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Generalized $s\ell(2)$ Gaudin algebra and corresponding Knizhnik–Zamolodchikov equation

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Abstract

The Gaudin model has been revisited many times, yet some important issues remained open so far. With this paper we aim to properly address its certain aspects, while clarifying, or at least giving a solid ground to some other. Our main contribution is establishing the relation between the off-shell Bethe vectors with the solutions of the corresponding Knizhnik–Zamolodchikov equations for the non-periodic $s\ell(2)$ Gaudin model, as well as deriving the norm of the eigenvectors of the Gaudin Hamiltonians. Additionally, we provide a closed form expression also for the scalar products of the off-shell Bethe vectors. Finally, we provide explicit closed form of the off-shell Bethe vectors, together with a proof of implementation of the algebraic Bethe ansatz in full generality.

1. Introduction

Historically, Gaudin model was first proposed almost half a century ago [1–3], and has promptly gained attention primarily due to its long-range interactions feature [4,5]. It was shortly

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generalized to different underlining simple Lie algebras, as well as to trigonometric and elliptic types, cf. [6–9] and the references therein. The non-periodic boundary conditions were treated somewhat later [10–17], while in [18,19] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms and obtained the spectrum of the generating function with the corresponding Bethe equations. The very latest developments are taking the field in various new directions, e.g. [20–23] which shows that the topic is still very attractive.

However, in spite of the substantial interest for the topic, certain issues have not yet been, to our knowledge, fully addressed. First and foremost, we note that the relation of the Knizhnik–Zamolodchikov (KZ) equations [24] with the Gaudin $s\ell(2)$ model [25,26] with non-periodic boundary was not yet established for arbitrary spins. Hikami comes close to this goal in his paper [10], but does not tackle the issue in full generality – namely, he constrains his analysis to a special case of equal spins at all nodes, moreover fixing these spins to the value $\frac{1}{2}$. He also does not provide the expression for the norms of the eigenvectors of the Gaudin Hamiltonians, which can be obtained from the KZ approach. One of our goals here is to improve on both of these points: we successfully establish the relation between solutions of the corresponding KZ equations with the off-shell Bethe vectors in the case of arbitrary spins and derive the norm formula.

Superior to the formula for norm of the on-shell Bethe vectors is a formula for scalar product of arbitrary off-shell Bethe vectors. Following an approach laid in [27], we derive such an expression pertinent to the non-periodic $s\ell(2)$ case for arbitrary spins, in a closed form. The expression involves a sum of certain matrix determinants and its significance stems from the fact that it represents the first step towards the correlation functions.

En route to our treatment of the KZ equations, we present a closed form expression for the off-shell Bethe vectors and prove the implementation of the algebraic Bethe ansatz in full generality (for arbitrary reflection matrices and to arbitrary number of excitations). Such a development was a result of a suitable change of generalized Gaudin algebra basis (as compared to the one used in [19]), combined with observation of certain algebraic relations that we came across. The resulting simplifications have also facilitated calculations related to KZ equations.

The paper is structured as follows. In the next section, we introduce some standard notions while nevertheless relying heavily on the notation and conclusions of our previous paper [19], to which we direct the reader as a preliminary. The third section is devoted to the task of deriving the general off-shell form of the Bethe vectors and to proving its validity. As a key step to this end we, within the same section, first present a new basis of the generalized Gaudin algebra [28,29], and point to its advantages. In the fourth section we finally turn to KZ equations, establishing their relation to the previously derived Bethe vectors and obtaining the norm formula. In the same section we also present the novel formula for the scalar product of off-shell Bethe vectors. Finally, we summarize our results in the last section.

2. Preliminaries

The generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms was derived in [19]. Besides, the suitable Lax operator, accompanied by the corresponding linear bracket and an appropriate non-unitary r-matrices, as well as the transfer matrix, were also obtained. In this section we will briefly review only the most relevant of these results, while for the details of the notations and derivation we refer to the [19].

We study the $s\ell(2)$ Gaudin model with N sites, characterised by the local space $V_m = \mathbb{C}^{2s_m+1}$ and inhomogeneous parameter α_m, implying non-periodic boundary conditions. The relevant classical r-matrix was given e.g. in [6], $r(\lambda) = -\frac{\mathcal{P}}{\lambda}$, where $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

In the case of periodic boundary conditions, this structure is essentially sufficient (after proceeding in the standard manner) to obtain the complete solution of the system [6], together with the corresponding correlation functions [30]. However, the non-periodic case which is the subject of our present consideration is substantially more involved. In this case, of relevance is the classical reflection equation [31–33]:

$$\begin{aligned} &r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = \\ &= K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu). \end{aligned} \tag{2.1}$$

In [19] we have derived the general form of the K-matrix solution, and have shown that it can be, without any loss of generality, brought into the upper triangular form:

$$K(\lambda) = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix}, \tag{2.2}$$

where neither of the parameters ξ, ψ, ν depends on the spectral parameter λ.

In the course of our analysis in [19] we arrived to the generalized $s\ell(2)$ Gaudin algebra [28, 29] with generators $\widetilde{e}(\lambda)$, $\widetilde{h}(\lambda)$ and $\widetilde{f}(\lambda)$. To facilitate later comparison with the new basis, we give the three nontrivial relations:

$$\left[\widetilde{h}(\lambda), \widetilde{e}(\mu)\right] = \frac{2}{\lambda^2-\mu^2}\left(\widetilde{e}(\mu) - \widetilde{e}(\lambda)\right), \tag{2.3}$$

$$\begin{aligned} \left[\widetilde{h}(\lambda), \widetilde{f}(\mu)\right] = &\frac{-2}{\lambda^2-\mu^2}\left(\widetilde{f}(\mu) - \widetilde{f}(\lambda)\right) - \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\widetilde{h}(\mu) - \lambda^2\widetilde{h}(\lambda)\right) \\ &- \frac{\psi^2}{(\lambda^2-\mu^2)\xi^2}\left(\mu^2\widetilde{e}(\mu) - \lambda^2\widetilde{e}(\lambda)\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left[\widetilde{e}(\lambda), \widetilde{f}(\mu)\right] = &\frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\widetilde{e}(\mu) - \lambda^2\widetilde{e}(\lambda)\right) \\ &- \frac{4}{\lambda^2-\mu^2}\left((\xi^2-\mu^2\nu^2)\widetilde{h}(\mu) - (\xi^2-\lambda^2\nu^2)\widetilde{h}(\lambda)\right), \end{aligned} \tag{2.5}$$

as well as the form of generating function of the Gaudin Hamiltonians in [19]:

$$\begin{aligned} \tau(\lambda) = &2\lambda^2\left(\widetilde{h}^2(\lambda) + \frac{2\nu^2}{\xi^2-\lambda^2\nu^2}\widetilde{h}(\lambda) - \frac{\widetilde{h}'(\lambda)}{\lambda}\right) \\ &- \frac{2\lambda^2}{\xi^2-\lambda^2\nu^2}\left(\widetilde{f}(\lambda) + \frac{\psi\lambda^2\nu}{\xi}\widetilde{h}(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2}\widetilde{e}(\lambda) - \frac{\psi\nu}{\xi}\right)\widetilde{e}(\lambda). \end{aligned} \tag{2.6}$$

In [19] we tried to implement the algebraic Bethe ansatz based on these generators. Although the approached looked promising and resulted in the conjecture for the spectra of the generating function $\tau(\lambda)$ and the corresponding Gaudin Hamiltonians, the expression for the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, was missing. It turned out, as we show in the following section, that the full implementation of the algebraic Bethe ansatz in this case requires to define a new set of generators which will enable explicit expressions for the Bethe vectors as well as the algebraic proof of the off shell action of the generating function $\tau(\lambda)$ and its spectrum.

3. New generators and the eigenvectors

In the algebraic Bethe ansatz it is essential to find the commutation relations between the generating function and a product of the creation operators in a closed form. To this end, with the aim to simplify the relations (2.4) and (2.5) as well as the expression (2.6), we introduce new generators $e(\lambda)$, $h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the previous ones:

$$e(\lambda) = \widetilde{e}(\lambda), \quad h(\lambda) = \widetilde{h}(\lambda) + \frac{\psi}{2\xi\nu}\widetilde{e}(\lambda), \quad f(\lambda) = \widetilde{f}(\lambda) + \frac{\psi\xi}{\nu}\widetilde{h}(\lambda) + \frac{\psi^2}{4\nu^2}\widetilde{e}(\lambda). \tag{3.1}$$

It is straightforward to check that in the new basis we still have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = [f(\lambda), f(\mu)] = 0, \tag{3.2}$$

while the key simplification occurs in the three nontrivial relations which are now given by

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2}\left(e(\mu) - e(\lambda)\right), \tag{3.3}$$

$$[h(\lambda), f(\mu)] = \frac{-2}{\lambda^2 - \mu^2}\left(f(\mu) - f(\lambda)\right), \tag{3.4}$$

$$[e(\lambda), f(\mu)] = \frac{-4}{\lambda^2 - \mu^2}\Big((\xi^2 - \mu^2\nu^2)\, h(\mu) - (\xi^2 - \lambda^2\nu^2)\, h(\lambda)\Big). \tag{3.5}$$

By using these generators the expression for the generating function of the Gaudin Hamiltonians with boundary terms (2.6) also simplifies. We invert the relations (3.1) and obtain the expression for the generating function in terms of the new generators

$$\tau(\lambda) = 2\lambda^2\left(h^2(\lambda) + \frac{2\nu^2}{\xi^2 - \lambda^2\nu^2}h(\lambda) - \frac{h'(\lambda)}{\lambda}\right) - \frac{2\lambda^2}{\xi^2 - \lambda^2\nu^2} f(\lambda)e(\lambda). \tag{3.6}$$

Evidently we have achieved our first objective, as the relations (3.3)–(3.5) and the expression (3.6) are much simple than before. Below we will demonstrate how these new results facilitate the study of the Bethe vectors.

As in [19], we define the vacuum Ω_+ which is annihilated by $e(\lambda)$, while being an eigenstate for $h(\lambda)$:

$$h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \quad \text{with} \quad \rho(\lambda) = \frac{1}{\lambda}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda - \alpha_m} + \frac{s_m}{\lambda + \alpha_m}\right) = \sum_{m=1}^{N}\frac{2s_m}{\lambda^2 - \alpha_m^2}. \tag{3.7}$$

The next relevant remark is that the vector Ω_+ is an eigenvector of the generating function $\tau(\lambda)$. To show this we use (3.6) and the action (3.7):

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2\left(\rho^2(\lambda) + \frac{2\nu^2\,\rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda}\right)\Omega_+. \tag{3.8}$$

Our main aim in this section it to prove that the generator $f(\lambda)$ (3.1) defines the Bethe vectors naturally, that is, to show that the Bethe vector in the general case is given by the following symmetric function of its arguments:

$$\varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = f(\mu_1)\cdots f(\mu_M)\Omega_+. \tag{3.9}$$

We stress that this was not possible in the old basis (of tilde operators), and thus the general form of the Bethe vector lacked in [19].

The action of the generating function of the Gaudin Hamiltonians $\tau(\lambda)$ on $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is given by

$$\tau(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) = [\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\,\Omega_+ + \chi_0(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M). \tag{3.10}$$

The key part of the proof will be to determine the commutator in the first term of the righthand side. Due to the simplicity of the new commutation relations (3.3)–(3.5) we will show that it is now possible to evaluate this commutator in an algebraically closed form. As the first step we will calculate the commutator between the generating function (3.6) and a single generator $f(\lambda)$. A straightforward calculation yields

$$\begin{aligned}[\tau(\lambda), f(\mu)] = &-\frac{8\lambda^2}{\lambda^2-\mu^2} f(\mu)\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\\ &+\frac{8\lambda^2}{\lambda^2-\mu^2}\frac{\xi^2-\mu^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda)\left(h(\mu)+\frac{\nu^2}{\xi^2-\mu^2\nu^2}\right).\end{aligned} \tag{3.11}$$

For the general case, we assert that the following holds:

$$\begin{aligned}&[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\\ &= f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)\\ &+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda)f(\mu_2)\cdots f(\mu_M)\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\\ &\vdots\\ &+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)\\ &\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right).\end{aligned} \tag{3.12}$$

Our proof of this statement is based on the induction method: we assume that, for some integer $M \geq 1$, the above formula (i.e. the induction hypothesis) is satisfied and proceed to show that this assumption implicates the same relation for the product of $M+1$ operators. To this end we write

$$\begin{aligned}\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)f(\mu_{M+1})\right] = &[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\, f(\mu_{M+1})\\ &+ f(\mu_1)\cdots f(\mu_M)\left[\tau(\lambda), f(\mu_{M+1})\right].\end{aligned} \tag{3.13}$$

To evaluate the first term on the right-hand-side of (3.13) we use the induction assumption (3.12), while in the second term we apply (3.11) and obtain

$$\begin{aligned}&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]\\ &= f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right) f(\mu_{M+1})\end{aligned}$$

$$
\begin{aligned}
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_M)\\
&\times\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)f(\mu_{M+1})\\
&\vdots\\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)\\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M}\frac{2}{\mu_M^2-\mu_j^2}\right)f(\mu_{M+1})\\
&+f(\mu_1)\cdots f(\mu_M)\left(\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}f(\mu_{M+1})\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\right.\\
&\left.+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right)\right).
\end{aligned}
\tag{3.14}
$$

Then, using (3.4), we rearrange the terms having $f(\mu_{M+1})$ on the right

$$
\begin{aligned}
&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]\\
&=f(\mu_1)\cdots f(\mu_{M+1})\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)\\
&+f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(\frac{-2}{\lambda^2-\mu_{N+1}^2}\left(f(\mu_{M+1})-f(\lambda)\right)\right)\\
&+f(\mu_1)\cdots f(\mu_{M+1})\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_M)\left(\frac{-2}{\mu_1^2-\mu_{M+1}^2}\left(f(\mu_{M+1})-f(\mu_1)\right)\right)\\
&\vdots\\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)f(\mu_{M+1})\\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right)
\end{aligned}
$$

$$+ \frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1}) f(\lambda)\left(\frac{-2}{\mu_M^2-\mu_{M+1}^2}\left(f(\mu_{M+1})-f(\mu_M)\right)\right)$$
$$+ \frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_M) f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right). \tag{3.15}$$

The next step is to add similar terms appropriately

$$\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]$$
$$= f(\mu_1)\cdots f(\mu_{M+1})\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M+1}\frac{1}{\lambda^2-\mu_j^2}\right)$$
$$+ f(\mu_1)\cdots f(\mu_{M+1})\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j=1}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)$$
$$+ \frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda) f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M+1}\frac{2}{\mu_1^2-\mu_j^2}\right)$$
$$\vdots$$
$$+ \frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1}) f(\lambda) f(\mu_{M+1})$$
$$\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M+1}\frac{2}{\mu_M^2-\mu_j^2}\right)$$
$$+ \frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_M) f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right)$$
$$+ f(\mu_1)\cdots f(\mu_M) f(\lambda)\sum_{i=1}^{M}\left(\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\frac{2}{\lambda^2-\mu_{M+1}^2}+\frac{8\lambda^2}{\lambda^2-\mu_i^2}\frac{2}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_i^2\nu^2}{\xi^2-\lambda^2\nu^2}\right). \tag{3.16}$$

Using the following identity

$$\frac{-\lambda^2}{\lambda^2-\mu_i^2}\frac{1}{\lambda^2-\mu_{M+1}^2}+\frac{\lambda^2}{\lambda^2-\mu_i^2}\frac{1}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_i^2\nu^2}{\xi^2-\lambda^2\nu^2}$$
$$=\frac{\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{1}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2}, \tag{3.17}$$

for $i = 1, \ldots, N$, we can bring together all the terms in the last two lines of (3.16) and obtain the final expression

$$
\begin{aligned}
&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right] \\
&= f(\mu_1)\cdots f(\mu_{M+1}) \sum_{i=1}^{M+1} \frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M+1}\frac{1}{\lambda^2-\mu_j^2}\right) \\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda) f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M+1}\frac{2}{\mu_1^2-\mu_j^2}\right) \\
&\vdots \\
&+\frac{8\lambda^2}{\lambda^2-\mu_N^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1}) f(\lambda) f(\mu_{M+1}) \\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M+1}\frac{2}{\mu_M^2-\mu_j^2}\right) \\
&+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_M) f(\lambda) \\
&\times\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}-\sum_{j=1}^{M}\frac{2}{\mu_{M+1}^2-\mu_j^2}\right).
\end{aligned}
\tag{3.18}
$$

Since we have already explicitly showed that the induction hypothesis is valid for $M = 1$ (the (3.11) is a special case of (3.12)), this completes our proof of (3.12) by induction.

Now, using the result (3.12), we finally find the off shell action (3.10) of the generating function $\tau(\lambda)$ on $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ to be:

$$
\tau(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) = \chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) \tag{3.19}
$$

$$
\begin{aligned}
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}\left(\rho(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\varphi_M(\lambda,\mu_2,\ldots,\mu_M) \\
&\vdots \\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}\left(\rho(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right)\varphi_M(\mu_1,\ldots,\mu_{M-1},\lambda),
\end{aligned}
\tag{3.20}
$$

and the eigenvalue is

$$
\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M) = \chi_0(\lambda) - \sum_{i=1}^{M}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right).
\tag{3.21}
$$

The above off shell action of the generating function also contains the M unwanted terms which vanish when the following Bethe equations are imposed on the parameters $\mu_1, \ldots, \mu_M$,

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{M} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \tag{3.22}$$

where $i = 1, 2, \ldots, M$.

Hence we have showed that the symmetric function $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ defined in (3.9) is the Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)$, stated above (3.21). With this proof we close the topic of the implementation of the algebraic Bethe ansatz for this model.

4. Solutions to the Knizhnik–Zamolodchikov equations

Finding the off-shell action on Bethe vectors in the previous section was, in this approach, a necessary prerequisite for solving of the corresponding Knizhnik–Zamolodchikov equations [25,27]. In this context the local realization of Gaudin algebra basis operators is also relevant:

$$e(\lambda) = -2 \sum_{m=1}^{N} \frac{\xi - \alpha_m \nu}{\lambda^2 - \alpha_m^2} S_m^+, \tag{4.1}$$

$$h(\lambda) = 2 \sum_{m=1}^{N} \frac{1}{\lambda^2 - \alpha_m^2} \left(S_m^3 - \frac{\psi}{2\nu} S_m^+ \right), \tag{4.2}$$

$$f(\lambda) = 2 \sum_{m=1}^{N} \frac{\xi + \alpha_m \nu}{\lambda^2 - \alpha_m^2} \left(S_m^- + \frac{\psi}{\nu} S_m^3 - \frac{\psi^2}{4\nu^2} S_m^+ \right), \tag{4.3}$$

where S_m^3, $S_m^\pm$, are the usual spin generators at the local node m (see [19]). In this local realization the vacuum vector Ω_+ has the form

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}, \tag{4.4}$$

where vector ω_m belongs to local node Hilbert space $V_m = \mathbb{C}^{2s+1}$ and:

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \tag{4.5}$$

The Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function $\tau(\lambda)$ at poles $\lambda = \pm\alpha_m$ [19] and in order to make the paper self contained, we state these result also here:

$$\mathrm{Res}_{\lambda=\alpha_m} \tau(\lambda) \;=\; 4\, H_m \quad \text{and} \quad \mathrm{Res}_{\lambda=-\alpha_m} \tau(\lambda) \;=\; (-4)\, \widetilde{H}_m, \tag{4.6}$$

yielding:

$$H_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\left(K_m(\alpha_m) \vec{S}_m K_m^{-1}(\alpha_m) \right) \cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(\alpha_m) \vec{S}_m K_m^{-1}(\alpha_m) \right)}{2(\alpha_m + \alpha_n)}, \tag{4.7}$$

and

$$\widetilde{H}_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\left(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\right)\cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\right)}{2(\alpha_m + \alpha_n)}. \tag{4.8}$$

It follows from the above relations and (3.21) that the eigenvalues of the Gaudin Hamiltonians (4.7) and (4.8) can be derived as the residues of $\chi_M(\lambda, \mu_1, \dots, \mu_M)$, obtained in the previous section, at the poles $\lambda = \pm\alpha_m$ [19]. It turns out that the respective eigenvalues of the Hamiltonians (4.7) and (4.8) coincide:

$$\begin{aligned}\mathcal{E}_{m,M} &= \frac{1}{4}\mathrm{Res}_{\lambda=\alpha_m}\chi_M(\lambda, \mu_1, \dots, \mu_M)\\ &= \widetilde{\mathcal{E}}_{m,M} = \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2\nu^2} + \sum_{n\neq m}^{N} \frac{2s_n}{\alpha_m^2 - \alpha_n^2}\right)\\ &- 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2}.\end{aligned} \tag{4.9}$$

When all the spin s_m are set to one half, these energies, as well as the Bethe equations, coincide with the expressions obtained in [14] (up to normalisation; for the connection of the corresponding notations, cf. [19]).

The key observation in what follows will be that by taking the residue of both sides of the equation (3.19) at $\lambda = \alpha_n$, using (4.6), (4.7) and (4.9), and dividing both sides of the equation by the factor of four one obtains

$$\begin{aligned}H_n\varphi_M(\mu_1, \mu_2, \dots, \mu_M) &= \mathcal{E}_{n,M}\, \varphi_M(\mu_1, \mu_2, \dots, \mu_M) + \sum_{j=1}^{M} \frac{2\alpha_n^2}{\alpha_n^2 - \mu_j^2}\, \frac{\xi^2 - \mu_j^2\nu^2}{\xi^2 - \alpha_n^2\nu^2} \times\\ &\times \left(\rho(\mu_j) + \frac{\nu^2}{\xi^2 - \mu_j^2\nu^2} - \sum_{k\neq j}^{M} \frac{2}{\mu_j^2 - \mu_k^2}\right) \frac{\xi + \alpha_n \nu}{\alpha_n}\\ &\times \left(S_n^- + \frac{\psi}{\nu} S_n^3 - \frac{\psi^2}{4\nu^2} S_n^+\right) \varphi_{M-1}(\mu_1, \dots, \widehat{\mu_j}, \dots, \mu_M)\,,\end{aligned} \tag{4.10}$$

here the notation $\widehat{\mu_j}$ means that the argument μ_j is not present.

The solutions to the Knizhnik–Zamolodchikov equations we seek in the form of contour integrals over the variables $\mu_1, \mu_2, \dots, \mu_M$ [25,27]:

$$\psi(\alpha_1, \alpha_2, \dots, \alpha_N) = \oint \cdots \oint \phi(\vec{\mu}|\vec{\alpha})\varphi_M(\vec{\mu}|\vec{\alpha})\, d\mu_1 \cdots d\mu_M, \tag{4.11}$$

where the integrating factor $\phi(\vec{\mu}|\vec{\alpha})$ is a scalar function

$$\phi(\vec{\mu}|\vec{\alpha}) = \exp\left(\frac{S(\vec{\mu}|\vec{\alpha})}{\kappa}\right) \tag{4.12}$$

obtained by exponentiating a function $S(\vec{\mu}|\vec{\alpha})$ [34]. As in [10], from now on, the K-matrix parameters take fixed values $\psi = \xi = 0$ and $\nu = 1$. For these values it is straightforward to check that i) the Gaudin Hamiltonians are Hermitian; and ii) Hamiltonians (4.7) and (4.8) coincide.

We find that the proper form of $S(\vec{\mu}|\vec{\alpha})$ in this case is:

$$\begin{aligned} S(\vec{\mu}|\vec{\alpha}) = &\sum_{n=1}^{N} \frac{s_n(s_n-1)}{2\ln(\alpha_n)} + \sum_{n<m}^{N} \alpha_n \alpha_m \ \ln(\alpha_n^2 - \alpha_m^2) + \sum_{j=1}^{M} \ln(\mu_j) \\ &+ \sum_{j<k}^{M} \ln(\mu_j^2 - \mu_k^2) - \sum_{j=1}^{M}\sum_{n=1}^{N} s_n \ln(\alpha_n^2 - \mu_j^2). \end{aligned} \tag{4.13}$$

In order to show this, it is important to notice that the function $\phi(\vec{\mu}|\vec{\alpha})$ as defined above also satisfies the following equations

$$\kappa\ \partial_{\alpha_n}\phi = \mathcal{E}_{n,M}\ \phi, \tag{4.14}$$

$$\kappa\ \partial_{\mu_j}\phi = \beta_M(\mu_j)\ \phi, \tag{4.15}$$

where

$$\beta_M(\mu_j) := -\mu_j \left(\rho(\mu_j) - \frac{1}{\mu_j^2} - \sum_{k \neq j}^{M} \frac{2}{\mu_j^2 - \mu_k^2} \right). \tag{4.16}$$

Introducing the notation

$$\widetilde{\varphi}_{M-1}^{(j,n)} := S_n^- \ \varphi_{M-1}(\mu_1, \ldots, \widehat{\mu_j}, \ldots, \mu_M) \tag{4.17}$$

the equation (4.10) can be expressed in the following form

$$H_n \varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{E}_{n,M}\ \varphi_M(\mu_1, \mu_2, \ldots, \mu_M) + \sum_{j=1}^{M} \frac{(-2)\mu_j}{\alpha_n^2 - \mu_j^2}\ \beta_M(\mu_j)\ \widetilde{\varphi}_{M-1}^{(j,n)}. \tag{4.18}$$

Using the definition of φ_M (3.9) and the local realisation of the generator $f(\mu)$ (4.3) it follows that

$$\partial_{\alpha_n} \varphi_M = (-2) \sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{\mu_j\ \widetilde{\varphi}_{M-1}^{(j,n)}}{\mu_j^2 - \alpha_n^2} \right). \tag{4.19}$$

Then it is straightforward to show that

$$\kappa\ \partial_{\alpha_n} (\phi \varphi_M) = H_n (\phi \varphi_M) + \kappa \sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{(-2)\mu_j}{\mu_j^2 - \alpha_n^2} \phi \widetilde{\varphi}_{M-1}^{(j,n)} \right). \tag{4.20}$$

A closed contour integration of $\phi\varphi_M$ with respect to the variables $\mu_1, \mu_2, \ldots, \mu_M$ will cancel the contribution from the terms under the sum in (4.20) and therefore $\psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ given by (4.11) satisfies the Knizhnik–Zamolodchikov equations

$$\kappa\ \partial_{\alpha_n} \psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = H_n \psi(\alpha_1, \alpha_2, \ldots, \alpha_N). \tag{4.21}$$

Moreover, the interplay between the Gaudin model and the Knizhnik–Zamolodchikov equations, once the Bethe equations are imposed

$$\frac{\partial S}{\partial \mu_j} = \beta_M(\mu_j) = -\mu_j \left(\sum_{m=1}^{N} \frac{2s_m}{\mu_j^2 - \alpha_m^2} - \frac{1}{\mu_j^2} - \sum_{k \neq j}^{M} \frac{2}{\mu_j^2 - \mu_k^2} \right) = 0, \tag{4.22}$$

enabled us to determine the on-shell norm of the Bethe vectors

$$||\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)||^2 = 2^M \det \left(\frac{\partial^2 S}{\partial \mu_j \partial \mu_k} \right). \tag{4.23}$$

It turns out to be possible to derive also a stronger formula than the one above for the norms [27]. Indeed, we calculate the following expression for the off-shell scalar product of arbitrary two Bethe vectors:

$$\Omega_+^* e(\lambda_1) e(\lambda_2) \cdots e(\lambda_M) f(\mu_M) \cdots f(\mu_2) f(\mu_1) \Omega_+ = 4^M \sum_{\sigma \in S_M} \det \mathcal{M}^\sigma , \tag{4.24}$$

where S_M is the symmetric group of degree M and the $M \times M$ matrix $\mathcal{M}^\sigma$ is given by

$$\mathcal{M}^\sigma_{jj} = -\frac{\lambda_j^2 \rho(\lambda_j) - \mu_{\sigma(j)}^2 \rho(\mu_{\sigma(j)})}{\lambda_j^2 - \mu_{\sigma(j)}^2} - \sum_{k \neq j} \frac{\lambda_k^2 + \mu_{\sigma(k)}^2}{(\lambda_j^2 - \lambda_k^2)(\mu_{\sigma(j)}^2 - \mu_{\sigma(k)}^2)} , \tag{4.25}$$

$$\mathcal{M}^\sigma_{jk} = -\frac{\lambda_k^2 + \mu_{\sigma(k)}^2}{(\lambda_j^2 - \lambda_{\sigma(k)}^2)(\mu_{\sigma(j)}^2 - \mu_{\sigma(k)}^2)} , \quad \text{for} \quad j, k = 1, 2, \ldots, M. \tag{4.26}$$

This formula (that can be proved by commuting $e(\lambda)$ operators to the right and using mathematical induction) has obvious potential applications as the first step towards the general correlation functions. It should be noted that in [13] a related problem was analysed in the trigonometric case and under certain restrictions: local spins were all fixed to the value $\frac{1}{2}$ and it was required that $N = 2M$ (in the notation of that paper). Our formula is more compact and valid for arbitrary spins and arbitrary number of excitations.

5. Conclusion

In this paper we addressed a number of open problems related to Gaudin model with non periodic boundary conditions.

First, we obtained a new basis of the generalized $s\ell(2)$ Gaudin algebra, in which the commutation relations and the generating function are manifestly simpler. This step allowed us to calculate Bethe vectors and off-shell action of the generating function upon them in a closed form, for arbitrary number of excitations. The obtained expressions we have proved by mathematical induction.

Once having the general expressions for the Bethe vectors and for the corresponding eigenvalues, we could proceed to relate KZ equations with the Bethe vectors. Taking residues of the off-shell action at poles $\pm\alpha_m$, we obtained both Gaudin Hamiltonians and their eigenvalues. By finding the appropriate form of the function S in (4.13), we managed to establish and prove relations (4.14) and (4.15) which led to solution to KZ equations. Proceeding in the same framework, we also obtained the expression for norms of Bethe vectors on shell. Moreover, we went a step further and provided a closed form formula for the scalar product of arbitrary two Bethe vectors.

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