coisas sim, mas o computador normal para alguns slides, mas nada que tenha a ver com *softwares*.

Diante desta afirmação é possível constatar que a escola já possui o Laboratório de Informática, mas geralmente, este espaço é restrito às aulas de informática e administrado pelo professor responsável, o que de certo modo, dificulta a sua utilização pelos professores de outras disciplinas. Isso reforça o que afirma Bittar (2006, p. 2)

Atualmente, muitas escolas, públicas e privadas, dos Ensinos Fundamental e Médio têm sido equipadas por laboratórios de informática e têm feito uso de tecnologia com seus alunos. Porém, o que temos visto, muitas vezes, são aulas sem ligação específica com o conteúdo das disciplinas e sem aproveitamento do que a informática pode trazer como benefício para o processo de aprendizagem do aluno.

Não basta levar os alunos ao laboratório de informática, é necessário que o professor tenha objetivos bem definidos para não fazer uso dos recursos de modo superficial e desvinculado do processo de construção de conhecimento do aluno (Bittar, 2006). Neste sentido, se questionou os professores em relação ao aplicativo de geometria apresentado neste trabalho, bem como em relação às atividades propostas para a exploração do conteúdo por meio da utilização do tablet.

> Professora 4: Eu acho que vai ser um sucesso na sala, eles vão, adorar. [...] Eles fizeram um texto citando os jogos e o que eles jogam é "jogo de tiro", não é jogo lógico, com esse propósito. [...] É um incentivo nosso, como professores.

> Professora 8: Legal! A questão do jogo como algo educativo, e não o jogar pelo jogar.

De fato, os alunos gostam e dominam estas tecnologias, mas cabe ao professor transformá-las em uma ferramenta pedagógica. Neste sentido, Moran (2013b, s/p) alerta: "Na medida em que entram na sala de aula o seu uso não pode ser só complementar. Podemos repensar a forma de ensinar e de aprender, colocando o professor como mediador, como organizador de processos mais abertos e colaborativos".

A Matemática, segundo Bittar (2006, p. 6) "é uma área privilegiada no que diz respeito à riqueza de *softwares*". A autora reforça esta afirmação, falando da variedade e da gratuidade desses materiais que "podem contribuir com a elaboração de situações didáticas significativas para o aluno". A partir do exposto, pode-se considerar que o êxito na adoção desses recursos nas aulas de Matemática, em especial os tablets, depende da escolha de aplicativos adequados aos conteúdos e aos objetivos pretendidos pelo professor e que sejam explorados por meio de atividades bem planejadas.

CONSIDERAÇÕES FINAIS

O grupo desta pesquisa considera que a formação continuada de professores pode ser um caminho para a integração de recursos tecnológicos nas aulas de Matemática. Porém, acredita-se que isto só será possível se o professor for parte ativa do percurso de formação continuada e os debates estiverem voltados às suas necessidades. Inúmeros são os desafios, pois o uso de tecnologias ainda provoca receio e insegurança aos professores. É fundamental que os mesmos acompanhem a sua evolução e consigam manuseá-las de modo produtivo, para poder atingir positivamente os alunos.

Observa-se, durante os encontros a motivação dos participantes no momento da exploração das atividades. As discussões estão sendo produtivas e demonstram que os professores estão começando a sentir-se mais seguros, bem como estão iniciando o uso de tablets em suas aulas. Aliado a isto, os encontros estão possibilitando problematizar conteúdos matemáticos, que os professores demonstram algumas dificuldades, em particular em relação a conceitos geométricos, tais como: propriedades de formas geométricas, diferenciação entre polígonos e poliedros. Portanto, pode-se inferir que a experiência vivenciada, por este grupo de professores, no curso de formação continuada, está sendo muito significativa, em particular por dois fatores: na aprendizagem de novos conteúdos e, como meio de motivação, para o professor integrar os aplicativos computacionais em sua prática pedagógica.

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A CONSOLIDAÇÃO DE UM GRUPO COLABORATIVO DE PROFESSORES DE MATEMÁTICA: UMA EXPERIÊNCIA DE FORMAÇÃO CONTINUADA PARA O USO PEDAGÓGICO DA WEB 2.0 / CONSOLIDATION OF A COLLABORATIVE GROUP OF MATHEMATICS TEACHERS: A CONTINUING EDUCATION EXPERIENCE FOR THE EDUCATIONAL USE OF WEB 2.0

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*This article describes part of the results of a doctoral research that aimed to investigate how a continuing education program with a collaborative approach can contribute to mathematics teachers' knowledge and reflective use of Web 2.0 resources in educational practice. Therefore, a training process was proposed and conducted, which excelled for collaboration as a motivating factor and for the possibility of the teacher to engage with the technological resources in real practice and subsequently share the experiences with the group. In this work, the results from the analysis of the group's constitution process and its consolidation as a collaborative group are presented and discussed. The results point to a longitudinal path of teachers towards the collectiveness, gradually establishing shared goals and having the group as a motivator to analyze their own practices.**Keywords: Grupos Colaborativos, Professores de Matemática, Web 2.0.*

INTRODUÇÃO

Historicamente, o desenvolvimento tecnológico sempre provocou medo e desconfiança nas pessoas não envolvidas diretamente na sua concepção. Com o advento dos computadores isso não foi diferente e, ao mesmo tempo em que provocou (e ainda provoca) fascínio, também despertou (e ainda desperta) receios quanto à sua integração ao dia-a-dia das pessoas. Assim como os computadores, o surgimento da internet também provocou esse misto de fascínio e receio, uma vez que apresentou novas formas de comunicação e acesso à informação. No início, a internet serviu apenas como um espaço para acesso a conteúdos. No final da década de 1990 e início dos anos 2000, surge o conceito de Web 2.0, criado por O'Reilly (2007) e que define a Web como um ambiente potencializador da interação, da colaboração e da cooperação entre seus usuários, agora muito mais do que meros leitores, uma vez que os usuários passaram também a ser produtores de conteúdo na rede. Esse conjunto de ferramentas e a facilidade de acesso econômico cada vez maior aos dispositivos que permitem o uso da internet desencadearam rápidas e profundas transformações no que diz respeito à produção e à disseminação de conteúdo na Web.

A necessidade de adequação da escola à nova sociedade que se apresenta direcionam as discussões e reflexões para formação inicial e continuada do professor que deve atuar nesse novo espaço. Freitas (2009) aponta que as iniciativas de formação inicial e continuada de professores para o uso das tecnologias têm sido poucas e insipientes se forem consideradas aquelas que visam propiciar um reflexivo dos novos recursos existentes. As reais necessidades de reflexão não são atendidas pela formação inicial e o que percebemos é a perpetuação de um modelo centrado na instrumentalização desprovida de uma análise relacionada à realidade na qual o professor está inserido. Isso distancia o que é apresentado ao professor e a realidade que o mesmo vivencia no seu cotidiano, caracterizando um processo formativo precário e com poucas contribuições para uma mudança de prática pedagógica. Retirado do seu espaço e fora do seu ambiente de trabalho, o professor entra em contato com recursos sem ter reais oportunidades de vivenciar e trocar experiências sobe o uso dos mesmos no seu contexto de trabalho.

Dentro deste contexto, este artigo descreve parte dos resultados de uma pesquisa de doutorado que teve como objetivo geral investigar como uma formação continuada com enfoque colaborativo pode contribuir para que professores de Matemática conheçam e façam uso reflexivo dos recursos da Web 2.0 na prática pedagógica. Para tanto, propusemos e realizamos uma formação continuada que primou pela coletividade e pela possibilidade de vivência dentro da realidade de cada professor.

Acreditando que um processo formativo com enfoque colaborativo tenha um potencial de contribuição maior para o uso reflexivo das tecnologias oferecidas pela Web 2.0, estabelecemos a colaboração como objeto e fio condutor da pesquisa aqui descrita. Buscamos como um diferencial da pesquisa o acompanhamento da constituição do grupo e as influências de tal processo para o uso reflexivo dos recursos tecnológicos definidos. Assim, objetivamos fazer com que a pesquisa – enquanto um processo de intervenção pedagógica – permitisse uma construção coletiva de saberes, constituindo-se num espaço de reflexão e transformação.

Concentraremos nossa discussão na análise da constituição do grupo, evidenciando como os professores de Matemática nele se organizam e como passam de uma perspectiva de trabalho individual para colaborativa.

COLABORAÇÃO E FORMAÇÃO DE PROFESSORES

O trabalho colaborativo entre professores é apontado por Antúnez (1999) como constituinte de um ou mais critérios determinantes de qualidade. Hargreaves (2003) também aponta as práticas colaborativas como pontos vitais para o aperfeiçoamento e o desenvolvimento da escola, pois: promovem o crescimento profissional e o aperfeiçoamento escolar impulsionado de dentro; constituem formas de garantir a implementação eficaz de mudanças introduzidas externamente; contribuem para a implementação de uma reforma curricular centralizada.

Antúnez (1999) afirma que a colaboração mediante o trabalho em equipe é um objetivo da educação escolar e concordamos que, ao menos, deveria ser assim. É necessário que os professores reconheçam a necessidade da troca, da discussão dos problemas que lhes são comuns. Daí a necessidade de se estruturar uma formação que promova o desenvolvimento do espírito colaborativo nos professores. Cristóvão (2009) defende o desencadeamento de um processo contínuo de formação de professores de Matemática que os desenvolva como profissionais críticos a ponto de não aceitarem mais serem meros reprodutores de resultados de pesquisas educacionais ou de imposições governamentais. Complementamos com Tardif & Lessard & Lahaye (1991), que acreditam que professores partilham seus saberes, materiais e informações sobre alunos, dividindo assim um saber prático sobre sua atuação. Ao analisar a constituição de um grupo colaborativo focando o professor de Matemática como o centro da investigação, Ferreira (2003) entende que a metacognição consiste em um processo de tomada de consciência e compreensão dos próprios saberes e prática, assim como a reflexão e a autorregulação da própria aprendizagem prática. Portanto, nesse processo, a consciência da própria cognição e a autorregulação são, em grande parte, influenciadas pelas crenças que o professor tem de si mesmo enquanto profissional, da natureza do processo de ensino e aprendizagem da Matemática e do papel que desempenha juntamente com os seus alunos nesse processo. Portanto, os processos metacognitivos permitem que o professor repense seus saberes e sua prática e, a partir do seu aprofundamento, tome decisões sobre como alcançar suas metas profissionais.

Imbernón (2010) acredita e defende a formação continuada de professores como um dos procedimentos que podem ajudar a romper com o individualismo dos professores. Assim, preconiza uma formação centrada num trabalho colaborativo que permita chegar à solução de situações problemáticas. Ao mesmo tempo, observa a complexidade inerente ao estabelecimento do trabalho colaborativo, considerando que cada um dos participantes do grupo vê-se responsável não apenas por sua aprendizagem, como também pela dos demais membros do grupo. Isso demanda um processo formativo que prime por provocar reflexões baseadas na participação, centrando-se em casos, trocas, debates, leituras, trabalho em grupo, situações problemáticas, dentre outras ações.

Partindo do que aqui foi exposto, acreditamos na formação continuada com enfoque colaborativo como um instrumento de crescimento dos professores de Matemática e concordamos com Ferreira (2003) ao afirmar que é necessário que seja criado um ambiente que possibilite ao professor reconhecer seus próprios saberes e práticas para que – a partir da sua avaliação, em termos de suas próprias metas e expectativas em relação ao ensino e a aprendizagem de Matemática – ressignifique os seus saberes, decidindo se deseja ou não reconstruí-los.

DELINEAMENTO DA PESQUISA E A FORMAÇÃO REALIZADA

Concordando com as ideias de Alves-Mazzotti (1998) e Esteban (2010), definimos a pesquisa qualitativa como alternativa necessária para o desenvolvimento do presente trabalho. Quanto à sua natureza, entendemos a pesquisa como descritivo-explicativa, de acordo com Gil (2010), mas também ressaltamos o caráter interventivo da mesma, uma vez que houve uma intensão de transformação da realidade pesquisada.

Buscamos como base para o desenvolvimento metodológico da pesquisa as ideias de Pimenta (2005) e Franco (2005) quanto às características e vantagens do uso da pesquisa-ação como uma modalidade de pesquisa que permite ao professor refletir sobre suas próprias práticas, sua condição de trabalhador, bem como os limites e possibilidades do seu trabalho. Também defendemos a importância de uma pesquisa que permita ao professor refletir sobre suas próprias práticas, sua condição de trabalhador, bem como os limites e possibilidades do seu trabalho.

A partir do delineamento estabelecido, o processo de coleta e análise de dados ocorreu durante uma formação proposta de realizada com seis (6) professores de Matemática da rede pública de ensino de uma cidade localizada no interior do Brasil durante o ano de 2013. Foram realizados 14 encontros presenciais, além do contato virtual estabelecido durante todo o processo. Além da observação de toda a interação, realizamos entrevistas no início e término da formação ocorrida.

O processo formativo proposto iniciou-se por meio de um momento que ofereceu aos professores o contato com ideias e conceitos acerca da Web 2.0 e um panorama geral das ferramentas existentes, discussões sobre o atual contexto de uso das tecnologias por parte dos professores envolvidos e leitura de material que sistematizasse relatos de experiências de uso de tais recursos no contexto da Matemática. Realizados os estudos conceituais iniciais, o processo formativo encaminhou-se para o que denominamos “Ciclo Formativo”, organizado de modo que cada “volta” implicasse num conjunto de atividades que se repetissem para toda e qualquer ferramenta Web 2.0 trabalhada até o momento em que não fossem mais definidos novos recursos, o que “finalizaria” a formação. A proposta foi que, a cada ciclo ocorresse: a escolha de um recurso; a exploração e aprendizagem técnica do recurso; análise das suas possibilidades pedagógicas no contexto do ensino de Matemática; elaboração, aplicação de atividades; e socialização das experiências vivenciadas com a ferramenta explorada. O “Ciclo Formativo” constituiu-se de cinco (5) atividades que se repetiram de acordo com o estudo das ferramentas, enquanto foram definidos novos recursos a serem trabalhados, sendo tais atividades: 1) Escolha do recurso; 2) Exploração técnica; 3) Discussão das possibilidades; 4) Elaboração e uso do recurso; 5) Socialização das experiências.

A *escolha do recurso* deveria ser consensual e envolver negociação entre os membros do grupo, partindo de suas reais necessidades e interesses de aprendizagem. Acreditamos na negociação como elemento motivador ao trabalho coletivo, pois tratou-se de um instrumento que permitiu ver, argumentar e ouvir argumentos dos demais colegas. Uma vez escolhido o recurso, iniciou-se a atividade de *exploração técnica*, na qual o foco esteve na aprendizagem da ferramenta, descobrindo seus comandos, recursos disponíveis, compreendendo sua usabilidade e desenvolvendo pequenas tarefas para seu domínio operacional.

Após o aprendizado das questões técnicas da ferramenta escolhida, os professores foram incentivados a *discutir suas possibilidades de uso* como apoio no processo de ensino e aprendizagem de conteúdos matemáticos. Nesta fase, promoveu-se a busca de materiais, relatos de experiência e tudo que estivesse disponível e que pudesse contribuir para essa reflexão.

Em meio às reflexões e discussões sobre as possibilidades, os professores *propuseram e elaboraram atividades* que foram aplicadas em sala de aula com seus estudantes. Cada professor teve liberdade no seu fazer e foi incentivado a compartilhar por si só suas experiências positivas e até mesmo por ele consideradas negativas em sala de aula. Mesmo que a implementação das atividades ocorresse individualmente, o grupo foi incentivado a manter contato virtual para sanar possíveis dúvidas técnicas entre si. Por fim, a fase de *socialização* representou o momento onde os professores trouxeram de volta para o grupo as experiências vivenciadas com as atividades planejadas. Foi o momento de partilhar os erros e acertos nas atividades previamente elaboradas e aplicadas, concluindo sobre a efetividade do uso do recurso trabalhado no ciclo.

Para cada recurso escolhido, todo o ciclo formativo se repetiu. Além dos encontros presenciais, buscamos também incentivar interações por meio de listas de discussão criadas especificamente para a formação, assim como o uso de ambientes virtuais de aprendizagem e redes sociais. Ao término dos ciclos, realizou-se uma reunião de fechamento, na qual os integrantes do grupo foram incentivados a avaliar criticamente o processo vivido.

Durante a formação, ocorreram três ciclos, nos quais foram trabalhados os seguintes recursos da Web 2.0: Google Drive®, Blogue e Wiki.

A CONSOLIDAÇÃO DO GRUPO COLABORATIVO

Para analisar o percurso dos professores durante a formação, buscamos apoio em Ferreira (2003) que, em sua tese, identificou no grupo com o qual trabalhou três movimentos nos quais a dinâmica e a forma de participação dos membros se diferenciaram durante o percurso de consolidação da colaboração. Não se tratam de momentos estáticos e com fronteiras delimitadas, mas sim diferenciações que em certos momentos convivem paralelamente até que um ganhe mais força e substitua o outro, gradativamente.

O *movimento constitutivo de um grupo de trabalho* é o primeiro e contempla os momentos de constituição inicial do grupo e realização dos primeiros encontros. Os membros do grupo vão se conhecendo gradativamente e, dessa maneira, estabelecem um convívio amigável. A participação ainda é pequena e requer considerável atuação dos formadores, que organizam, incentivam e valorizam o envolvimento dos professores no grupo que começa a interagir. O segundo movimento – chamado *movimento constitutivo de um grupo de trabalho colaborativo* – consiste no fortalecimento e sentimento coletivo e diminuição da dependência dos membros do grupo com relação aos formadores. Aos poucos, os professores assumem o protagonismo, mobilizando-se mais no sentido de participar das decisões, fazer as escolhas, criticar e ponderar. Estabelece-se então um verdadeiro relacionamento de colaboração. O *movimento de consolidação de um grupo de trabalho colaborativo* é o terceiro e último e, neste momento, os professores se mostram mais seguros quanto às escolhas e às decisões necessárias. O grupo assume uma postura autônoma maior ao ponto de não mais se sentir dependente das ideias dos formadores. Os professores mobilizam-se mais para ajudarem uns aos outros, alcançando um considerável grau de autonomia, reflexão e auto-regulação.

Com a conclusão do processo formativo, podemos afirmar que o grupo percorreu um caminho de intensas descobertas e consideráveis mudanças, tanto em termos de motivação quanto de interação e troca. Durante tal percurso, procuramos observar principalmente como os professores se comportaram num ambiente que motivou o compartilhamento de ideias, crenças, anseios, experiências e inseguranças. Mesmo tendo participado de outras formações, alguns dos professores inicialmente mantiveram uma postura mais de observadores e com pouca interação. Tal comportamento se evidenciou principalmente nos encontros de estudos conceituais, nos quais estes professores pouco expuseram suas ideias e opiniões, reservando-se à observação das falas dos demais colegas, mesmo quando foram por nós incentivados a participar. Procuramos não pressioná-los quanto à participação nas falas, deixando-os livres para se expressar quando sentissem mais segurança em fazê-lo.

Os professores começaram a interagir mais a partir do início dos ciclos formativos, num processo gradativo de estabelecimento de segurança e receptividade. Neste momento da formação nos ficou claro o *movimento constitutivo de um grupo de trabalho*, apontado por Ferreira (2003), com os membros do grupo estabelecendo os primeiros contatos e uma considerável necessidade de atuação por parte do formador, incentivando a interação no grupo que começa a se formar. No decorrer do processo formativo, os professores mais observadores foram aumentando sua participação, expondo com mais segurança suas ideias e opiniões e interagindo mais ativamente com os demais colegas. Inicialmente, observamos que os professores, em sua maioria, expunham apenas suas próprias impressões e pouco falavam sobre o que os demais colegas expressavam, denotando certo receio em discordar do outro. Assim, nas primeiras reuniões, ocorreu mais exposição do que troca, sem muito confronto de ideias entre os participantes da formação. Concordando com Fullan & Hargreaves (2000) de que "discordância e diferença individuais devem, às vezes, ser propiciadas pelo grupo, ao invés de reprimidas" (p. 25), aos poucos fomos contribuindo para que tais diferenças fossem entendidas como elementos motivadores do processo de interação.

Percebemos que os professores – acostumados com um modelo receptivo de formação continuada – em princípio não souberam lidar com o espaço que foi dado às suas vozes e, principalmente, com a possibilidade de argumentar com os demais a favor daquilo que acreditavam como certo em termos de práticas pedagógicas. Não demonstraram perceber a troca como um elemento que pudesse contribuir para suas próprias ideias e temiam pela não aceitação de suas opiniões pelo outro. Podíamos perceber tal comportamento principalmente nos momentos em que discutimos os textos introdutórios, observando a hesitação nas falas de alguns professores e no silêncio de outros. Quanto a isso, concordamos com Ferreira (2003) quando esta afirma que "colaborar é co-responsabilizar-se pelo processo. É ter vez, ter voz e ser ouvido, é sentir-se membro de algo que só funciona porque todos se empenham e constroem coletivamente o caminho para alcançar os seus objetivos" (p. 326).

Vimos no decorrer dos ciclos que, aos poucos, os professores passaram a expor mais o que pensavam e sentiam não apenas sobre si mesmos, mas também a respeito do que era exposto pelos outros. Assim, gradativamente, a troca passou a se consolidar como elemento estruturante das interações. Aos poucos, o modelo de fala expositivo, do "ouvir sem falar", deu lugar à conversa, à verdadeira interação, à comunicação em via dupla, sem receios do que se falava e, principalmente, sem temer o que se ouviria. Reforçamos que tal transformação teve início a partir do momento em que os professores começaram a discutir as possibilidades de uso das ferramentas que foram trabalhadas no decorrer dos ciclos.

O levantamento e troca de materiais também foi gradativamente tendo sua dinâmica alterada durante a formação. Tivemos que intervir mais no início, compartilhando *links* e arquivos com os professores, que se limitavam a acessá-los. No decorrer dos ciclos, os próprios professores assumiram o levantamento de materiais complementares às discussões e análises realizadas, sentindo-se seguros quanto à sua socialização com seus pares. Ainda permanecemos compartilhando materiais interessantes, porém não nos vimos mais sozinhos em tal processo. Costa (2006) considera que os papeis dos parceiros nos processos colaborativos "[...] podem ser diferenciados, [...] mas não deve haver um chefe a centralizar as decisões que são cumpridas pelos demais: todos participam democraticamente das tomadas de decisão e são responsáveis pelas ações" (p. 176). Concordamos mas ressaltamos, assim como Fiorentini (2010), que tal processo de assimilação de papéis requer tempo e complementamos com Imbernón (2010), ao apresentar os seguintes fatores como necessários para a promoção de uma cultura colaborativa: Explicar o que nos acontece e escutar a todos da mesma forma; Praticar e compartilhar a reflexão individual e coletiva; Assumir o risco da inovação; Comprometer-se com o trabalho na instituição e com os demais; Não batalhar por coisas insignificantes; Pedir ajuda aos colegas; Equilibrar trabalho docente e vida; Tornar o projeto compreensível a todos; Considerar que o mais importante são os alunos e que seu desenvolvimento é paralelo ao dos professores.

Como já descrevemos, na medida em que os ciclos foram ocorrendo, os professores passaram a necessitar cada vez menos da nossa intervenção. Isso nos fez identificar o *movimento constitutivo de um grupo de trabalho colaborativo* estabelecido por Ferreira (2003), caracterizado pelo fortalecimento do sentimento coletivo e diminuição da dependência com relação ao formador. Entendemos que, a partir do momento em que os professores começaram a viver os ciclos, compreenderam melhor a nossa proposta e, dessa maneira, começaram a interagir de forma que o grupo foi, gradativamente, se consolidando como colaborativo.

Os interesses dos professores também se modificaram no decorrer da formação. Em princípio, não havia metas compartilhadas pelo grupo, a não ser a vontade de participar no projeto e compreender como utilizar as ferramentas. Entretanto, os questionamentos denotavam demandas particulares, de maneira que cada um se interessava apenas por aquilo que lhe serviria num contexto específico, sem prestar muita atenção às necessidades dos demais professores. Aos poucos, essa necessidade individual foi se atenuando, dando espaço para o estabelecimento de metas compartilhadas por todos. Não queremos aqui defender que o coletivo deva prevalecer sempre, mas sim que a coletividade deve ser entendida como um instrumento para o crescimento individual. O professor deve buscar sempre soluções para questões que se apresentem nas suas práticas e defendemos isso como elemento motivador para a sua formação continuada. Entretanto, acrescentamos a essa postura a necessidade de enxergar o grupo como possibilidade de busca colaborativa de soluções. Ao mesmo tempo, concordamos com as ideias de Hargreaves (2003), que defende a individualidade, diferenciando-a do individualismo presente no ambiente escolar.

Cabe também analisarmos aqui a interação virtual dos professores no Facebook®. Para Miskulin et al. (2011) "ao teorizarmos a colaboração e a prática docente, não podemos deixar de mencionar a virtualidade como um possível espaço formativo de colaboração entre professores" (p. 176). Assim, uso da rede social Facebook® se mostrou eficiente durante a formação, mesmo tal recurso não tendo sido objeto de nenhum dos ciclos realizados. Foram muitas as interações, principalmente durante os ciclos, como já relatado.

Concordamos com o que defendem Miskulin et al. (2011) e acrescentamos que a aprendizagem socialmente compartilhada proporcionada pelo espaço virtual não se restringe apenas aos alunos. Quando vivem essa experiência de uso da virtualidade como espaço de interação, os professores também podem ser beneficiados pelos seus resultados em termos de redução do isolamento e aumento da troca. Entendemos que as ferramentas da Web 2.0 se constituem não apenas um objeto de uso pedagógico a ser apropriado pelos professores de modo reflexivo como também um instrumento de sua própria formação.

Podemos afirmar que o grupo, no decorrer do processo formativo desenvolvido, passou de uma postura passiva e receptiva para uma de trabalho cooperativo e, posteriormente, consolidou características de um grupo colaborativo, assumindo com mais autonomia a sua própria formação e esperando menos soluções externas prontas e não relacionadas às suas próprias necessidades de formação. Nos movimentos apontados por Ferreira (2003), identificamos, ao término da formação, o *movimento de consolidação de um grupo de trabalho colaborativo*, no qual o grupo assume seu protagonismo com uma postura autônoma e não mais dependente do formador.

CONSIDERAÇÕES FINAIS

A análise da formação vivida pelos professores nos permite afirmar que a atuação deles enquanto grupo foi se modificando ao longo do processo. No início, as necessidades individuais prevaleceram de tal forma que não havia um pensamento coletivo em busca de uma meta comum a todos os participantes. Apesar de reunidos, os professores ainda não pensavam como um grupo e buscavam respostas para questões essencialmente individuais. Com o passar dos ciclos e a vivência do modelo formativo em todas as suas etapas, tal comportamento deu lugar, primeiramente, a uma postura cooperativa e, após a consolidação de um pensamento coletivo, passou a apresentar características colaborativas.

Enfatizamos que não há como desenvolvermos ações formativas que realmente busquem um uso reflexivo das tecnologias se não tivermos clareza de que necessitamos de tempo para que tais resultados se estabeleçam. As mudanças se apresentaram gradativamente, de modo cumulativo, porém mais efetivo do que os já por nós criticados modelos fechados de formação, apresentando "soluções" previamente elaboradas por agentes externos para problemas que sequer foram ouvidos ou discutidos e, muito menos, emergiram das diversas realidades escolares que se apresentam.

Ao longo do caminho percorrido pelo grupo, as estratégias coletivas foram, aos poucos, sendo incorporadas de maneira que as discussões contribuíram tanto para a busca coletiva por soluções e possibilidades de experimentação com os recursos da Web 2.0, quanto na verbalização de opiniões acordantes ou discordantes. A interação passou a ocorrer num ambiente sem hierarquia, onde todos se viram como iguais e com potencial para contribuir no crescimento uns dos outros.

A análise do percurso do grupo também nos permite afirmar que o esforço coletivo para o alcance de bons resultados está diretamente relacionado ao sentimento de grupo por parte dos professores que o integraram. Tal sentimento é uma construção contínua, decorrente principalmente da segurança que é transmitida aos professores e ultrapassa o isolamento e o individualismo hoje praticamente onipresentes no ambiente escolar. Entendemos que uma formação continuada que se proponha a trabalhar numa perspectiva colaborativa deve, prioritariamente, ter suas bases fincadas na concepção do professor como um profissional em constante desenvolvimento, capaz de refletir sobre o que faz desde que lhe sejam dadas as devidas condições para que tal comportamento se consolide.

Os dados que analisamos nos permitem concluir que a formação realizada propiciou aos professores envolvidos um ambiente no qual a colaboração, uma vez estimulada, aos poucos foi sendo incorporada às práticas destes professores, numa dinâmica de troca que se consolidou no decorrer do processo. Pudemos verificar que, longitudinalmente, os professores conseguiram se organizar enquanto grupo e as perspectivas de suas ações passaram de uma busca individual para um trabalho coletivo cujas metas compartilhadas respeitaram a individualidade das necessidades pedagógicas identificadas por meio da troca de ideias, crenças, angústias e experiências.

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PERSPETIVAS DE PROFESSORES DE MATEMÁTICA SOBRE O USO DE COMPUTADORES NAS PRÁTICAS DE ENSINO / MATH TEACHER PERSPECTIVES ABOUT COMPUTER USE IN TEACHING PRACTICES

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This article describes some results of a research carried out in schools in the city of Braga, located in northern Portugal, on the use of computers by mathematics teachers in their classes. Based on data collected through a questionnaire, considerations are presented on this use according to the perspectives of 44 teachers participating in this research. The results show that most of the teachers surveyed (84.1%) integrates the computer in their teaching practices, while few (15.9%) did not use it. Show up, too, the prospects of these teachers on this integration to their teaching practices, indicating advantages, disadvantages and the reasons for the use of computers in their classes. The results demonstrate the satisfaction of these teachers with the use of computers in their teaching practices, recognizing the relevance of such use.

Keywords: Computers; perspectives; uses; Math teachers.

CONSIDERAÇÕES INICIAIS

O avanço científico e tecnológico em nossa sociedade exige uma maior atualização e profissionalização do professor, bem como a superação de paradigmas da educação vigentes. Neste contexto, é necessário refletirmos sobre o processo de formação de professores e redefinir o processo de ensino de forma que englobe, entre outras coisas, as novas tecnologias digitais. O ensino passa a ter uma nova proposta baseada nos contributos que a integração das tecnologias pode oferecer à aprendizagem.

Essa evolução da sociedade foi sentida no sistema educacional português. Em Portugal, iniciou-se na segunda metade dos anos oitenta uma reforma educativa como consequência dessa evolução. As potencialidades do computador puderam, então, ser reconhecidas no ensino da Matemática e, assim, foram criados projetos com o objetivo de promover a utilização educativa do computador (Costa & Canavarro, 2008).

Esses projetos tinham como objetivo, entre tantos outros, preparar e dar suporte aos professores na adaptação a essa tecnologia. Era e é preciso estar atento às conceções e práticas dos professores sobre a utilização dos computadores. A integração do computador na sala de aula não é algo simples. Pelo contrário, exige muito da escola, mas principalmente do professor que tem que mudar toda a sua prática e, para alguns, fazer isso após longos anos de carreira não é nada simples e pode mesmo não ser agradável.

Hoje a utilização do computador nas aulas de matemática tornou-se uma recomendação fortemente sustentada pelos programas curriculares, tendo em vista suas potencialidades no que se refere à compreensão, visualização, representação e aplicação dos conteúdos. Estas potencialidades e implicações já foram previstas por Ball, Higgo, Oldknow, Straker e Wood (1991), que acreditam que o uso de computadores nas aulas de Matemática está diretamente ligado à aprendizagem de formas de trabalho e aptidões específicas. Parafraseando Papert, Ponte (1997) diz que "o computador facilita extraordinariamente uma abordagem experimental e intuitiva da Matemática. Deste modo, poderá caminhar-se para que o aluno assuma cada vez mais a condução do seu próprio processo de aprendizagem", ressaltando também que é necessário "desenvolver noções de organização e estrutura, o sentido de rigor e a aptidão para realizar demonstrações" (p. 34). Para concretizar estas orientações, é preciso que o professor conheça as potencialidades e limitações do *software*, avalie a sua adequação aos objetivos de aprendizagem estabelecidos e os utilize em suas aulas (Oliveira & Domingos, 2008).

Essa utilização do computador pelos professores tem merecido um olhar atento de alguns autores. Para Ponte (1992), "o professor tem de estar constantemente a aprender e a renovar-se. Professores e alunos passam a ser companheiros de um mesmo processo de aprendizagem" (p. 107). Complementarmente, de acordo com Larsen (2012), "os professores devem ensinar os alunos a avaliar e gerir na prática a informação que lhes chega" (p. 2). Para Thomas e Cooper (2000, citado por Amado & Carreira, 2008), existe uma inconsistência entre aquilo que os professores aprendem em sua formação educacional ou em formações continuadas ao longo de sua carreira e aquilo que se faz em sala de aula.

Frente a este cenário, é preciso analisar o que esta evolução global da tecnologia tem causado e contribuído para o sistema educacional. Nesse sentido, esse trabalho tem como objetivo relatar alguns dados de uma investigação que foi realizada junto de professores que lecionam em escolas públicas (ensino básico/fundamental e secundário/médio[1]) na cidade de Braga, localizada na região norte de Portugal, na qual se procurou averiguar a adequação dessa evolução tecnológica nas aulas de matemática. Para tal, estudou-se a utilização do computador no ensino da matemática centrando-se o olhar nos professores, nas suas conceções sobre o uso do computador em suas aulas, enquanto recurso do processo de ensino/aprendizagem, bem como se esses professores integram o computador em suas práticas docentes e quais são suas perspetivas face a essa integração.

ENQUADRAMENTO CONTEXTUAL E TEÓRICO

Em Portugal, desde a década de 90, os documentos curriculares têm incluído as TIC nas suas orientações para o ensino, e nos últimos anos tem-se intensificado as iniciativas para estimular os professores a criarem ambientes de aprendizagem apoiados na utilização de software (Oliveira & Domingos, 2008).

Viseu (2007) relata que, segundo um estudo feito em 1997, 76% das escolas públicas não possuíam computadores e que, a partir de então, as escolas começaram a receber equipamentos informáticos através de projetos de âmbito nacional criados com a finalidade de promover a utilização educativa desses recursos. Nesse mesmo trabalho, Viseu cita várias iniciativas ao longo de todo esse período, que tinham um objetivo comum: implantar o computador nas escolas e permitir seu acesso por alunos de todas as idades. Também foram criados programas pensando na formação de professores em TIC, como o Programa Utilização Pedagógica das TIC no 1º CEB e Formação Contínua de Professores na área das TIC. Além dessas, muitas outras iniciativas ocorreram contribuindo para o mesmo objetivo. Hoje, os programas de Matemática, tanto do ensino básico quanto do ensino secundário, recomendam a utilização do computador em sala de aula.

1 Usamos correspondência do sistema de ensino português ao sistema brasileiro, em que o ensino básico em Portugal corresponde ao ensino fundamental no Brasil e o ensino secundário ao ensino médio.

No Programa de Matemática do Ensino Básico, de 2007, programa ainda em vigor quando se coletou os dados aqui apresentados, o computador é recomendado em todos os ciclos de ensino, preconizando-se nas Orientações Metodológicas Gerais uma utilização consciente do computador, respeitando o momento de aprendizagem do aluno.

Quanto aos programas do ensino secundário, tanto na Matemática A, como na Matemática B e nas Ciências Sociais, salientam-se as potencialidades do computador, sendo a sua utilização considerada obrigatória no programa, nomeadamente ao nível do 10º ano, tanto para a Matemática A como para a Matemática B. Esse recurso informático é tido como uma indicação metodológica e um recurso facilitador da aprendizagem, compreensão e visualização por parte do aluno, possibilitando, também, mais recursos para esses processos de aprendizagem.

Com esse reconhecimento das potencialidades do computador no ensino em Portugal, procurou-se pensar na prática docente, reconhecendo a investigação que existem grandes dificuldades na sua integração curricular:

> Mesmo quando motivados para o uso dos computadores e da Internet, os professores deparam-se com grandes dificuldades, sobretudo porque não tiveram a preparação específica e adequada para o fazerem, dificilmente conseguindo concretizar propostas para além do que habitualmente fazem com os seus alunos. Usam, geralmente, as tecnologias como suporte de tarefas rotineiras, não acrescentando nada em termos cognitivos, ou seja, falham precisamente, por exemplo, em termos de estimulação e desenvolvimento de competências de nível superior (Costa, 2007, p. 15).

Amado & Carreira (2008) afirmam que "A mudança do papel do professor e do aluno na aula com tecnologias constitui uma das maiores resistências à sua utilização" (p. 287). Borba e Penteado (2003) acreditam que muitos professores não mudam suas práticas, não inserem o computador em suas aulas por falta de domínio do mesmo. Santos (2000) adverte-nos de que uma experiência mal sucedida com o computador em sala de aula pode contribuir para que o professor não volte a tentar usá-lo.

Canavarro (1993), numa investigação realizada em Portugal, identificou três perspetivas dos professores sobre a utilização do computador no ensino da Matemática:

> (a) Como instrumento de animação, permitindo melhorar o ambiente da sala de aula; (b) como instrumento facilitador, permitindo realizar tarefas habitualmente realizadas à mão; (c) como instrumento de possibilidade, permitindo realizar atividades que seriam difíceis de realizar de outra maneira (Costa & Canavarro, 2008, p. 303).

Frente a isso, o inquérito desenvolvido para a pesquisa que gerou os dados que aqui serão apresentados e analisados teve suas perguntas estruturadas de forma a identificar as perspetivas dos professores de matemática das escolas públicas de Braga sobre a utilização do computador no ensino da Matemática, incluindo suas vantagens e desvantagens. Este nosso trabalho traz resultados, que mostram as perspetivas desses professores referentes a integração do computador em suas práticas de ensino.

METODOLOGIA

Tendo por propósito a realização de um estudo analítico-descritivo acerca das práticas e perspetivas de professores de matemática sobre o uso de computadores nas práticas de ensino, fez-se um mapeamento através de um questionário planejado em termos do uso do computador no ensino da matemática, aplicado aos professores de matemática das escolas públicas da cidade de Braga, Portugal, que estavam lecionando no ano letivo 2013/2014. A metodologia adotada, investigação por questionário, baseado em Hill e Hill (2002), foi ao encontro do objetivo de identificar os professores de matemática atuantes nessas escolas que trabalham com os recursos das TIC. Ainda, buscou-se coletar as opiniões e perceções desses professores referentes ao uso do computador em suas aulas. Participaram da pesquisa 44 professores de matemática que se dispuseram a responder o questionário que foi enviado às escolas de Braga sem qualquer seleção prévia. É importante ressaltar que não foram todos os professores de matemática que lecionam no município de Braga que responderam ao inquérito. Por esse motivo, os dados que serão aqui apresentados não representam em sua totalidade os professores de matemática de Braga.

Na Tabela 1 apresenta-se a caracterização dos professores de matemática que responderam ao inquérito, segundo as variáveis: idade, sexo, número de anos de serviço docente, nível ou níveis de ensino em que tinha lecionado e nível de escolaridade que lecionava na altura.

Variáveis	Frequência (%)
Idade	
Até 40 anos	7 (15,9)
Mais de 40 anos	37 (84,1)
Sexo	
Feminino	37 (84,1)
Masculino	7 (15,9)
Número de anos de serviço docente	
Até 25 anos	19 (43,2)
Mais de 25 anos	20 (45,4)
Sem Informação	5 (11,4)
Nível ou níveis de ensino em que lecionou	
Ensino Básico	28 (63,6)
Ensino Secundário	16 (36,4)
Nível de escolaridade que leciona atualmente	
Ensino Básico	28 (63,6)
Ensino Secundário	16 (36,4)

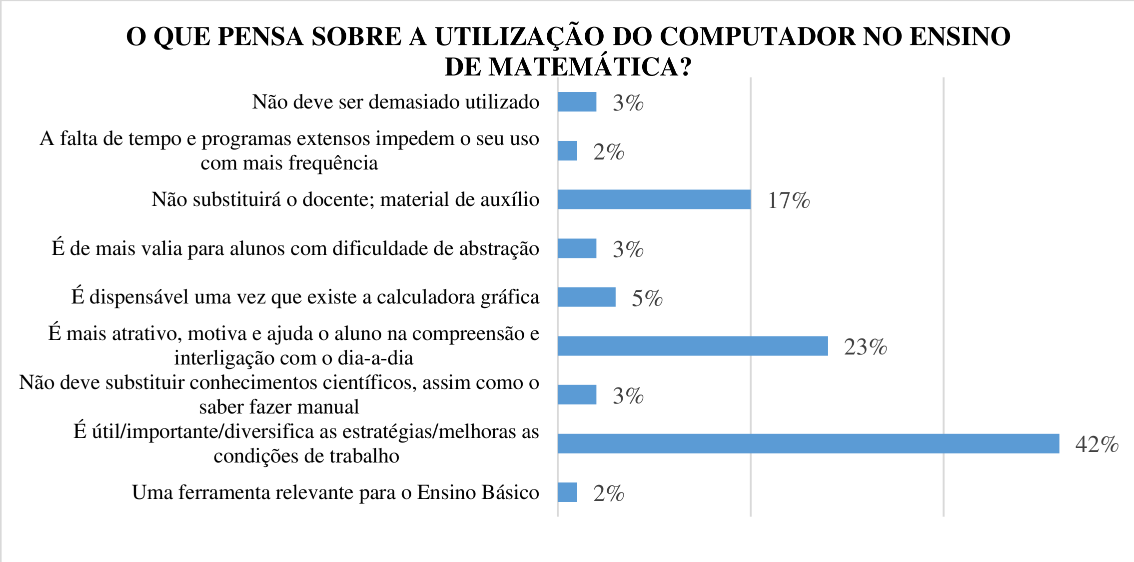
Table 1 – Characteristics of mathematics teachers (n=44)

Em termos de tratamento de dados, utilizou-se o *software* de análise estatística Statistical Package for the Social Sciences (SPSS), versão 22 para Windows, e recorreu-se a métodos de estatística descritiva, essencialmente a determinação de frequências, médias e desvios padrão, eventualmente resumidos em tabelas.

APRESENTAÇÃO DOS RESULTADOS

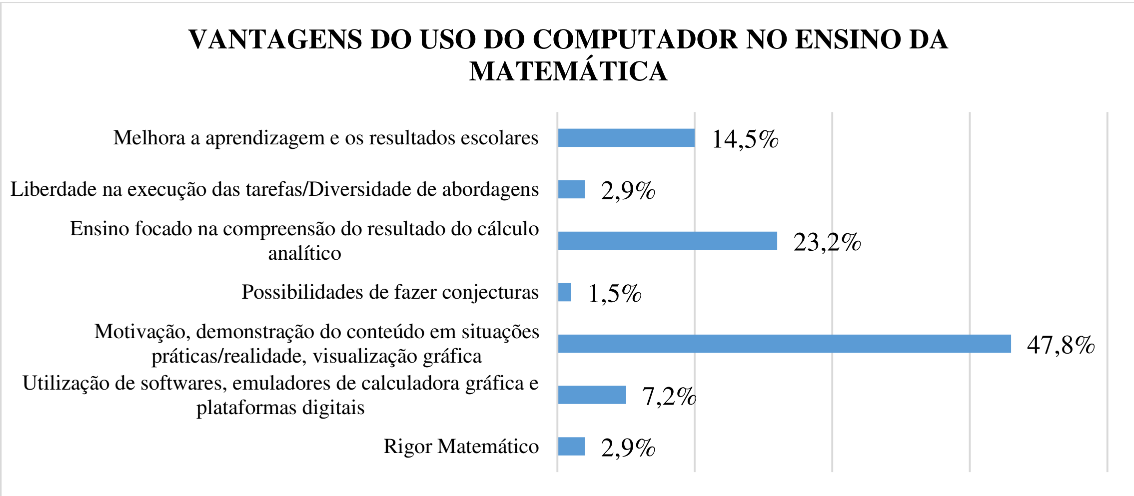
A maioria dos professores inquiridos possui uma opinião favorável sobre a utilização do computador no ensino de matemática (95,5%), enquanto poucos (4,5%) possuem uma opinião desfavorável. A maioria (84,1%) dos professores inquiridos utiliza o computador em suas práticas de ensino para ensinar matemática, enquanto alguns (15,9%) não o utilizam.

Para melhor entender esses indicadores, sintetizou-se no Gráfico 1 (percentagem de professores) a opinião dos professores sobre a utilização do computador no ensino de matemática, destacando-se a opinião de que o computador é útil/importante/diversifica as estratégias/melhora as condições de trabalho.

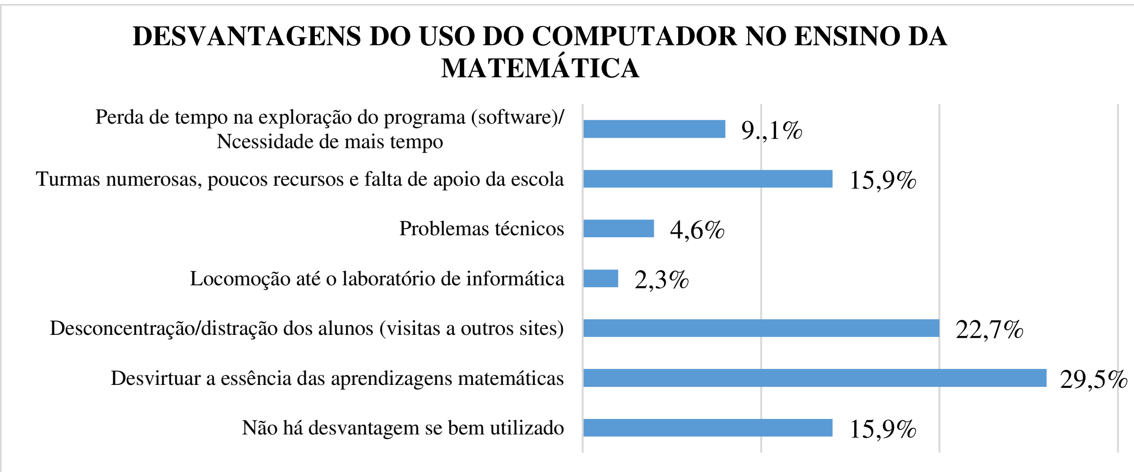


Graphic 1: Opinion of mathematics teachers on the use of computers in teaching

Apresentam-se nos Gráficos 2 e 3 as perspetivas dos professores inquiridos sobre as vantagens e desvantagens, respetivamente, do uso do computador no ensino da matemática.

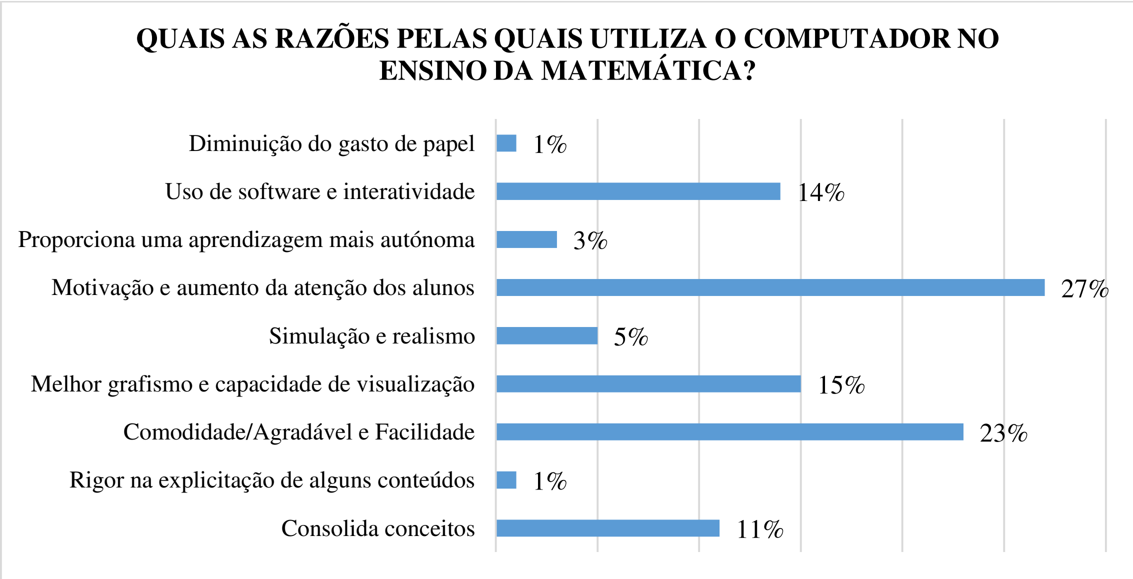


Graphic 2: Teacher perspectives about the advantages of computer use



Graphic 3: Teacher perspectives about the disadvantages of computer use

No Gráfico 4 indicam-se as razões pelas quais os professores integram o computador em suas práticas de ensino.



Graphic 4: Reasons why teachers integrate the computer in teaching practices.

A síntese das razões apontadas mostra vantagens centradas na motivação, demonstração do conteúdo em situações práticas da realidade e visualização gráfica (47,8%). Sobre essas razões, mostram-se, a seguir, os comentários de alguns professores:

> [O computador] ajuda a consolidar os conceitos que pretendo lecionar. O grande obstáculo, nessas circunstâncias, são as considerações logísticas, porque entendo que não basta que o professor mostre, é preciso que o aluno experimente (P2)

> Algumas áreas da matemática requerem uma forte componente de capacidade de visualização. Assim, o computador e *softwares* específicos facilitam a tarefa. (P1)

> Permite a visualização de abordagens essenciais à compreensão de conceitos matemáticos indispensáveis a resolução de exercícios. Facilita a aprendizagem de conteúdos mais abstratos que exigem muita atenção e concentração que os alunos nem sempre têm. (P23)

> [O computador] é motivador para os alunos. Proporciona uma aprendizagem mais autónoma, maior visualização dos conceitos e proporciona uma aprendizagem pela descoberta. (P27)

Ainda sobre suas opiniões sobre o uso do computador no ensino da matemática, alguns professores são claros ao dizerem:

> O computador permite proporcionar aos alunos situações que estimulam a curiosidade e o gosto pela construção do conhecimento. Apresenta grandes potencialidades na visualização e capacidade de tratamento de informações. (P28)

> Penso que é benéfico desde que seja bem articulado com os conteúdos a lecionar. Por vezes, a falta de tempo e programas extensos impede o seu uso com mais frequência. (P30)

As desvantagens se revelam no âmbito de o computador desvirtuar a essência das aprendizagens matemáticas (29,5%) e da desconcentração/distração dos alunos, como por exemplo, visitas a outros *sites* (22,7%). Finalmente, a motivação e aumento da atenção dos alunos (27%) e comodidade e facilidade (23%) são as razões que levam mais professores a utilizar o computador em suas práticas de ensino.

Através de uma escala de concordância, **DT** – Discordo Totalmente, **D** – Discordo, **C** – Concordo, **CT** – Concordo Totalmente, foram avaliados pelos professores vários aspetos relativos ao uso do computador incorporado às suas práticas de ensino em sala de aula. Recorrendo a percentagens e aos valores da média e do desvio-padrão, neste último caso depois de codificados os valores da escala (DT – 1, D – 2, C – 3 e CT – 4), obtiveram-se os resultados que constam na Tabela 2 (Nota: DT/D – Discordo Totalmente ou Discordo; C/CT – Concordo ou Concordo Totalmente; NR – Não Respondeu).

Item	% de respostas			$\overline{x}$	s
	DT/D	C/CT	NR		
Tive computadores disponíveis sempre que pretendi utilizá-los em minhas aulas	36,4	**50,0**	13,6	2,7	0,97
Tenho conhecimento suficiente sobre o computador para poder usá-lo em minhas aulas	2,3	**84,1**	13,6	3,2	0,47
Tenho conhecimento suficiente sobre como usar o computador para ensinar matemática.	18,2	**68,2**	13,6	3,0	0,70
Estou satisfeito/a com a forma como tenho usado o computador em minhas aulas	9,1	**77,3**	13,6	3,1	0,54
O uso do computador tem-me permitido explorar com os meus alunos uma matemática mais realista	13,6	**72,7**	13,6	3,1	0,61
O uso do computador nas minhas aulas tem-me permitido centrar menos o ensino nos aspectos de cálculo.	**56,8**	29,5	13,6	2,3	0,78
O uso do computador tem-me permitido centrar mais o ensino nas questões de interpretação e significado.	22,7	**61,4**	15,9	2,7	0,63
Estou satisfeito/a com a forma como os meus alunos têm usado o computador em minhas aulas.	36,3	**45,4**	18,2	2,6	0,70
Os meus alunos preferem aprender matemática usando o computador do que usando apenas outros meios	27,3	**54,6**	18,2	2,8	0,69
Os meus alunos envolvem-se mais na aprendizagem quando usam o computador.	31,8	**52,3**	15,9	2,8	0,68
Os meus alunos trabalham mais em grupo quando usam o computador.	**52,3**	29,6	18,2	2,4	0,60

Table 2 – Aspects related to computer use in teaching practices of mathematics teachers

Verifica-se que, face ao maior valor possível da média (4, no caso de todos os professores concordarem totalmente), os professores de matemática que participaram do estudo manifestaram um nível alto de concordância sobre possuírem conhecimentos suficientes sobre o computador para poder utilizá-lo em suas aulas e como usá-lo para ensinar matemática. Concordam que o computador tem-lhes permitido explorar uma matemática contextualizada com seus alunos e estão satisfeitos com a forma como têm usado o computador em suas aulas. Em um nível de concordância um pouco inferior, os professores concordam que tiveram computadores disponíveis sempre que os pretenderam utilizar, estão satisfeitos com a forma como os seus alunos têm usado o computador nas aulas e o computador tem permitido centrar *mais* o ensino nas questões de interpretação e significado e envolver os alunos mais na aprendizagem. Em contraponto, não concordam que o uso do computador tem permitido centrar *menos* o ensino nos aspetos de cálculo e nem mesmo que os alunos trabalham mais em grupo quando usam o computador.

CONSIDERAÇÕES FINAIS

A maioria dos professores inquiridos possui uma opinião favorável sobre a utilização do computador no ensino de matemática, o que facilita a sua integração às práticas de ensino, como está refletido nas respostas obtidas na Tabela 2. A falta de espaço não nos permitiu colocar em detalhes os comentários dos professores inquiridos em relação aos dados que compõem os Gráficos 1, 2, 3 e 4 mas, brevemente, identifica-se as mesmas perspetivas identificadas na investigação feita por Canavarro (1993) referida no enquadramento contextual e teórico desse trabalho e uma satisfação dos professores inquiridos sobre a integração do computador em suas práticas de ensino.

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Theme: Students

A VISUALIZAÇÃO DE VALORES MÁXIMOS E MÍNIMOS DE FUNÇÕES DE DUAS VARIÁVEIS / VISUALIZATION OF MAXIMUM AND MINIMUM VALUES OF FUNCTIONS OF TWO VARIABLES

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This article is focused in extending the study of Duval in order to analyze the content of a graphic register of functions of two variables, that is to say, to study the visualization of mathematical objects represented in a R^3 Cartesian system, especially the visualization of maximum and minimum values of functions of two variables. We highlight the study of visual variables in the graphic register in R^3 because the visualization requires starting the chart reading for some visual values that indicate the characteristics of the represented phenomenon and thus relate them to theorems, which are presented in another register. This is why we propose the following question: how to discriminate visual variables in the graphic register of a function of two variables in order to understand the idea of maximum and minimum local values? In order to distinguish the visual variables in that graphic register, we make use of the CAS Mathematica since it constitutes an important means to visualization of the maximum and minimum local values.

Keywords: visualization, visual variables, calculus of two variables, CAS Mathematica.

INTRODUÇÃO

A atividade matemática apresenta uma grande riqueza de conteúdos visuais, representáveis graficamente ou geometricamente, cuja representação resulta favorável para compreendê-la. Segundo Duval (2004), a atividade matemática necessita de modos de funcionamento cognitivos que demandam a mobilização de sistemas específicos de representação visto que, sua integração à arquitetura cognitiva dos sujeitos é uma condição necessária para ter uma compreensão em Matemática.

Dado que para o autor, a representação semiótica e a visualização estão no núcleo dessa compreensão, vale a pena nos questionar sobre o papel da representação e da visualização no pensamento matemático e na aprendizagem da matemática, particularmente, no que diz respeito às funções de duas variáveis porque, um dos problemas encontrados no ensino dessas funções é a dificuldade de visualização de suas representações gráficas, isto é, as representações gráficas no sistema cartesiano R^3. Nesse sentido, Imafuku (2008), constatou que muitos alunos, mesmo aqueles bem-sucedidos nas disciplinas de Cálculo Diferencial e Integral, em que se estudam funções de uma variável, não obtiveram o mesmo sucesso quando se depararam com funções de mais de uma variável, principalmente, na interpretação de seu significado e de sua representação gráfica. De maneira semelhante, Trigueiros e Martinez (2010), perceberam dificuldade na compreensão de funções de duas variáveis, em particular, na sua representação gráfica, o que pode estar relacionado, segundo as autoras, com a construção própria dos alunos do sistema cartesiano R^3.

Buscando trabalhos relacionados aos valores máximos e mínimos locais de funções de duas variáveis, constatamos que Alves (2011) fez um estudo desses valores por meio de mapas de contorno, para investigar como o aluno percebe ou intui o comportamento das curvas de nível de altura k e a conjecturar sobre a localização de todos os valores máximo, mínimo local e os pontos de sela. No entanto, acreditamos que a visualização não é apenas perceber, intuir ou representar mentalmente esses valores, pois, de acordo com Duval (1999), a visualização é uma atividade cognitiva intrinsecamente semiótica, sendo esta atividade de representação e não apenas de percepção. Para o autor, a visualização ao contrário da visão, que proporciona um acesso direto ao objeto, é baseada na produção de uma representação semiótica, pois mostra relações, ou melhor, uma organização de relações entre unidades representacionais.

Assim, este artigo tem por objetivo estender o estudo de Duval (1988; 1999) para analisar o conteúdo de um registro gráfico de funções de duas variáveis, ou seja, estudar a visualização de objetos matemáticos representados no sistema cartesiano R^3, particularmente, a visualização dos valores máximo e mínimo de funções de duas variáveis. Destacamos o estudo das variáveis visuais no registro gráfico em R^3 dado que, a visualização requer partir do registro gráfico por alguns valores visuais que apontam para as características do fenômeno representado e desta maneira relacioná-las com os teoremas, que são apresentados no registro algébrico. Em relação ao *software*, o Sistema Algébrico Computacional (*CAS) Mathematica* será utilizado por ser um CAS que permite a construção, a visualização e a manipulação de representações gráficas no sistema cartesiano R^3, preservando propriedades e permitindo modificações no objeto representado.

VISUALIZAÇÃO

As ideias básicas do Cálculo para funções de duas variáveis, por exemplo, derivadas parciais, plano tangentes, taxa de variação, nascem de problemas específicos, por exemplo problemas da física, e visuais. Segundo Guzmán (1997), todo profissional na área de Matemática conhece a utilidade de prestar atenção a esses problemas quando quer manipular os objetos abstratos correspondentes. Para o autor, essa forma de prestar atenção às possíveis representações concretas enquanto elas revelam as relações abstratas que interessam ao matemático constitui o que o autor denomina visualização em matemática. Assim, o autor afirma que a visualização aparece como algo totalmente natural tanto no nascimento do pensamento matemático, quanto na descoberta de relações entre os objetos matemáticos, e também, na transmissão e comunicação próprias da prática matemática.

Nesse sentido, Duval (1999) afirma que, a visualização é baseada na produção de uma representação semiótica que mostra relações, ou melhor, uma organização de relações entre unidades significativas. Estas unidades podem ser, por exemplo, as coordenadas de um ponto, quando se trata das representações gráficas no sistema cartesiano R^2. Segundo o autor, a partir do ponto de vista da aprendizagem da Matemática, a visualização, modalidade cognitiva relevante na Matemática, não pode ser usada como uma confirmação imediata e óbvia de compreensão; a visualização torna visível tudo o que não é acessível à visão, e por isso que a visualização não pode ser reduzida à visão.

Concordamos com o autor quando afirma que, frequentemente acreditamos que ensinar como construir registros gráficos ou figuras geométricas é suficiente para aprender a visualizar em Matemática. Entretanto essa construção requer somente uma sucessão de apreensões locais, isto é, focar nas unidades significativas e não na configuração final, o que permite que o sujeito não tenha a capacidade de olhar para a configuração final de um registro gráfico a não ser como representação icônica. O pesquisador afirma que construir um gráfico requer somente computar algumas coordenadas e traçar uma reta, uma curva, sempre partindo de tabelas de dados, ou pela lei de formação de uma função linear afim. Mas a visualização requer o oposto: deve partir de um registro gráfico para alguns valores visuais que apontam para características do fenômeno representado ou que corresponde a um tipo de lei de formação e a alguns valores característicos dessa lei.

O autor interessou-se, particularmente, por analisar o conteúdo de um registro gráfico da função linear afim representada algebricamente por $f(x)=ax+b$; $a\neq 0$ e por estudar a visualização nesse registro gráfico. Vamos explorar, neste artigo, seu trabalho e estender esse estudo para analisar o conteúdo de um registro gráfico de funções de duas variáveis, ou seja, estudar a visualização de objetos matemáticos representados no sistema cartesiano R^3, particularmente, a visualização dos valores máximo e mínimo de funções de duas variáveis.

Visualização em um Registro Gráfico no Sistema Cartesiano R^2.

Para analisar qualquer forma de visualização, segundo Duval (1999), é importante considerar a existência de diversos registros de representação, porque fornece maneiras específicas de tratar cada um, e a articulação desses registros. Assim, para criar condições para a articulação de registros de representação, entre os sujeitos, é necessária a identificação de unidades significantes no registro de saída e de chegada visto que,

> a discriminação das unidades significantes próprias a cada registro é a condição necessária para toda atividade de conversão e, em consequência, para o desenvolvimento da coordenação dos registros de representação. [...] A discriminação das unidades significantes de uma representação e, portanto, a possibilidade de uma apreensão do que ela representa, depende da apreensão de um campo de possíveis variações no que diz respeito ao significado de um registro (Duval, 2005, p. 77, tradução nossa).

Nesse sentido, acreditamos que a discriminação das unidades significantes do registro gráfico no sistema cartesiano e do registro algébrico merece nossa atenção porque, esses dois registros são de naturezas diferentes (registro não discursivo/discursivo) e como afirma Duval (2004), as unidades significantes do registro gráfico são não separáveis dado que estão integradas em uma única forma percebida, enquanto as unidades significantes do registro algébrico são discretas.

Segundo o autor, a leitura das representações gráficas pressupõe a discriminação de variáveis visuais pertinentes e a percepção de variações correspondentes no registro algébrico. Entendemos por variáveis visuais as variáveis significantes no registro gráfico. Duval (1988), distingue dois tipos de abordagem para os registros gráficos, ou seja, a passagem do registro gráfico para o registro algébrico, que levam em conta diferentes aspectos do registro gráfico: a abordagem *ponto a ponto* e a abordagem de interpretação global de propriedades figurais, para nosso estudo, esta segunda abordagem é de interpretação global de propriedades do gráfico.

A primeira abordagem nos permite observar associações entre pontos e pares ordenados o que propicia encontrar a lei de formação mediante uma regra de codificação, para o autor esta regra de codificação limita-se a uma apreensão local e não permite encontrar a lei de formação de uma função linear afim a partir de seu registro gráfico, dado que tira a atenção das variáveis visuais, conforme se mostra na figura 1, por exemplo.

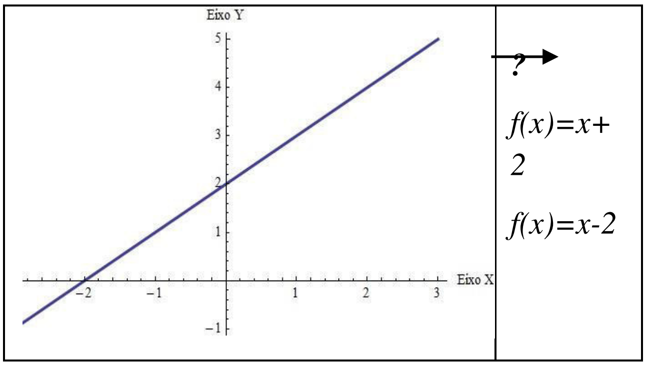

Figura 1. A apreensão local do registro gráfico.

Por outro lado, a abordagem de interpretação global de propriedades do gráfico facilita observar associações entre variável visual e unidade significante do registro algébrico, ou seja, identificar uma relação entre duas variáveis definidas sobre dois conjuntos de valores. Quando se trata de encontrar a lei de formação da função linear afim representada graficamente na Figura 1, por exemplo, utilizando a noção de inclinação ou de direção, é essa abordagem que se torna necessária. Segundo o autor essa abordagem é deixada de lado no ensino, porque depende de uma análise semiótica visual e algébrica. O autor, destacou as variáveis visuais e as unidades significantes correspondentes da função linear afim representada algebricamente por $f(x) = ax + b;\ a \neq 0$. Na Figura 2 apresentamos o que é observado no registro gráfico dessa função na abordagem de interpretação global de propriedades do gráfico.

O que é observado	Variáveis visuais	Unidades significantes
A direção respeito ao eixo horizontal positivo.	O sentido de inclinação do traçado com o eixo horizontal positivo.	Parâmetro *a>0*
		Parâmetro *a<0*
	Ângulo de inclinação do traçado com o eixo horizontal positivo.	*a=1* *a>1*
		0<a<1
Posição em relação ao eixo vertical.	Posição do traçado em relação à origem do eixo vertical.	*b>0*
		b<0
		b=0

Figura 2. Interpretação global de propriedades do gráfico.

Além disso Duval (2004), afirma que existe outro tipo de abordagem de um registro gráfico, a percepção icônica. Esta percepção evoca o alto e o baixo, as subidas suaves ou abruptas a partir do nível de base. Por exemplo, conforme mostra a figura 3, pela percepção icônica observamos o valor máximo e o valor mínimo locais de uma função de uma variável real dado que, fazemos uma analogia com o valor mais alto em relação com o nível horizontal e com os termos crescente, decrescente, pois como afirma o autor, o que se mobiliza na percepção icônica é uma analogia com mudanças de posição no espaço físico: o mais alto, o mais baixo.

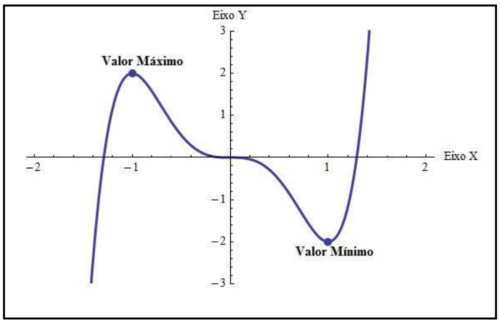

Figura 3. Percepção icônica do valor máximo e mínimo local.

Assim, como essas variáveis visuais estão ligadas a casos mais simples de funções reais de uma variável real, ou seja, o caso da função linear afim, nos interessamos por problemas referentes a valores máximo e mínimo de funções de duas variáveis reais porque ajuda-nos a analisar seus registros gráficos, além de fornecer informações importantes sobre seu comportamento além de solucionar problemas de otimização.

Visualização em um Registro Gráfico no Sistema Cartesiano R^3.

Conhecemos que representar graficamente funções de duas variáveis no sistema cartesiano R^3 é mais complicado do que representá-las no sistema R^2, porque há mais aspectos que devem ser considerados, por exemplo, a própria construção do sistema de coordenadas retangulares R^3, pois como afirma Trigueiros e Martinez (2010), os alunos apresentam dificuldades para representar graficamente funções de duas variáveis reais. Exceto nos casos mais simples, planos e superfícies quadráticas, as representações gráficas dessas funções podem ser difíceis de visualizar sem ajuda de um *software* desenvolvido especificamente para representar graficamente funções de duas variáveis. O *CAS* (*Computer Algebra System*) *Mathematica*, por meio de seus comandos, constitui um meio importante para a visualização desse tipo de funções, particularmente, dos valores máximos e mínimos locais.

Segundo Ingar (2014), a formação de uma representação gráfica de uma função de duas variáveis, com a utilização do *software Mathematica* é feita por meio de seu próprio menu de comandos, ou seja, manda instruções ao seu núcleo para exibir na tela do computador, especificamente no caderno do *Mathematica*, a representação gráfica de uma função de duas variáveis reais. Por exemplo, escrevemos o comando com suas opções respectivas: $Plot3D\ [x3 + 3xy2 - 15x - 12y, \{x, -3,3\}, \{y, -3,3\}, AxesLabel \rightarrow \{``X", "Y", "Z"\}]$, a seguir, pressionamos a tecla *shift* e *enter,* gerando dessa maneira uma representação gráfica como se mostra na Figura 4.

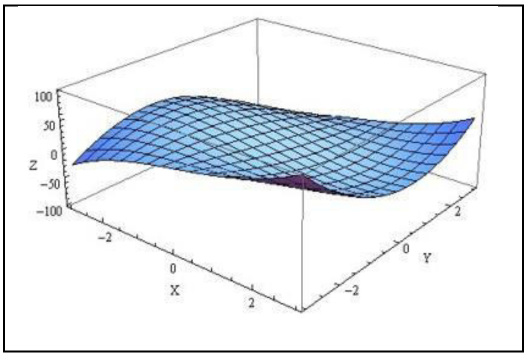

Figura 4. Formação de uma representação gráfica em R^3.

Observamos que essa representação gráfica é feita em função de regras de conformidade próprias do *software Mathematica*, porque o próprio software representa o sistema de coordenadas retangulares R^3 como uma caixa definida pelos eixos coordenados que não passam pela origem, em que se observa os oito oitantes do sistema cartesiano R^3.

Acrescentando duas opções ao comando *Plot3D*, formamos outra representação gráfica da mesma função de duas variáveis reais.

Por exemplo, $Plot3D[x3 + 3xy2 - 15x - 12y, \{x, -3,3\}, \{y, -3,3\}, AxesLabel \rightarrow \{“X”, ”Y”, ”Z”\}, AxesOrigin \rightarrow \{0,0,0\}, Boxed \rightarrow False]$, a seguir tecleamos *shift* e *enter*, conforme mostra a Figura 5.

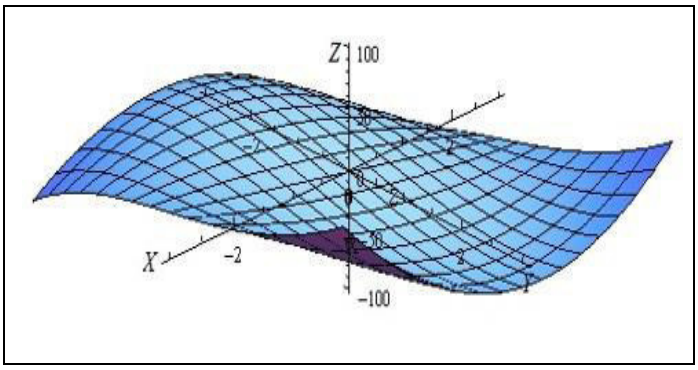

Figura 5. Outra formação de uma representação gráfica em R^3

Notamos que, neste caso, as regras de conformidade são próprias do sistema cartesiano R^3, em que a maneira de representar o R^3 é a canônica.

As duas representações do sistema de coordenadas retangulares R^3 cumprem a regra de mão direita, ou seja, quando os dedos da mão direita são fechados de tal modo que se curvam do eixo *x* positivo em direção do eixo *y* positivo, então o polegar aponta na direção do eixo *z* positivo. A orientação das duas representações gráficas é a mesma, positiva. Para Ingar (2014), as duas representações são registros gráficos, o apresentado na Figura 4 chama-se Registro Gráfico CAS_MATH e o da Figura 5, Registro Gráfico CAS.

Aplicando para objetos representados graficamente no sistema cartesiano R^3, particularmente o objeto matemático valores máximo e mínimo local de funções de duas variáveis, a ideia de variáveis visuais consideradas por Duval (1988), dado que, para o autor, visualizar um registro gráfico requer a interpretação desse registro e, para essa interpretação é importante a descriminação das variáveis visuais, apontamos duas variáveis visuais gerais relativas ao caso em que o gráfico é uma superfície e quatro variáveis específicas.

Em relação às variáveis visuais gerais, Ingar (2014) distingue, em relação ao que se destaca como representação gráfica: a curva e a superfície cuja representação algébrica pode ser conhecida ou não. Por exemplo, na Figura 6, observamos que o que se destaca é uma superfície chamada paraboloide em que seu eixo corresponde à variável *z* e o traço no plano *xy* é a origem.

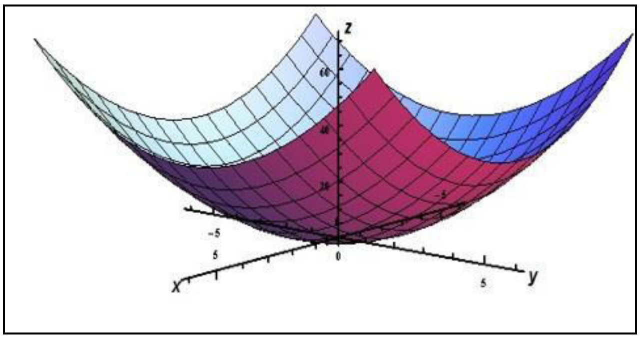

Figura 6. Variável visual: Uma superfície

Na sequência, distinguimos a segunda variável visual geral: a curva traçada correspondente a um dos cortes verticais, podem ser retas ou curvas abertas, já para os cortes horizontais, podem ser curvas fechadas ou abertas. Por exemplo, na Figura 7, destacamos a curva cuja representação gráfica é a interseção da função representada algebricamente por $z = f(x, 3)$ no plano *y=3*, ou seja, uma curva aberta. Observamos que os traços nos planos *yz* e *xz*, bem como em planos paralelos a eles são parábolas.

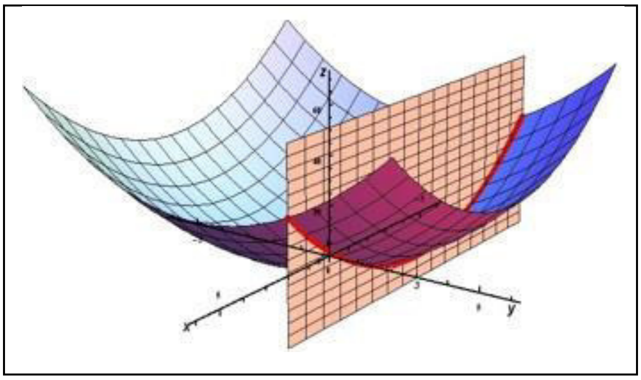

Figura 7. Variável visual: a curva traçada corresponde ao corte vertical.

No tocante às variáveis específicas, Ingar (2014) distingue quatro variáveis:

- Representação gráfica do sistema cartesiano R^3: registro gráfico CAS_MATH e registro gráfico CAS, como mostramos na Figura 8, observamos nesta figura a representação gráfica de uma função de duas variáveis em duas diferentes representações do sistema de coordenadas retangulares R^3.

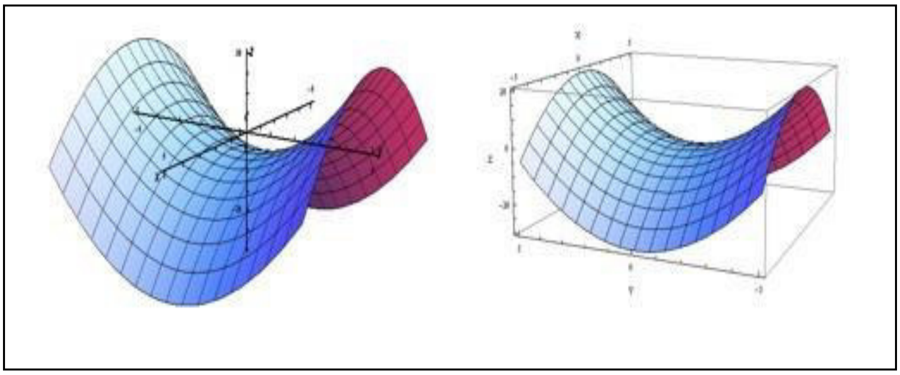

Figura 8. Variável visual: Registro CAS_MATH, Registro CAS.

- Os diferentes valores do eixo z, são comparados sucessivamente até obter o valor maior ou menor possível da representação gráfica. Por exemplo, na Figura 9, podemos observar os valores de z em que se pode encontrar o mínimo valor.

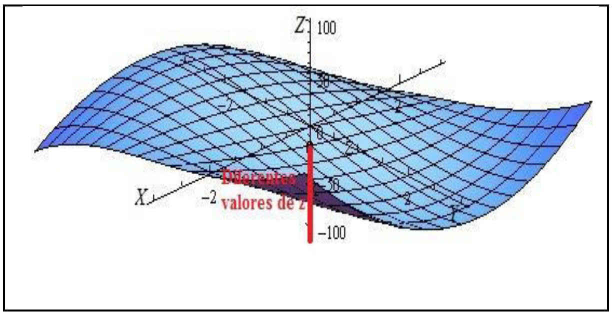

Figura 9. Relação dos pontos da superfície em relação com o eixo z.

- A posição da superfície em relação ao plano perpendicular ao eixo *z* (está sobre o plano, abaixo dele ou o atravessa). Por exemplo, Figura 10 a superfície é tangente ao plano $z=0$ e está completamente sobre o plano $z=0$.

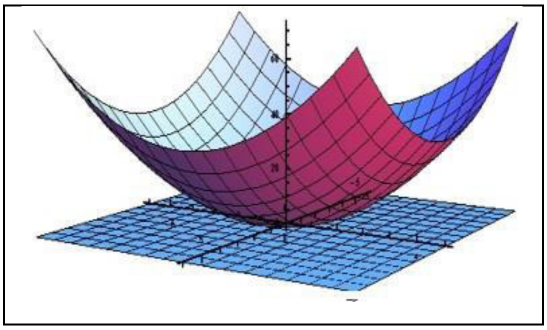

Figura 10. Posição do registro gráfico em relação com o plano XY.

- Variação do valor de *z* em relação aos valores de *x* e *y* da curva de interseção da superfície com o plano perpendicular ao eixo *z*. Por exemplo, na Figura 11 mostramos os traços em planos paralelos ao plano *xy*, isto é, os traços nos planos $\boldsymbol{z = 10, z = 7\ e\ z = 2}$. Observamos que conforme o valor de z decresce o diâmetro dos traços (curvas fechadas) diminuem até que o traço seja um ponto.

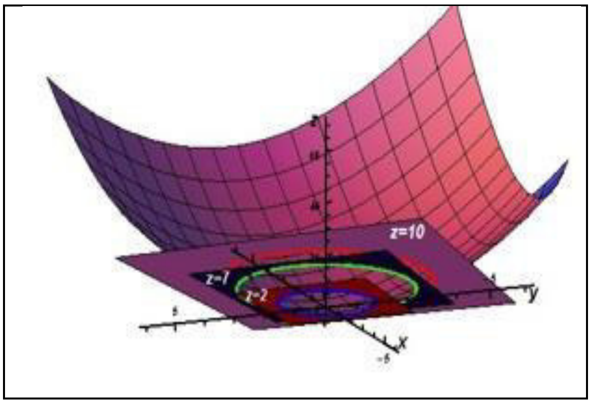

Figura 11. Variação do valor z

A autora afirma que, essas variáveis visuais desempenham um papel importante na interpretação do registro gráfico no sistema cartesiano R^3 para que o aluno compreenda a noção de valor máximo e mínimo local de funções de duas variáveis, bem como na conversão entre registros de representação (o registro gráfico e o registro algébrico) e na coordenação do registro gráfico com as expressões algébricas presentes no teorema que enunciamos a seguir: Se f tiver um extremos relativo em um ponto (x_0, y_0) e se as derivadas parciais da primeira ordem de f existirem neste ponto, então $f_x(x_0, y_0) = 0$ e $f_y(x_0, y_0) = 0$ Destacamos que esse teorema é desenvolvido pelo aluno a partir da coordenação desses registros.

CONSIDERAÇÕES FINAIS

As modificações no registro gráfico e a articulação entre este registro e o registro algébrico, essenciais no processo de visualização, não são consideradas pelos livros didáticos como fundamental para que o aluno compreenda e construa seus conhecimentos matemáticos a respeito de valores máximos e mínimos locais de funções de duas variáveis. Além disso, os processos de construção de representações gráficas, se limitam a uma apreensão local ou a uma apreensão icônica dado que as situações problemas que precisam da utilização de gráficos consideram apenas essas duas abordagens. Isto é, os registros gráficos no sistema cartesiano R^3 não são visualmente trabalhados pela maioria dos sujeitos exceto sua leitura. Acreditamos que o CAS *Mathematica* ajuda-nos para identificar e discriminar as variáveis visuais no registro gráfico nesse sistema cartesiano e, assim, compreender o valor máximo e mínimo local de funções de duas variáveis, uma vez que o aluno pode ver, manipular, visualizar esse registro e fazer conjecturas sobre o que visualiza.

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SOFTWARES MATEMÁTICOS NAS AULAS DE MATEMÁTICA: UM ESTUDO SOB A ANÁLISE DO PROGRAMA ACESSA ESCOLA / MATHEMATICAL SOFTWARES IN MATH CLASSES: A STUDY UNDER ANALYSIS OF ACCESS SCHOOL PROGRAM

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Under the perspective of a larger project entitled Mapping the use of information technology in Mathematics classes in State of São Paulo, supported by CAPES, Notice 049/2012 / CAPES / INEP, using data survey of state public schools of Presidente Prudente - SP, registered in the Access School Program, and having a qualitative research study, the aim of this work is to identify the software that math teachers use as GeoGebra, Cabri, Winplot, etc.; prioritized content, being in the majority, study of graphics and geometry, and what the teachers say about the importance of using ICT for learning, as are the Mathematics classes with the help of ICT and the conditions provided by Access School Programme. Data were collected through interviews with professionals from schools and visits to these laboratories.

Keywords: ICT, Math classes; softwares.

INTRODUÇÃO

Este trabalho apresenta dados de uma investigação realizada junto às escolas estaduais vinculadas à Diretoria Regional de Ensino de Presidente Prudente, desenvolvida dentro do contexto de um projeto de pesquisa intitulado *Mapeamento do uso de tecnologias da informação nas aulas de Matemática no Estado de São Paulo, financiado pela CAPES, Edital 049/2012/CAPES/INEP*, cujo objetivo é realizar um mapeamento do uso de tecnologias informáticas, mais especificamente, o uso do computador nas aulas de Matemática do Ensino Fundamental II das escolas públicas paulistas.

Os dados que serão apresentados nesse trabalho referem-se ao uso dos laboratórios de informática nas escolas públicas do município de Presidente Prudente, localizado na região Oeste do estado de São Paulo, Brasil, que possuem Ensino Fundamental II e pertencem ao Programa Acessa Escola. Foram coletados através de entrevistas com professores de matemática e visitas aos laboratórios de informática de cada escola envolvida no estudo.

O Programa Acessa Escola, desenvolvido pela Secretaria da Educação do Estado de São Paulo (SEESP), visa implantar nas escolas públicas do estado, laboratórios de informática equipados para que professores possam integrar as Tecnologias da Informação e Comunicação (TIC) às suas práticas de ensino. O Programa oferece acesso à internet e outros recursos, como *softwares*, para que professores possam utilizá-los visando a aprendizagem dos alunos.

Este trabalho apresenta a utilização desses recursos por professores de matemática em contexto de aula, nomeando cada software utilizado e os temas matemáticos que são trabalhados com esses recursos. Utilizando as recomendações de Gomes et. al. (2012), questionamos os professores de matemática participantes do estudo sobre quais softwares e temas do currículo de matemática, eles indicam para serem trabalhados no laboratório disponibilizados nas escolas pelo Programa Acessa Escola, de forma a melhorar o ambiente de aprendizagem dos alunos.

JUSTIFICATIVA

As inovações tecnológicas estão cada vez mais presentes na realidade de crianças e jovens do Brasil, mas infelizmente a exclusão social também é uma realidade, principalmente nas áreas de periferia. Borba e Penteado (2003) comentam a opinião da sociedade a respeito da inserção de computadores e também outras tecnologias na escola, ressaltando que dado o dinamismo e a importância social dos computadores, este poderia ser a solução para a falta de motivação dos alunos, que a inserção de novas tecnologias nas escolas pode estimular o aperfeiçoamento profissional, fazendo que professores busquem capacitações para estarem aptos a lidar com informática, mas, de qualquer forma, o uso contínuo, seja de um determinado software ou mesmo de giz, pode tornar as aulas cansativas e até mesmo não motivar. O objetivo principal da inserção de computadores na escola deve ser colocar todo o potencial do software educativo a serviço do aperfeiçoamento do processo educacional, fundamentando-se na proposta pedagógica da escola (Gomes et. al., 2002).

> O computador, o rádio, a tevê, a internet e as mídias digitais precisam estar presentes na escola, concorrendo para que essa deixe de ser mera consumidora de informações produzidas alhures e passe a se transformar – cada escola, cada professor e cada criança – em produtores de culturas e conhecimentos. Cada escola, assim, começa a ser um espaço de produção, ampliação e multiplicação de culturas, apropriando-se das tecnologias (Pettro & Assis, 2008, p. 81).

Não é de hoje que iniciativas têm sido criadas e concretizadas visando a criação desse ambiente tecnológico nas escolas do país. Segundo Tavares (2002) e Borba e Penteado (2003), os governos federal e estadual tentam, ao longo dos anos, proporcionar o acesso às tecnologias por meio, por exemplo, do Programa Nacional de Informática Educativa, PRONINFE, criado em 1989, que colaborou para a criação de novos laboratórios e formação de professores, bem como do Programa Nacional de Informática na Educação, ProInfo, criado em 1997 pelo Ministério da Educação, para promover o uso de tecnologias como recurso pedagógico, levando recursos digitais educacionais e computadores para o ensino público básico de todo o país.

Ao longo dos anos houve a criação de novos programas e a disponibilização de novos recursos, como softwares educacionais. Atualmente, está implantado nas escolas públicas do Estado de São Paulo o Programa Acessa Escola, um programa da Secretária Estadual da Educação de São Paulo, Brasil, que visa a inclusão social e digital de toda comunidade escolar, oportunizando acesso às TIC através de Salas Ambiente de Informática (SAI), disponibilizando e preparando monitores para auxiliar o professor nas aulas realizadas nessas Salas Ambientes de Informática.

O Programa está em vigor desde 2008 e conta com 644 municípios atendidos, 3773 escolas cadastradas, mais de 60 mil estagiários do nível médio trabalhando como monitores nos laboratórios de informática, mais de 4 milhões de usuários entre alunos, professores e funcionários da escola. O programa permite fazer atividades e trabalhos, interdisciplinares ou não, com o uso de tecnologias e contribui para a aquisição de conhecimento básico em tecnologias, como é previsto pela Proposta Curricular do Estado de São Paulo (São Paulo, 2008).

Os monitores são alunos que devem estar matriculados no 1.º ou 2.º ano do Ensino Médio/ secundário[2] regular, e ter pelo menos 16 anos. A seleção dos monitores acontece por meio de uma prova, e se aprovado, o monitor trabalhará durante 4 horas, em período contrário ao que estiver matriculado, no

2 Usamos correspondência do sistema de ensino brasileiro ao sistema português, em que o ensino médio no Brasil corresponde ao ensino secundário em Portugal.

laboratório de informática da escola em que estuda. Não havendo candidatos da mesma escola, é chamado um candidato da escola mais próxima, seguindo a classificação da prova de seleção.

Os monitores devem fazer cursos online e presenciais, pois ficam responsáveis pelo cadastro dos alunos no sistema utilizado pelo Programa, pelo monitoramento dos computadores dos alunos, por garantir o tempo estabelecido de 30 minutos de uso, pelo cumprimento das regras da utilização da sala e por projetos, como projeto de capacitação de professores ou criação de uma página virtual para escola. Até o momento há 894 projetos cadastrados no site do Programa.

Com toda essa estrutura que garante a utilização do computador por professores, em particular, por professores de matemática, o próximo passo é a escolha do software educativo a ser utilizado. Esse processo de escolha é fundamental para que todo o potencial disposto no laboratório contribua de forma efetiva para o processo de aprendizagem dos alunos. Essa escolha "depende da forma como este se insere nas práticas de ensino, das dificuldades dos alunos indentificadas pelo professor e por uma análise das situações realizadas com alunos para os quais o software é destinado" (Gomes et. al., 2012).

A utilização dos softwares educacionais "podem sugerir caminhos para a realização de demonstrações desconhecidas, proprondo artifícios que, muitas vezes, em demonstrações formais são necessárias e de dificil compreensão" (Lourenço, 2002 citado por Scheffer e Sachet, 2010, p. 105). Barreto, Souza e Loureiro (n.d.) relatam que, segundo Piccoli (2006), "os softwares contribuem na construção dos conceitos matemáticos devido à dinamicidade possibilitada pela simulação e variação de situações" (p. 2), confirmando, portanto, a importância do uso de softwares nas aulas de matemática.

OBJETIVO

Pretende-se relatar o trabalho de professores de matemática durante as aulas nas SAI do Programa Acessa Escola, os softwares e conteúdos que os professores participantes do estudo fazem uso, bem como divulgar os softwares instalados nos computadores desses laboratórios, que poucos têm conhecimento, e tratar das possibilidades do uso das TIC para a aprendizagem.

METODOLOGIA

Primeiramente foi realizado um estudo a cerca do Programa Acessa Escola e das escolas jurisdicionadas à Diretoria de Ensino de Presidente Prudente, cadastradas no Programa. Houve visitas à Diretoria de Ensino para confrontar os dados encontrados no site do Programa com os dados mais atualizados da Diretoria, verificando, por exemplo, o número de computadores que cada escola possui, se há nessas escolas um espaço físico adequado para os laboratórios e se há acesso a internet, uma vez que o programa garante computadores, instalação de uma sala própria para o laboratório e acesso a internet

Em seguida foram agendadas visitas nas escolas para serem feitas entrevistas com diretores ou coordenadores, monitores do programa, professores de matemática e analisar as condições dos laboratórios de informática nas escolas. As entrevistas foram tomadas por escrito ou por áudios e tendo em vista uma pesquisa qualitativa, como vemos em Borba e Araujo (2004), uma pesquisa que da ênfase ao particular e as características citadas pelo entrevistado, buscou-se analisar as falas dos entrevistados e tentar fazer transparecer as experiências e opiniões destes profissionais, a respeito das aulas de matemática com o auxílio de TIC, sobre os softwares e sobre a sala de informática.

A Diretoria de Ensino de Presidente Prudente abrange escolas de 11 municípios, incluindo o Município de Presidente. Na totalidade, 35 escolas estão sob sua jurisdição. Para o estudo, visitamos 16 escolas que pertencem ao perímetro urbano de Presidente Prudente e que estão cadastradas no Programa Acessa Escola.

RESULTADOS

Foi possível conversar com 9 professores de matemática de escolas diferentes, que deram seus pareceres quanto a importância da utilização de TIC para aprendizagem, citaram os softwares que já usaram e os conteúdos que trabalham com esses softwares.

Dos professores entrevistados, apenas um afirmou que não utilizava o laboratório, tanto pelo mau comportamento dos alunos quanto pela burocracia da escola e pela responsabilidade sobre os danos que ocasionalmente aconteçam. Já os demais professores afirmaram utilizar, como mostra a tabela a seguir, na qual P1 representa o primeiro professor a dar entrevista, P2 o segundo professor, sucessivamente.

Como um mesmo professor leciona em mais de uma escola, nos dados da Tabela 1 consta a utilização do laboratório, pelo professor de matemática entrevistado, na escola em que ele estava no momento da entrevista, e a frequência refere-se a utilização do laboratório pelo professor nas escolas que leciona de forma geral, ou seja, não se limita a escola em que ele estava no momento da entrevista.

A frequência de uso foi separada em três categorias sendo: *Nunca utilizou*, na qual o professor nunca preparou uma aula para o uso de TIC; *Poucas vezes*, na qual o professor afirma utilizar duas ou no máximo três vezes ao longo do ano e, *Frequentemente,* quando o professor faz uso pelo menos uma vez por mês, dependendo apenas do conteúdo que estudam.

Professor	**Utiliza a SAI da Escola que Estás no Momento da Entrevista**	**Frequência**
P1	Não.	Poucas vezes
P2	Não.	Nunca Utilizou
P3	Não.	Frequentemente
P4	Sim.	Frequentemente
P5	Sim.	Poucas vezes
P6	Sim.	Frequentemente.
P7	Não.	Poucas vezes.
P8	Sim.	Poucas vezes.
P9	Sim.	Frequentemente.

Tabela 1. Professores de matemática e o uso do IAS do Programa Acessa Escola

Observamos que os professores de matemática entrevistados preferem usar o laboratório de informática na escola em que ele possui a maior carga horária de trabalho e, por isso, alguns afirmam não usar o laboratório da escola em que se encontravam no momento da entrevista, mas mesmo assim, afirmaram usar com alguma frequência.

Os professores afirmam que usam TIC para complementar ou introduzir conceitos que são vistos em sala de aula e, portando, usam conforme o conteúdo. Vejamos o infográfico a seguir, que mostra os conteúdos e a quantidade de professores, dentre os entrevistados, que já prepararam aulas com os recursos dos laboratórios com cada um dos temas listados.

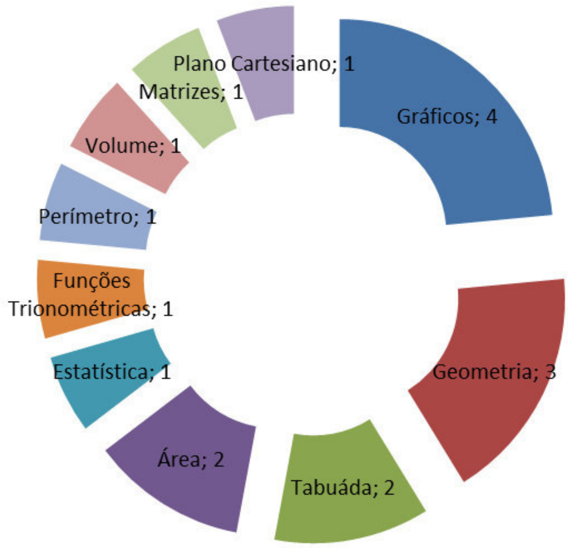

Gráfico 1. Conteúdos trabalhados com recurso às TIC

Metade dos professores que utilizam o laboratório já usou algum software para ensino ou realização de atividades, como estudo de gráfico de funções e, como afirma Pettro e Assis (2008), citado como referencial teórico na justificativa deste trabalho, todas as formas de mídias são potenciais para aquisição de conhecimento e costumes.

Em geometria, por exemplo, estudam a visualização e planificação de sólidos geométricos, ou com geometria analítica. Ainda, há preenchimento de tabuada e de tabelas com multiplicações montadas no computador. É possível fazer cálculos de área e perímetro com o uso, por exemplo, do software *Logo* ou do *Cabri*, e os cálculos estatísticos, construções de gráficos de barra, coluna ou setor, encontrar média, moda e mediana podem ser realizados no próprio *Excel*.

O professor P6 iniciou o uso da TIC no contexto de um projeto interdisciplinar, no qual os alunos separam o lixo reciclável de suas casas e nos dias programados levam para a escola, onde fazem anotações em seus cadernos distinguindo quantidades, tipo de material, ou seja, se era papel, plástico, entre outros dados, montando tabelas e construindo gráficos de barras e de setores, e utilizando régua, transferidor e tudo o que é necessário. Depois, o lixo das escolas era levado pra reciclagem, mas com os dados em mãos, os alunos, no laboratório de informática, recriaram as tabelas, aprenderam como montar os diversos gráficos e fizeram cálculos simples. O professor relatou que infelizmente o tempo disponível não foi suficiente para a conclusão do projeto, mas ele espera dar continuidade nos anos seguintes.

O conteúdo de matrizes foi abordado utilizando um CD-ROM disponibilizado por um sistema privado de ensino. Segundo o professor, o uso do CD-ROM dispensa o uso de livros didáticos e o aluno pode ir resolvendo exercícios nos computadores. Já o estudo de gráficos e funções trigonométricas foi indicado num contexto de uso no qual os alunos podem interagir com o computador, alterando valores e coeficientes para observar e compreender o que acontece com a função, por exemplo, se multiplicamos a função seno por qualquer número real, podemos verificar que a amplitude do gráfico varia conforme o número real escolhido.

Para esclarecer melhor os conteúdos e os recursos/métodos de trabalhá-los, sugeridos pelos professores na entrevista, apresentamos a Tabela 2.

Professor	Conteúdos	Softwares
P1	Gráficos e funções trigonométricas.	Microsoft Mathematics, Cabri e Winplot.
P3	Tabuada e matrizes.	CD-ROM COC
P4	Gráficos, estatística, área e volume.	Cabri
P5	Geometria.	Sites para pesquisa
P6	Tabuada e gráficos.	Excel, sites de jogos e vídeos
P7	Plano cartesiano.	Não soube dizer.
P8	Geometria.	Poly.
P9	Gráficos, geometria, área e perímetro.	Cabri, Logo e GeoGebra.

Tabela 2. Conteúdos e *softwares* explorados na aula no SAI

Dentre os softwares utilizados, o *Microsoft Mathematics, Cabri* e *Winplot* são programas matemáticos que plotam gráficos de funções e proporcionam o estudo detalhado dos gráficos; *Logo* é um software que pautado na linguagem de programação, com o qual pode-se estudar geometria plana e formas geométricas, plano cartesiano, etc. e o *Cabri*, é um software de geometria dinâmica. Os sites e vídeos são selecionados pelos professores para serem acessados durante as aulas no laboratório e a partir do que é explorado, no geral, os professores solicitam a realização de uma atividade, que pode ser uma síntese escrita ou um seminário.

A saber, a plataforma na qual funciona o Programa Acessa Escola nos computadores das escolas já traz diversos outros softwares instalados, não só para a matemática como, também, para as outras disciplinas. São os de matemática:

- *GeoGebra* – permite gerar gráficos, montar animações, trabalhar conceitos de geometria e figuras geométricas, e até mesmo derivadas;
- *Graphmatica* – gerador de gráficos de funções de uma variável e, ainda, calcula derivadas, integrais, etc.;
- *Poly1.12* – citado por professores, permite visualizar poliedros e suas planificações, podendo imprimir estas planificações e reconstruí-los;
- *Scratch* – é uma versão mais recente do *Logo*, introduz uma linguagem de programação mais simples e permite explorar geometria, lógica, entre outros;
- *Tess1.75* – trabalha simetrias e diferentes superfícies. Por exemplo, é possível criar figuras simétricas que remetem às pinturas de Escher;
- *Torre de Hanói* – um problema que consiste em jogar para resolvê-lo ou pode-se usar propriedades de Progressão Geométrica para resolver o problema;
- *Wolfram CDF Player 9* – Visualização de gráficos, análise de dados estatísticos, teoria dos números, entre inúmeros outros.

Ainda sobre a utilização desses softwares pelos professores, destacamos a fala de dois professores durante a entrevista: "[Faltam] mais cursos para aprender a mexer com os novos softwares" (P4, Outubro, 2013); "Sei que existem vários programas que nos ajudam a trabalhar conteúdos matemáticos como: funções, geometria, o 'Logo', por exemplo, o 'GeoGebra', porém utilizo o 'Cabri', pois tenho mais habilidade e competência para trabalhar com ele" (P9, Dezembro, 2013).

CONCLUSÃO

Os professores concordam com a importância do uso de TIC para a aprendizagem matemática, justificando que contribuem para a visualização e pela facilidade de manipulação pelos alunos ou mesmo por proporcionar habilidades informáticas, motivar os alunos por ser uma aula diferente da tradicional, indo ao encontro dos respaldos teóricos mencionados nesse trabalho. Os professores acreditam que o Programa Acessa Escola é um potencial para o ensino e inclusão digital, no entanto, observa-se que a maioria dos professores e monitores não tem conhecimento dos softwares já instalados nas máquinas do laboratório, citam diversos conteúdos, como estudo de gráfico e geometria, que são os mais citados, que podem ser tratados com os softwares já presentes nos computadores, sem depender de internet ou instalação de outros programas, mas ainda não sabem como utilizá-los em contexto de aula, como pode-se observar nas falas dos professores apresentadas.

Observa-se que foram poucos professores que aceitaram dar seu parecer. Talvez os professores que não utilizam este recurso ainda estejam em sua zona de conforto, pois como afirmam Borba e Penteado (2003, p. 57), "[...] Esses professores nunca avançam para o que chamamos de zona de risco, na qual é preciso avaliar constantemente as consequências das ações propostas", o que mostra a necessidade de formação e motivação para que professores aprendam os novos programas matemáticos e uma maior preparação dos monitores para que possam auxiliar professores.

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PRODUÇÃO DE CONHECIMENTO ACERCA DO TEOREMA DE PITAGÓRAS EM AMBIENTE INFORMATIZADO / PRODUCTION OF KNOWLEDGE ABOUT THE PYTHAGOREAN THEOREM ON A COMPUTER ENVIRONMEN

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This paper presents part of the activities carried out during a Master's research, in which the role of digital technologies were investigated in a group of humans-with-media for the production of knowledge about the Pythagorean Theorem. The research subjects were elementary school students. This qualitative study is theoretically based on the theoretical construct of humans-with-media and the role of dialogue regarded as a conversation that aims at learning. The results indicate that the GeoGebra software contributed to the creation of a learning environment, which favored the students' actions in the construction of mathematical knowledge and provided rich possibilities of visualization of concepts and properties, enhanced by the dynamism of the trials from the constructions performed in GeoGebra.

Keywords: Mathematics Education; Pythagorean Theorem; Dialogues: Digital Technologies; GeoGebra.

INTRODUÇÃO

O uso de tecnologias digitais aparece com frequência nas pesquisas em Educação Matemática. A inserção da informática nas escolas é realidade, assim como se reconhece sua contribuição para os processos de ensino e aprendizagem. Essa realidade leva os envolvidos com o contexto escolar a repensarem suas concepções sobre educação e estratégias de ensino.

Apresentamos, neste artigo, parte de uma pesquisa realizada pela primeira autora deste trabalho e orientada pela segunda, na qual buscamos compreender como se dá a produção de conhecimento acerca do Teorema de Pitágoras, num coletivo de seres-humanos-com-mídias, tendo como atores a informática e os alunos em grupo. Para tanto desenvolvemos, aplicamos e analisamos atividades com o uso da informática, mais especificamente do *software* GeoGebra, com 15 alunos do 9.º ano do Ensino Fundamental de uma escola pública de Minas Gerais, Brasil. As atividades aconteceram em nove encontros de 90 minutos cada. Neste trabalho discutiremos o quarto e o quinto encontro.

Concebemos as atividades considerando, como Borba (2001), que o conhecimento é produzido com uma determinada mídia, ou com uma tecnologia da inteligência, e não por seres humanos solitários ou coletivos formados apenas por seres humanos. Essa é a perspectiva teórica dos seres-humanos-com-mídias (Borba & Villarreal, 2005). A informática pode ser usada para criar ambientes de experimentação, interações, diálogos e elaboração de conjecturas acerca do conhecimento matemático. Os recursos da tecnologia, como, por exemplo, as possibilidades de manipulação numérica e gráfica e as ferramentas de visualização dinâmica, podem contribuir para que os alunos façam antecipações e simulações, confrontando com suas ideias iniciais, levando à verificação de relações, regularidades ou propriedades.

As possibilidades de manipulação e visualização do GeoGebra foram essenciais para a exploração dos conceitos nas atividades, assim como as possibilidades de interação e diálogo entre os participantes. A importância do coletivo também é ressaltada por Araújo (2002; 2004) que, baseada nas ideias de Alro e Skovsmose (1996), afirma que o significado matemático emerge entre os participantes nas interações no processo de ensino e aprendizagem, não sendo transmitido de professor para aluno nem construído por cada aluno individualmente, e ainda que por meio do diálogo os participantes de um ambiente educacional podem negociar suas perspectivas, buscar compreendê-las e compartilhá-las, para negociar os significados das atividades, dos conceitos e dos resultados. Assim, consideramos como principais referenciais teóricos para a pesquisa o constructo teórico dos seres-humanos-com-mídias (Borba, 2011; Borba & Villarreal, 2005; Levy, 1993), a visualização (Borba & Villarreal, 2005; Gúzman, 2002) e os diálogos (Alro & Skovsmose, 2010; Araújo, 2002; 2004).

Na pesquisa, de cunho qualitativo, adotamos uma metodologia baseada no *design experiments* (Collins, Joseph & Bielaczye, 2004) para analisar o conhecimento relativo ao Teorema de Pitágoras, produzido pelo coletivo constituído por seres humanos e mídias. Buscamos observar se esse coletivo favoreceu a produção do conhecimento sobre o teorema e de outros conceitos ligados a ele, como, por exemplo, a classificação de triângulos quanto aos ângulos. Cada atividade foi elaborada a partir das observações feitas durante a realização da atividade anterior. Para coletar os dados usamos gravações de áudio e vídeo da realização das atividades, produções escritas dos participantes e anotações do diário de campo da pesquisadora. Buscamos descrever, de forma narrativa e minuciosa, as situações de sala de aula, e assim foi possível focar no processo do qual os dados emergiram e não simplesmente no resultado apresentado ao final de determinada intervenção ou mesmo do tempo todo de observação. Apresentaremos a seguir os referenciais teóricos utilizados para concepção e análise das atividades.

TECNOLOGIAS DIGITAIS, VISUALIZAÇÃO E DIÁLOGOS PARA PRODUÇÃO DE CONHECIMENTO

Para a realização da pesquisa, seja na concepção ou na análise das atividades, nos referenciamos na produção de conhecimento que se constitui a partir de um coletivo de seres-humanos-com-mídias, focando especialmente na visualização e experimentação proporcionadas pelo software GeoGebra e nos diálogos que emergem nesse coletivo, em que os alunos trabalham em grupo.

Borba (2001) apresenta o construto teórico seres-humanos-com-mídias, apoiado nas noções de reorganização do pensamento de Tikhomirov (1981) e na relação entre técnica, conhecimento e história de Lévy (1993). Para Borba e Villarreal (2005) "os seres humanos são constituídos por tecnologias que transformam e modificam o seu raciocínio e, ao mesmo tempo, esses humanos estão constantemente transformando essas tecnologias". Sendo assim, o conhecimento é construído sempre vinculado a alguma mídia, suportando a noção de que o conhecimento é produzido por um coletivo composto por seres-humanos-com-mídias, sendo esta a unidade básica de conhecimento.

Um ponto-chave para a produção de conhecimentos é a visualização. Segundo Guzmán (2002), a visualização matemática consiste, basicamente, na atenção dedicada às possibilidades da representação concreta dos objetos, que estão sendo manipulados, para se ter uma abordagem das relações abstratas de forma mais eficiente. Segundo esse autor, a visualização não consiste na visão imediata de uma relação e sim na interpretação que é possível, a partir da contemplação de uma situação. Para Borba (2011) os *softwares* educacionais têm a capacidade de atribuir um papel importante à visualização, realçando o componente visual da matemática. Dessa forma, "neste coletivo a mídia adquire novo *status*, vai além de mostrar uma imagem. Mais que isso, é possível dizer que o *software* torna-se ator no processo de fazer matemática" (p.3). A visualização parece ser o principal meio de feedback fornecido pelos computadores (Borba & Vilarreal, 2005). Manipulando imagens e experimentando é possível seguir por caminhos nem sempre previstos inicialmente, provocando o sujeito a interpretar as respostas e imagens geradas durante o processo, o que está de acordo com a ideia de Guzmán (2002) acima exposta.

É importante dar aos alunos oportunidades de explorar situações na busca do conhecimento. Um instrumento que acreditamos ser indicado para essa exploração é o *software* de geometria dinâmica GeoGebra. Uma característica relevante desse *software* é o tratamento de "desenhos em movimento", fazendo com que as particularidades da representação física do objeto mudem, mantendo os invariantes, ou seja, as reais propriedades geométricas da construção (Gravina, 1996). As figuras em movimento são obtidas arrastando pontos que as compõem (como os vértices de um polígono). É possível ainda obter medidas relativas aos objetos manipulados como, por exemplo, amplitudes de ângulos, comprimento de segmentos, áreas de figuras planas, entre outras.

Outros aspectos importantes a serem considerados são a interação e os diálogos entre os participantes das atividades. Santos (2009) afirma que "no ensino e aprendizagem da Matemática, os aspectos linguísticos precisam ser considerados inseparáveis dos aspectos conceituais para que a comunicação e, por extensão, a aprendizagem aconteça" (p.119). Essa ideia está de acordo com o construto teórico dos seres-humanos-com-mídias, pois segundo Alro e Skovsmose (2010) "seres humanos" aparecem no plural porque é importante considerar a aprendizagem como um processo de interação de várias pessoas, o que pressupõe comunicação e diálogo. Para esses autores, um diálogo é entendido como "uma conversação que visa aprendizagem", não sendo concebido como uma conversação qualquer. Só mediante o diálogo é que se estabelece a verdadeira comunicação e o mais importante no diálogo é a "natureza da conversação e a relação entre os participantes" (Alro & Skovsmose, 2010).

Alro e Skovsmose (2010) interpretam o diálogo relacionado à aprendizagem, focando em termos de elementos ideais, são eles: realizar uma investigação; correr riscos e promover a igualdade. Os autores abordam a noção de investigação a partir da coletividade e da colaboração, no qual os participantes do diálogo podem expressar suas perspectivas, não existindo espaço para o comodismo. A negociação de significados se dá quando os alunos abrem mão de algumas de suas perspectivas, mesmo que por alguns momentos, permitindo-se explorar novos pressupostos a partir de novos ângulos, para que, em algumas situações, venham a construir novas perspectivas. Expor perspectivas implica em correr riscos, pois, muitas vezes, os alunos ficam vulneráveis a críticas, causando certo desconforto. Um diálogo deve estar embasado no princípio da igualdade, no qual todos os participantes se encontram num mesmo lugar, tendo o direito de apresentar suas colocações e de serem respeitados por isso. "Promover a igualdade não significa promover o acordo" (Alro & Skovsmose, 2010, p. 133). Em suma, num diálogo, deve-se buscar ser coerente, entender e respeitar a visão do outro, argumentar e construir o conhecimento a partir do confronto de ideias.

Tendo apresentado resumidamente os referenciais teóricos que embasaram a pesquisa, apresentaremos a seguir atividades referentes à parte da pesquisa que teve como objetivo levar os alunos a perceber e enunciar o Teorema de Pitágoras, a partir da visualização e experimentação que é possibilitada pelo uso do *software* GeoGebra e dos diálogos estabelecidos entre os participantes.

ATIVIDADES VISANDO À PERCEPÇÃO E ENUNCIADO DO TEOREMA DE PITÁGORAS: APRESENTAÇÃO E ANÁLISE

Trataremos aqui de atividades que foram desenvolvidas em dois encontros de noventa minutos, com o objetivo de que os alunos percebessem as relações que caracterizam o Teorema de Pitágoras e conseguissem expressar essas relações enunciando o teorema. Buscamos construir um ambiente no qual os alunos tivessem a oportunidade de manipular e explorar as imagens do GeoGebra livremente, elaborando e testando suas conjecturas a respeito das relações.

No primeiro encontro disponibilizamos aos alunos uma construção (Figura 1) na qual havia um triângulo qualquer, quadrados construídos sobre os lados deste triângulo, as amplitudes dos ângulos do triângulo e as medidas das áreas dos quadrados. Pedimos aos alunos que movimentassem os vértices do triângulo de modo a obter diferentes tipos de triângulo e observassem as correspondentes alterações dos valores das áreas dos quadrados. Intencionávamos que os alunos procurassem estabelecer alguma relação entre os valores das áreas dos quadrados para cada tipo de triângulo: acutângulo, retângulo e obtusângulo.

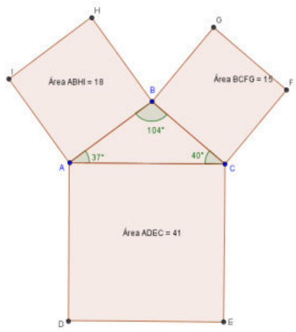

Figura 1: Triângulo qualquer e quadrados construídos sobre seus lados

Os valores obtidos para os diferentes triângulos deveriam ser registrados em uma tabela que fornecemos (Figura 2).

	A	B	C	D	E
1	Maior ângulo	Tipo de triângulo	Menor área	Área mediana	Maior área
2	83°	Acutângulo	16	18	30
3					

Figura 2: Tabela da atividade do primeiro encontro

Inicialmente os alunos apresentaram certa dificuldade para manipular os objetos no GeoGebra, possivelmente pelo hábito de trabalhar apenas com objetos estáticos, usualmente utilizados em trabalhos com a mídia lápis e papel. Posteriormente conseguiram fazer o que foi pedido. Para instigá-los a pensar sobre os diferentes valores registrados, pedimos que observassem se havia alguma relação entre as medidas das áreas dos quadrados para cada tipo de triângulo e que escrevessem suas conclusões. Esperávamos que os alunos dialogassem na tentativa de encontrar respostas. Esse diálogo não aconteceu de imediato, sendo possível observar certa ansiedade dos alunos por uma solução rápida. É possível que esse tipo de postura passiva dos alunos decorra das experiências escolares nas quais o aluno permanece na condição de receptor, sendo o professor aquele que responde a todas as dúvidas.

Por outro lado o professor não está acostumado a não responder imediatamente as perguntas dos alunos, o que também gera ansiedade por não saber como conduzir as atividades de modo a orientar as reflexões sem direcionar demais. Na situação em questão, pedimos que somassem valores de duas áreas comparando com a outra e também que relacionassem a relação percebida com o tipo de triângulo. Alguns alunos conseguiram perceber e verbalizar de forma satisfatória a relação percebida, mas não todos.

Tendo refletido sobre o ocorrido, julgamos necessário retomar esse ponto no encontro seguinte, construindo no quadro uma tabela similar à da Figura 2, preenchendo-a com valores fornecidos oralmente pelos alunos dos diferentes grupos. Os dados da tabela foram discutidos com a sala e alguns alunos conseguiram perceber relações. No caso dos triângulos não retângulos não houve dúvidas: os alunos concluíram que para os acutângulos a área do maior quadrado é sempre menor que a soma das áreas dos dois outros quadrados e que para obtusângulos a área do maior quadrado é sempre maior que a soma das áreas dos dois outros quadrados.

Um fato muito interessante aconteceu no caso dos triângulos retângulos: por conta de arredondamentos feitos pelo *software* nas amplitudes dos ângulos e nas medidas das áreas, a construção trabalhada não permitiu concluir sobre a relação entre as áreas, uma vez que em alguns casos a área do maior quadrado era menor, em outros maior e em outros era igual à soma das áreas menores. Com a Figura 3 exemplificamos as respostas dos alunos.

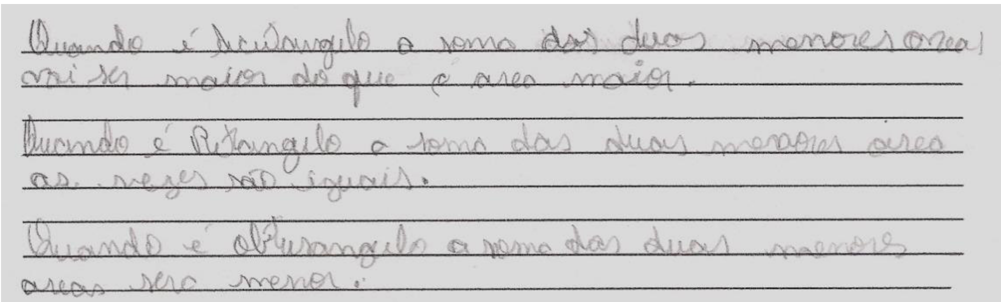
Quando é Acutângulo a soma das duas menores area vai ser maior do que a area maior.
Quando é Retangulo a soma das duas menores area as vezes são iguais.
Quando é obtusangulo a soma das duas menores areas sera menor.

Figura 3: Resposta do grupo A

Os alunos suspeitaram que no caso do triângulo retângulo essa relação seria de equivalência, mas ficou evidente a necessidade de entender o que estava acontecendo e usar outras formas para comprovar suas conjecturas. Houve uma discussão com toda a turma a respeito das aproximações e arredondamentos feitos pelo software. Pedimos que habilitassem um maior número de casas decimais no GeoGebra. Nesse momento os alunos perceberam, e ficaram surpresos, que os ângulos aparentemente retos de alguns dos triângulos obtidos tinham amplitudes inferiores ou superiores a 90º.

Na atividade seguinte os alunos exploraram outra construção no GeoGebra na qual o triângulo era retângulo, sendo o ângulo reto garantido por construção. Isso foi planejado para que independentemente das movimentações realizadas nos seus vértices, o triângulo sempre se mantivesse retângulo, alterando apenas as amplitudes dos ângulos agudos e as medidas dos lados e, consequentemente, das áreas dos quadrados. Os alunos foram solicitados a apresentar por escrito a relação que haviam percebido, de maneira organizada, clara, e em consenso com todos os membros do grupo. O objetivo com isso era que os alunos argumentassem e discutissem com os colegas a respeito das conjecturas que tinham elaborado e encontrassem uma forma de expressar o resultado obtido. Estimulamos que eles o escrevessem, mesmo que não fosse usando uma linguagem matemática simbólica. Tivemos também o objetivo de estimular o trabalho coletivo e o diálogo entre os elementos dos grupos, que não foi tão presente na atividade anterior. A resposta de um dos grupos é apresentada a seguir:

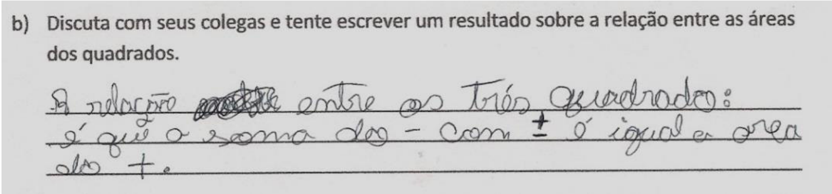

Figura 4: Resposta Grupo A

Ao final da aula as respostas de todos os grupos foram transcritas no quadro. Discutimos sobre a relação que expressavam e os alunos concluíram que, mesmo que escritas de maneiras diferentes, todas retratavam o mesmo resultado observado. Esse resultado foi denominado "Teorema de Pitágoras". Procuramos uma forma mais precisa para escrever o resultado e o teorema foi enunciado como: "Num triângulo retângulo a área do quadrado construído sobre a hipotenusa é igual à soma das áreas dos quadrados construídos sobre os catetos".

A estas atividades se seguiram diálogos e outras atividades com os objetivos de enunciar o Teorema usando notação algébrica, de provar o Teorema a partir das equivalências das áreas dos quadrados e também de possivelmente generalizar considerando as áreas de outros tipos de polígonos construídos sobre os lados do triângulo. Estas não serão aqui apresentadas por limitação de espaço para o artigo. A seguir discorreremos sobre a análise dos resultados relativos às atividades aqui apresentadas.

O SOFTWARE GEOGEBRA, OS DIÁLOGOS E A PRODUÇAO DO CONHECIMENTO ACERCA DO TEOREMA

Trazemos para este artigo considerações sobre dois dos aspectos considerados na análise da pesquisa de modo mais amplo: a influência dos recursos do *software* GeoGebra e a dos diálogos na produção do conhecimento acerca do Teorema de Pitágoras no coletivo de seres-humanos-com-mídias. A experimentação e a visualização tiveram importante papel no desenvolvimento das atividades com o GeoGebra, possibilitando diálogos entre participantes nos quais conhecimentos foram produzidos. Acreditamos, assim como Borba e Villarreal (2005) que a unidade básica de conhecimento é formada por humanos e mídias, trabalhando juntos.

Destacamos dois recursos do GeoGebra que contribuíram para que o conhecimento produzido fosse significativamente diferente do que aquele que seria possível sem a presença dessa mídia: as possibilidades de manipulação das figuras e de obtenção de valores numéricos referentes às figuras geométricas construídas como, por exemplo, as medidas dos segmentos que compõem os lados dos triângulos e os valores das áreas dos quadrados construídos sobre esses lados. No caso que analisamos foi fundamental não apenas a visualização da figura em si como também os valores das áreas correspondentes. Associar a manipulação dinâmica e a obtenção dos valores numéricos das áreas nas nossas atividades deu aos alunos a possibilidade de alterar a posição do vértice de um triângulo e observar as transformações dos ângulos, das medidas dos lados, dos valores numéricos referentes às áreas dos quadrados. Com isso, em um curto espaço de tempo, obtivemos grande quantidade de valores numéricos das áreas dos quadrados construídos para triângulos acutângulos, retângulos e obtusângulos, e pudemos estimular os estudantes a observar relações. Algo semelhante poderia ser feito com a mídia lápis e papel, no entanto modificar a figura no *software* de geometria dinâmica nos parece significativamente diferente de desenhar várias figuras. Modificar no *software* pode levar a conjecturas do tipo: o que acontece com as áreas dos quadrados construídos sobre os lados quando mudamos os ângulos e consequentemente o tipo de triângulo? Isso nos parece não ser facilitado com a observação de diferentes figuras de modo estático.

O GeoGebra possibilita o tratamento de “desenhos em movimento”, fazendo com que seja possível modificar a representação física do objeto mantendo as reais propriedades geométricas da construção. Isso foi importante para podermos discutir sobre as relações aparentemente diferentes para o caso do triângulo retângulo, decorrentes das aproximações feitas pelo *software* nas amplitudes dos ângulos. Pudemos explorar triângulos garantidamente retângulos por construção, movimentando seus vértices e mantendo sua característica principal de ser retângulo. Assim as questões dos arredondamentos foram discutidas e a relação entre as áreas foi percebida.

Na pesquisa, chamamos de perceber a relação do Teorema de Pitágoras, o fato dos estudantes perceberem aquilo que é específico dos triângulos retângulos, no que diz respeito às áreas dos quadrados construídos sobre seus lados, a saber: a soma das áreas dos quadrados construídos sobre os catetos é igual à área do quadrado construído sobre a hipotenusa. A busca por padrões e regularidades caracteriza um dos objetivos do ensino da geometria e, no nosso caso, buscou-se por um padrão para cada tipo de triângulo. A percepção da relação não foi algo imediato para os estudantes e as atividades, inicialmente planejadas, não foram suficientes, tendo que ser complementadas com a elaboração posterior de uma planilha pela professora-pesquisadora, em conjunto com os alunos. Os diálogos aconteceram, inicialmente, entre os estudantes e, posteriormente, entre professora e estudantes. Ao final dessa etapa vemos que os alunos perceberam a regularidade para o caso do triângulo retângulo e, no excerto abaixo, vemos a aluna Dária dialogando com seu colega e expressando oralmente essa relação:

> José: Qual a relação entre os três quadrados no retângulo? Nenhum é maior que o outro?
>
> Dária: Não..., mas é pra escrever a relação entre as áreas dos quadrados. Ahhh...eu já sei...a soma do quadrado maior com o mediano é idêntica com a soma do... [corrigindo] o menor com o mediano é idêntico à soma do maior.

Expressar-se sobre as relações observadas é o que estamos chamando de enunciar o que foi percebido. Os alunos demonstraram algumas dificuldades para escrever, porém conseguiram fazê-lo usando a linguagem que julgaram conveniente e que, embora nem sempre totalmente correta do ponto de vista matemático, deixava claro o entendimento que tiveram. O grupo A (Figura 4), por exemplo, utilizou os sinais de + para indicar o quadrado maior e – para indicar o quadrado menor, numa tentativa de simplificar a escrita da relação.

CONSIDERAÇOES FINAIS

As atividades desenvolvidas na pesquisa se deram em um coletivo de seres-humanos-com-mídias (Borba & Villarreal, 2005), em que os alunos em grupos tiveram oportunidades de dialogar (Alro & Skovsmose, 2010) uns com os outros, favorecendo um ambiente de colaboração entre eles.

O uso do *software* GeoGebra, por meio da visualização vinculada à dinamicidade do software, permitiu que os alunos fizessem questionamentos que resultaram em ricas discussões matemáticas. As ferramentas do *software* favoreceram o diálogo entre os alunos, numa relação de respeito entre seus participantes, em que os alunos assumiam os riscos que o diálogo oferece e, dessa forma, realizavam investigações que possibilitaram a produção de conhecimento acerca do Teorema de Pitágoras.

Por estas razões, e por outras analisadas na pesquisa de modo mais amplo, somos levados a crer que atividades realizadas num coletivo de seres-humanos-com-mídias podem proporcionar aos alunos diversas oportunidades de dialogar com seus pares e também, que o uso da tecnologia, baseado na experimentação, pode proporcionar aos estudantes a oportunidade de investigar, testar possibilidades e levantar hipóteses. O momento de teorização e generalização de hipóteses, não menos importante, pode vir depois desse momento proporcionado pela experimentação. A apresentação e a discussão das demais atividades mencionadas e que não foram apresentadas nesse trabalho, bem como o roteiro de todas as atividades realizadas, podem ser encontradas em Sette (2013).

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POSTERS

UMA PROPOSTA DE MATERIAL DIDÁTICO PARA O ENSINO DE ISOMETRIA E HOMOTETIA MEDIADO POR SOFTWARE DE GEOMETRIA DINÂMICA / A DIDACTICAL PROPOSAL FOR THE TEACHING OF ISOMETRY AND DILATION MEDIATED BY DYNAMIC GEOMETRY SOFTWARE

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This work offers a didactical proposal for the teaching of isometry and dilation based on the theoretical frameworks of Investigative Activities and Educational Technologies. As theoretical perspectives we adopted the contributions from Ponte (2006), Borba (1999) and Zeichener (2008). As a teaching method we used the Van Hiele (1957) model. The presented activities are analysed from a qualitative research approach to find out evidences of effective learning.

Keywords: Informática Educativa, Atividade Investigativa, Homotetias.

CONSIDERAÇÕES GERAIS SOBRE TECNOLOGIA E INFORMÁTICA EDUCATIVA

Atualmente a sociedade tem passado por grandes e rápidas transformações, devido à influência da tecnologia que a vem impulsionando. Como as instituições educacionais acabam por aderir a tais transformações, faz-se necessária uma reflexão sobre o uso destas tecnologias por alunos e professores.

Mediante ao ambiente apresentado e através de observações cotidianas em sala de aula no Brasil, pode-se perceber que o interesse do jovem está relacionado a suas percepções mais imediatas. Elas estão relacionadas às situações sociais e tecnológicas da atualidade e refletem a necessidade de mudança de paradigmas da educação, em especial aos relacionados à utilização da tecnologia como mecanismo de auxílio na aprendizagem de matemática.

Visto que a Escola não sofreu, ao longo dos tempos, transformações radicais, enquanto que a sociedade cria novas formas de transmitir o saber, a escola parecia alheia às mudanças na percepção humana que a realidade tecnológica esta provocando. Tais afirmativas confirmam a necessidade de repensar o método com o qual se ensina matemática, na expectativa que os alunos tenham mais satisfação, curiosidade e interesse pelo conhecimento, construindo, assim, uma aprendizagem com significado.

Como forma apropriada de introduzir os computadores nas salas de aula, Papert (1994) defende a abordagem construcionista. Nela o aluno deve passar pelo ciclo descrição-execução-reflexão-depuração de solução de um problema (ciclo **D-E-R-D**). Ou seja, o aluno descreve a resolução do problema, reflete sobre os resultados obtidos e depura suas ideias por intermédio da busca de conteúdos e de novas estratégias. Nelas, o aluno poderá construir seus conhecimentos através da exploração, testando suas conjecturas, ou seja, o aluno poderá aprender fazendo.

Descobrindo a Homotetia

Nesta atividade espera-se a construção do conceito de homotetia, bem como o cálculo da razão de crescimento/decrescimento, a descrição de regularidades e a demonstração da verdade matemática.

Em atividades investigativas, é necessário o envolvimento ativo do aluno na construção do conhecimento, sequenciado por um conjunto de questionamentos em níveis cada vez mais complexos, onde esteja consciente do seu processo de aprendizagem.

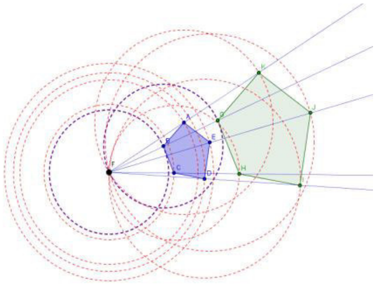

Fig. 1: Homotetia entre pentágonos.

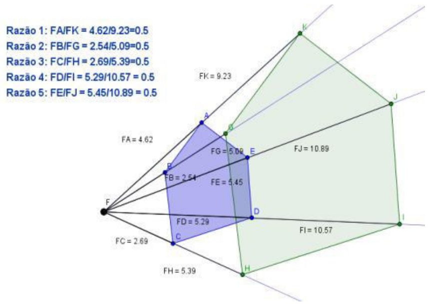

Fig. 2: Calculando a razão.

Com o computador este aluno pode testas suas hipóteses e, assim, respeitar os caminhos diferenciados, de acordo com o seu nível de aprendizagem. Aqui, destaca-se a interseção entre as três teorias adotadas nesta proposta de material de trabalho.

A possibilidade de múltiplas experimentações auxilia na Análise e Orientação, bem como na Dedução das verdades matemáticas da atividade proposta. Estas ações devem ser orientadas por um modelo de atividade que insira o aluno no centro do processo educativo e incentive a participação individual e de grupos.

Na proposta verificaram-se os níveis de aprendizagem de van Hiele nas seguintes situações:

1: Visualização – Quando os alunos visualizam o pentágono na tela do computador e registram características relacionadas a sua aparência geral;

2: Análise – Quando os alunos entendem o pentágono e seu conjunto de propriedades. Cabe destacar a necessidade de movimentação do objeto e a verificação do Princípio da Propriedade Mantida (PPM).

3: Ordenação – Quando os alunos ordenam de forma lógica as propriedades da figura apresentada. Aqui a manipulação do objeto é fundamental na busca de regularidades. Cabe salientar que os questionamentos da atividade devem auxiliar esta ação e na reflexão ativa da ação (Zeichener);

4: Dedução – Quando os alunos entendem a propriedade de ampliação e redução da figura de forma mais geral e conseguem descrever, na língua materna, este sistema dedutivo;

5: Rigor – Quando os alunos conseguem utilizar-se dos diversos sistemas axiomáticos para a demonstrar matematicamente a relação lógica anterior e vislumbrar desdobramentos desta verdade.

CONSIDERAÇÕES FINAIS

Desta forma, o aluno se comporta como investigador e, portanto, pode haver a superação do desejo imediato de resposta visto a percepção dos caminhos complexos da aprendizagem. Acredita-se que as utilizações de tais recursos tecnológicos e teóricos, aliados aos caminhos didáticos, formem um sujeito mais reflexivo e de acordo com os anseios da sociedade e mercado da atualidade.

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REFLEXÕES SOBRE EDUCAÇÃO A DISTÂNCIA / REFLECTIONS ON DISTANCE EDUCATION

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Even though the research on Educational Technologies and on Distance Education has grown over the last years, there is still the need to conduct research involving the analyses of resources that may promote an effective, critical, dialogical and collaborative learning. The production of knowledge must surpass the one-to-one and one-to-many interactions in view of the many-to-many interactions in socio-interactionist environments aiming for learning.

Keywords: Educação a distância (EAD), Ambiente Virtual de Aprendizagem (AVA), Tecnologia Educacional

INTRODUÇÃO

O avanço tecnológico possibilitou a expansão da Internet e a dinâmica da Educação a Distância (EAD), que inicialmente ocorria por correspondência impressa, por transmissão via satélite, por materiais em áudio e vídeo, por telefone e por videoconferências. Hoje, os cursos na modalidade a distância apresentam uma estrutura diferente, utilizando-se de plataformas virtuais de aprendizagem e de variados ambientes computacionais tais como: TelEduc, Moodle, WebCT, Blackboard, AVA, entre outros.

Frente a estas necessidades, as mudanças tecnológicas da atualidade revelam um mundo marcado pelo uso crescente da tecnologia e da comunicação virtual, assim, é frequente encontrarmos o uso destas no dia-a-dia das pessoas. Diante deste patamar, das necessidades educacionais do Brasil e das práticas vigentes é preciso encontrar soluções para os problemas da educação de um país de dimensões continentais.

> A humanidade, hoje, testemunha um momento histórico complexo e marcado pelo crescente uso das tecnologias digitais, principalmente no que diz respeito à comunicação e informação, o que tem encurtado distâncias, influenciado e acelerado o tomadas de decisões e os processos históricos. Esse uso tecnológico crescente vem adentrando o cotidiano das pessoas nos mais diversos campos da atividade humana. A educação, por sua vez, não poderia se encontrar apartada desse processo (Carvalho, Matta, 2007, p.2).

O ensino presencial encontra-se enraizado em nossas vivências e práticas de ensino, visto que em sua maioria, somos fruto desta modalidade de ensino. Acredita-se que parte do temor, referente à negação da EAD, esteja ligada ao pouco conhecimento da proposta pela grande maioria dos educadores brasileiros. É imprescindível esclarecer à comunidade educacional os pressupostos filosóficos da EAD e suas finalidades, para que instituições educacionais de EAD que possuam pouco compromisso com a causa educativa não sejam tomadas como exemplo geral.

A impessoalidade pode ser entendida como falta de presença física. No entanto, a relação professor aluno pode ser ampliada de acordo com a utilização apropriada dos recursos tecnológicos disponíveis. Assim, a cooperação entre alunos e professores supera as barreiras do contato físico e fortalece a relação humanística.

Como se pode perceber, estes temores referentes a EAD são superados pela própria consolidação de um trabalho consciente e de qualidade. Assim, a qualidade é ampliada na medida em que expandimos o diálogo entre as modalidades presencial e a distância. Pode-se arriscar a dizer que, a superação das dificuldades atuais acarretará no surgimento de novos temores e estas conquistas serão determinantes na busca de uma formação consistente, seja ela presencial ou não.

AMBIENTES VIRTUAIS DE APRENDIZAGEM (AVA)

Observando o desenvolvimento da tecnologia, dos pressupostos teóricos da EAD e as diversidades de possibilidades, podemos estabelecer dois grandes cenários: O primeiro revela o avanço tecnológico como apenas recurso didático, neste caso o processo de ensino e aprendizagem é centrado na figura do professor. O segundo apresenta-se como avanço na forma de se ensinar e de aprender e, assim, é direcionado uma prática pedagógica centrada na aprendizagem.

Partindo do segundo cenário, ocorre à mudança de paradigmas da EAD e novos desafios e oportunidades se apresentam. Podem-se destacar alguns que são imprescindíveis no planejamento de um Ambiente virtual de aprendizagem, tais como: o didático-pedagógico, o metodológico, o tecnológico, o gerencial e o estratégico.

Estes elementos encerram uma boa possibilidade para a transformação das práticas educativas tradicionais e se consolida como caminho viável e efetivo para uma educação consistente. Sob este olhar, novas competências são exigidas do profissional de EAD, em especial aos relacionados a tutoria a distância. Abaixo são descritas algumas delas.

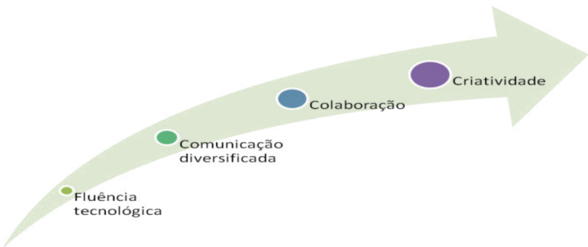

Imagem 1: Competências em EAD

Desta forma, segundo (Garrido, 2009), procura-se oferecer instrumentos efetivos para uma multiplicidade de significações que se originam nos esquemas mentais dos sujeitos. Os professores passam a ter a função de orientadores, articuladores, problematizadores, pesquisadores e especialistas na comunidade de aprendizagem virtual. Assim, abre espaço para novas possibilidades de interação e do diálogo complexo e reflexivo sobre o conhecimento produzido.

Considerações Finais

A utilização dos AVA e as construções coletivas reforçam a ideia de cooperação da informação. Logo devemos estar preparados para os desafios que se estabelecem a partir da inserção de tais ferramentas em ambientes virtuais de aprendizagem. Trata-se de planejar, coordenar, orientar, mediar e buscar atingir os objetivos definidos. Portanto, a exploração de possibilidades, de oportunidades e a otimização de tempo deve conceber um espaço bem estruturado.

Acredita-se que a EAD, baseada nos pressupostos da colaboração, pode ampliar a capacidade de analisar situações, de sintetizar ideias, de avaliar pontos de vista, de pensar criativamente, de resolver problemas e de justificar tomadas de decisões e opiniões individuais ou de grupo, formando sujeitos críticos, reflexivos e de acordo com as necessidades da sociedade da atualidade.

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A UTILIZAÇÃO DE AMBIENTE VIRTUAL DE APRENDIZAGEM PARA ENSINAR MATEMÁTICA NO ENSINO MÉDIO POR MEIO DO AMBIENTE KHAN ACADEMY / THE USE OF VIRTUAL LEARNING ENVIRONMENT TO TEACH MATHEMATICS IN HIGH SCHOOLS THROUGH THE KHAN ACADEMY ENVIRONMENT

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The chosen platform is the Khan Academy, which is a non-profit organization that aims at "changing education for better"(KHAN, 2014) independently of the user being a student, a teacher, a parent or anyone interested in learning something new. This platform is entirely free. The results of an experiment with a high school student from a public school in Penápolis, Sao Paulo, Brazil will be presented.

Keywords: Mathematic Education, New Technologies, Virtual Learning Environment.

BREVE JUSTIFICATIVA E VISÃO GERAL DO CONTEÚDO

Distintos projetos como EDUCOM (Computadores na Educação), FORMAR (Formação de Professores) e PRONINFE (Programa Nacional de Informática na Educação), desde o ano de 1983, foram criados pelo Ministério da Educação e Cultura (MEC) brasileiro juntamente com a Secretaria Especial de Informática (SEI), com o intuito de desenvolver pesquisas sobre as diversas aplicações do computador na educação, formar recursos humanos para o trabalho na área de informática educativa, bem como a criação de laboratórios e centros para a capacitação de professores, respectivamente. Em 1997 a Secretaria de Educação a Distância em parceria com o MEC criou o PROINFO (Programa Nacional de Informática na Educação) que, na concepção de Borba (2012, p. 20), visava estimular e dar suporte para a introdução de tecnologia informática nas escolas do nível fundamental e médio de todo o país.

São apresentadas como inquietações as seguintes questões: Como transformar essa tecnologia cada vez mais presente na vida das pessoas em algo bom para a educação? Como mudar a ideia que os jogos provenientes de tecnologia não trazem produtividade ao ensino?

Para tanto, descreve-se como se deu a pesquisa realizada com uma aluna da escola estadual supramencionada com a utilização do *Khan Academy* que oferece desde exercícios de nível básico até o avançado, abrangendo desde as primeiras noções de Matemática até a Matemática Pura e Aplicada, por exemplo. O site foca o Ensino de Matemática, mas também aborda muitos outros conteúdos.

A aluna, aqui chamada de Aluna A chegou à sala de aula meio cabisbaixa informando que não conseguiu fazer todos os exercícios. Ao entrar na conta dela na plataforma A.V.A. foi possível observar que, de fato, ela não havia conseguido completar a sequência de cinco acertos seguidos, apesar de já ter anteriormente acertado oito exercícios, mas não na ordem correta. Iniciou-se o processo de ajuda para o primeiro exercício que foi feito em conjunto, do jeito da aluna, para que esta o compreendesse. Houve um erro conjunto pelo fato de a aluna não saber explicar o que o exercício pedia, já que ela não o leu por inteiro, restando dúvidas sobre o procedimento a ser feito. Com as dicas da plataforma relidas foi realizado novamente o exercício em conjunto, agora com uma maior compreensão por parte da aluna que depois conseguiu fazer sozinha. Ao acompanhar a aluna lhe foi permitido perguntar e responder as próprias perguntas. Contudo, a aluna errou o exercício, por estar confundindo as etapas.

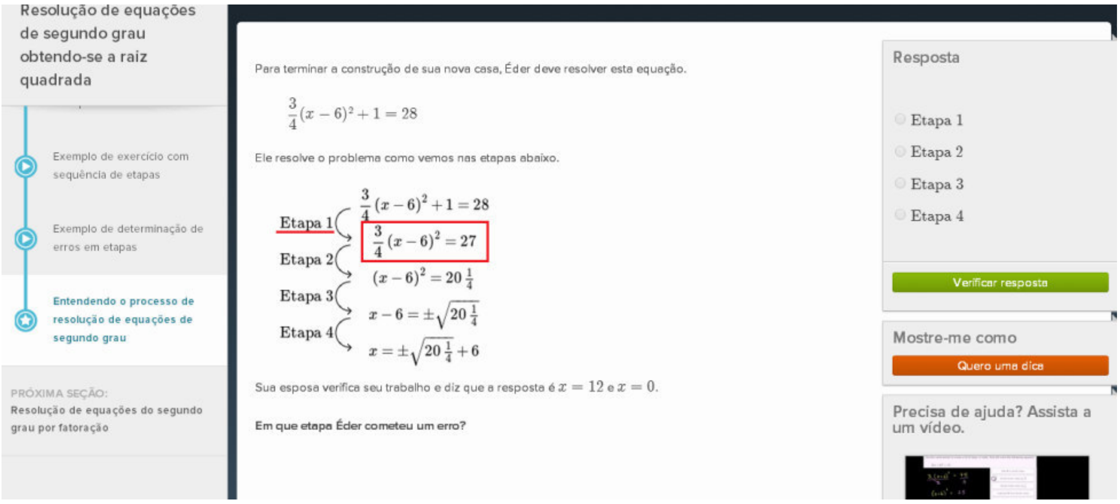

Figura 1: Etapas necessárias para a resolução do problema.

A Figura 1 mostra as etapas para a resolução do problema mostrando onde encontrar o erro. Foi explicado à aluna que a linha 2 é referente à "Etapa 1".

Na sequência foi explicado que a linha 3 é referente à "Etapa 2" e assim por diante. Com isso, ela conseguiu resolver os próximos exercícios e foi prosseguindo as atividades. Foi possível perceber uma certa dependência do professor, visto que mesmo o exercício estando certo, ela ainda requisitava a ajuda do professor para ter a certeza, com medo de errar e ter que começar novamente.

No final da aula foi proposto à Aluna A que fizesse mais alguns exercícios em casa. Como faltavam poucos minutos para o término da aula o tempo restante foi aproveitado para mostrar à aluna o progresso dela durante o projeto. Foi mostrada a planilha que aparece para a professora, na qual se observa a quantidade de minutos que ela utilizou para fazer as atividades. Igualmente foi mostrada a planilha que aparece no perfil dela, por meio da qual a aluna pode perceber a quantidade de habilidades que ela praticou, dominou e as que não foram muito bem assimiladas. A Aluna A gostou da possibilidade de ver o próprio desempenho e começou a se interessar por mexer no site para entender melhor o que cada gráfico significava.

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I2CALC: UM APLICATIVO ANDROID PARA A APRENDIZAGEM DE NÚMEROS COMPLEXOS / I2CALC: AN ANDROID APP FOR LEARNING COMPLEX NUMBERS

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The scenario of the teaching and learning of mathematics has changed significantly in recent years, due to the insertion of mobile technologies to access information. In this context, and considering the difficulties presented in the learning of complex numbers, was proposed the development and use of an Android application, in Portuguese, to broaden educational possibilities and stimulate the autonomy of students.

A APRENDIZAGEM DE NÚMEROS COMPLEXOS

Os números complexos têm menos relação com o mundo real que os outros números já nossos conhecidos. Um número imaginário não serve para medir a quantidade ou realizar a contagem de algo. Esse conjunto numérico está presente na explicação matemática para a sustentação de um avião durante o voo, na análise de circuitos de corrente alternada presente na eletrônica e na eletricidade, além de descrever sistemas formadores de geometria fractal. Isso justifica sua preservação no currículo escolar básico e o legitima como conteúdo que coordena diferentes saberes.

Os PCN+ (2002, p.119), quando tratam sobre como o conteúdo dos números complexos deveriam ser trabalhados, não valoriza esse conjunto como necessário:

> Tradicionalmente, a Matemática do ensino médio trata da ampliação do conjunto numérico, introduzindo os números complexos. Como esse tema isolado da resolução de equações perde seu sentido para os que não continuarão seus estudos na área, ele pode ser tratado na parte flexível do currículo das escolas.

É importante salientar que, mesmo para quem não seguirá essa área em seus estudos futuros, o conjunto dos números complexos está diretamente relacionado ao estudo de funções, se ampliado para além do universo de variáveis reais. Diante dessas informações, é evidente a necessidade da permanência desses estudos no ensino básico e superior, e ainda mais, que esse estudo seja qualificado de maneira a agrupar diferentes áreas do conhecimento matemático e cotidiano.

A INSERÇÃO DE DISPOSITIVOS MÓVEIS NA SALA DE AULA

A partir do início do ano de 2012 está sendo implantada uma nova fase do Proinfo – uma iniciativa do governo federal brasileiro com o objetivo de promover o uso pedagógico da informática na rede pública de educação básica – caracterizada pela distribuição de tablets para professores, inicialmente do ensino médio e posteriormente para os dos anos finais do ensino fundamental. As experiências realizadas pelos programas federais – bem-sucedidas ou não – mostraram que, se o professor não se apropriar das tecnologias e perceber seus reais ganhos para a prática pedagógica, elas se tornam apenas uma “pilha de caixas” nas escolas.

Equipamentos tecnológicos, como os tablets, permitem a conversão de livros, apostilas, materiais e inclusive provas em arquivos digitais que podem ser visualizados e editados a qualquer momento e lugar. Realizar uma pesquisa na internet ficou muito mais simples, localizar trechos ou referências tornou-se um processo mais veloz e o acesso a quaisquer informações, ainda que em outros idiomas, mais fácil. É importante também considerar que a ferramenta será utilizada para outras funções, o que faz com que o custo-benefício seja ainda mais vantajoso.

APP INVENTOR 2

A plataforma App Inventor – aplicação web originalmente fornecida pela Google e atualmente mantida pelo Massachusetts Institute of Technology (MIT) – permite que o usuário desenvolva aplicativos para o sistema operacional Android usando somente o navegador e um smartphone ou tablet conectado – ou um emulador, caso o usuário não tenha acesso a algum equipamento com esse sistema operacional. Os servidores da plataforma armazenam seus projetos e permitem que haja maior acompanhamento deles.

O encaixe dos blocos ocorre de maneira visual, unindo peças como partes de um quebra-cabeça. O aplicativo é atualizado na tela a cada parte adicionada, o que possibilita a verificação do projeto conforme ele é construído. Quando ele está finalizado, pode-se realizar o *download* do aplicativo para instalação posterior em qualquer dispositivo que suporte o sistema Android.

I2CALC

Durante o segundo semestre do ano de 2014, na disciplina de Variáveis Complexas, percebeu-se que muitos dos matriculados na disciplina não haviam sequer sido apresentados aos números complexos no ensino médio. Utilizando-se disso como motivação, buscou-se desenvolver um aplicativo para que o aluno pudesse verificar se havia desenvolvido o solicitado corretamente, e ainda, explorar a ferramenta para que o usuário por si só pudesse investigar e perceber características recorrentes a partir do que o aplicativo retorna ao que foi informado. Além disso, em uma busca no local disponível para *download* de aplicativos para Android, notou-se a ausência de alguma ferramenta em português que tratasse sobre os números complexos e suas operações.

O desenvolvimento do aplicativo ampliou o aprendizado na disciplina e a respeito dos números complexos como um todo. O envolvimento completo em cada parte do processo fez com que, para cada operação a ser executada pelo i2Calc, fosse pensada, dentre as possíveis formas de efetuá-la, qual seria a mais clara e desprovida de erros na programação e montagem do aplicativo.

Ainda, organizar um projeto de forma integral propiciou, à estudante do curso de licenciatura envolvida, uma grande experiência empreendedora, na qual há o planejamento e execução de diferentes ângulos – funcionalidades, *layout*, divulgação, aplicação e avaliação, por exemplo – além de despertar a criatividade e espírito inovador no desenvolvimento do aplicativo. Isso corrobora com a metodologia construcionista apresentada, na qual o aluno torna-se agente construtor do próprio conhecimento.

Nesse contexto foi projetado o i2Calc, um aplicativo que fornece uma calculadora de números complexos. Ele envolve, em sua versão 1.0, as operações de adição, subtração, multiplicação e divisão de números complexos, além do cálculo do valor de seu módulo. Durante o primeiro semestre letivo do ano de 2015, o aplicativo será aplicado na disciplina de Variáveis Complexas e Polinômios, ofertada no terceiro semestre do curso de Licenciatura em Matemática do IFRS – Câmpus Bento Gonçalves, para testes e qualificação do aplicativo, ampliando suas funcionalidades.

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WORKSHOPS

FERRAMENTAS GRÁFICAS, DINÂMICAS E INTERATIVAS PARA O ESTUDO DE FUNÇÕES REAIS DE VARIÁVEL REAL / GRAPHIC, DYNAMIC, AND INTERACTIVE TOOLS FOR THE STUDY OF REAL FUNCTIONS

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Our main goal is to present the F-Tool concept, an interactive Mathematica notebook, designed specifically to explore the concept of real function, by analyzing the effects caused by changing the values of the parameters present in general analytical expressions. Each F-Tool allows the study of a typical class of functions providing graphical and analytical information in real time. It will be discussed the teaching possibilities offered by this dynamic educational software.

Keywords: F-Tool, Educational Software, Mathematica Computer Algebra System, CDF Player

DURAÇÃO, EQUIPAMENTO E SOFTWARE

O Workshop, apresentado em Português, terá a duração de 3 horas e realizar-se-á numa sala de informática com acesso à internet sendo necessário instalar, antecipadamente, o Wolfram *CDF Player* e as F-Tool:

Wolfram *CDF Player*: http://www.wolfram.com/cdf-player/

F-Linear: https://sapientia.ualg.pt/handle/10400.1/2735

F-Quadratic: https://sapientia.ualg.pt/handle/10400.1/2736

F-Sine: https://sapientia.ualg.pt/handle/10400.1/2737

F-Cosine: https://sapientia.ualg.pt/handle/10400.1/2738

F-Exponential: https://sapientia.ualg.pt/handle/10400.1/2739

F-Logarithm: https://sapientia.ualg.pt/handle/10400.1/2740

O CONCEITO F-TOOL

No Ensino da Matemática e, em particular no estudo de funções, a recomendação para a utilização de *software* educacional tem como objetivo, entre outros, ajudar a ultrapassar as dificuldades que os alunos têm em associar as representações algébricas com as representações numéricas e/ou gráficas. O conceito F-Tool foi criado (Conceição, Pereira, Silva, & Simão, 2012)[1] com o objetivo de melhorar o processo de estudo de classes de funções reais de variável real. O *software* educacional implementado com base neste conceito permite estudar dinâmica e interactivamente, e em tempo real, conceitos e propriedades fundamentais do pré-cálculo e cálculo diferencial. A sua utilização permite estabelecer um contexto de ensino-aprendizagem onde alunos e professores são igualmente convidados a contribuir. Todas as ferramentas foram implementadas com recurso ao sistema de álgebra computacional *Mathematica*, o que permitiu criar aplicações autónomas, que podem ser

[1] Artigo distinguido com um prémio atribuído pela empresa Timberlake Consultants, especializada em software científico.

obtidas gratuitamente e utilizadas por qualquer pessoa com acesso a um computador. Todas as F-Tool têm uma interface muito intuitiva que permite que até mesmo o utilizador mais inexperiente, sem nenhum conhecimento anterior em *software* educacional, possa começar a usar todos os recursos de uma forma eficiente.

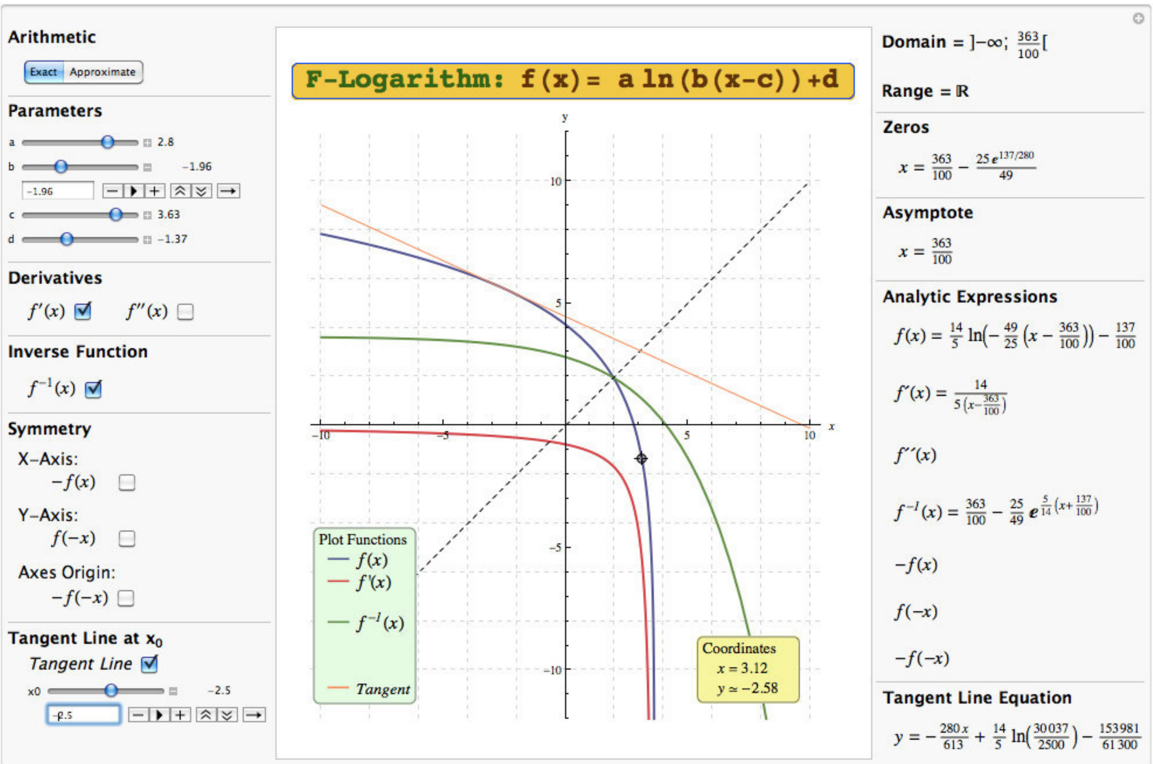

Figura 1. Imagem ilustrativa da F-Logarithm.

O Workshop inclui informação sobre todas as F-Tool atualmente implementadas em computador, desde o modo de obtenção até aos vários modos de utilização, com explicação e exemplificação dos diversos conceitos matemáticos que cada ferramenta permite explorar. Os exemplos fornecidos foram concebidos para ilustrar como a utilização das aplicações pode inovar de forma positiva a experiência didática dos alunos e professores, promovendo uma abordagem de aprendizagem ativa onde a apropriação de vários conceitos fundamentais é realizada de uma forma dinâmica e interativa.

CONCLUSÕES

Quando aplicado em sala de aula (Conceição, Pereira, Silva, & Simão, 2013; Pereira & Conceição, 2013) o conceito F-Tool promove novas formas de raciocinar/pensar, ensinar e aprender.

Acreditamos que o conceito F-Tool, ao dotar professores e estudantes com novas ferramentas para explorar os conceitos fundamentais das áreas de pré-cálculo e cálculo diferencial, desenvolverá positivamente o processo de ensino e de aprendizagem da Matemática.

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MODELAGEM COMPUTACIONAL PARA O ENSINO DE EQUAÇÕES DIFERENCIAIS ORDINÁRIAS / COMPUTATIONAL MODELING FOR TEACHING ORDINARY DIFFERENTIAL EQUATIONS

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In this work presents a proposal for exploration of ordinary differential equations in the context of problem situations using as a tool of support the software called Powersim. The study is part of a doctoral thesis and was developed with students enrolled in Engineering courses. The theoretical approach that is based this study are the meaningful learning theory of Ausubel and the social interactionist theory of Vygotsky.

Keywords: Ordinary differential equations, Software Powersim, Undergraduate teaching.

APRESENTAÇÃO

Estudos apontam que a metodologia dominante no contexto do ensino de equações diferenciais (EDs) está fortemente voltada para a resolução analítica, mas os recursos computacionais hoje disponíveis permitem ir além da mera aplicação de técnicas, podendo auxiliar os alunos na interpretação das equações diferenciais e suas soluções.

Trabalhando com o ensino de Cálculo Diferencial e Integral nos cursos de Engenharia (de Computação, de Automação e Controle, de Produção e Ambiental) e Química Industrial notamos a insatisfação dos alunos por não perceberem importância desse conteúdo para o seu curso. A cada nova turma, repetem-se os questionamentos: por que fazer a mão essas contas enormes se existem máquinas para isso? Por que "decorar" tantas fórmulas, se o dia que precisar posso buscar em livros ou na *internet*? Por que preciso saber tudo isto, afinal?

Comparando o contexto de ensino das EDs hoje em dia com o que se tinha na metade do século passado, percebemos que os tipos de alunos são outros, as necessidades e exigências do mercado de trabalho não são as mesmas, assim como as ferramentas disponíveis, mas a maioria das aulas continuam, em essência, sendo ministradas da mesma forma. Os currículos precisam ser repensados e os avanços tecnológicos considerados. Em função dessa problemática, nos propusemos a elaborar, aplicar e avaliar uma abordagem pedagógica que auxilie os alunos na superação das dificuldades e proporcione condições favoráveis à aprendizagem significativa de EDs.

Para o desenvolvimento do trabalho elaboramos uma proposta de ensino focada na solução de situações-problema com o uso de recursos computacionais, buscando trabalhar as EDOs de forma contextualizada e com abordagens analítica, numérica e gráfica, contando com a ajuda de recursos computacionais para facilitar o processo (Dullius, Araujo e Veit, 2011).

Na elaboração do material instrucional levou-se em conta os pressupostos da Teoria da Aprendizagem Significativa de Ausubel (2003). A metodologia empregada na prática pedagógica teve como suporte a Teoria Sócio-interacionista de Vygotsky (2000 e 2003) especialmente no que diz respeito à interação professor-aluno-material didático no ambiente com recursos computacionais.

Na abordagem do conteúdo de EDOs, nos concentramos nos seguintes pontos: a) representar matematicamente, por meio de EDOs, situações-problemas; b) favorecer o domínio de técnicas de soluções analíticas de EDOs, sabendo classificá-las segundo critérios de ordem e linearidade; e c) obter informações sobre o comportamento das soluções de EDOs sem resolvê-las analiticamente, por meio da análise semiquantitativa de variáveis e parâmetros. Nesse último ponto, foi explorado o impacto da alteração de valores de variáveis e parâmetros em representações gráficas das soluções das EDOs trabalhadas.

Em nossa proposta consideramos as EDs como um instrumento para explorar modelos e resolver problemas e procuramos abordar, equilibrada e simultaneamente, representações gráficas, numéricas e simbólicas das equações e respectivas soluções. Buscamos uma abordagem mais qualitativa das EDs, trabalhando o conteúdo com maior ênfase na contextualização através de situações-problema passíveis de serem representadas por meio de equações diferenciais. No delineamento das atividades, procuramos explorar também questões conceituais, de modo a auxiliá-los a dar significado às EDOs e às suas soluções. Nosso intuito foi estimular os estudantes a mudarem o foco da simples manipulação analítica das equações, para a compreensão de seu caráter representativo. Inicialmente exploramos a interpretação das EDOs e o comportamento das soluções, contando com a ajuda de recursos computacionais para facilitar e agilizar o processo e somente depois abordamos as técnicas de solução analítica.

Após a abordagem qualitativa da ED, passávamos à resolução analítica, e explorávamos diversos problemas que podem ser tratados com a ED em estudo. Por exemplo, discutíamos diversas situações em que a taxa de variação da quantidade em função do tempo é proporcional à quantidade existente num determinado instante de tempo t, como situações de decaimento radioativo, absorção de medicamentos, juros compostos e reações químicas.

Em relação à proposta das atividades, podemos observar que os alunos, de modo geral, mostraram-se satisfeitos com a resolução de situações-problema em sala de aula, pelo fato de poderem ver as aplicações e implicações dos aspectos teóricos, facilitando o estabelecimento de relações entre o conhecimento novo e os subsunçores adequados, em sua estrutura cognitiva. Porém, sentiram muitas dificuldades para as interpretações que lhes eram requeridas e quatro meses, que é a duração de uma disciplina, é pouco tempo para desenvolverem esta capacidade de tal forma a produzir resultados satisfatórios em termos de aprendizagem significativa do conteúdo.

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CONSTRUINDO TRÊS MODELOS PLANOS PARA A GEOMETRIA HIPERBÓLICA E ISOMORFISMOS ENTRE ELES, POR MEIO DO GEOGEBRA 2D E 3D / BUILDING THREE TWO-DIMENSIONAL MODELS FOR HYPERBOLIC GEOMETRY AND ISOMORPHISMS BETWEEN THEM, USING 2D AND 3D GEOGEBRA

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The construction of models that satisfy the axioms established in hyperbolic geometry contributed to the acceptance of such geometry. Felix Klein built a plane model and Henry Poincaré two models plans. In this workshop these three models will be built in GeoGebra 2D, using predefined tools that enable construction of the Poincaré disk model and others. These models also allow the exploration of several results that are relevant to hyperbolic geometry. Using the 3D GeoGebra we will show the isomorphism between the three 2D models.

Keywords: GeoGebra 2D and 3D; Hiperbolic Geometry; model plane; isomorphism between models.

OS MODELOS PLANOS PARA A GEOMETRIA HIPERBÓLICA

Nas diretrizes curriculares para a Educação Básica-Matemática, Paraná (2008), para o Ensino Médio, consta que "Para abordar os conceitos elementares da geometria hiperbólica, uma possibilidade é através do postulado de Lobachevsky (partindo do conceito de pseudo-esfera, pontos ideais, triângulo hiperbólico e a soma de seus ângulos internos)" (p. 57). O estudo de modelos dessa geometria, auxilia a compreensão de vários dos seus conceitos e resultados.

Felix Klein construiu um modelo plano para a Geometria Hiperbólica e Henri Poincaré construiu dois modelos para essa Geometria. O modelo de Klein e um dos modelos de Poincaré utilizam ferramentas existentes no GeoGebra 2D para suas construções, já o modelo do disco de Poincaré, pode ser construído por meio de uma ferramenta nova feita especificamente para ele. No workshop serão construídos esses três modelos e as vantagens pedagógicas de cada um deles.

A figura 1 mostra o modelo de Felix Klein, em que o plano é o interior do círculo com contorno Γ. Neste modelo as retas são cordas do círculo e os pontos são os pontos euclidianos do seu interior. A circunferência Γ, não pertence ao plano e é chamada de horizonte, assim os extremos das retas hiperbólicas neste modelo são pontos ideais.

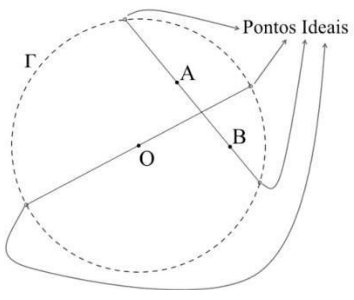

Figura 1 – reta por A e B, e reta por O

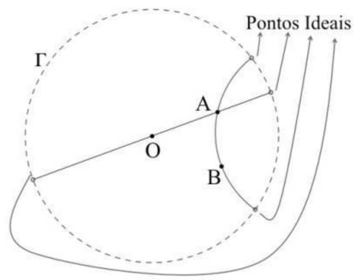

Figura 2 – reta por A e B, e reta por O

Na figura 2, está a representação do modelo plano do disco de Poincaré, em que o plano, como no modelo de Klein, também é o interior do círculo com contorno Γ. Neste modelo as retas são diâmetros ou arcos ortogonais à Γ. Os pontos são os pontos euclidianos do seu interior. A circunferência Γ, não pertence ao plano e assim como no modelo de Klein, é chamada de horizonte, assim os extremos das retas hiperbólicas neste modelo são pontos ideais.

A figura 3, a seguir, mostra outro modelo plano construído por Poincaré para a Geometria Hiperbólica. Neste modelo, o plano é um dos semiplanos determinado pela reta Γ, que não faz parte do plano de Poincaré neste modelo. As retas são semirretas euclidianas perpendiculares à Γ e semicircunferências com centro em Γ, e os pontos são os pontos euclidianos do semiplano.

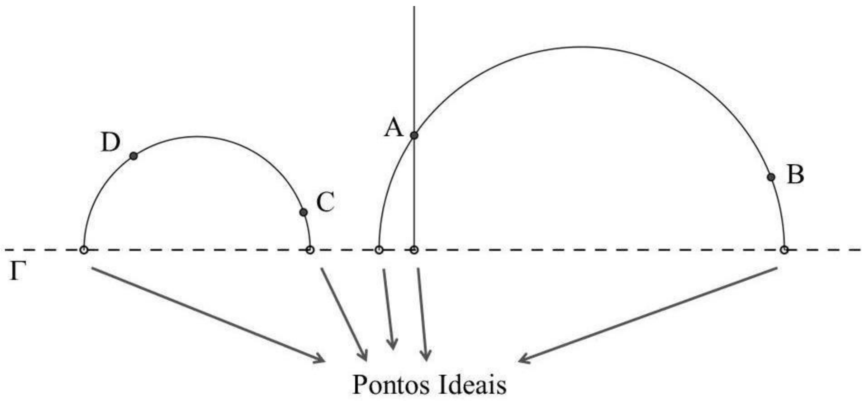

Figura 3 – três retas: por C e D, por A e B e por A.

ISOMORFISMO ENTRE OS MODELOS

No workshop será construído e discutido com auxílio do GeoGebra 3D, isomorfismos entre os três modelos. Esses isomorfismos podem ser encontrados em Greenberg (1980).

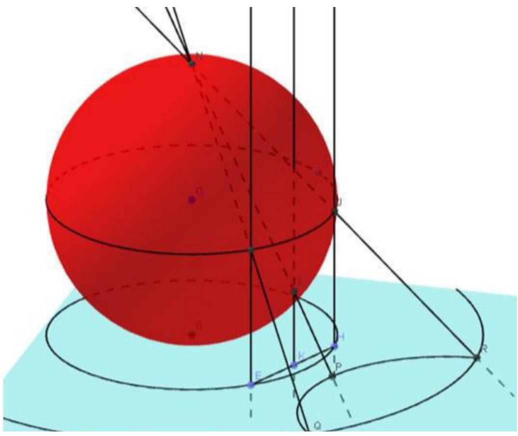

Figura 4 – isomorfismo entre os modelos planos de Klein e o disco de Poincaré.

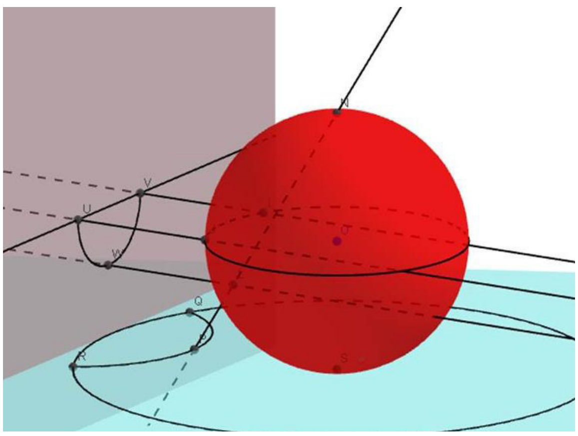

Figura 5 – isomorfismo entre os dois modelos planos de Poincaré

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ATRMINI / ATRMINI

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AtrMini is a collection of games aimed at young children that can be freely downloaded from the Atractor Website (http://www.atractor.pt/mat/AtrMini). It is a useful tool for teaching elementary mathematics, combining play with the acquisition of several competencies: as mental arithmetic, money use, combinatorial reasoning, etc.

Keywords: AtrMini, Computer software, Primary School mathematics

WORKSHOP SOBRE O PROGRAMA ATRMINI

O *workshop* centrar-se-á no programa informático AtrMini, da autoria do Atractor (http://www.atractor.pt), uma associação sem fins lucrativos, cujo objetivo principal é atrair o público para a Matemática. Neste *workshop* iremos explorar todos os jogos que compõem este programa.

ATRMINI

Com o AtrMini, as crianças podem não só treinar o cálculo mental através dos jogos "Adição", "Subtração", "Multiplicação" e "Divisão", como também comparar números em "Maior, menor ou igual" ou "observar" a comutatividade da multiplicação. Tarefas quotidianas como pagar um objeto e calcular o respetivo troco ou calcular a porção (fração/percentagem) de uma fatia escolhida num bolo podem ser treinadas e aperfeiçoadas no AtrMini, de forma interativa e lúdica.

Uma versão elementar da linguagem Logo encontra-se ainda disponível neste programa, em que o utilizador é convidado a recolher um número pré-definido de bolas, usando algumas instruções simples do Logo. Num outro jogo, "Procurar o tesouro", o utilizador é confrontado com uma caça ao tesouro obrigando a algum raciocínio.

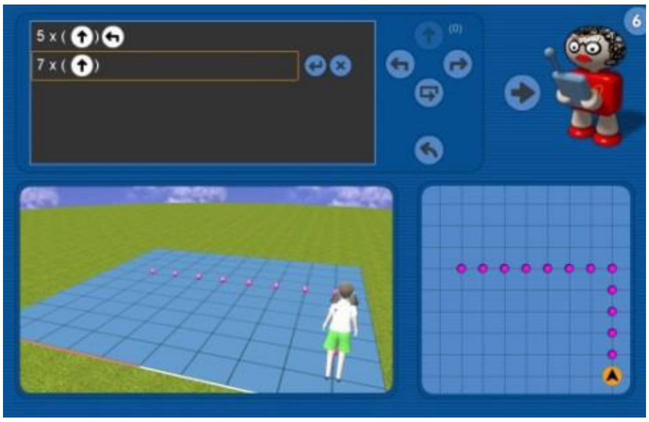

Fig 1. Apanha Bolas

Em "Quantas escolhas?", as crianças têm um primeiro contacto com questões simples de combinatória, mais concretamente, com a pesquisa de todas as combinações/arranjos possíveis entre objetos.

Fig 2. Quantas escolhas?

O capítulo da Simetria não é esquecido, tendo o utilizador à sua disposição um “caleidoscópio” com a particularidade de não operar apenas com reflexões, mas também com rotações, translações e reflexões deslizantes.

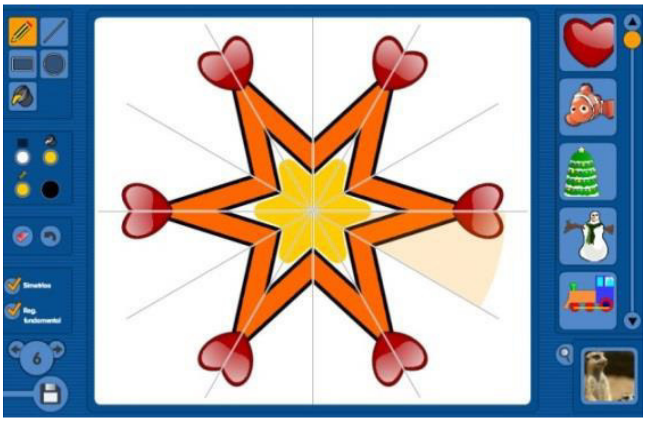

Fig 3. Desenhar com simetria

A pedido de professores do Ensino Básico, o Atractor incorporou o jogo “Frações de chocolate” que está relacionado com a escrita de números em forma de fração e também em percentagem e em forma de número decimal.

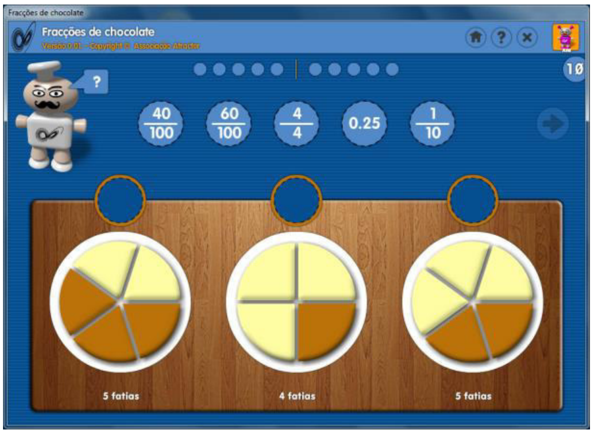

Fig 4. Frações de chocolate

REFERÊNCIAS

http://www.atractor.pt/

http://www.atractor.pt/mat/AtrMini

GECLA / GECLA

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GeCla is a tool that concerns the mathematical study of symmetry of figures, especially in geometric patterns, boarder patterns and rosaces. It can be freely downloaded from the Atractor Website (http://www.atractor.pt/mat/GeCla). It is a useful tool to teach symmetry and it has a recreational character in allowing organizing competitions with students in different school levels.

Keywords: GeCla, Computer software, Symetry, Competition

WORKSHOP SOBRE O PROGRAMA GECLA

O *workshop* centrar-se-á no programa informático GeCla, da autoria do Atractor (http://www.atractor.pt), uma associação sem fins lucrativos, cujo objetivo principal é atrair o público para a Matemática. Neste workshop, iremos explorar o programa e desenvolver uma competição entre os participantes.

GECLA

O GeCla, cujo nome corresponde a uma abreviatura de "Gerador e Classificador", pode ser uma ferramenta útil no ensino da Simetria, permitindo também uma utilização lúdica, através da realização de competições entre alunos da mesma ou de diferentes escolas, via internet. Este programa centra-se no estudo matemático de simetrias de figuras planas, mais precisamente, padrões, frisos e rosáceas.

Neste *workshop* iremos explorar as diversas potencialidades do programa e, na parte final, dedicaremos algum tempo à realização de uma competição entre os participantes usando o GeCla.

GECLA

Este programa pode ser importado gratuitamente do *site* do Atractor.

O programa tem diversas secções:

–"Procurar simetrias", que permite a pesquisa das simetrias de uma dada figura, i.e., a procura de um certo tipo de transformações do plano, as isometrias (funções que preservam distâncias), que levam essa figura nela mesma.

–"Classificação", onde o utilizador é convidado a descobrir todas as simetrias.

–"Geração de rosáceas, frisos e padrões", através da qual o utilizador pode criar as suas próprias imagens com simetrias previamente escolhidas (padrões/frisos ou rosáceas).

Para alunos mais novos, por exemplo do 1º ciclo, o Atractor concebeu uma outra versão simplificada do GeCla, o GeCla Mini. No GeCla Mini, o aspeto gráfico,

as imagens que servem como motivo e o tipo de simetria (mais simples) existente nas rosáceas/frisos/padrões foram pensados tendo em atenção o público-alvo.

Uma das secções interessantes do GeCla, do ponto de vista lúdico, e que será explorada durante a sessão do Atractor, é a possibilidade de realização de competições entre duas equipas. Esta é uma ferramenta com potencial didático, pois permite, por exemplo, a promoção de competições entre alunos de uma mesma escola ou (via Internet) de escolas diversas, situadas inclusive em países diferentes (o que já aconteceu: envolvendo uma escola italiana e uma portuguesa).

REFERÊNCIAS

http://www.atractor.pt/

http://www.atractor.pt/mat/GeCla

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