



Existence and Uniqueness of Weak Solutions for the Generalized Stochastic Navier-Stokes-Voigt Equations

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Received: 12 April 2025 / Accepted: 4 August 2025

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Abstract

In this work, we consider the incompressible generalized Navier-Stokes-Voigt equations in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 2$, driven by a multiplicative Gaussian noise. The considered momentum equation is given by:

$$d(\mathbf{u} - \kappa \Delta \mathbf{u}) = [\mathbf{f} + \operatorname{div}(-\pi \mathbf{I} + \nu |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) - \mathbf{u} \otimes \mathbf{u})] dt + \Phi(\mathbf{u}) dW(t).$$

In the case of $d = 2, 3$, \mathbf{u} accounts for the velocity field, π is the pressure, \mathbf{f} is a body force and the final term represents the stochastic forces. Here, κ and ν are given positive constants that account for the kinematic viscosity and relaxation time, and the power-law index p is another constant (assumed $p > 1$) that characterizes the flow. We use the usual notation \mathbf{I} for the unit tensor and $\mathbf{D}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$ for the symmetric part of velocity gradient. For $p \in (\frac{2d}{d+2}, \infty)$, we first prove the existence of a *martingale solution*. Then we show the *pathwise uniqueness of solutions*. We employ the classical *Yamada-Watanabe theorem* to ensure the existence of a unique *probabilistic strong solution*.

Keywords Stochastic generalized Navier-Stokes-Voigt equations · Gaussian noise · Martingale solution · Strong solution

Mathematics Subject Classification Primary 60H15, 35R60 · Secondary 35Q35, 76D03, 76A05

Communicated by Ivan Corwin.

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1 Introduction

1.1 The Model

We study flows of incompressible fluids with elastic properties governed by the incompressible generalized Navier-Stokes-Voigt (NSV) equations in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, with its boundary denoted by $\partial\mathcal{O}$, during the time interval $[0, T]$, with $0 < T < \infty$. The momentum equation is assumed here to be perturbed by a stochastic term, that is, infinite-dimensional multiplicative Gaussian noise which is the product of $\Phi(\mathbf{u})$ and a Wiener process $W(\cdot)$. The problem under consideration is posed by the following system of equations:

$$d(\mathbf{u} - \kappa \Delta \mathbf{u}) = [\mathbf{f} + \operatorname{div}(-\pi \mathbf{I} + \nu |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) - \mathbf{u} \otimes \mathbf{u})] dt + \Phi(\mathbf{u}) dW(t), \text{ in } \mathcal{O}_T, \quad (1.1)$$

$$\operatorname{div} \mathbf{u} = 0, \text{ in } \mathcal{O}_T, \quad (1.2)$$

$$\mathbf{u} = \mathbf{u}_0, \text{ in } \mathcal{O} \times \{0\}, \quad (1.3)$$

$$\mathbf{u} = \mathbf{0}, \text{ on } \Gamma_T, \quad (1.4)$$

where $\mathcal{O}_T := \mathcal{O} \times (0, T)$ and $\Gamma_T := \partial\mathcal{O} \times [0, T]$. Here, $\mathbf{u} = (u_1, \dots, u_d)$ and $\mathbf{f} = (f_1, \dots, f_d)$ are vector-valued functions, π is a scalar-valued function, whereas ν and κ are positive constants. In real-world applications, space dimensions of interest are $d = 2$ and $d = 3$, and in that case \mathbf{u} accounts for the velocity field, π is the pressure, \mathbf{f} represents an external forcing field. Greek letters ν and κ stay for positive constants that, in the dimensions of physical interest, account for the kinematics viscosity and for the relaxation time, that is, the time required for a viscoelastic fluid relax from a deformed state back to its equilibrium configuration. The power-law index p is a constant that characterizes the flow and is assumed to be such that $1 < p < \infty$. In particular, for $1 < p < 2$, the model describes shear-thinning fluids, when $p = 2$, we recover the model that governs Newtonian fluids, while for $p > 2$, the model describes shear-thickening fluids. The capital letters stay for tensor-valued functions, in particular \mathbf{I} is the unit tensor and $\mathbf{D}(\mathbf{u})$ is the symmetric part of the velocity gradient, that is, $\mathbf{D}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$. The idea behind the presence of the noise $\Phi(\mathbf{u})$ is an interaction between the solution \mathbf{u} and the random perturbation caused by the Wiener process $W(\cdot)$, and the product of these two terms represents the stochastic part in the momentum equation (1.1). This stochastic part can be interpreted in different ways. It can be understood as a turbulence in the fluid motion or it can be interpreted as a perturbation from the physical model. On the other hand, apart from the forcing term \mathbf{f} , there might be further quantities with influence on the motion which can be characterized by this stochastic part as well.

1.2 Literature Review

We start with the physical significance of NSV equations, also named Kelvin-Voigt (KV) equations, as we have considered the generalized stochastic NSV equations (1.1)-(1.4). KV equations are used in the applications to model the response of materials that exhibit all intermediate range of properties between an elastic solid and a viscous fluid, as, for instance, polymers. These materials have some memory in the sense that they can come back to some previous state when the shear stress is removed (see [10]). The deterministic KV equations were extensively studied in the article [56], whose author coined the name Kelvin-Voigt for the incompressible Navier-Stokes equations perturbed by the relaxation term $\kappa \partial_t \Delta \mathbf{u}$ (see [52] also). However, and as observed in the work [68], neither Kelvin nor Voigt have suggested

any stress-strain relation, or system of governing equations for viscoelastic fluids (see [37]). Currently, the Navier-Stokes-Voigt name for the associated system of equations seems to be the most accepted by the researchers in this field. Mathematically speaking, and was noticed by Ladyzhenskaya (see [42]), the most important property of NSV equations is that the term $\kappa \partial_t \Delta \mathbf{u}$ works as a regularization of the 3D incompressible Navier-Stokes (NS) equations ensuring the unique global solvability of the corresponding problem. The existence and uniqueness of the weak and strong solutions of NSV equations have been established in the work [56] (cf. [43]). A local in time existence and uniqueness result of an inverse problem for NSV equations is established in [41]. Several other authors have also studied some variants of the NSV equations, in many settings and under different assumptions, with respect to the well-posedness and long time behavior. It is worth to mention that the works [40, 57] established a connection between NSV and the Oldroyd models, and proved the existence of classical global solutions for the models. Also, in the works [54, 68, 69], the authors proved the existence and uniqueness of weak solutions in case of domain varying with time and with the optimal control. The authors in [4–6, 8, 9] discussed the existence, uniqueness and some qualitative properties of the solutions to some variants of NSV equations, more specifically the case with power-laws and anisotropic diffusion and several other properties. In [3], the authors considered the generalized NSV equations for non-homogeneous and incompressible fluid flows with the diffusion and relaxation terms described by two distinct power-laws and proved the existence of weak solutions as well as large-time behavior of the solutions. The works [20, 38, 46, 60] established a clear connection between NSV equations and the turbulence modeling, to be more specific, with Bardina turbulence models. The same type of connection was firstly studied in the work [44] and followed by the work [45]. The authors in [1] connects NSV equations with the Prandtl-Smagorinsky and turbulent kinematic energy models in order to compute the turbulent viscosity. Several questions on NSV equations as a regularization of both NS and Euler equations have been tackled in [12, 13, 38]. Also, the author in the work [25] considered generalized NS equations (NS equations with power-law) with the presence of damping term in the momentum equation which governs isothermal flows of incompressible and homogeneous non-Newtonian fluids and established the existence of weak solutions using regularization techniques, the monotone theory, compactness arguments and truncation method. In a recent work [7], the authors considered the classical NSV problem for incompressible fluids with unknown non-constant density and established the existence of weak solutions (velocity and density), unique pressure and some regularity results of the solutions on the smoothness of the given data.

We are interested in the generalized stochastic NSV equations (1.1)-(1.4). As mentioned above, from several physical point of view it is important to have a stochastic part in the equation of motion:

- (1) It gives a clear understanding of turbulence in the fluid motion (see [51]).
- (2) It could be viewed as a deviation from the physical model.
- (3) We are seeing that there may be additional values, generally minor ones, that have an impact on the motion in addition to the force \mathbf{f} .

Several works are available in the literature dealing with the existence of weak solutions of stochastic NS equations starting from the article [11]. One can see the work [28] for an overview. However, only a few works are available dealing with the generalized ($p \neq 2$) NS problem. In the article [21], the authors discussed the bipolar shear thinning fluid. In that work, the presence of additional bi-Laplacian term $\Delta^2 \mathbf{u}$ makes the computations easy. Since, the additional term gives enough initial regularity which leads to the proof using monotone operators without any truncation arguments.

In [64, 66], the authors considered the stochastic power-law fluids and established the existence and uniqueness of weak solutions and strong convergence of the Galerkin approximation and energy equality, respectively. There were no interaction between the solution and Brownian motion in [64, 66] (additive noise). Further, in [14], the author extended the existence theory of the stochastic power-law fluids and improved the results of [64, 66]. He proved the existence of weak solutions (in both analytic and probabilistic sense), using the standard procedure as the pressure term disappear in the weak formulation and later it reconstructed into the formulation. He borrowed the idea from [65] (for deterministic setting), which relates each term in the equation a pressure part. Moreover, he stabilized the auxiliary equations by adding a large power of \mathbf{u} and the approach was based on Galekin method. Later, he extended the deterministic approach of the work [65] to the stochastic case and combined the techniques from nonlinear partial differential equations with stochastic calculus for martingales. Since the author cannot apply test functions in this situation, he instead employed Itô's formula to obtain energy estimates and other stochastic tools. Finally, in order to justify the limit in the nonlinear terms, he used the monotone operator theory together with the L^∞ -truncation.

Let us mention some works available for stochastic NSV equations. In [48], the authors considered 3D stochastic NSV equations in bounded domains perturbed by a finite-dimensional Brownian motion and established an large deviation principle for the solution using weak convergence approach. The existence and long-time behavior of the solutions (more specifically mean square exponentially stability and the almost sure exponential stability of the stationary solutions) to 3D stochastic NSV equations in bounded domains perturbed by infinite-dimensional Wiener process obtained in [2]. In a recent work [67], the author studied the stability of pullback random attractors for 3D stochastic NSV equations with delays.

1.3 Objectives and Novelties of the Paper

The main aim of this work is to prove the existence of martingale solution and unique probabilistic strong solution of generalized stochastic NSV equations perturbed by infinite-dimensional Wiener process. There seems to be a very scarce knowledge about the model presented in this work, and to the authors best knowledge the analysis presented here has not yet been investigated in the available literature.

- As we have mentioned above, there are only a few works available in the literature dealing with stochastic power-law fluids [14, 64, 66]. In the first work [14], the author proved the existence theory for stochastic power-law fluids driven by an infinite-dimensional multiplicative Gaussian noise. The latter two works [64, 66] discussed the existence and uniqueness of weak solutions for stochastic power-law fluids driven by a finite-dimensional Brownian motion.
- In this work, we are following the stochastic approach considered in the work [14], which in turn was an extension to the stochastic case of the deterministic problem studied in the work [65]. We are also considering the perturbation by an infinite-dimensional multiplicative Gaussian noise.

In addition to the stochastic situation, the main conclusion of this work is that the Voigt term $\kappa \partial_t \Delta \mathbf{u}$ regularizes the power-law model of the NS equations in such a way that the existence result can be proven up to the smallest possible value (in the sense that the Gelfand triple can still be used) of the power-law exponent p ($p > \frac{2d}{d+2}$), (see Theorem 2.6 below). The NS power-law model, which corresponds to considering $\kappa = 0$ in the momentum equation (1.1),

was considered in [14]. But there the author used the L^∞ -truncation method, and obtained the existence result for values of $p > \frac{2(d+1)}{d+2}$.

1.4 Organization

This article is organized as follows: We start with the basic formulation of the stochastic background, as well as Definitions 2.3 and 2.4 of martingale solution and probabilistically strong solution, respectively, and then we state our main Theorem 2.6 in Section 2. In Section 3, we discuss the decomposition of pressure terms. As in the weak formulation of the problem, the pressure term disappears (see Definition 2.3), later it can be recovered by using a version of de Rham’s theorem. The idea of decomposing the pressure term that we use in the sequel is borrowed from the work [14], but its first application dates back to [65]. In Section 4, we consider an approximate system with a stabilization term (a large power of \mathbf{u}), and, by using Galerkin approximations, we obtain the uniform energy estimate, followed by the existence of martingale solution for this approximate system, and for which we still have to use the Skorokhod representation theorem. Later, we reconstruct the pressure term and using monotone operator theory leads to existence of martingale solution to the system (1.1)-(1.4). We conclude by establishing the pathwise uniqueness of the solution to the system (1.1)-(1.4) and hence the existence of a unique probabilistic strong solution by an application of the classical Yamada-Watanabe theorem.

2 Auxiliary Results

2.1 Function Spaces

Let us introduce the functions spaces used to characterize the solutions of Fluid Dynamics problems. Let \mathcal{O} be a domain in \mathbb{R}^d , that is, an open and connected subset of \mathbb{R}^d . By $C_0^\infty(\mathcal{O})^d$, we denote the space of all infinitely differentiable \mathbb{R}^d -valued functions with compact support in \mathcal{O} . For $1 \leq p \leq \infty$, we denote by $L^p(\mathcal{O})^d$, the Lebesgue space consisting of all \mathbb{R}^d -valued measurable (equivalence classes of) functions that are p -summable over \mathcal{O} . By $|\mathcal{O}|$, we denote the d -Lebesgue measure of \mathcal{O} . The corresponding Sobolev spaces are represented by $W^{k,p}(\mathcal{O})^d$, where $k \in \mathbb{N}$. When $p = 2$, $W^{k,2}(\mathcal{O})^d$ are Hilbert spaces that we denote by $H^k(\mathcal{O})^d$. We define

$$\begin{aligned} \mathcal{V} &:= \{ \mathbf{u} \in C_0^\infty(\mathcal{O})^d : \nabla \cdot \mathbf{u} = 0 \}, \\ \mathbf{H} &:= \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^2(\mathcal{O})^d, \\ L_\sigma^p(\mathcal{O})^d &:= \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^p(\mathcal{O})^d, \\ W_\sigma^{p,k}(\mathcal{O})^d &:= \text{the closure of } \mathcal{V} \text{ in the Sobolev space } W^{k,p}(\mathcal{O})^d, \end{aligned}$$

for $1 \leq p < \infty$ and $k \in \mathbb{N}$. In the particular case of $p = 2$, we denote the space $W_\sigma^{p,k}(\mathcal{O})^d$ by \mathbf{V}^k , and if $k = 1$, we denote it by \mathbf{V}_p . If both $p = 2$ and $k = 1$, we denote $W_\sigma^{p,k}(\mathcal{O})^d$ solely by \mathbf{V} . Let (\cdot, \cdot) stand for the inner product of the Hilbert space $L^2(\mathcal{O})^d$, and we denote by $\langle \cdot, \cdot \rangle$, the induced duality product between the space $W_0^{1,p}(\mathcal{O})^d$ and its dual $W^{-1,p'}(\mathcal{O})^d$, as well as between $L^p(\mathcal{O})^d$ and its dual $L^{p'}(\mathcal{O})^d$, where $\frac{1}{p} + \frac{1}{p'} = 1$. In the sequel, the L^p , $W^{k,p}$ and $W^{-k,p'}$ norms will be denoted in short by $\| \cdot \|_p$, $\| \cdot \|_{k,p}$ and $\| \cdot \|_{-k,p'}$. On \mathbf{V} , we consider the equivalent norm $\| \mathbf{u} \|_{\mathbf{V}} := \| \nabla \mathbf{u} \|_2$, $\mathbf{u} \in \mathbf{V}$.

2.2 Stochastic Setting

In this subsection, we start by recalling some notions of stochastic processes necessary to study stochastic partial differential equations. We consider the scalar-valued case, being the vectorial case a straightforward generalization. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω denotes the set of all possible outcomes ω , \mathcal{F} is the set of events and $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ accounts for the probability measure function. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, that is, with a nondecreasing family of sub-sigma fields of \mathcal{F} ($\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$), such that \mathcal{F}_0 contains all the null elements (elements $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$) and $\{\mathcal{F}_t\}_{t \in [0, T]}$ is right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$). In this case, we denote a filtered probability space by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$. Letting $1 \leq p \leq \infty$ and given a Banach space \mathbf{B} , we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{B})$, the space of all measurable mappings $v : (\Omega, \mathcal{F}) \rightarrow (\mathbf{B}, \mathcal{B}(\mathbf{B}))$, where $\mathcal{B}(\mathbf{B})$ denotes the Borel σ -algebra on \mathbf{B} , such that

$$\mathbb{E}[\|v\|_{\mathbf{B}}^p] < \infty,$$

where the expectation \mathbb{E} is taken with respect to $(\Omega, \mathcal{F}, \mathbb{P})$. A \mathbf{B} -valued stochastic process can be interpreted as a mapping $X : [0, T] \times \Omega \rightarrow (\mathbf{B}, \mathcal{B}(\mathbf{B}))$, denoted by $X(t, \omega)$. For a fixed $\omega \in \Omega$, the mapping $t \mapsto X(t, \omega)$ is called the *path or trajectory* of X . We say that a \mathbf{B} -valued stochastic process X is *adapted to a filtration* $\{\mathcal{F}_t\}_{t \in [0, T]}$, or just \mathcal{F}_t -adapted, if the mapping $X(t) : \Omega \rightarrow (\mathbf{B}, \mathcal{B}(\mathbf{B}))$, defined by $\omega \mapsto X(t, \omega)$, is measurable for all $t \in [0, T]$. The \mathbf{B} -valued stochastic process X is *progressively measurable* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, that is, is $\{\mathcal{F}_t\}_{t \in [0, T]}$ -measurable, if the mapping $(s, \omega) \mapsto X(s, \omega)$ is measurable on $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ for all $t \in [0, T]$. Note that every progressively measurable stochastic process is \mathcal{F}_t -adapted.

We are now ready to define the notion of integral of a stochastic process with values in specific Hilbert spaces. To this end, we first consider a bounded linear operator $Q \in \mathcal{L}(\mathbf{U}, \mathbf{U})$, where \mathbf{U} is a separable Hilbert space with the inner product denoted by $(\cdot, \cdot)_{\mathbf{U}}$ and the associated norm by $\|\cdot\|_{\mathbf{U}}$. In fluid dynamics, \mathbf{U} typically represents the space of velocity fields. To model the effect of random external influences, such as thermal fluctuations, unresolved turbulence, or stochastic forcing from wind or pressure variations, one uses stochastic processes defined over infinite-dimensional function spaces. The covariance operator $Q \in \mathcal{L}(\mathbf{U}, \mathbf{U})$ describes the spatial structure of the noise. It determines how random energy is distributed across different spatial modes. For example, the eigenvectors $\{\mathbf{e}_k\}_{k \in \mathbb{N}}$ of Q correspond to spatial modes, and the associated eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ represent the intensity (or variance) of the noise in each mode. If Q is nonnegative, symmetric, and has finite trace, then there exists an orthonormal basis $\{\mathbf{e}_k\}_{k \in \mathbb{N}}$ of \mathbf{U} such that

$$Q\mathbf{e}_k = \lambda_k \mathbf{e}_k, \quad \lambda_k \geq 0 \text{ for all } k \in \mathbb{N},$$

with 0 as the only accumulation point of $\{\lambda_k\}_{k \in \mathbb{N}}$. Moreover, Q is called a trace-class operator if it has finite trace, that is,

$$\sum_{k \in \mathbb{N}} \lambda_k < \infty.$$

The condition that Q is nonnegative, symmetric, and of finite trace ensures that the total energy injected by the noise is finite, which is essential for the well-posedness of the stochastic Navier-Stokes-Voigt equations. If Q is of trace-class, then the resulting noise is spatially smooth and the stochastic process remains regular. For a fixed $Q \in \mathcal{L}(\mathbf{U}, \mathbf{U})$ that is nonnegative, symmetric, and trace-class, a \mathbf{U} -valued stochastic process $\{W(t)\}_{t \in [0, T]}$ on the

probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard Q-Wiener process if and only if

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) \mathbf{e}_k, \quad t \in [0, T],$$

where β_k , with $k \in \{n \in \mathbb{N} : \lambda_n > 0\}$, are independent real-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ ([24, Proposition 4.3]). A Q-Wiener process $\{W(t)\}_{t \in [0, T]}$ is said to be adapted to a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ if $W(t) \in \mathcal{F}_t$ for every $t \in [0, T]$ and $W(t) - W(s)$ is independent of \mathcal{F}_s for all $0 \leq s \leq t \leq T$ (for more details one can see [24]).

When the covariance operator $Q \in \mathcal{L}(U, U)$ is not of trace-class, the above series does not converge in the Hilbert space U . This implies that the noise is too rough to be interpreted as taking values in U . To overcome this, we define the space $U_0 := Q^{1/2}U$, equipped with the inner product

$$(\mathbf{u}, \mathbf{v})_0 := (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v})_U, \quad \mathbf{u}, \mathbf{v} \in U_0,$$

where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$. The norm associated with this inner product is denoted by $\|\cdot\|_0$ and the square root of the operator Q is defined via

$$Q^{1/2}\mathbf{e}_k = \sqrt{\lambda_k}\mathbf{e}_k.$$

The space U_0 contains U , that is, $U \subset U_0$, and is strictly larger when Q is not a trace-class operator. Intuitively, U_0 allows for functions with less regularity than those in U , and is thus better suited to accommodate rougher noise. In this setting, the stochastic process $\{W(t)\}_{t \in [0, T]}$ is well-defined as a U -valued process in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ because the inclusion $U_0 \hookrightarrow U$ is a Hilbert-Schmidt embedding. This guarantees that the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ even though it may not converge in U itself pointwise. In this case, $\mathcal{L}_2(U_0, U)$ denotes the space of Hilbert-Schmidt operators from U_0 to U . That is, for any orthonormal basis $\{\mathbf{e}_k^0\}_{k \in \mathbb{N}}$ of U_0 , an operator T belongs to $\mathcal{L}_2(U_0, U)$ if

$$\sum_{k \in \mathbb{N}} \|T\mathbf{e}_k^0\|_U^2 < \infty.$$

Despite the previous construction, in situations commonly encountered when modeling rougher noise or more realistic physical scenarios, Q is not a trace-class operator. For instance, in turbulent flows, forcing terms often simulate energy input from a broad spectrum of scales. If forcing is not strongly localized in frequency, the covariance operator Q does not decay fast enough to be trace-class. Another example, is the Dirac forcing that models impulses at points and leads to highly singular (non-trace-class) noise. In such cases, the process $\{W(t)\}_{t \in [0, T]}$ defined above may not converge in U . To address this, we introduce an auxiliary Hilbert space U_1 and a Hilbert-Schmidt embedding $J : U_0 \rightarrow U_1$ that smooths the process just enough to regain convergence. Denoting by J^* , the adjoint operator of J , we then define a new covariance operator

$$Q_1 := JJ^* \in \mathcal{L}(U_1, U_1),$$

which is symmetric, nonnegative, and of finite trace. In this setting, we define a new Q_1 -Wiener process on U_1 by

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) J\mathbf{e}_k^0, \quad t \in [0, T], \tag{2.1}$$

which converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U_1)$ ([49, Proposition 2.5.2]). This lifting of the noise process allows us to model the Navier-Stokes-Voigt equations with lower regularity while preserving well-posedness. Moreover, we have that $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and $J : U_0 \rightarrow Q_1^{\frac{1}{2}}(U_1)$ is an

isometry. Since the Hilbert space \mathbf{U}_1 and the Hilbert-Schmidt embedding \mathbf{J} considered above always exist, (2.1) defines a Wiener process, even though \mathbf{Q} is not of finite trace. The stochastic process defined this way is usually referred to as a *cylindrical Wiener process* in \mathbf{U} . In this case, we say that a process $\phi(t)$, $t \in [0, T]$, is integrable with respect to $\mathbf{W}(t)$, if it takes values in $\mathcal{L}_2(\mathbf{Q}^{\frac{1}{2}}(\mathbf{U}_1), \mathbf{U})$, is predictable and if

$$\mathbb{P}\left(\int_0^T \|\phi(s)\|_{\mathcal{L}_2(\mathbf{Q}^{\frac{1}{2}}(\mathbf{U}_1), \mathbf{U})}^2 ds < \infty\right) = 1,$$

where $\mathcal{L}_2(\mathbf{Q}^{\frac{1}{2}}(\mathbf{U}_1), \mathbf{U})$ denotes the set of all Hilbert-Schmidt operators from $\mathbf{Q}^{\frac{1}{2}}(\mathbf{U}_1)$ to \mathbf{U} . Hence, we can define the stochastic integral

$$\int_0^t \phi(s) d\mathbf{W}(s) = \int_0^t \phi(s) \circ \mathbf{J}^{-1} d\mathbf{W}(s) = \sum_{k \in \mathbb{N}} \int_0^t \phi(s) \mathbf{e}_k d\beta_k(s),$$

which is independent of the choice of \mathbf{U}_1 and \mathbf{J} . For more details on stochastic integration, we are referring to [24, 49, 59].

Next, we define some probabilistic spaces of time-dependent functions that we deal throughout this work. Given $1 \leq q \leq \infty$ and a separable Banach space \mathbf{B} , we consider the Bochner space $L^q(0, T; \mathbf{B})$ formed by all the \mathbf{B} -valued measurable functions \mathbf{u} on $[0, T]$ such that

$$\|\mathbf{u}\|_{L^q(0, T; \mathbf{B})} = \left(\int_0^T \|\mathbf{u}(t)\|_{\mathbf{B}}^q dt\right)^{\frac{1}{q}} < \infty.$$

In the particular case of $\mathbf{B} = L^q(\mathcal{O})^d$, we shall denote $L^q(0, T; \mathbf{B})$ in short by $L^q(\mathcal{O}_T)$. Let $L_w^q(0, T; \mathbf{B})$ denote a space $L^q(0, T; \mathbf{B})$ with the weak topology w .

The space $C([0, T]; \mathbf{B})$ represents the Banach space of all continuous functions from $[0, T]$ to \mathbf{B} with the norm

$$\|\mathbf{u}\|_{C([0, T]; \mathbf{B})} = \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbf{B}}.$$

Also, $C_w([0, T]; \mathbf{B})$ denotes the subspace of $L^\infty(0, T; \mathbf{B})$ formed by weakly continuous functions from $[0, T]$ into \mathbf{B} .

For $1 \leq p < \infty$, $1 \leq q \leq \infty$ and a separable Banach space \mathbf{B} , we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; L^q(0, T; \mathbf{B}))$, the space of all functions $\mathbf{u} = \mathbf{u}(x, t, \omega)$ defined on $\mathcal{O} \times [0, T] \times \Omega$ and with values in \mathbf{B} such that \mathbf{u} is measurable with respect to (t, ω) and \mathbf{u} is \mathcal{F}_t -measurable for a.e. t , and

$$\|\mathbf{u}\|_{L^p(\Omega, \mathcal{F}, \mathbb{P}; L^q(0, T; \mathbf{B}))} = \begin{cases} \left\{ \mathbb{E} \left[\left(\int_0^T \|\mathbf{u}(t)\|_{\mathbf{B}}^q dt \right)^{\frac{p}{q}} \right] \right\}^{\frac{1}{p}} < \infty, & 1 \leq q < \infty, \\ \left\{ \mathbb{E} \left[\operatorname{ess\,sup}_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbf{B}}^p \right] \right\}^{\frac{1}{p}} < \infty, & q = \infty. \end{cases}$$

By $L^p(\Omega, \mathcal{F}, \mathbb{P}; C([0, T]; \mathbf{B}))$, with $1 \leq p < \infty$, we denote the space of all continuous and progressively $\{\mathcal{F}_t\}_{t \in [0, T]}$ -measurable \mathbf{B} -valued stochastic processes \mathbf{u} such that

$$\|\mathbf{u}\|_{L^p(\Omega, \mathcal{F}, \mathbb{P}; C([0, T]; \mathbf{B}))} = \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbf{B}}^p \right] \right\}^{\frac{1}{p}} < \infty.$$

We recall an important result on stochastic ordinary differential equations in finite-dimensions, where we denote by $\mathcal{M}^{d \times d}(\mathbb{R})$, the vector space of all real $d \times d$ -matrices.

Lemma 2.1 *Let the map $\mathbf{M} : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathcal{M}^{d \times d}(\mathbb{R})$, defined by $(t, x, \omega) \mapsto \mathbf{M} = \mathbf{M}(t, x, \omega)$, and $\mathbf{b} : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$, defined by $(t, x, \omega) \mapsto \mathbf{b} = \mathbf{b}(t, x, \omega)$ be both continuous in $x \in \mathbb{R}^d$ for $t \in [0, T]$ and $\omega \in \Omega$, and be progressively measurable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. Assume the following conditions are verified for all $R \in (0, \infty)$:*

- (1) $\int_0^T \sup_{|x| \leq R} (|\mathbf{b}(t, x)| + |\mathbf{M}(t, x)|^2) dt < \infty$;
- (2) $2(x - y) \cdot (\mathbf{b}(t, x) - \mathbf{b}(t, y)) + |\mathbf{M}(t, x) - \mathbf{M}(t, y)|^2 \leq K_t(R), \forall t \in [0, T], \forall x, y \in \mathbb{R}^d : |x|, |y| \leq R$;
- (3) $2x \cdot \mathbf{b}(t, x) + |\mathbf{M}(t, x)|^2 \leq K_t(1)(1 + |x|^2), \forall t \in [0, T], \forall x \in \mathbb{R}^d : |x| \leq R$;

where $K_t(R)$ is an \mathbb{R}^+ -valued process adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ and such that

$$\alpha_S(R) := \int_0^S K_t(R) dt < \infty, \forall S \in [0, T].$$

Then for any \mathcal{F}_0 -measurable map $X_0 : \Omega \rightarrow \mathbb{R}^d$, there exists a unique solution to the stochastic differential equation

$$dX(t) = \mathbf{b}(t, X(t))dt + \mathbf{M}(t, X(t))dW(t).$$

Proof See [59, Theorem 3.1.1]. □

We recall also the following inequalities which are classical in the theory of p -Laplace equations.

Lemma 2.2 *For all $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{d \times d}$, the following assertions hold true:*

$$2 \leq p < \infty \Rightarrow \frac{1}{2^{p-1}} |\mathbf{M} - \mathbf{N}|^p \leq (|\mathbf{M}|^{p-2} \mathbf{M} - |\mathbf{N}|^{p-2} \mathbf{N}) : (\mathbf{M} - \mathbf{N}); \tag{2.2}$$

$$1 < p < 2 \Rightarrow (p - 1) |\mathbf{M} - \mathbf{N}|^2 \leq (|\mathbf{M}|^{p-2} \mathbf{M} - |\mathbf{N}|^{p-2} \mathbf{N}) : (\mathbf{M} - \mathbf{N}) (|\mathbf{M}|^p + |\mathbf{N}|^p)^{\frac{2-p}{p}}. \tag{2.3}$$

Proof The proof combines [32, Lemmas 5.1 and 5.2].

From (2.2) and (2.3), one easily gets

$$(|\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) - |\mathbf{D}(\mathbf{v})|^{p-2} \mathbf{D}(\mathbf{v})) : (\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})) \geq 0, \forall \mathbf{u}, \mathbf{v} \in W_0^{1,p}(\mathcal{O})^d, \tag{2.4}$$

and whenever $1 < p < \infty$. As a consequence of (2.4), the operator

$$\mathbf{A}(\mathbf{u}) := |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) \tag{2.5}$$

is said to be *monotone* for any p such that $1 < p < \infty$.

2.3 Probabilistic Weak Formulation

We are interested in martingale solutions, that is, weak solutions in the probabilistic sense, to the stochastic problem (1.1)-(1.4). This means that when seeking martingale solutions of (1.1)-(1.4), constructing a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ and a cylindrical Wiener process W on it are both part of the problem. We shall prove that a martingale solution typically exists for given Borel measures Λ_0 and Λ_f that account for initial and forcing laws as follows,

$$\Lambda_0 = \mathbb{P} \circ \mathbf{u}_0^{-1}, \quad \text{that is, } \mathbb{P}(\mathbf{u}_0 \in U) = \Lambda_0(U), \quad \forall U \in \mathcal{B}(\mathbf{V}), \quad (2.6)$$

$$\Lambda_f = \mathbb{P} \circ \mathbf{f}^{-1}, \quad \text{that is, } \mathbb{P}(\mathbf{f} \in U) = \Lambda_f(U), \quad \forall U \in \mathcal{B}(\mathbf{L}^2(\mathcal{O}_T)). \quad (2.7)$$

It should be noted that even if the initial datum \mathbf{u}_0 and the forcing term \mathbf{f} are given, they might live on different probability spaces, and therefore \mathbf{u}_0 and $\mathbf{u}(0)$ from one hand, and $\mathbf{k}(t)$ and $\mathbf{f}(t)$ on the other, can only coincide in law. In particular, the underlying probability space is not a priori known but becomes part of the solution. Next, we define martingale solutions for the stochastic problem (1.1)-(1.4) starting with an initial law defined on \mathbf{V} and a forcing law defined on $\mathbf{L}^2(\mathcal{O}_T)$.

Definition 2.3 Let Λ_0, Λ_f be Borel probability measures on \mathbf{V} and $\mathbf{L}^2(\mathcal{O}_T)$, respectively. We say that

$$((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}), \mathbf{u}, \mathbf{u}_0, \mathbf{f}, W)$$

is a *martingale solution* to the stochastic problem (1.1)-(1.4), with initial datum Λ_0 and forcing term Λ_f , if:

- (1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$;
- (2) W is a cylindrical $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted Wiener process;
- (3) \mathbf{u} is a progressively $\{\mathcal{F}_t\}_{t \in [0, T]}$ -measurable stochastic process with paths $t \mapsto \mathbf{u}(t, \omega) \in \mathbf{L}^\infty(0, T; \mathbf{V}) \cap \mathbf{L}^p(0, T; \mathbf{W}_0^{1,p}(\mathcal{O})^d)$, \mathbb{P} -a.s. with a continuous modification having \mathbb{P} -a.s. paths in $C([0, T]; \mathbf{V})$;
- (4) $\mathbf{u}(0)$ ($:= \mathbf{u}_0$) is progressively $\{\mathcal{F}_t\}_{t \in [0, T]}$ -measurable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with \mathbb{P} -a.s. paths $\mathbf{u}(0, \omega) \in \mathbf{V}$ and $\Lambda_0 = \mathbb{P} \circ \mathbf{u}_0^{-1}$ in the sense of (2.6);
- (5) \mathbf{f} is an $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted stochastic process \mathbb{P} -a.s. paths $\mathbf{f}(t, \omega) \in \mathbf{L}^2(\mathcal{O}_T)$ and $\Lambda_f = \mathbb{P} \circ \mathbf{f}^{-1}$ in the sense of (2.7);
- (6) for every $\varphi \in C_0^\infty(\mathcal{O})^d$ with $\text{div } \varphi = 0$ and all $t \in [0, T]$, the following identity holds \mathbb{P} -a.s.

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}(t) \cdot \varphi \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}(t) : \nabla \varphi \, dx - \int_0^t \int_{\mathcal{O}} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \, dx ds \\ & + \nu \int_0^t \int_{\mathcal{O}} |\mathbf{D}(\mathbf{u})|^{p-2} \mathbf{D}(\mathbf{u}) : \mathbf{D} \varphi \, dx ds \\ & = \int_{\mathcal{O}} \mathbf{u}_0 \cdot \varphi \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0 : \nabla \varphi \, dx + \int_0^t \int_{\mathcal{O}} \mathbf{f} \cdot \varphi \, dx ds + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \varphi \, dx. \end{aligned} \quad (2.8)$$

We are now interested to know whether it is possible to find a probabilistically strong solution for the problem (1.1)-(1.4), for a given initial velocity \mathbf{u}_0 and forcing \mathbf{f} (which are random variables rather than probability laws) on a given probability space.

Definition 2.4 (*Probabilistically strong solution*) We are given a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, initial datum \mathbf{u}_0 and a forcing term \mathbf{f} . Then, the problem (1.1)-(1.4) has a pathwise *strong probabilistic solution* if and only if there exists a $\mathbf{u} : [0, T] \times \Omega \rightarrow \mathbf{V}$ with \mathbb{P} -a.s., paths

$$\mathbf{u}(\cdot, \omega) \in L^\infty(0, T; \mathbf{V}) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d),$$

with a continuous modification having \mathbb{P} -a.s. paths in $C([0, T]; \mathbf{V})$, and (2.8) holds for all $\phi \in \mathbf{V}$.

Definition 2.5 (*Pathwise uniqueness*) For $i = 1, 2$, let \mathbf{u}_i be any solution on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ to the system (1.1)-(1.4) with initial datum \mathbf{u}_0 and forcing term \mathbf{f} . Then, the solutions of the system (1.1)-(1.4) are *pathwise unique* if and only if

$$\mathbb{P}\{\mathbf{u}_1(t) = \mathbf{u}_2(t), \forall t \geq 0\} = 1.$$

2.4 Assumptions and Main Result

The existence of martingale solutions to the stochastic problem (1.1)-(1.4) shall be carried out in the forthcoming pages under suitable assumptions on the initial datum \mathbf{u}_0 , forcing term \mathbf{f} and on the noise coefficient $\Phi(\mathbf{u})$, and for a suitable range of the summability Lebesgue exponent p . On the initial and forcing laws, we assume that for some constant $\gamma = \gamma(p, d)$ (to be determined further on)

$$\int_{\mathbf{V}} \|\mathbf{z}\|_{\mathbf{V}}^\gamma d\Lambda_0(\mathbf{z}) < \infty, \tag{2.9}$$

$$\int_{L^2(\mathcal{O}_T)} \|\mathbf{g}\|_{L^2(\mathcal{O}_T)}^\gamma d\Lambda_{\mathbf{f}}(\mathbf{g}) < \infty. \tag{2.10}$$

Let \mathbf{U} be a Hilbert space with orthonormal basis $\{\mathbf{e}_k\}_{k \in \mathbb{N}}$ and let $\mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d)$ be the space of Hilbert-Schmidt operators from \mathbf{U} to $L^2(\mathcal{O})^d$ (to simplify writing, in the sequel we denote the operator norm $\|\cdot\|_{\mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d)}$ solely by $\|\cdot\|_{\mathcal{L}_2}$). Moreover, we define an auxiliary space $\mathbf{U}_0 \supset \mathbf{U}$ as

$$\mathbf{U}_0 := \left\{ \mathbf{v} = \sum_{k \in \mathbb{N}} \alpha_k \mathbf{e}_k : \sum_{k \in \mathbb{N}} \frac{\alpha_k^2}{k^2} < \infty \right\}$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{U}_0}^2 := \sum_{k \in \mathbb{N}} \frac{\alpha_k^2}{k^2}, \quad \mathbf{v} = \sum_{k \in \mathbb{N}} \alpha_k \mathbf{e}_k.$$

Throughout the sequel, we consider a cylindrical $\{\mathcal{F}_t\}$ -Wiener process $\mathbf{W} = \{\mathbf{W}(t)\}_{t \geq 0}$ which has the form

$$\mathbf{W}(t) = \sum_{k \in \mathbb{N}} \mathbf{e}_k \beta_k(t),$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ are independent real-valued Brownian motions. The embedding $\mathbf{U} \hookrightarrow \mathbf{U}_0$ is Hilbert-Schmidt and trajectories of \mathbf{W} are \mathbb{P} -a.s. continuous with values in \mathbf{U}_0 (see Subsection 2.2). In this article, we suppose that the noise coefficient $\Phi(\mathbf{u})$ satisfies linear growth and Lipschitz conditions. More precisely, we assume that for each $\mathbf{w} \in L^2(\mathcal{O})^d$ there is a mapping

$$\Phi(\mathbf{w}) : \mathbf{U} \longrightarrow L^2(\mathcal{O})^d$$

$$e_k \mapsto \Phi(\mathbf{w})e_k = \phi_k(\mathbf{w}),$$

where $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of \mathbf{U} , such that $\phi_k \in C(\mathbb{R}^d)$ and the following conditions hold for some constants $K, L > 0$:

$$\sum_{k \in \mathbb{N}} |\phi_k(\xi)| \leq K(1 + |\xi|), \text{ and } \sum_{k \in \mathbb{N}} |\phi_k(\xi) - \phi_k(\zeta)| \leq L|\xi - \zeta|, \quad \xi, \zeta \in \mathbb{R}^d. \quad (2.11)$$

Moreover, we are assuming that the following condition holds for some constant $C > 0$,

$$\sup_{k \in \mathbb{N}} k^2 |\phi_k(\xi)|^2 \leq C(1 + |\xi|^2), \quad \xi \in \mathbb{R}^d. \quad (2.12)$$

Theorem 2.6 *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\mathcal{O}$ of class C^2 , and assume that conditions (2.9) and (2.10) hold for*

$$\gamma \geq \max \left\{ \frac{pd}{d-2}, 2 + \frac{2p}{d-2}, \frac{2d}{d-2} \right\} \text{ (} d \neq 2 \text{) and } \gamma \geq 2 \text{ (} d = 2 \text{)}, \quad (2.13)$$

and (2.11), (2.12) are fulfilled. If $2 \leq d \leq 4$ and

$$p > \frac{2d}{d+2}, \quad (2.14)$$

then there exists, at least, a martingale solution

$$((\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}, \bar{\mathbb{P}}), \bar{\mathbf{u}}, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}, \bar{\mathbf{W}})$$

in the sense of Definition 2.3 to the stochastic problem (1.1)-(1.4).

Theorem 2.7 *Under the assumptions of Theorem 2.6, there exists a unique probabilistically strong solution of the system (1.1)-(1.4) in the sense of Definition 2.4.*

The proof of the above theorems are presented in the subsequent sections.

3 Pressure Decomposition

In this section, our focus will be on the pressure term coming in our model and we decompose it in such a way that each part of the pressure term corresponds to one term in the equation. The following theorem generalizes the idea of [65, Theorem 2.6] to the stochastic case and a similar result has been established in [14, Section 3] for the stochastic case.

Theorem 3.1 *Let us consider a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\mathbf{u} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathbf{V}))$, $\mathcal{H} \in L^r(\mathcal{O}_T)$ for some $r > 1$, both adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Moreover, if the initial data $\mathbf{u}_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{V})$ and $\Phi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))$) progressively measurable such that*

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}(t) \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}(t) : \nabla \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla \boldsymbol{\phi} \, dx \, ds \\ & = \int_{\mathcal{O}} \mathbf{u}_0 \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0 : \nabla \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) d\mathbf{W}(s) \cdot \boldsymbol{\phi} \, dx, \end{aligned} \quad (3.1)$$

for all $\boldsymbol{\phi} \in \mathcal{V}$ and all $t \in [0, T]$. Then there are functions $\pi_{\mathcal{H}}, \pi_{\Phi}$ and π_h adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ such that

(1) We have $\Delta\pi_h = 0$ and there holds for $m := \min\{2, r\}$

$$\begin{aligned} \mathbb{E} \left[\int_0^T \|\pi_{\mathcal{H}}(t)\|_r^r dt \right] &\leq C \mathbb{E} \left[\int_0^T \|\mathcal{H}(t)\|_r^r dt \right], \\ \mathbb{E} \left[\sup_{t \in [0, T]} \|\pi_{\Phi}(t)\|_2^2 \right] &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} \|\Phi(\mathbf{u}(t))\|_{\mathcal{L}_2}^2 \right], \\ \mathbb{E} \left[\sup_{t \in [0, T]} \|\pi_h(t)\|_m^m \right] &\leq C \mathbb{E} \left[1 + \sup_{t \in [0, T]} \left\{ \|\mathbf{u}(t)\|_2^2 + \kappa \|\nabla \mathbf{u}(t)\|_2^2 \right\} \right. \\ &\quad \left. + \|\mathbf{u}_0\|_2^2 + \kappa \|\nabla \mathbf{u}_0\|_2^2 + \sup_{t \in [0, T]} \|\Phi(\mathbf{u}(t))\|_{\mathcal{L}_2}^2 + \int_0^T \|\mathcal{H}(t)\|_r^r dt \right]. \end{aligned}$$

(2) There holds

$$\begin{aligned} &\int_{\mathcal{O}} (\mathbf{u}(t) + \nabla \pi_h(t)) \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}(t) : \nabla \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla \boldsymbol{\phi} \, dx ds \\ &\quad - \int_0^t \int_{\mathcal{O}} \pi_{\mathcal{H}} \operatorname{div} \boldsymbol{\phi} \, dx ds \\ &= \int_{\mathcal{O}} \mathbf{u}_0 \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0 : \nabla \boldsymbol{\phi} \, dx + \int_{\mathcal{O}} \pi_{\Phi}(t) \operatorname{div} \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \boldsymbol{\phi} \, dx, \end{aligned}$$

for all $\boldsymbol{\phi} \in C^\infty(\mathcal{O})^d$. Moreover $\pi_h(0) = \pi_{\mathcal{H}}(0) = \pi_{\Phi}(0) = 0$, \mathbb{P} -a.s.

Remark 3.2 If we place all the decomposed terms of pressure together, we have

$$\pi(t) = \pi_h(t) + \pi_{\Phi}(t) + \int_0^t \pi_{\mathcal{H}}(s) ds,$$

then there holds $\pi \in L^m(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; L^m(\mathcal{O})))$.

Before proceeding with the proof of Theorem 3.1, let us recall some important auxiliary results. The following lemma is a variant of the well-known de Rham’s theorem.

Lemma 3.3 Assume $1 < m < \infty$.

(1) For each $\mathbf{v}^* \in W^{-1,m}(\mathcal{O})^d$ such that

$$\langle \mathbf{v}^*, \mathbf{v} \rangle = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{m'},$$

there exists a unique $\pi \in L^m(\mathcal{O})$, with $\int_{\mathcal{O}} \pi \, dx = 0$, such that

$$\langle \mathbf{v}^*, \mathbf{v} \rangle = \int_{\mathcal{O}} \pi \operatorname{div} \mathbf{v} \, dx, \quad \forall \mathbf{v} \in W_0^{1,m'}(\mathcal{O})^d.$$

Moreover, there exists a positive constant C such that

$$\|\pi\|_m \leq C \|\mathbf{v}^*\|_{-1,m}.$$

(2) For each $\xi \in L^m(\Omega)$ with $\int_{\Omega} \xi \, dx = 0$, there exists, at least, one solution $\mathbf{w} \in W_0^{1,m}(\mathcal{O})^d$ to the problem

$$\operatorname{div} \mathbf{w} = \xi \quad \text{in } \mathcal{O},$$

and such that for some positive constant C ,

$$\|\nabla \mathbf{w}\|_m \leq C \|\xi\|_m.$$

Proof The proof is due to Bogovskiĭ [18] (see also [31, Theorems III.3.1 and III.5.3] and [63, Proposition I.1.1]).

Proof Step 1: Define a bilinear form \mathcal{S} with the domain $\mathcal{D}(\mathcal{S}) \subseteq L^2(\mathcal{O})^d$ by setting $\mathcal{D}(\mathcal{S}) = W_0^{1,2}(\mathcal{O})^d$ and

$$\mathcal{S}(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx.$$

Since $W_0^{1,2}(\mathcal{O})^d$ is complete with respect to the norm

$$\mathcal{S}(\mathbf{u}, \mathbf{u})^{\frac{1}{2}} = (\|\mathbf{u}\|_2^2 + \kappa \|\nabla \mathbf{u}\|_2^2)^{\frac{1}{2}},$$

the form \mathcal{S} is closed. As $\kappa > 0$, it is easy to see that \mathcal{S} is positive and symmetric. Therefore by [61, Lemma II.3.2.1], there exists a uniquely determined positive self-adjoint operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow L^2(\mathcal{O})^d$, with dense domain $\mathcal{D}(\mathcal{A}) \subseteq W_0^{1,2}(\mathcal{O})^d$, such that

$$\int_{\mathcal{O}} \mathcal{A}(\mathbf{u}) \cdot \mathbf{v} \, dx = \mathcal{S}(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx.$$

Setting $\mathbf{v} \in C_0^\infty(\mathcal{O})^d$, we can see that $\mathcal{A}(\mathbf{u}) = (\mathbf{I} - \kappa \Delta)(\mathbf{u}) = (\mathbf{I} - \kappa \operatorname{div} \nabla)(\mathbf{u})$ holds in the distribution sense, and so we can set $\mathcal{A} = \mathbf{I} - \kappa \Delta$, where \mathbf{I} denotes the identity operator. In this way, the operator $\mathbf{I} - \kappa \Delta : \mathcal{D}(\mathbf{I} - \kappa \Delta) \rightarrow L^2(\mathcal{O})^d$ is defined by

$$\int_{\mathcal{O}} (\mathbf{I} - \kappa \Delta)(\mathbf{u}) \cdot \mathbf{v} \, dx = \mathcal{S}(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx, \tag{3.2}$$

for $\mathbf{u} \in \mathcal{D}(\mathbf{I} - \kappa \Delta)$, $\mathbf{v} \in W_0^{1,2}(\mathcal{O})^d$, and

$$\mathcal{D}(\mathbf{I} - \kappa \Delta) = \left\{ \mathbf{u} \in W_0^{1,2}(\mathcal{O})^d : \mathbf{v} \mapsto \mathcal{S}(\mathbf{u}, \mathbf{v}) \text{ is continuous in the } L^2\text{-norm} \right\}.$$

Given the above, and observing that $m \leq \min\{r, 2\}$, we can write (3.1) in the following form:

$$\begin{aligned} & \int_{\mathcal{O}} (\mathbf{I} - \kappa \Delta)(\mathbf{u}(t)) \cdot \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla \boldsymbol{\phi} \, dx \, ds \\ &= \int_{\mathcal{O}} (\mathbf{I} - \kappa \Delta)(\mathbf{u}_0) \cdot \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \boldsymbol{\phi} \, dx, \end{aligned} \tag{3.3}$$

for all $\boldsymbol{\phi} \in W_0^{1,m'}(\mathcal{O})^d$ with $\operatorname{div} \boldsymbol{\phi} = 0$ in \mathcal{O} , and all $t \in [0, T]$.

Step 2: Using (3.3) and Lemma 3.3-(1), we can argue as in the proof of [65, Theorem 2.6] to prove the existence of a unique pressure $\pi \in C_w([0, T]; L^m(\mathcal{O}))$, with

$$\int_{\mathcal{O}} \pi(t) \, dx = 0, \quad \forall t \in [0, T], \tag{3.4}$$

such that

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}(t) \cdot (\mathbf{I} - \kappa \Delta)(\boldsymbol{\phi}) \, dx + \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla \boldsymbol{\phi} \, dx \, ds \\ &= \int_{\mathcal{O}} \mathbf{u}_0 \cdot (\mathbf{I} - \kappa \Delta)(\boldsymbol{\phi}) \, dx + \int_{\mathcal{O}} \pi(t) \operatorname{div} \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \boldsymbol{\phi} \, dx, \end{aligned} \tag{3.5}$$

for all $\boldsymbol{\phi} \in W_0^{1,m'}(\mathcal{O})^d$ (note that $m = \min\{2, r\}$), and all $t \in [0, T]$.

Now, we will prove that

$$\pi \in L^m(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; L^m(\mathcal{O}))). \tag{3.6}$$

From (3.5), we can say that π is a progressively measurable process, since all other terms in this equation are. By Lemma 3.3-(2), with m' in the place of m , given $\psi \in L^{m'}(\mathcal{O})$, there exists $\phi \in W_0^{1,m'}(\mathcal{O})^d$ such that

$$\operatorname{div} \phi = \psi - \psi_{\mathcal{O}}, \quad \psi_{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \psi \, dx. \tag{3.7}$$

Combining (3.4) and (3.7) with (3.5), one has

$$\begin{aligned} \int_{\mathcal{O}} \pi(t) \psi \, dx &= \int_{\mathcal{O}} \pi(t) (\psi - \psi_{\mathcal{O}}) \, dx \\ &= \int_{\mathcal{O}} (\mathbf{u}(t) - \mathbf{u}_0) \cdot (\mathbf{I} - \kappa \Delta) \mathcal{B}(\psi) \, dx + \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla \mathcal{B}(\psi) \, dx \, ds \\ &\quad - \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \mathcal{B}(\psi) \, dx, \end{aligned}$$

for all $\psi \in L^{m'}(\mathcal{O})$, and where $\mathcal{B} : L^{m'}(\mathcal{O}) \rightarrow W_0^{1,m'}(\mathcal{O})^d$ is the Bogovskiĭ operator (see [58, Appendix 4.2]). Denoting by $\mathcal{B}^* : W_0^{1,m'}(\mathcal{O})^d \rightarrow L^{m'}(\mathcal{O})$, the Bogovskiĭ adjoint operator, this implies

$$\pi(t) = ((\mathbf{I} - \kappa \Delta) \mathcal{B})^* (\mathbf{u}(t) - \mathbf{u}_0) + \int_0^t (\nabla \mathcal{B})^* \mathcal{H} \, ds - \int_0^t \mathcal{B}^* \Phi(\mathbf{u}) dW(s), \tag{3.8}$$

with respect to the $L^2(\mathcal{O})$ -inner product. From (3.8), one immediately has $\pi(0) = 0$. Averaging (3.8) in the $L^m(\mathcal{O})$ -norm, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \|\pi(t)\|_m^m \right] &= \mathbb{E} \left[\sup_{t \in [0, T]} \left\| ((\mathbf{I} - \kappa \Delta) \mathcal{B})^* (\mathbf{u}(t) - \mathbf{u}_0) - \int_0^t (\nabla \mathcal{B})^* \mathcal{H}(\mathbf{u}(s)) \, ds \right. \right. \\ &\quad \left. \left. + \int_0^t \mathcal{B}^* \Phi(\mathbf{u}(s)) dW(s) \right\|_m^m \right] \\ &\leq C \left(1 + \mathbb{E} \left[\sup_{t \in [0, T]} \|((\mathbf{I} - \kappa \Delta) \mathcal{B})^* (\mathbf{u}(t) - \mathbf{u}_0)\|_2^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \|(\nabla \mathcal{B})^* \mathcal{H}(\mathbf{u}(t))\|_m^m \, dt \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} \left\| \int_0^t \mathcal{B}^* \Phi(\mathbf{u}(s)) dW(s) \right\|_2^2 \right] \right) =: C \left(1 + \sum_{i=1}^3 I_i \right), \tag{3.9} \end{aligned}$$

where we have used the Minkowski, Hölder and Young inequalities, together with the fact that $m \leq 2$. Using the continuity of the operator \mathcal{B}^* from $L^2(\mathcal{O})^d$ to $L^2(\mathcal{O})$, observing first that $(\mathbf{I} - \kappa \Delta)$ is self-adjoint and therefore $((\mathbf{I} - \kappa \Delta) \mathcal{B})^* = \mathcal{B}^* (\mathbf{I} - \kappa \Delta)$, and still using (3.2) and Minkowski's inequality, we have

$$\begin{aligned} I_1 &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} \|(\mathbf{I} - \kappa \Delta) (\mathbf{u}(t) - \mathbf{u}_0)\|_2^2 \right] \\ &= C \mathbb{E} \left[\sup_{t \in [0, T]} \|\mathbf{u}(t) - \mathbf{u}_0\|_2^2 + \kappa \|\nabla (\mathbf{u}(t) - \mathbf{u}_0)\|_2^2 \right] \end{aligned}$$

$$\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\left\{\|\mathbf{u}(t)\|_2^2 + \kappa \sup_{t\in[0,T]}\|\nabla\mathbf{u}(t)\|_2^2\right\} + \|\mathbf{u}_0\|_2^2 + \kappa\|\nabla\mathbf{u}_0\|_2^2\right].$$

Since $\mathcal{H} \in L^r(\mathcal{O}_T)$, $m = \min\{2, r\}$ and still using the fact that $(\nabla\mathcal{B})^*$ is a continuous operator from $L^r(\mathcal{O})^{d\times d}$ to $L^r(\mathcal{O})$, a similar calculation yields

$$I_2 \leq C\mathbb{E}\left[1 + \int_0^T \|\mathcal{H}(t)\|_r^r dt\right].$$

The stochastic integral term I_3 appearing in (3.9) can be handled with the help of Burkholder-Davis-Gundy (BDG) inequality, using also here the continuity of \mathcal{B}^* from $L^2(\mathcal{O})^d$ to $L^2(\mathcal{O})$,

$$I_3 \leq C\mathbb{E}\left[\int_0^T \|\Phi(\mathbf{u}(t))\|_{\mathcal{L}_2}^2 dt\right].$$

Combining the above estimates in (3.9), we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]}\|\pi(t)\|_m^m\right] &\leq \mathbb{E}\left[1 + \sup_{t\in[0,T]}\left\{\|\mathbf{u}(t)\|_2^2 + \kappa\|\nabla\mathbf{u}(t)\|_2^2\right\} + \|\mathbf{u}_0\|_2^2 + \kappa\|\nabla\mathbf{u}_0\|_2^2\right. \\ &\quad \left. + \int_0^T \|\mathcal{H}(t)\|_r^r dt + \int_0^T \|\Phi(\mathbf{u}(t))\|_{\mathcal{L}_2}^2 dt\right]. \end{aligned} \tag{3.10}$$

On the other hand, by [65, Corollary 2.5], there exist unique functions

$$\begin{aligned} \pi_0 &\in C_w([0, T]; \Delta W_0^{2,m}(\mathcal{O})), & \Delta W_0^{2,m}(\mathcal{O}) &:= \left\{\Delta v : v \in W_0^{2,m}(\mathcal{O})\right\}, \\ \pi_h &\in C_w([0, T]; L_\Delta^m(\mathcal{O})), & L_\Delta^m(\mathcal{O}) &:= \left\{v \in L^m(\mathcal{O}) : \Delta v = 0, \int_{\mathcal{O}} v \, dx = 0\right\}, \end{aligned}$$

such that

$$\begin{aligned} \pi &:= \pi_0 + \pi_h \quad \text{in } \mathcal{O}_T, \\ \sup_{t\in[0,T]}\|\pi_0(t)\|_m + \sup_{t\in[0,T]}\|\pi_h(t)\|_m &\leq C \sup_{t\in[0,T]}\|\pi(t)\|_m, \end{aligned} \tag{3.11}$$

for some positive constant C . Therefore, we can decompose the pressure term pointwise on \mathcal{O}_T as follows:

$$\pi := \pi_0 + \pi_h, \quad \pi_0 := \Delta\Delta^{-2}\Delta\pi, \quad \pi_h := \pi - \pi_0,$$

where Δ^{-2} denotes the solution operator to the bi-Laplace equation with respect to zero boundary values for the function and gradient. As a consequence of (3.11), inequality (3.10) holds true for π_0 and π_h as well. Into (3.5), replacing π by $\pi_0 + \pi_h$, inserting $\phi = \nabla\psi$, $\psi \in W_0^{2,m'}(\mathcal{O})$, and observing that $\operatorname{div} \mathbf{u} = 0$, $\operatorname{div} \mathbf{u}_0 = 0$ and $\Delta\pi_h = 0$, one has

$$\int_{\mathcal{O}} \pi_0(t)\Delta\psi \, dx = \int_0^t \int_{\mathcal{O}} \mathcal{H} : \nabla^2\psi \, dx ds - \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u})dW(s) \cdot \nabla\psi \, dx, \tag{3.12}$$

for all $\psi \in W_0^{2,m'}(\mathcal{O})$. From (3.12), it is clear that $\pi_0(0) = 0$. And once that $\pi(0) = 0$, we also have $\pi_h(0) = 0$. As $\operatorname{div}^2 \mathcal{H} = \operatorname{div}(\operatorname{div} \mathcal{H}) \in W^{-2,m}(\mathcal{O})$, by [65, Lemma 2.4], there exists a function $\pi_{\mathcal{H}}(t) \in \Delta W_0^{2,m}(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \pi_{\mathcal{H}}(t)\Delta\psi \, dx = \int_{\mathcal{O}} \mathcal{H}(t) : \nabla^2\psi \, dx,$$

for all $\psi \in C_0^\infty(\mathcal{O})$. The measurability of $\pi_{\mathcal{H}}$ follows from the measurability of the right hand side. On account of the solvability of the bi-Laplace equation, one has

$$\|\pi_{\mathcal{H}}(t)\|_m^m \leq C \|\mathcal{H}(t)\|_m^m \quad \mathbb{P} \otimes \lambda - \text{a.e.},$$

for some positive constant C , where λ is the Lebesgue measure on $(0, T)$. For more details, see [53], or [65, Lemma 2.1] and [15, Lemma 2.2]. This in turn gives

$$\int_{\Omega \times \mathcal{O}_T} \|\pi_{\mathcal{H}}(t)\|_m^m dt d\mathbb{P} \leq C \int_{\Omega \times \mathcal{O}_T} \|\mathcal{H}(t)\|_m^m dt d\mathbb{P}.$$

Next, we define

$$\pi_\Phi(t) := \pi_0(t) + \int_0^t \pi_{\mathcal{H}}(s) ds,$$

as the unique solution to the following integral equation

$$\int_{\mathcal{O}} \pi_\Phi(t) \Delta \psi \, dx = \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}(s)) dW(s) \cdot \nabla \psi \, dx, \quad \psi \in C_0^\infty(\mathcal{O}). \tag{3.13}$$

Since $\pi_\Phi(t) \in \Delta W_0^{2,m}(\mathcal{O})$, (3.13) can be written as follows

$$\int_{\mathcal{O}} \pi_\Phi(t) \psi \, dx = \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \nabla (\Delta_{\mathcal{O}}^{-2} \Delta \psi) \, dx, \quad \psi \in C_0^\infty(\mathcal{O}). \tag{3.14}$$

From here, one immediately has $\pi_\Phi(0) = 0$. Defining the operator $\mathcal{D} := \nabla \Delta_{\mathcal{O}}^{-2} \Delta : W_0^{2,m}(\mathcal{O}) \rightarrow W_0^{1,m}(\mathcal{O})^d$, we can write (3.13) as

$$\int_{\mathcal{O}} \pi_\Phi(t) \psi \, dx = \int_0^t \int_{\mathcal{O}} \mathcal{D}^* \Phi(\mathbf{u}) dW(s) \psi \, dx, \quad \psi \in C_0^\infty(\mathcal{O}),$$

and hence

$$\pi_\Phi(t) = \int_0^t \mathcal{D}^* \Phi(\mathbf{u}(s)) dW(s), \quad \mathbb{P} \times \lambda^{d+1} \text{-a.e.},$$

where λ^{d+1} is the Lebesgue measure on $\mathcal{O} \times (0, T)$. Using the BDG inequality, together with the fact that \mathcal{D}^* is a continuous operator from $L^2(\mathcal{O})^d$ to $L^2(\mathcal{O})$, we can show that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\pi_\Phi(t)\|_2^2 \right] = \mathbb{E} \left[\sup_{t \in [0, T]} \left\| \int_0^t \mathcal{D}^* \Phi(\mathbf{u}(s)) dW(s) \right\|_2^2 \right] \leq C \mathbb{E} \left[\int_0^T \|\Phi(\mathbf{u}(t))\|_{L^2}^2 dt \right].$$

Finally, we can see that

$$\bar{\pi}_0(t) := \pi_\Phi(t) + \int_0^t \pi_{\mathcal{H}}(s) ds$$

solves (3.12), and $\bar{\pi}_0(t) \in \Delta W_0^{2,m}(\mathcal{O})$. This implies

$$\pi_0(t) = \pi_\Phi(t) + \int_0^t \pi_{\mathcal{H}}(s) ds,$$

and hence we obtained the required result. □

For the following result, it is worth keeping in mind that we are assuming a smooth boundary $\partial\mathcal{O}$, in fact of class C^2 . See [22, Section 2.2] and [23, Section 2.2] in the case of non-smooth boundary $\partial\mathcal{O}$.

Corollary 3.4 *Assume the conditions of Theorem 3.1 are satisfied. Then there exists $\Phi_\pi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})))$ progressively measurable such that*

$$\int_{\mathcal{O}} \pi_{\Phi}(t) \operatorname{div} \boldsymbol{\phi} \, dx = \int_0^t \int_{\mathcal{O}} \Phi_\pi dW(s) \cdot \boldsymbol{\phi} \, dx, \quad \forall \boldsymbol{\phi} \in C_0^\infty(\mathcal{O}). \tag{3.15}$$

and $\|\Phi_\pi \mathbf{e}_j\|_2 \leq C(\mathcal{O})\|\Phi \mathbf{e}_j\|_2$, for all j , that is, we have $\mathbb{P} \otimes \lambda$, a.e.

$$\|\Phi_\pi\|_{\mathcal{L}_2} \leq C(\mathcal{O})\|\Phi\|_{\mathcal{L}_2}.$$

Furthermore, if Φ satisfies (2.11), then there holds

$$\|\Phi_\pi(\mathbf{u}_1) - \Phi_\pi(\mathbf{u}_2)\|_{\mathcal{L}_2} \leq C(L, \mathcal{O})\|\mathbf{u}_1 - \mathbf{u}_2\|_2,$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in L^2(\mathcal{O})^d$.

Proof From (3.14) of the proof of Theorem 3.1, we have for any $\boldsymbol{\phi} \in C_0^\infty(\mathcal{O})^d$

$$\begin{aligned} \int_{\mathcal{O}} \pi_{\Phi}(t) \operatorname{div} \boldsymbol{\phi} \, dx &= \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) dW(s) \cdot \nabla(\Delta_{\mathcal{O}}^{-2} \Delta \operatorname{div} \boldsymbol{\phi}) \, dx \\ &= \sum_j \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}) \mathbf{e}_j d\beta_j(s) \cdot \nabla(\Delta_{\mathcal{O}}^{-2} \Delta \operatorname{div} \boldsymbol{\phi}) \, dx \\ &= \sum_j \int_0^t \int_{\mathcal{O}} \nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi(\mathbf{u}) \mathbf{e}_j d\beta_j(s) \cdot \boldsymbol{\phi} \, dx \\ &= \int_0^t \int_{\mathcal{O}} \nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi(\mathbf{u}) dW(s) \cdot \boldsymbol{\phi} \, dx, \end{aligned}$$

which proves (3.15) by setting $\Phi_\pi = \nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi$. For the second part, we use the local regularity theory for the bi-Laplace equation in the following manner:

$$\begin{aligned} |(\Phi_\pi \mathbf{e}_k, \boldsymbol{\psi})| &= |(\nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi \mathbf{e}_k, \boldsymbol{\psi})| \leq \|\nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi \mathbf{e}_k\|_2 \|\boldsymbol{\psi}\|_2 \\ &\leq C(\mathcal{O})\|\Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi \mathbf{e}_k\|_{-1,2} \|\boldsymbol{\psi}\|_2 \leq C(\mathcal{O})\|\Delta_{\mathcal{O}}^{-2} \operatorname{div} \Phi \mathbf{e}_k\|_{3,2} \|\boldsymbol{\psi}\|_2 \\ &\leq C(\mathcal{O})\|\operatorname{div} \Phi \mathbf{e}_k\|_{-1,2} \|\boldsymbol{\psi}\|_2 \leq C(\mathcal{O})\|\Phi \mathbf{e}_k\|_2 \|\boldsymbol{\psi}\|_2 \leq C(\mathcal{O})\|\Phi \mathbf{e}_k\|_2 \|\boldsymbol{\psi}\|_2, \end{aligned}$$

for $\boldsymbol{\psi} \in L^2(\mathcal{O})^d$. Similarly, we can conclude the final part as follows:

$$\begin{aligned} \|(\Phi_\pi(\mathbf{u}_1) - \Phi_\pi(\mathbf{u}_2)) \mathbf{e}_k\|_2 &= \|\nabla \Delta \Delta_{\mathcal{O}}^{-2} \operatorname{div}(\Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_2)) \mathbf{e}_k\|_2 \\ &\leq C(\mathcal{O})\|(\Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_2)) \mathbf{e}_k\|_2 = C(\mathcal{O})\|\phi_k(\mathbf{u}_1) - \phi_k(\mathbf{u}_2)\|_2 \\ &\leq C(L, \mathcal{O})\|\mathbf{u}_1 - \mathbf{u}_2\|_2, \end{aligned}$$

where we have used (2.11). □

Corollary 3.5 *Let the conditions of Theorem 3.1 be satisfied. Then, for all $\gamma \in [1, \infty)$*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \|\pi_h(t)\|_m^m \right]^\gamma &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} \{ \|\mathbf{u}(t)\|_2^2 + \kappa \|\nabla \mathbf{u}(t)\|_2^2 \} + \sup_{t \in [0, T]} \|\Phi(t)\|_{\mathcal{L}_2}^2 \right]^\gamma \\ &\quad + C \mathbb{E} \left[1 + \|\mathbf{u}_0\|_2^2 + \kappa \|\nabla \mathbf{u}_0\|_2^2 + \int_0^T \|\mathcal{H}(t)\|_r^r dt \right]^\gamma, \end{aligned}$$

provided the right hand side of above inequality is finite.

Proof The proof of this corollary is straightforward from Theorem 3.1-(1). □

Corollary 3.6 *Let the conditions of Theorem 3.1 be satisfied, and assume that the following decomposition holds:*

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

where $\mathcal{H}_1 \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(0, T; L^1(\mathcal{O})^{d \times d}))$, $\mathcal{H}_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; L^2(\mathcal{O})^{d \times d}))$ and $\text{div } \mathcal{H}_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; L^2(\mathcal{O})^d))$. Then, we have

$$\pi \mathcal{H} = \pi_1 + \pi_2,$$

and there holds for all $\gamma \in [1, \infty)$,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \|\pi_1(t)\|_{r_1}^{r_1} dt \right]^\gamma &\leq C \mathbb{E} \left[\int_0^T \|\mathcal{H}_1(t)\|_{r_1}^{r_1} dt \right]^\gamma, \\ \mathbb{E} \left[\int_0^T \|\pi_2(t)\|_{r_2}^{r_2} dt \right]^\gamma &\leq C \mathbb{E} \left[\int_0^T \|\mathcal{H}_2(t)\|_{r_2}^{r_2} dt \right]^\gamma, \\ \mathbb{E} \left[\int_0^T \|\nabla \pi_2(t)\|_{r_2}^{r_2} dt \right]^\gamma &\leq C \mathbb{E} \left[\int_0^T \{ \|\mathcal{H}_2(t)\|_{r_2}^{r_2} + \|\text{div } \mathcal{H}_2(t)\|_{r_2}^{r_2} \} dt \right]^\gamma. \end{aligned}$$

Proof Here π_1 and π_2 are the unique solutions of the following equations defined on $\mathbb{P} \otimes \lambda$, a.e.:

$$\begin{aligned} \int_{\mathcal{O}} \pi_1(t) \Delta \psi \, dx &= - \int_{\mathcal{O}} \mathcal{H}_1(t) : \nabla^2 \psi \, dx, & \pi_1 &\in \Delta W_0^{2,r_1}(\mathcal{O}), \\ \int_{\mathcal{O}} \pi_2(t) \Delta \psi \, dx &= - \int_{\mathcal{O}} \mathcal{H}_2(t) : \nabla^2 \psi \, dx, & \pi_2 &\in \Delta W_0^{2,r_2}(\mathcal{O}). \end{aligned}$$

By [65, Lemma 2.3], and once that $\mathcal{H}_1 \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^1(0, T; L^1(\mathcal{O})^{d \times d}))$, $\mathcal{H}_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; L^2(\mathcal{O})^{d \times d}))$ and $\text{div } \mathcal{H}_2 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; L^2(\mathcal{O})^d))$, the estimates are straightforward. \square

4 Solvability of The Approximate System

In this section, we establish the solvability of an auxiliary problem that regularizes the problem (1.1)-(1.4) with the following stabilization term in the momentum equation:

$$\alpha \mathbf{a}(\mathbf{u}), \quad \mathbf{a}(\mathbf{u}) := |\mathbf{u}|^{q-2} \mathbf{u}, \quad \alpha > 0, \quad 1 < q < \infty. \tag{4.1}$$

Given $\alpha > 0$, we consider the problem

$$\begin{cases} d(\mathbb{I} - \kappa \Delta) \mathbf{u} = \{ \text{div } \mathbf{A}(\mathbf{u}) - \text{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi - \alpha \mathbf{a}(\mathbf{u}) + \mathbf{f} \} dt + \Phi(\mathbf{u}) dW(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{4.2}$$

depending on the initial Λ_0 and forcing Λ_f laws in the conditions of (2.9) and (2.10), respectively, and for the operator $\mathbf{A}(\mathbf{u})$ defined in (2.5). The exponent q in (4.1) is chosen in such a way that the convective term becomes a compact perturbation (see (4.24) below). For that purpose, we choose

$$q \geq \max\{2p', 3\}, \tag{4.3}$$

and thus a solution \mathbf{u} is expected in the following space:

$$\begin{aligned} \mathcal{V}_{p,q} &:= L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathbf{V})) \cap L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; W_0^{1,p}(\mathcal{O})^d) \\ &\quad \cap L^q(\Omega, \mathcal{F}, \mathbb{P}; L^q(0, T; L^q(\mathcal{O})^d)). \end{aligned}$$

4.1 Approximate Solutions

We construct a solution to the problem (4.2) as a limit of suitable Galerkin approximations. Let s be the smallest positive integer such that $W_0^{s,2}(\mathcal{O})^d \hookrightarrow W_0^{1,\infty}(\mathcal{O})^d$, for $s > 1 + \frac{d}{2}$, and let us consider the space V_s associated to $W_0^{s,2}(\mathcal{O})^d$ defined in Section 2. By means of separability, there exists a basis $\{\psi_k\}_{k \in \mathbb{N}}$ of V_s , formed by the eigenfunctions of a suitable spectral problem, that is, orthogonal in $L^2(\mathcal{O})^d$ and that can be made orthonormal in $W_0^{s,2}(\mathcal{O})^d$ (see [50, Theorem A.4.11]). Given $n \in \mathbb{N}$, let us consider the n -dimensional space $\mathbf{X}^n = \text{span}\{\psi_1, \dots, \psi_n\}$.

For each $n \in \mathbb{N}$, we search for approximate solutions of the form

$$u_n(x, t) = \sum_{k=1}^n c_k^n(t) \psi_k(x), \quad \psi_k \in \mathbf{X}^n, \tag{4.4}$$

where the coefficients $c_1^n(t), \dots, c_n^n(t)$ are solutions of the following n stochastic ordinary differential equations

$$\begin{aligned} & d\left[(u_n(t), \psi_k) + \kappa (\nabla u_n(t), \nabla \psi_k) \right] \\ &= \left[(u_n(t) \otimes u_n(t) : \nabla \psi_k) - \nu (|\mathbf{D}(u_n(t))|^{p-2} \mathbf{D}(u_n(t)) : \mathbf{D}(\psi_k)) + (f(t), \psi_k) \right] dt \\ &\quad - \alpha (\mathbf{a}(u_n(t)), \psi_l) + \Phi(u_n(t)) dW_n(t), \quad k = 1, \dots, n, \end{aligned} \tag{4.5}$$

supplemented with the initial conditions

$$u_n(0) = u_0^n, \quad \text{in } \mathcal{O}, \tag{4.6}$$

where $u_0^n = P^n(u_0)$, with P^n denoting the orthogonal projection $P^n : \mathbf{V} \rightarrow \mathbf{X}^n$ so that

$$u_n(0, x) = \sum_{k=1}^n c_k^n(0) \psi_k(x), \quad c_k^n(0) = c_{k,0}^n := (u_0, \psi_k), \quad k = 1, \dots, n.$$

In (4.5), we assume that the approximate cylindrical Wiener process $\{W_n(\cdot)\}$ has the form

$$W_n(t) = \sum_{k=1}^n \mathbf{e}_k \beta_k(t),$$

and note that (4.5) is to be understood \mathbb{P} -a.s., and for all $t \in (0, T]$. Observe that the stochastic system (4.5)-(4.6) can be written in the matrix form as follows,

$$\mathbf{C} d\mathbf{c}(t) = \mathbf{b}(t) dt + \mathbf{G}(t) d\boldsymbol{\beta}(t), \quad \mathbf{c}(0) = \mathbf{c}_n, \tag{4.7}$$

where $\mathbf{C} = \{a_{lm}^n\}_{l,m=1}^n$, $\mathbf{b}(t) = \{b_l^n(t)\}_{l=1}^n$, $\mathbf{c}(t) = \{c_m^n(t)\}_{m=1}^n$, $\mathbf{G}(t) = \{g_{lm}^n(t)\}_{l,m=1}^n$ and $\boldsymbol{\beta}(t) = \{\beta_m^n(t)\}_{m=1}^n$, with

$$\begin{aligned} a_{lm}^n &:= (\psi_l, \psi_m) + \kappa (\nabla \psi_l, \nabla \psi_m), \\ b_l^n(t) &:= (f(t), \psi_l) + (u_n^n(t) \otimes u_n^n(t) : \nabla \psi_l) - \nu (|\mathbf{D}(u_n(t))|^{p-2} \mathbf{D}(u_n(t)) : \mathbf{D}(\psi_l)) \\ &\quad - \alpha (\mathbf{a}(u_n(t)), \psi_l), \\ \mathbf{c}(t) &:= (c_1^n(t), \dots, c_n^n(t)) \quad \text{and} \quad \mathbf{c}_0 = (c_{1,0}^n, \dots, c_{n,0}^n), \\ g_{lm}^n(t) &:= (\Phi(u_n(t)) \mathbf{e}_l, \psi_m), \\ \boldsymbol{\beta}(t) &:= (\beta_1^n(t), \dots, \beta_n^n(t)). \end{aligned}$$

Taking into account that the family $\{\psi_k\}_{k \in \mathbb{N}}$ is linearly independent in \mathbf{V} , $(\mathbf{C}\xi, \xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, we can write (4.7) in the form

$$d\mathbf{c}(t) = \overline{\mathbf{b}(t)}dt + \overline{\mathbf{G}(t)}d\beta(t), \quad \mathbf{c}(0) = \mathbf{c}_{n,0}, \tag{4.8}$$

where $\overline{\mathbf{b}(t)} = \mathbf{C}^{-1}\mathbf{b}(t)$ and $\overline{\mathbf{G}(t)} = \mathbf{C}^{-1}\mathbf{G}(t)$.

If both the coefficients $\mathbf{b}(\cdot)$ and $\mathbf{G}(\cdot)$ are globally Lipschitz-continuous, one can use the classical existence results for stochastic differential equations. Our case does not fall in that category, so we have to use the monotonicity method and verify the conditions of the vectorial version of Lemma 2.1 as follows:

$$\begin{aligned} & \langle \mathbf{b}(\cdot, \mathbf{u}_n) - \mathbf{b}(\cdot, \mathbf{v}_n), \mathbf{u}_n - \mathbf{v}_n \rangle \\ &= \langle \mathbf{u}_n \otimes \mathbf{u}_n - \mathbf{v}_n \otimes \mathbf{v}_n : \nabla(\mathbf{u}_n - \mathbf{v}_n) \rangle \\ & \quad - \nu \langle |\mathbf{D}(\mathbf{u}_n)|^{p-2}\mathbf{D}(\mathbf{u}_n) - |\mathbf{D}(\mathbf{v}_n)|^{p-2}\mathbf{D}(\mathbf{v}_n) : \mathbf{D}(\mathbf{u}_n - \mathbf{v}_n) \rangle \\ & \quad - \alpha \langle |\mathbf{u}_n|^{q-2}\mathbf{u}_n - |\mathbf{v}_n|^{q-2}\mathbf{v}_n, \mathbf{u}_n - \mathbf{v}_n \rangle \\ & \leq \langle \mathbf{u}_n \otimes \mathbf{u}_n - \mathbf{v}_n \otimes \mathbf{v}_n : \nabla(\mathbf{u}_n - \mathbf{v}_n) \rangle, \end{aligned}$$

where we have used (2.4). If $\|\mathbf{u}_n\|_2 \leq R$ and $\|\mathbf{v}_n\|_2 \leq R$, there exists a constant $C(R, n) > 0$ such that

$$\langle \mathbf{b}(t, \mathbf{u}_n) - \mathbf{b}(t, \mathbf{v}_n), \mathbf{u}_n - \mathbf{v}_n \rangle \leq C(R, n)\|\mathbf{u}_n - \mathbf{v}_n\|_2^2.$$

This, together with the Lipschitz-continuity of \mathbf{G} (cf. (2.11)), gives us the weak monotonicity property in the sense of Lemma 2.1-(2). Using the fact $\langle \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{u}_n \rangle = 0$, we find

$$\begin{aligned} \langle \mathbf{b}(\cdot, \mathbf{u}_n), \mathbf{u}_n \rangle &= -\nu \langle |\mathbf{D}(\mathbf{u}_n)|^{p-2}\mathbf{D}(\mathbf{u}_n) : \mathbf{D}(\mathbf{u}_n) \rangle - \alpha \langle |\mathbf{u}_n|^{q-2}\mathbf{u}_n, \mathbf{u}_n \rangle + \langle \mathbf{f}, \mathbf{u}_n \rangle \\ &\leq C(1 + \|\mathbf{f}\|_2\|\mathbf{u}_n\|_2) \leq C(1 + \|\mathbf{f}\|_2)(1 + \|\mathbf{u}_n\|_2^2). \end{aligned}$$

Using the linear growth of \mathbf{G} cf. (2.11), we find

$$\langle \mathbf{b}(t, \mathbf{u}_n), \mathbf{u}_n \rangle + \|\mathbf{G}\|_{\mathcal{L}_2}^2 \leq C(1 + \|\mathbf{f}\|_2)(1 + \|\mathbf{u}_n\|_2^2).$$

Since the term $\int_0^T (1 + \|\mathbf{f}\|_2)dt$ is finite, \mathbb{P} -a.s., this yields the weak coercivity property in the sense of Lemma 2.1-(3). By Lemma 2.1, we obtain the existence of a unique strong solution $\mathbf{C}\mathbf{c} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0, T^*]; \mathbb{R}^n))$ to the stochastic system (4.7), which implies

$$\sup_{t \in [0, T^*]} \|\mathbf{C}\mathbf{c}(t)\|_{\mathbb{R}^n} < +\infty.$$

Our aim is to show that $\sup_{t \in [0, T^*]} \|\mathbf{c}(t)\|_{\mathbb{R}^n} < +\infty$, which we obtain as follows:

$$\sup_{t \in [0, T^*]} \|\mathbf{c}(t)\|_{\mathbb{R}^n} = \sup_{t \in [0, T^*]} \|\mathbf{C}^{-1}\mathbf{C}\mathbf{c}(t)\|_{\mathbb{R}^n} \leq \|\mathbf{C}^{-1}\|_{\mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)} \sup_{t \in [0, T^*]} \|\mathbf{C}\mathbf{c}(t)\|_{\mathbb{R}^n} < +\infty,$$

since \mathbf{C}^{-1} is a bounded operator. Thus, we obtain the existence of a unique strong solution to the stochastic system (4.6).

4.2 Uniform Estimates

In this subsection, we establish the uniform energy estimate for the solution of the finite-dimensional approximate system corresponding to the system (4.2).

Theorem 4.1 *Let $p \in (1, \infty)$ and assume that (2.11) and (4.3) are verified. Assume, in addition, that (2.9) and (2.10) hold with $\gamma = 2$. Then there exists a positive constant C , neither depending on n nor on α such that*

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in (0, T)} \left\{ \|\mathbf{u}_n(t)\|_2^2 + 2\kappa \|\nabla \mathbf{u}_n(t)\|_2^2 \right\} + 4C(p, \mathcal{O})\nu \int_0^T \|\nabla \mathbf{u}_n(t)\|_p^p dt + 2\alpha \int_0^T \|\mathbf{u}_n(t)\|_q^q dt \right] \\ & \leq C(\kappa) \left\{ \left(\frac{1}{\eta_1} + \kappa \right) \int_{\mathbf{V}} \|z\|_{\mathbf{V}}^2 d\Lambda_0(z) + \int_{\mathbf{L}^2(\mathcal{O}_T)} \|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2 d\Lambda_f(\mathbf{g}) + C(K)T \right\} e^{\frac{C(K)T}{\eta_1}}, \end{aligned} \tag{4.9}$$

where C is independent of α and η_1 is the first eigenvalue of the Dirichlet Laplacian.

Proof Define a sequence of stopping times $\{\tau_N^n\}_{N \in \mathbb{N}}$ as follows:

$$\tau_N^n := \inf_{t \in [0, T]} \{t : \|\nabla \mathbf{u}_n(t)\|_2 \geq N\}. \tag{4.10}$$

Applying finite-dimensional Itô’s formula to the process $\|(I - \kappa \Delta)^{\frac{1}{2}} \mathbf{u}_n(\cdot)\|_2^2$, we obtain, \mathbb{P} -a.s.,

$$\begin{aligned} & \|\mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 \\ & = \|\mathbf{u}_n(0)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(0)\|_2^2 - 2\nu \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \mathbf{A}(\mathbf{u}_n) : \mathbf{D}(\mathbf{u}_n) \, dx \, ds \\ & \quad - 2\alpha \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \mathbf{a}(\mathbf{u}_n) \cdot \mathbf{u}_n \, dx \, ds + 2 \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \Phi(\mathbf{u}_n) dW_n(s) \cdot \mathbf{u}_n \, dx \\ & \quad + \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\mathbf{u}_n) dW_n \right\rangle_s \, dx \\ & \leq \left(\frac{1}{\eta_1} + \kappa \right) \|\nabla \mathbf{u}_n(0)\|_2^2 - 2\nu \int_0^t \|\mathbf{D}(\mathbf{u}_n(s))\|_p^p \, ds - 2\alpha \int_0^{t \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_q^q \, ds \\ & \quad + 2 \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \mathbf{f} \cdot \mathbf{u}_n \, dx \, ds + 2 \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \Phi(\mathbf{u}_n) dW_n(s) \cdot \mathbf{u}_n \, dx \\ & \quad + \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\mathbf{u}_n) dW_n \right\rangle_s \, dx, \end{aligned} \tag{4.11}$$

where we have used Poincaré’s inequality. Taking expectation on both sides and then using Korn’s inequality (cf. [39] or [3, Lemma 2.1]), we find

$$\begin{aligned} & \mathbb{E} \left[\|\mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 \right. \\ & \quad \left. + 2C(p, \mathcal{O})\nu \int_0^{t \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_p^p \, ds + 2\alpha \int_0^{t \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_q^q \, ds \right] \\ & \leq \mathbb{E} \left[\left(\frac{1}{\eta_1} + \kappa \right) \|\nabla \mathbf{u}_n(0)\|_2^2 + 2 \underbrace{\int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \mathbf{f} \cdot \mathbf{u}_n \, dx \, ds}_{:= I_1(t \wedge \tau_N^n)} + \underbrace{\int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\mathbf{u}_n) dW_n \right\rangle_s \, dx}_{:= I_2(t \wedge \tau_N^n)} \right. \\ & \quad \left. + 2 \underbrace{\int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} \Phi(\mathbf{u}_n) dW_n(s) \cdot \mathbf{u}_n \, dx}_{:= I_3(t \wedge \tau_N^n)} \right] \end{aligned}$$

$$\leq \left(\frac{1}{\eta_1} + \kappa\right) \mathbb{E}[\|\nabla \mathbf{u}_n(0)\|_2^2] + 2\mathbb{E}[I_1(t \wedge \tau_N^n)] + \mathbb{E}[I_2(t \wedge \tau_N^n)] + 2\mathbb{E}[I_3(t \wedge \tau_N^n)].$$

We estimate $\mathbb{E}[|I_1(t \wedge \tau_N^n)|]$ using Young’s and Poincaré’s inequalities as follows:

$$\begin{aligned} \mathbb{E}[|I_1(t \wedge \tau_N^n)|] &\leq \varepsilon \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_2^2 ds\right] + C(\varepsilon) \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\mathbf{f}(s)\|_2^2 ds\right] \\ &\leq \frac{\varepsilon}{\eta_1} \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_2^2 ds\right] + C(\varepsilon) \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\mathbf{f}(s)\|_2^2 ds\right]. \end{aligned}$$

We know that $\mathbb{E}[I_3(t \wedge \tau_N^n)] = 0$, since the term $I_3(t \wedge \tau_N^n)$ is a martingale with zero expectation. Now, we consider the term $\mathbb{E}[I_2(t \wedge \tau_N^n)]$ and estimate it with the help of Itô’s isometry, (2.11) and Poincaré’s inequality as,

$$\begin{aligned} \mathbb{E}[I_2(t \wedge \tau_N^n)] &= \mathbb{E}\left[\sum_{k=1}^n \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} |\Phi(\mathbf{u}_n) \mathbf{e}_k|^2 dx ds\right] \leq \mathbb{E}\left[\sum_{k \in \mathbb{N}} \int_0^{t \wedge \tau_N^n} \int_{\mathcal{O}} |\phi_k(\mathbf{u}_n)|^2 dx ds\right] \\ &\leq C(K) \mathbb{E}\left[T + \frac{1}{\eta_1} \int_0^{t \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_2^2 ds\right]. \end{aligned}$$

Combining the above estimates and using it in (4.11), we find

$$\begin{aligned} &\mathbb{E}\left[\|\mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(t \wedge \tau_N^n)\|_2^2 + 2C(p, \mathcal{O})\nu \int_0^{t \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_p^p ds + 2\alpha \int_0^{t \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_q^q ds\right] \\ &\leq \left(\frac{1}{\eta_1} + \kappa\right) \mathbb{E}[\|\nabla \mathbf{u}_n(0)\|_2^2] + C(K, T) + \frac{1}{\eta_1} (C(K) + \varepsilon) \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_2^2 ds\right] \\ &\quad + C(\varepsilon) \mathbb{E}\left[\int_0^{t \wedge \tau_N^n} \|\mathbf{f}(s)\|_2^2 ds\right]. \end{aligned}$$

An application of Gronwall’s inequality with a proper choice of ε in the above inequality yields

$$\begin{aligned} \mathbb{E}\left[\|\nabla \mathbf{u}_n(t \wedge \tau_N^n)\|_2^2\right] &\leq C(\kappa) \left\{ \left(\frac{1}{\eta_1} + \kappa\right) \mathbb{E}[\|\nabla \mathbf{u}_n(0)\|_2^2] \right. \\ &\quad \left. + C \mathbb{E}\left[\int_0^T \|\mathbf{f}(s)\|_2^2 ds\right] + C(K, T) \right\} e^{\frac{C(K)T}{\eta_1}}. \end{aligned} \tag{4.12}$$

Taking limit $N \rightarrow \infty$ in (4.12) and using the monotone convergence theorem, we get

$$\mathbb{E}\left[\|\nabla \mathbf{u}_n(t)\|_2^2\right] \leq C(\kappa) \left\{ \left(\frac{1}{\eta_1} + \kappa\right) \mathbb{E}[\|\nabla \mathbf{u}_0\|_2^2] + C \mathbb{E}\left[\int_0^T \|\mathbf{f}(s)\|_2^2 ds\right] + C(K, T) \right\} e^{\frac{C(K)T}{\eta_1}}, \tag{4.13}$$

for all $t \in [0, T]$.

Taking supremum from 0 to $T \wedge \tau_N^n$ and then expectation in (4.11), we find

$$\mathbb{E}\left[\sup_{t \in [0, T \wedge \tau_N^n]} \left\{ \|\mathbf{u}_n(t)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(t)\|_2^2 \right\} + 2C(p, \mathcal{O})\nu \int_0^{T \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_p^p ds\right]$$

$$\begin{aligned}
 & + 2\alpha \int_0^{T \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_q^q ds \Big] \\
 \leq & \left(\frac{1}{\eta_1} + \kappa \right) \mathbb{E}[\|\nabla \mathbf{u}_0\|_2^2] + C(\varepsilon) \mathbb{E} \left[\int_0^T \|\mathbf{f}(s)\|_2^2 ds \right] + C(K)T \\
 & + \frac{1}{\eta_1} (C(K) + \varepsilon) \mathbb{E} \left[\int_0^{T \wedge \tau_N^n} \|\nabla \mathbf{u}_n(s)\|_2^2 ds \right] + 2\mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} I_3(t) \right]. \tag{4.14}
 \end{aligned}$$

We estimate the final term from the right hand side of (4.14) using the BDG (see [19, Theorem 1.1]), Hölder’s and Young’s inequalities as follows:

$$\begin{aligned}
 \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} |I_3(t)| \right] &= \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} \left| \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}_n) d\mathbf{W}_n(s) \cdot \mathbf{u}_n(s) \, d\mathbf{x} \right| \right] \\
 &= \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} \left| \int_0^t \sum_{k=1}^n \int_{\mathcal{O}} \Phi(\mathbf{u}_n) \mathbf{e}_k d\beta_k(s) \cdot \mathbf{u}_n \, d\mathbf{x} \right| \right] \\
 &= \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} \left| \int_0^t \sum_{k=1}^n \int_{\mathcal{O}} \phi_k(\mathbf{u}_n) d\beta_k(s) \cdot \mathbf{u}_n \, d\mathbf{x} \right| \right] \\
 &\leq C \mathbb{E} \left[\int_0^{T \wedge \tau_N^n} \sum_{k=1}^n \left(\int_{\mathcal{O}} \phi_k(\mathbf{u}_n) \cdot \mathbf{u}_n \, d\mathbf{x} \right)^2 ds \right]^{\frac{1}{2}} \\
 &\leq C \mathbb{E} \left[\int_0^{T \wedge \tau_N^n} \sum_{k=1}^n \left(\int_{\mathcal{O}} |\mathbf{u}_n|^2 d\mathbf{x} \int_{\mathcal{O}} |\phi_k(\mathbf{u}_n)|^2 d\mathbf{x} \right) ds \right]^{\frac{1}{2}} \\
 &\leq C(K) \mathbb{E} \left[1 + \int_0^{T \wedge \tau_N^n} \left(\int_{\mathcal{O}} |\mathbf{u}_n|^2 d\mathbf{x} \right)^2 ds \right]^{\frac{1}{2}} \\
 &\leq \varepsilon \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} \|\mathbf{u}_n(s)\|_2^2 \right] + C(\varepsilon, K) \mathbb{E} \left[T + \int_0^{T \wedge \tau_N^n} \|\mathbf{u}_n(s)\|_2^2 ds \right]. \tag{4.15}
 \end{aligned}$$

Substituting the above estimate in (4.14) with a proper choice of ε , and an application of Gronwall’s inequality yields

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_N^n]} \left\{ \|\mathbf{u}_n(t)\|_2^2 + 2\kappa \|\nabla \mathbf{u}_n(t)\|_2^2 \right\} + 4C(p, \mathcal{O})v \int_0^{T \wedge \tau_N^n} \|\nabla \mathbf{u}_n(t)\|_p^p dt \right. \\
 & \quad \left. + 2\alpha \int_0^{T \wedge \tau_N^n} \|\mathbf{u}_n(t)\|_q^q dt \right] \\
 & \leq C(\kappa) \left\{ \left(\frac{1}{\eta_1} + \kappa \right) \int_{\mathbf{V}} \|\mathbf{z}\|_{\mathbf{V}}^2 d\Lambda_0(\mathbf{z}) + 2C \int_{\mathbf{L}^2(\mathcal{O}_T)} \|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2 d\Lambda_f(\mathbf{g}) + C(K, T) \right\} e^{\frac{C(K)T}{\eta_1}}.
 \end{aligned}$$

Finally, passing $N \rightarrow \infty$ in the above inequality, we arrive at

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{t \in [0, T]} \left\{ \|\mathbf{u}_n(t)\|_2^2 + 2\kappa \|\nabla \mathbf{u}_n(t)\|_2^2 \right\} + 4C(p, \mathcal{O})v \int_0^T \|\nabla \mathbf{u}_n(t)\|_p^p dt + 2\alpha \int_0^T \|\mathbf{u}_n(t)\|_q^q dt \right] \\
 & \leq C(\kappa) \left\{ \left(\frac{1}{\eta_1} + \kappa \right) \int_{\mathbf{V}} \|\mathbf{z}\|_{\mathbf{V}}^2 d\Lambda_0(\mathbf{z}) + \int_{\mathbf{L}^2(\mathcal{O}_T)} \|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2 d\Lambda_f(\mathbf{g}) + C(K, T) \right\} e^{\frac{C(K)T}{\eta_1}},
 \end{aligned}$$

by assuming the right-hand side is finite, which is true due to (2.9) and (2.10). □

Theorem 4.2 *Let $p \in (2d/d+2, \infty)$ and assume the conditions (2.11) and (4.3) are satisfied. Assume, in addition, that (2.9) and (2.10) hold with $\gamma = 2$. Then there exists a martingale solution to the approximate system (4.2),*

$$((\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}, \bar{\mathbb{P}}), \bar{\mathbf{u}}, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}, \bar{\mathbf{W}}),$$

in the following sense:

- (1) $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}, \bar{\mathbb{P}})$ is a stochastic basis, with a complete right-continuous filtration $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$;
- (2) $\bar{\mathbf{W}}$ is a cylindrical $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -adapted Wiener process;
- (3) $\bar{\mathbf{u}}$ is a progressively $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -measurable stochastic process with paths $t \mapsto \bar{\mathbf{u}}(t, \omega) \in \mathbf{L}^\infty(0, T; \mathbf{V}) \cap \mathbf{L}^p(0, T; \mathbf{W}_0^{1,p}(\mathcal{O})^d) \cap \mathbf{L}^q(0, T; \mathbf{L}^q(\mathcal{O})^d)$, $\bar{\mathbb{P}}$ -a.s. with a continuous modification having $\bar{\mathbb{P}}$ -a.s. paths in $\mathbf{C}([0, T]; \mathbf{V})$;
- (4) $\bar{\mathbf{u}}_0$ is progressively $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -measurable on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, with $\bar{\mathbb{P}}$ -a.s. paths $\bar{\mathbf{u}}_0(\omega) \in \mathbf{V}$ and $\Lambda_0 = \bar{\mathbb{P}} \circ \bar{\mathbf{u}}_0^{-1}$ in the sense of (2.6);
- (5) $\bar{\mathbf{f}}$ is a $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -adapted stochastic process, with $\bar{\mathbb{P}}$ -a.s. paths $\bar{\mathbf{f}}(t, \omega) \in \mathbf{L}^2(\mathcal{O}_T)$ and $\Lambda_{\bar{\mathbf{f}}} = \bar{\mathbb{P}} \circ \bar{\mathbf{f}}^{-1}$ in the sense of (2.7);
- (6) for every $\varphi \in \mathbf{C}_0^\infty(\mathcal{O})^d$, with $\text{div } \varphi = 0$ in \mathcal{O}_T , and for all $t \in [0, T]$, the following identity holds $\bar{\mathbb{P}}$ -a.s.:

$$\begin{aligned} & \int_{\mathcal{O}} \bar{\mathbf{u}}(t) \cdot \varphi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{\mathbf{u}}(t) : \nabla \varphi \, dx - \int_0^t \int_{\mathcal{O}} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} : \nabla \varphi \, dx ds \\ & + \alpha \int_0^t \int_{\mathcal{O}} |\bar{\mathbf{u}}|^{q-2} \bar{\mathbf{u}} \cdot \varphi \, dx ds \end{aligned} \tag{4.16}$$

$$\begin{aligned} & + \nu \int_0^t \int_{\mathcal{O}} |\mathbf{D}(\bar{\mathbf{u}})|^{p-2} \mathbf{D}(\bar{\mathbf{u}}) : \mathbf{D} \varphi \, dx ds \\ & = \int_{\mathcal{O}} \bar{\mathbf{u}}_0 \cdot \varphi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{\mathbf{u}}_0 : \nabla \varphi \, dx + \int_0^t \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \varphi \, dx ds + \int_0^t \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}}(s) \cdot \varphi \, dx \end{aligned} \tag{4.17}$$

In addition, there exists a positive constant C , not depending on α , such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \{ \|\bar{\mathbf{u}}(t)\|_2^2 + 2\kappa \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \} + 4\nu \int_0^T \|\nabla \bar{\mathbf{u}}(t)\|_p^p dt + 2\alpha \int_0^T \|\bar{\mathbf{u}}(t)\|_q^q dt \right] \\ & \leq C \left\{ \left(\frac{1}{\lambda_1} + \kappa \right) \int_{\mathbf{V}} \|\bar{\mathbf{z}}\|_{\mathbf{V}}^2 d\Lambda_0(\bar{\mathbf{z}}) + \int_{\mathbf{L}^2(\mathcal{O}_T)} \|\bar{\mathbf{g}}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2 d\Lambda_{\bar{\mathbf{f}}}(\bar{\mathbf{g}}) + C(K, T) \right\}. \end{aligned} \tag{4.18}$$

Proof For the sake of better comprehension, the proof of Theorem 4.2 shall be split through the following sections:

4.3 Weak Convergence

In view of (4.9), the Banach-Alaoglu theorem gives the existence of functions \mathbf{u} , \mathbf{w} , \mathbf{S} and Ψ such that, for some subsequences still labeled by the same subscript,

$$\mathbf{u}_n \xrightarrow[n \rightarrow \infty]{*} \mathbf{u}, \quad \text{in } \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{L}^\infty(0, T; \mathbf{V})), \tag{4.19}$$

$$\mathbf{u}_n \xrightarrow[n \rightarrow \infty]{} \mathbf{u}, \quad \text{in } \mathbf{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{L}^p(0, T; \mathbf{W}_0^{1,p}(\mathcal{O})^d)), \tag{4.20}$$

$$\mathbf{u}_n \xrightarrow{n \rightarrow \infty} \mathbf{u}, \quad \text{in } L^q(\Omega, \mathcal{F}, \mathbb{P}; L^q(0, T; L^q(\mathcal{O})^d)), \tag{4.21}$$

$$\mathbf{a}(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \mathbf{s}, \quad \text{in } L^{q'}(\Omega, \mathcal{F}, \mathbb{P}; L^{q'}(0, T; L^{q'}(\mathcal{O})^d)), \tag{4.22}$$

$$\mathbf{u}_n \otimes \mathbf{u}_n \xrightarrow{n \rightarrow \infty} \mathbf{w}, \quad \text{in } L^{\frac{q}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^{\frac{q}{2}}(0, T; L^{\frac{q}{2}}(\mathcal{O})^{d \times d})), \tag{4.23}$$

$$\mathbf{A}(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \mathbf{S}, \quad \text{in } L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; L^{p'}(\mathcal{O})^{d \times d})), \tag{4.24}$$

$$\mathbf{A}(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \mathbf{S}, \quad \text{in } L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; W^{-1, p'}(\mathcal{O})^{d \times d})), \tag{4.25}$$

$$\Phi(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \Psi, \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))). \tag{4.26}$$

Our aim is to establish that

$$\mathbf{s} = \mathbf{a}(\mathbf{u}), \quad \mathbf{w} = \mathbf{u} \otimes \mathbf{u}, \quad \mathbf{S} = \mathbf{A}(\mathbf{u}), \quad \text{and } \Psi = \Phi(\mathbf{u}).$$

4.4 Compactness

From the Subsection 4.3, we have the weak convergence results (4.19)-(4.26), which are not enough to pass to the limit in the nonlinear terms and in the noise coefficient appearing in our model. In order to pass these terms to the limit, we need some compactness arguments.

We test (4.5) with $\phi \in \mathcal{V}$, so that \mathbb{P} -a.s.,

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}_n(t) \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_n(t) : \nabla \phi \, dx \equiv \int_{\mathcal{O}} \mathbf{u}_n(t) \cdot P_s^n(\phi) \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_n(t) : \nabla P_s^n(\phi) \, dx \\ &= \int_{\mathcal{O}} \mathbf{u}_0 \cdot P_s^n(\phi) \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0(t) : \nabla P_s^n(\phi) \, dx + \int_0^t \int_{\mathcal{O}} \mathbf{G}_n : \nabla P_s^n(\phi) \, dx \, ds \\ & \quad + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}_n) dW_n(s) \cdot P_s^n(\phi) \, dx, \end{aligned} \tag{4.27}$$

where P_s^n denotes the projection into the n -dimensional space \mathbf{X}^n with respect to the \mathbf{V}^s inner product, and

$$\mathbf{G}_n := \mathbf{u}_n \otimes \mathbf{u}_n + \nabla \Delta^{-1} \mathbf{a}(\mathbf{u}_n) - \nu \mathbf{A}(\mathbf{u}_n) + \mathbf{F}, \tag{4.28}$$

with \mathbf{F} chosen in $L^2(0, T; W^{1,2}(\mathcal{O})^{d \times d})$ in such a way that $\text{div } \mathbf{F} = -\mathbf{f}$ in the weak sense.

Claim 1 $\mathbf{w} = \mathbf{u} \otimes \mathbf{u}$ and $\Psi = \Phi(\mathbf{u})$.

Proof of Claim 1 Using the energy estimate obtained in (4.9) and the definitions of \mathbf{A} and \mathbf{a} (see (2.5) and (4.1)), and observing (4.3), we find

$$\mathbf{G}_n \in L^{q_0}(\Omega, \mathcal{F}, \mathbb{P}; L^{q_0}(0, T; L^{q_0}(\mathcal{O})^{d \times d})), \quad q_0 := \min\{p', q'\} > 1, \tag{4.29}$$

uniformly in n . Note that in fact it should be $q_0 := \min\{p', q', \frac{q}{2}\} > 1$, but once that $q \geq 3$ (see assumption (4.3)), we have $q_0 = \min\{p', q'\}$. Let us define the functional

$$\mathcal{H}(t, \phi) := \int_0^t \int_{\mathcal{O}} \mathbf{G}_n : \nabla P_s^n(\phi) \, dx \, ds, \quad \phi \in \mathcal{V}.$$

As $1 + \frac{2}{q_0} > \frac{1}{q_0} - \frac{1}{2}$ implies the embedding $W^{\tilde{s}, q_0}(\mathcal{O}) \hookrightarrow W^{s, 2}(\mathcal{O})$ for $\tilde{s} \geq s + d(1 + \frac{2}{q_0})$, we can use (4.29) to show that

$$\mathbb{E} \left[\|\mathcal{H}\|_{W^{1, q_0}(0, T; W^{\tilde{s}, q_0}(\mathcal{O}))} \right] \leq C. \tag{4.30}$$

The above estimate can be justified as follows (cf. [16, Section 4]):

$$\begin{aligned}
 \left\| \frac{d}{dt} \mathcal{H}(t, \cdot) \right\|_{L^{q_0}(0, T; W_{\sigma}^{-\bar{s}, q_0}(\mathcal{O}))} &= \left\| \sup_{\|\phi\|_{\bar{s}, q'_0} \leq 1} \frac{d}{dt} \mathcal{H}(t, \phi) \right\|_{L^{q_0}(0, T)} \\
 &= \left\| \sup_{\|\phi\|_{\bar{s}, q'_0} \leq 1} \int_{\mathcal{O}} \mathbf{G}_n(t) : \nabla P_{\bar{s}}^n(\phi) \, dx \right\|_{L^{q_0}(0, T)} \\
 &\leq \left\| \sup_{\|\phi\|_{\bar{s}, q'_0} \leq 1} \|\mathbf{G}_n(t)\|_{q_0} \|\nabla P_{\bar{s}}^n(\phi)\|_{q'_0} \right\|_{L^{q_0}(0, T)} \\
 &\leq C \left(\int_0^T \|\mathbf{G}_n(t)\|_{q_0}^{q_0} dt \right)^{\frac{1}{q_0}}.
 \end{aligned}$$

For all $0 \leq s < t \leq T$, we have

$$\begin{aligned}
 &\mathbb{E} \left[\left\| \int_0^t \Phi(\mathbf{u}_n(\ell)) dW_n(\ell) - \int_0^s \Phi(\mathbf{u}_n(\ell)) dW_n(\ell) \right\|_2^q \right] \\
 &= \mathbb{E} \left[\left\| \int_s^t \Phi(\mathbf{u}_n(\ell)) dW_n(\ell) \right\|_2^q \right] = \mathbb{E} \left[\left\| \int_s^t \sum_{j=1}^{\infty} \Phi(\mathbf{u}_n(\ell)) \mathbf{e}_j d\beta_j^n \right\|_2^q \right] \\
 &\leq C \mathbb{E} \left[\left(\int_s^t \sum_{j=1}^{\infty} \|\phi_k(\mathbf{u}_n(\ell))\|_2^2 d\ell \right)^{\frac{q}{2}} \right] \leq C |t - s|^{\frac{q}{2}} \left\{ \mathbb{E} \left[\sup_{\ell \in [0, T]} \{1 + \|\mathbf{u}_n(\ell)\|_2^2\} \right] \right\}^{\frac{q}{2}} \\
 &\leq C |t - s|^{\frac{q}{2}},
 \end{aligned}$$

where we have used the BDG and Young’s inequalities and the energy estimate (4.9). By the Kolmogorov continuity criterion, there exists a modification which has \mathbb{P} -a.s. Hölder continuous paths such that

$$\mathbb{E} \left[\left\| \int_0^t \Phi(\mathbf{u}_n(s)) dW_n(s) \right\|_{C^{\mu}([0, T]; L^2(\mathcal{O})^d)} \right] \leq C, \quad \mu := \theta - \frac{1}{q}, \tag{4.31}$$

for $\frac{1}{q} < \theta < \frac{1}{2}$ if $q > 2$, which is the case due to assumption (4.3). Observing that $W_0^{\bar{s}, q'_0}(\mathcal{O})^d \hookrightarrow W_0^{1, 2}(\mathcal{O})^d \hookrightarrow L^2(\mathcal{O})^d$, $1 < q_0 < \infty$, implies $L^2(\mathcal{O})^d \hookrightarrow W^{-\bar{s}, q_0}(\mathcal{O})^d$, by the above inequality, we have

$$\mathbb{E} \left[\left\| \int_0^t \Phi(\mathbf{u}_n(s)) dW_n(s) \right\|_{C^{\mu}([0, T]; W_0^{-\bar{s}, q_0}(\mathcal{O})^d)} \right] \leq C. \tag{4.32}$$

Collecting the information from (4.30) and (4.32) in (4.27), and still using (2.9), we arrive at

$$\mathbb{E} \left[\|\mathbf{I} - \kappa \Delta\|_{C^{\mu}([0, T]; W_{\sigma}^{-\bar{s}, q_0}(\mathcal{O})^d)} \|\mathbf{u}_n\| \right] \leq C,$$

for some positive constant C that does not depend on n . This in turn implies for some $\eta > 0$ that

$$\mathbb{E} \left[\|\mathbf{I} - \kappa \Delta\|_{W^{\eta, q_0}(0, T; W_{\sigma}^{-\bar{s}, q_0}(\mathcal{O})^d)} \|\mathbf{u}_n\| \right] \leq C. \tag{4.33}$$

From (4.33), we conclude that

$$\mathbb{E} \left[\|\mathbf{u}_n\|_{W^{\eta, q_0}(0, T; W_{\sigma}^{2-\bar{s}, q_0}(\mathcal{O})^d)} \right] \leq C. \tag{4.34}$$

Observing that $\tilde{s} \geq s + d(1 + \frac{2}{q_0})$ implies $2 - \tilde{s} < 0$.

Let

$$\mathbf{Z}_{p,q} := L^\infty(0, T; \mathbf{V}) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d) \cap L^q(0, T; L^q(\mathcal{O})^d).$$

By the Sobolev embedding theorem, we infer

$$\mathbf{Z}_{p,q} \hookrightarrow L^\infty(0, T; L_\sigma^\rho(\mathcal{O})^d) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d), \quad \rho \leq \min\{2^*, q\}.$$

By a version of the Aubin-Lions compactness lemma (see [29, Theorem 2.1]), we further have

$$\begin{aligned} W^{\eta,q_0}(0, T; W_\sigma^{2-\tilde{s},q_0}(\mathcal{O})^d) \cap L^\infty(0, T; L_\sigma^\rho(\mathcal{O})^d) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d) \\ \hookrightarrow \hookrightarrow L^\rho(0, T; L_\sigma^\rho(\mathcal{O})^d), \end{aligned}$$

for $q_0 \leq \rho < p^*$. In view of this, and attending to the definition of q_0 in (4.29), one has

$$W^{\eta,q_0}(0, T; W_\sigma^{2-\tilde{s},q_0}(\mathcal{O})^d) \cap \mathbf{Z}_{p,q} \hookrightarrow \hookrightarrow L^\rho(0, T; L_\sigma^\rho(\mathcal{O})^d), \tag{4.35}$$

for

$$\min\{p', q'\} < \rho < \min\{2^*, p^*, q\}, \quad p > \frac{2d}{d+2}. \tag{4.36}$$

Now, we define the space

$$\mathfrak{X} := L^\rho(0, T; L_\sigma^\rho(\mathcal{O})^d) \otimes C([0, T]; \mathbf{U}_0) \otimes \mathbf{V} \otimes L^2(\mathcal{O}_T).$$

In the sequel, we use the following notations:

- (1) $\varrho_{\mathbf{u}_n}$ denotes the law of \mathbf{u}_n on $L^\rho(0, T; L_\sigma^\rho(\mathcal{O})^d)$;
- (2) ϱ_W denotes the law of W on $C([0, T]; \mathbf{U}_0)$;
- (3) ϱ_n denotes the joint law of $\mathbf{u}_n, W, \mathbf{u}_0$ and \mathbf{f} on the space \mathfrak{X} .

Let us consider a ball B_N in the space $W^{\eta,q_0}([0, T]; W_\sigma^{2-\tilde{s},q_0}(\mathcal{O})^d) \cap \mathbf{Z}_{p,q}$ and denote its complement by B_N^c . Using the uniform estimates (4.9) and (4.34), we find

$$\begin{aligned} \varrho_{\mathbf{u}_n}(B_N^c) &= \mathbb{P}\left(\|\mathbf{u}_n\|_{W^{\eta,q_0}(0,T;W_\sigma^{2-\tilde{s},q_0}(\mathcal{O})^d)} + \|\mathbf{u}_n\|_{\mathbf{Z}_{p,q}} \geq N\right) \\ &\leq \frac{1}{N} \mathbb{E}\left[\|\mathbf{u}_n\|_{W^{\eta,q_0}(0,T;W_\sigma^{2-\tilde{s},q_0}(\mathcal{O})^d)} + \|\mathbf{u}_n\|_{\mathbf{Z}_{p,q}}\right] \leq \frac{C}{N}. \end{aligned}$$

For any fixed $\xi > 0$, we can find $N(\xi)$ such that

$$\varrho_{\mathbf{u}_n}(B_{N(\xi)}) \geq 1 - \frac{\xi}{4}.$$

Since, the law ϱ_W is tight as being a Radon measure on the Polish space $C([0, T]; \mathbf{U}_0)$, then there exists a compact subset $K_\xi \subset C([0, T]; \mathbf{U}_0)$ such that $\varrho_{\mathbf{u}_n}(K_\xi) \geq 1 - \frac{\xi}{4}$. By the same reasoning, we are able to find compact subsets of \mathbf{V} and $L^2(\mathcal{O}_T)$, such that their measures Λ_0 and Λ_f are greater than $1 - \frac{\xi}{4}$. Therefore, we can find a compact subset $\mathfrak{X}_\xi \subset \mathfrak{X}$ such that $\varrho_n(\mathfrak{X}_\xi) \geq 1 - \xi$. Hence, $\{\varrho_n\}_{n \in \mathbb{N}}$ is tight in the same space. An application of Prokhorov’s theorem (see [35, Theorem 2.6]) yields ϱ_n is relatively weakly compact, which implies that ϱ_n has a weakly convergent subsequence with weak limit ϱ .

4.5 Existence of a Probability Space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$

Applying Skorohod’s representation theorem (see [35, Theorem 2.7]) to ensure the existence of another probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a random sequence $(\bar{\mathbf{u}}_n, \bar{\mathbf{W}}_n, \bar{\mathbf{u}}_0^n, \bar{\mathbf{f}}_n)$ and a random variable $(\bar{\mathbf{u}}, \bar{\mathbf{W}}, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})$ on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, taking values in \mathfrak{U} such that the following holds:

- (1) The laws of sequence of random variables $(\bar{\mathbf{u}}_n, \bar{\mathbf{W}}_n, \bar{\mathbf{u}}_0^n, \bar{\mathbf{f}}_n)$ and the random variable $(\bar{\mathbf{u}}, \bar{\mathbf{W}}, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})$ under the new probability measure $\bar{\mathbb{P}}$ coincide with ϱ_n and $\varrho := \lim_{n \rightarrow \infty} \varrho_n$, respectively;
- (2) The following convergence results hold true

$$\begin{aligned} \bar{\mathbf{u}}_n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}, && \text{in } L^\rho(0, T; L^\rho_\sigma(\mathcal{O})^d), && (4.37) \\ \bar{\mathbf{W}}_n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{W}}, && \text{in } C([0, T]; \mathbf{U}_0), \\ \bar{\mathbf{u}}_0^n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}_0, && \text{in } \mathbf{V}, \\ \bar{\mathbf{f}}_n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{f}}, && \text{in } L^2(0, T; L^2(\mathcal{O})^d), \end{aligned}$$

$\bar{\mathbb{P}}$ -a.s.;

- (3) The convergence results (4.20) and (4.23) still hold for the functions defined on the newly constructed probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. Moreover, for any finite β , we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{\mathbf{W}}_n(t)\|_{\mathbf{U}_0}^\beta \right] = \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{\mathbf{W}}(t)\|_{\mathbf{U}_0}^\beta \right].$$

Applying Vitali’s convergence theorem, we have for some subsequences still labeled by the same subscript

$$\bar{\mathbf{W}}_n \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{W}}, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; C([0, T]; \mathbf{U}_0)), \quad (4.38)$$

$$\bar{\mathbf{u}}_n \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}, \quad \text{in } L^\rho(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; L^\rho_\sigma(\mathcal{O})^d)), \quad (4.39)$$

$$\bar{\mathbf{u}}_0^n \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}_0, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; \mathbf{V}), \quad (4.40)$$

$$\bar{\mathbf{f}}_n \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{f}}, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^2(0, T; L^2(\mathcal{O})^d)), \quad (4.41)$$

for ρ and p in the conditions of (4.36). Now, we need to define the filtration on the newly constructed probability space. First, we define an operator of restriction to the interval $[0, T]$ denoted by \mathbf{h}_t acting on various path spaces. More precisely, let \mathbf{X} denote any of the spaces $L^\rho(0, T; L^\rho(\mathcal{O})^d)$, $L^2(0, T; L^2(\mathcal{O})^d)$, or $C([0, T]; \mathbf{U}_0)$, and for $t \in [0, T]$, we define

$$\mathbf{h}_t : \mathbf{X} \rightarrow \mathbf{X}|_{[0, t]}, \quad g \mapsto g|_{[0, t]}. \quad (4.42)$$

The mapping \mathbf{h}_t is continuous. Let us define the $\bar{\mathbb{P}}$ -augmented canonical filtration by $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ of the process $(\bar{\mathbf{u}}, \bar{\mathbf{f}}, \bar{\mathbf{W}})$, that is,

$$\bar{\mathcal{F}}_t = \sigma(\sigma(\mathbf{h}_t \bar{\mathbf{u}}, \mathbf{h}_t \bar{\mathbf{f}}, \mathbf{h}_t \bar{\mathbf{W}})) \cup \{M \in \bar{\mathcal{F}}; \bar{\mathbb{P}}(M) = 0\}, \quad t \in [0, T].$$

Next, we are going to show that our approximate equations also hold in the new probability space.

4.6 Validity of the Approximate Equations in $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$

The method we employ in this subsection has been already discussed in several works (see for e.g. [14, 34, 55]). The main aim of this method is to identify the quadratic variation of the martingale as well as cross variation with the limit Wiener process obtained through compactness. Note that the laws of \bar{W}_n and \bar{W} are the same. As a consequence of the new probability space, we can find a collection of mutually independent real-valued $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -Wiener processes $\{\bar{\beta}_j^n\}_{j \in \mathbb{N}}$ such that $\bar{W}_n = \sum_{j \in \mathbb{N}} \mathbf{e}_j \bar{\beta}_j^n$, that is, there exists a collection of mutually independent real-valued $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -Wiener processes $\{\bar{\beta}_j\}_{j \in \mathbb{N}}$ such that $\bar{W} = \sum_{j \in \mathbb{N}} \mathbf{e}_j \bar{\beta}_j$.

We used the abbreviation $\bar{W}_{n,n}$ for $\sum_{j=1}^n \mathbf{e}_j \bar{\beta}_j^n$.

Let us consider $\phi \in \mathcal{V}$ and define the functionals for $t \in [0, T]$

$$\begin{aligned}
 N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_t &= \int_{\mathcal{O}} \mathbf{u}_n(t) \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_n(t) : \nabla \phi \, dx - \int_{\mathcal{O}} \mathbf{u}_0^n \cdot \phi \, dx - \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0^n : \nabla \phi \, dx \\
 &\quad + \int_0^t \int_{\mathcal{O}} \mathbf{f} \cdot P^n(\phi) \, dx ds + \int_0^t \int_{\mathcal{O}} \mathbf{u}_n \otimes \mathbf{u}_n : \nabla P^n(\phi) \, dx ds \\
 &\quad + \nu \int_0^t \int_{\mathcal{O}} \mathbf{A}(\mathbf{u}_n) : \mathbf{D}(P^n(\phi)) \, dx ds, \\
 M(\mathbf{u}_n)_t &= \sum_{j=1}^n \int_0^t \left(\int_{\mathcal{O}} \phi_j(\mathbf{u}_n) \cdot P^n(\phi) \, dx \right)^2 ds, \\
 M_j(\mathbf{u}_n)_t &= \int_0^t \int_{\mathcal{O}} \phi_j(\mathbf{u}_n) \cdot P^n(\phi) \, dx ds.
 \end{aligned}$$

Let us denote the increment $N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_t - N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_s$ by $N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_{s,t}$ and similarly for $M(\mathbf{u}_n)_{s,t}$ and $M_j(\mathbf{u}_n)_{s,t}$. Observe that our proof will be over once we prove that the process $N(\bar{\mathbf{u}}_n)$ is an $\{\bar{\mathcal{F}}_t\}_{t \in [0, T]}$ -martingale and its quadratic and cross variations satisfy

$$\langle N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}) \rangle = M(\bar{\mathbf{u}}_n), \quad \text{and} \quad \langle N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}), \bar{\beta}_j \rangle = M_j(\bar{\mathbf{u}}_n), \tag{4.43}$$

respectively. In that situation, we have

$$\left\langle N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}) - \int_0^\cdot \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}_n) d\bar{W}_{n,n} \cdot P^n(\phi) \, dx \right\rangle = 0, \tag{4.44}$$

which implies the required result in the new probability space. Let us establish (4.43). To finish this proof, we use the uniform estimate and we claim that the mappings

$$(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f}) \mapsto N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_t, \quad \mathbf{u}_n \mapsto M(\mathbf{u}_n)_t, \quad \text{and} \quad \mathbf{u}_n \mapsto M_j(\mathbf{u}_n)_t,$$

are well defined and measurable on a subspace of the path space, where the joint law of $(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})$ is supported, that is, the uniform estimate (4.9) of Theorem 4.1 holds. In the case of $M(\mathbf{u}_n)_t$, we have by (2.11) and the continuity of P^n in the space $L^2(\mathcal{O})^d$

$$\begin{aligned}
 \sum_{j=1}^n \int_0^t \left(\int_{\mathcal{O}} \phi_j(\mathbf{u}_n) \cdot P^n(\phi) \, dx \right)^2 ds &\leq C(\phi) \sum_{j=1}^n \int_0^t \int_{\mathcal{O}} |\phi_j(\mathbf{u}_n)|^2 \, dx ds \\
 &\leq C(\phi, K) \left(T + \int_0^T \|\mathbf{u}_n(t)\|_2^2 dt \right),
 \end{aligned}$$

which is finite in view of the uniform estimate (4.9). The rest two mappings $N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_t$ and $M_j(\mathbf{u}_n)_t$ can be tackled in the same manner. Therefore the following random variables have the same laws:

$$\begin{aligned} N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f}) &\sim N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}}), \\ M(\mathbf{u}_n) &\sim M(\bar{\mathbf{u}}_n), \\ M_j(\mathbf{u}_n) &\sim M_j(\bar{\mathbf{u}}_n). \end{aligned}$$

For any fixed times $s, t \in [0, T]$ with $s < t$, let us define a continuous function by

$$h : \mathfrak{A}|_{[0,s]} \rightarrow [0, 1].$$

Since

$$N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_t = \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}_n) dW_n(s) \cdot P^n(\boldsymbol{\phi}) \, dx = \sum_{j=1}^n \int_0^t \int_{\mathcal{O}} \phi_j(\mathbf{u}_n) d\beta_j \cdot P^n(\boldsymbol{\phi}) \, dx,$$

is a square integrable $\{\mathcal{F}_t\}$ -martingale, we conclude that

$$[N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})]^2 - M(\mathbf{u}_n), \quad \text{and} \quad N(\mathbf{u}_n)\beta_j - M_j(\mathbf{u}_n),$$

are $\{\mathcal{F}_t\}$ -martingales. Again, consider the restriction of a function to the interval $[0, s]$ which is denoted by \mathbf{h}_s . Due to the equality of laws, we find

$$\begin{aligned} &\mathbb{E} \left[h(\mathbf{h}_s \bar{\mathbf{u}}_n, \mathbf{h}_s \bar{W}_n, \mathbf{h}_s \bar{\mathbf{f}}, \bar{\mathbf{u}}_0) N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})_{s,t} \right] \\ &= \mathbb{E} \left[h(\mathbf{h}_s \mathbf{u}_n, \mathbf{h}_s W, \mathbf{h}_s \mathbf{f}, \mathbf{u}_0) N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})_{s,t} \right] = 0, \\ &\mathbb{E} \left[h(\mathbf{h}_s \bar{\mathbf{u}}_n, \mathbf{h}_s \bar{W}_n, \mathbf{h}_s \bar{\mathbf{f}}, \bar{\mathbf{u}}_0) ([N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})]_{s,t}^2 - M(\bar{\mathbf{u}}_n)_{s,t}) \right] \\ &= \mathbb{E} \left[h(\mathbf{h}_s \mathbf{u}_n, \mathbf{h}_s W, \mathbf{h}_s \mathbf{f}, \mathbf{u}_0) ([N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})]_{s,t}^2 - M(\mathbf{u}_n)_{s,t}) \right] = 0, \\ &\mathbb{E} \left[h(\mathbf{h}_s \bar{\mathbf{u}}_n, \mathbf{h}_s \bar{W}_n, \mathbf{h}_s \bar{\mathbf{f}}, \bar{\mathbf{u}}_0) ([N(\bar{\mathbf{u}}_n, \bar{\mathbf{u}}_0, \bar{\mathbf{f}})]_{s,t} \bar{\beta}_j^n - M_j(\bar{\mathbf{u}}_n)_{s,t}) \right] \\ &= \mathbb{E} \left[h(\mathbf{h}_s \mathbf{u}_n, \mathbf{h}_s W, \mathbf{h}_s \mathbf{f}, \mathbf{u}_0) ([N(\mathbf{u}_n, \mathbf{u}_0, \mathbf{f})]_{s,t} \beta_j - M_j(\mathbf{u}_n)_{s,t}) \right] = 0. \end{aligned}$$

Thus, we proved (4.43) and hence (4.44), which implies that on the new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, we have the equations for $j = 1, \dots, n$,

$$\begin{aligned} &\int_{\mathcal{O}} d\bar{\mathbf{u}}_n \cdot \boldsymbol{\psi}_j \, dx + \kappa \int_{\mathcal{O}} d\nabla \bar{\mathbf{u}}_n : \nabla \boldsymbol{\psi}_j \, dx + \nu \int_{\mathcal{O}} \mathbf{A}(\bar{\mathbf{u}}_n) : \mathbf{D}(\boldsymbol{\psi}_j) \, dx dt \\ &\quad + \alpha \int_{\mathcal{O}} \mathbf{a}(\bar{\mathbf{u}}_n) \cdot \boldsymbol{\psi}_j \, dx dt \\ &= \int_{\mathcal{O}} \bar{\mathbf{u}}_n \otimes \bar{\mathbf{u}}_n : \nabla \boldsymbol{\psi}_j \, dx dt + \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \boldsymbol{\psi}_j \, dx dt + \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}_n) d\bar{W}_{n,n} \cdot \boldsymbol{\psi}_j \, dx, \end{aligned} \tag{4.45}$$

$$\bar{\mathbf{u}}_n(0) = P^n \bar{\mathbf{u}}_0, \tag{4.46}$$

and the following convergence results:

$$\bar{\mathbf{u}}_n \xrightarrow[n \rightarrow \infty]{*} \bar{\mathbf{u}}, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\infty(0, T; \mathbf{V})), \tag{4.47}$$

$$\bar{u}_n \xrightarrow[n \rightarrow \infty]{} \bar{u}, \quad \text{in } L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^p(0, T; W_0^{1,p}(\mathcal{O})^d)), \tag{4.48}$$

$$\bar{u}_n \xrightarrow[n \rightarrow \infty]{} \bar{u}, \quad \text{in } L^q(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^q(0, T; L^q(\mathcal{O})^d)), \tag{4.49}$$

$$\mathbf{a}(\bar{u}_n) \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{s}}, \quad \text{in } L^{q'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q'}(0, T; L^{q'}(\mathcal{O})^d)) \tag{4.50}$$

$$\bar{u}_n \otimes \bar{u}_n \xrightarrow[n \rightarrow \infty]{} \bar{u} \otimes \bar{u}, \quad \text{in } L^{\frac{q}{2}}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{\frac{q}{2}}(0, T; L^{\frac{q}{2}}(\mathcal{O})^{d \times d})), \tag{4.51}$$

$$\mathbf{A}(\bar{u}_n) \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{S}}, \quad \text{in } L^{p'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{p'}(0, T; L^{p'}(\mathcal{O})^{d \times d})), \tag{4.52}$$

$$\mathbf{A}(\bar{u}_n) \xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{S}}, \quad \text{in } L^{p'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{p'}(0, T; W^{-1,p'}(\mathcal{O})^{d \times d})), \tag{4.53}$$

$$\Phi(\bar{u}_n) \xrightarrow[n \rightarrow \infty]{} \Phi(\bar{u}), \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^2(0, T; \mathcal{L}_2(\mathbf{U}; L^2(\mathcal{O})^d))). \tag{4.54}$$

Using (4.38)-(4.41) and (4.47)-(4.54), we obtain the following limit equation, for all $t \in [0, T]$, $\bar{\mathbb{P}}$ -a.s.:

$$\begin{aligned} & \int_{\mathcal{O}} \bar{u}(t) \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{u}(t) : \nabla \phi \, dx + \nu \int_0^t \int_{\mathcal{O}} \bar{\mathbf{S}} : \mathbf{D}(\phi) \, dx ds + \alpha \int_0^t \int_{\mathcal{O}} \bar{\mathbf{s}} \cdot \phi \, dx ds \\ &= \int_{\mathcal{O}} \bar{u}_0 \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{u}_0 : \nabla \phi \, dx + \int_0^t \int_{\mathcal{O}} \bar{u} \otimes \bar{u} : \nabla \phi \, dx ds + \int_0^t \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \phi \, dx ds \\ &+ \int_{\mathcal{O}} \int_0^t \Phi(\bar{u}) d\bar{W}(s) \cdot \phi \, dx, \end{aligned} \tag{4.55}$$

for all $\phi \in C_{0,\sigma}^\infty(\mathcal{O})^d$. The stochastic terms need some justification while passing to the limit. From (4.38), from one hand, and (2.11) and (4.39), on the other, we have

$$\begin{aligned} \bar{W}_n &\xrightarrow[n \rightarrow \infty]{} \bar{W}, \quad \text{in } C([0, T]; \mathbf{U}_0), \\ \Phi(\bar{u}_n) &\xrightarrow[n \rightarrow \infty]{} \Phi(\bar{u}), \quad \text{in } L^2(0, T; \mathcal{L}_2(\mathbf{U}; L^2(\mathcal{O})^d)), \end{aligned}$$

in probability. By [26, Lemma 2.1], these convergence results imply

$$\int_0^t \Phi(\bar{u}_n) d\bar{W}_n(s) \xrightarrow[n \rightarrow \infty]{} \int_0^t \Phi(\bar{u}) d\bar{W}(s), \quad \text{in } L^2(0, T; L^2(\mathcal{O})^d),$$

in probability. Thus, we are able to pass to the limit in the stochastic term. This concludes the proof of Claim 1. □

Claim 2

$$\bar{\mathbf{s}} = \mathbf{a}(\bar{u}). \tag{4.56}$$

Proof Claim 2 Let us recall the convergence (4.37)

$$\bar{u}_n \xrightarrow[n \rightarrow \infty]{} \bar{u}, \quad \text{in } L^\rho(0, T; L_\sigma^\rho(\mathcal{O})^d),$$

$\bar{\mathbb{P}}$ -a.s., for the permissible ρ given in (4.36). As a consequence of the above convergence and Riesz-Fischer’s theorem, we have upto some subsequence still labeled as same subscript

$$\bar{u}_n \xrightarrow[n \rightarrow \infty]{} \bar{u}, \quad \bar{\mathbb{P}} \times \lambda^{d+1} - \text{a.e.}$$

With the help of above convergence we can pass the limit $n \rightarrow \infty$ in the nonlinear term $\mathbf{a}(\bar{u}_n)$, to find

$$\mathbf{a}(\bar{u}_n) \xrightarrow[n \rightarrow \infty]{} \mathbf{a}(\bar{u}), \quad \bar{\mathbb{P}} \times \lambda^{d+1} - \text{a.e.},$$

and due to (4.9), $\mathbf{a}(\bar{\mathbf{u}}_n)$ is uniformly bounded in the space $L^{q'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q'}(0, T; L^{q'}(\mathcal{O})^d))$. By [47, Lemma 1.3], we deduce that

$$\mathbf{a}(\bar{\mathbf{u}}_n) \xrightarrow{n \rightarrow \infty} \mathbf{a}(\bar{\mathbf{u}}), \quad \text{in } L^{q'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q'}(0, T; L^{q'}(\mathcal{O})^d)),$$

and hence by the uniqueness of weak limit, we obtain $\bar{\mathbf{s}} = \mathbf{a}(\bar{\mathbf{u}})$, which is the required Claim 2. □

The proof of the next claim shall be done by using the theory of monotone operators.

Claim 3

$$\bar{\mathbf{S}} = \mathbf{A}(\bar{\mathbf{u}}). \tag{4.57}$$

Proof Claim 3 Applying infinite-dimensional Itô’s formula (see [33, Theorem 2.1] (see Sub-section 4.8 below for a proper justification) to the process $\|(I - \kappa \Delta)^{\frac{1}{2}} \bar{\mathbf{u}}(\cdot)\|_2^2$ and using the fact $\int_{\mathcal{O}} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} : \nabla \bar{\mathbf{u}} dx = 0$ in (4.55), we get for all $t \in [0, T]$, $\bar{\mathbb{P}}$ -a.s.,

$$\begin{aligned} & \|\bar{\mathbf{u}}(t)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \\ & + 2\nu \int_0^t \int_{\mathcal{O}} \bar{\mathbf{S}} : \mathbf{D}(\bar{\mathbf{u}}) dx ds + \alpha \int_0^t \int_{\mathcal{O}} \mathbf{a}(\bar{\mathbf{u}}) \cdot \bar{\mathbf{u}} dx ds \\ & = \|\bar{\mathbf{u}}_0\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}_0\|_2^2 + 2 \int_0^t \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \bar{\mathbf{u}} dx ds + 2 \int_0^t \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}) d\bar{\mathbb{W}}(s) \cdot \bar{\mathbf{u}} dx \\ & + \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}) d\bar{\mathbb{W}} \right\rangle_s dx. \end{aligned}$$

Applying finite-dimensional Itô’s formula now to the process $\|(I - \kappa \Delta)^{\frac{1}{2}} \bar{\mathbf{u}}_n(\cdot)\|_2^2$, subtracting the former from the later, and then taking expectation to the obtained equation, we get

$$\begin{aligned} & 2\nu \bar{\mathbb{E}} \left[\int_0^T \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \mathbf{A}(\bar{\mathbf{u}})) : \mathbf{D}(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dx ds \right] \\ & + 2\alpha \bar{\mathbb{E}} \left[\int_0^T \int_{\mathcal{O}} (\mathbf{a}(\bar{\mathbf{u}}_n) - \mathbf{a}(\bar{\mathbf{u}})) \cdot (\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dx ds \right] \\ & = \bar{\mathbb{E}} \left[- \{ \|\bar{\mathbf{u}}_n(T)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}_n(T)\|_2^2 \} + \|\bar{\mathbf{u}}(T)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}(T)\|_2^2 \right. \\ & \quad \left. + \|P^n \bar{\mathbf{u}}_0^n\|_2^2 + \kappa \|\nabla P^n \bar{\mathbf{u}}_0^n\|_2^2 - \|\bar{\mathbf{u}}_0\|_2^2 - \kappa \|\nabla \bar{\mathbf{u}}_0\|_2^2 \right] \\ & + 2\nu \bar{\mathbb{E}} \left[\int_0^T \int_{\mathcal{O}} (\bar{\mathbf{S}} - \mathbf{A}(\bar{\mathbf{u}}_n)) : \mathbf{D}(\bar{\mathbf{u}}) dx ds - \int_0^T \int_{\mathcal{O}} \mathbf{A}(\bar{\mathbf{u}}) : \mathbf{D}(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dx ds \right] \\ & + 2\alpha \bar{\mathbb{E}} \left[\int_0^T \int_{\mathcal{O}} (\mathbf{a}(\bar{\mathbf{u}}) - \mathbf{a}(\bar{\mathbf{u}}_n)) \cdot \bar{\mathbf{u}} dx ds - \int_0^T \int_{\mathcal{O}} \mathbf{a}(\bar{\mathbf{u}}) \cdot (\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dx ds \right] \\ & + \bar{\mathbb{E}} \left[2 \int_0^T \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot (\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dx ds + \int_0^T \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}_n) d\bar{\mathbb{W}}_{n,n} \right\rangle_s dx \right. \\ & \quad \left. - \int_0^T \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}) d\bar{\mathbb{W}} \right\rangle_s dx \right]. \end{aligned}$$

By (4.47) and the lower semicontinuity of the norm, one has

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\|\bar{\mathbf{u}}_n(T)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}_n(T)\|_2^2 - \|\bar{\mathbf{u}}(T)\|_2^2 - \kappa \|\nabla \bar{\mathbf{u}}(T)\|_2^2 \right] \geq 0.$$

Using this, together with (4.48), (4.50), (4.53), and the monotonicity of the operator $\mathbf{a}(\cdot)$, we can pass to the limit $n \rightarrow \infty$ in the previous equation to show that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \mathbf{A}(\bar{\mathbf{u}})) : \mathbf{D}(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) \, dx ds \right] \\ & \leq \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} d \left(\left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}_n) d\bar{\mathbf{W}}_{n,n} \right\rangle_s - \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}} \right\rangle_s \right) \, dx \right]. \end{aligned}$$

For the remaining integral, we use (4.38) and (4.39), together with (2.11), to find that

$$\mathbb{E} \left[\int_0^T \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}_n) d\bar{\mathbf{W}}_{n,n} \right\rangle_s \, dx \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} d \left\langle \int_0^\cdot \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}} \right\rangle_s \, dx \right].$$

Finally, we arrive at

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \mathbf{A}(\bar{\mathbf{u}})) : \mathbf{D}(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}) dt \right] \leq 0.$$

Using the monotonicity of the operator $\mathbf{A}(\cdot)$, we find (cf. [27, Eqn. (1.6)], [30] or [65, Appendix A])

$$\mathbf{D}(\bar{\mathbf{u}}_n) \xrightarrow{n \rightarrow \infty} \mathbf{D}(\bar{\mathbf{u}}), \quad \bar{\mathbb{P}} \otimes \lambda^{d+1} - \text{a.e.}$$

As a consequence of this, we obtain (4.57), which proves Claim 3. □

This finishes the proof of Theorem 4.2. □

Corollary 4.3 *Let the hypothesis of Theorem 4.2 be verified. In addition, assume that (2.9) and (2.10) hold with $\gamma \geq 2$. Then there exists a martingale solution to the system (4.2) such that*

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left\{ \|\bar{\mathbf{u}}(t)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \right\} \right]^{\frac{\gamma}{2}} + C(p, \mathcal{O}) \nu \mathbb{E} \left[\int_0^T \|\nabla \bar{\mathbf{u}}(t)\|_p^p dt \right]^{\frac{\gamma}{2}} \\ & + C\alpha \mathbb{E} \left[\int_0^T \|\bar{\mathbf{u}}(t)\|_q^q dt \right]^{\frac{\gamma}{2}} \\ & \leq C_1 \left\{ \left(\frac{1}{\eta_1} + \kappa \right)^{\frac{\gamma}{2}} \left(\int_{\mathbf{V}} \|\bar{\mathbf{z}}\|_{\mathbf{V}}^2 d\Lambda_0(\bar{\mathbf{z}}) \right)^{\frac{\gamma}{2}} + \left(\int_{L^2(\mathcal{O}_T)} \|\bar{\mathbf{g}}\|_{L^2(\mathcal{O}_T)}^2 d\Lambda_{\bar{\mathbf{f}}}(\bar{\mathbf{g}}) \right)^{\frac{\gamma}{2}} + C(\gamma, K, T) \right\}, \end{aligned}$$

where the constant C_1 is independent of α .

Proof Applying infinite-dimensional Itô’s formula to the process $\|(\mathbf{I} - \kappa \Delta)^{\frac{1}{2}} \bar{\mathbf{u}}(\cdot)\|_2^2$, and taking the supremum from 0 to T in the resultant, next raising to the power $\frac{\gamma}{2}$ and then taking expectation, we find

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left\{ \|\bar{\mathbf{u}}(t)\|_2^2 + \kappa \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \right\} + C(p, \mathcal{O}) \nu \int_0^T \|\nabla \bar{\mathbf{u}}(t)\|_p^p dt + 2\alpha \int_0^T \|\bar{\mathbf{u}}(t)\|_q^q dt \right]^{\frac{\gamma}{2}} \\ & \leq C(\gamma) \left\{ \left(\frac{1}{\eta_1} + \kappa \right)^{\frac{\gamma}{2}} \mathbb{E} [\|\nabla \bar{\mathbf{u}}_0\|_2^2]^{\frac{\gamma}{2}} + \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \bar{\mathbf{u}} \, dx dt \right]^{\frac{\gamma}{2}} \right\} \end{aligned}$$

$$+ \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}}(s) \cdot \bar{\mathbf{u}} \, d\mathbf{x} \right|^2 \right]^{\frac{\gamma}{2}} + \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}} \right\rangle_s \, d\mathbf{x} \right]^{\frac{\gamma}{2}} \}. \tag{4.58}$$

To estimate the second term in the right hand side of (4.58), we use the Cauchy-Schwarz and Young’s inequalities as follows:

$$\mathbb{E} \left[\int_0^T \int_{\mathcal{O}} \bar{\mathbf{f}} \cdot \bar{\mathbf{u}} \, d\mathbf{x} dt \right]^{\frac{\gamma}{2}} \leq \varepsilon \mathbb{E} \left[\int_0^T \|\bar{\mathbf{u}}(t)\|_2^2 dt \right]^{\frac{\gamma}{2}} + C(\varepsilon, \gamma) \mathbb{E} \left[\int_0^T \|\bar{\mathbf{f}}(t)\|_2^2 dt \right]^{\frac{\gamma}{2}}.$$

Let us consider the penultimate term of the right hand side of (4.58). Using the similar calculations to (4.15), we deduce

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}}(s) \cdot \bar{\mathbf{u}} \, d\mathbf{x} \right|^2 \right]^{\frac{\gamma}{2}} &\leq \varepsilon \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{\mathbf{u}}(t)\|_2^2 \right]^{\frac{\gamma}{2}} \\ &\quad + C(\varepsilon, \gamma, K) \mathbb{E} \left[T + \int_0^T \|\bar{\mathbf{u}}(t)\|_2^2 dt \right]^{\frac{\gamma}{2}}. \end{aligned}$$

Choosing ε small enough and using the above estimates in (4.58), we find

$$\begin{aligned} &\mathbb{E} \left[\sup_{t \in [0, T]} \left\{ \|\bar{\mathbf{u}}(t)\|_2^2 + 2\kappa \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \right\} \right]^{\frac{\gamma}{2}} + C(p, \mathcal{O}) \nu \mathbb{E} \left[\int_0^T \|\nabla \bar{\mathbf{u}}(t)\|_p^p dt \right]^{\frac{\gamma}{2}} \\ &\quad + C\alpha \mathbb{E} \left[\int_0^T \|\bar{\mathbf{u}}(t)\|_q^q dt \right]^{\frac{\gamma}{2}} \\ &\leq C(\gamma) \left\{ \left(\frac{1}{\eta_1} + \kappa \right)^{\frac{\gamma}{2}} \mathbb{E} \left[\|\nabla \bar{\mathbf{u}}_0\|_2^2 \right]^{\frac{\gamma}{2}} + C(\gamma, \eta_1, K) \mathbb{E} \left[\int_0^T \|\nabla \bar{\mathbf{u}}(t)\|_2^2 dt \right]^{\frac{\gamma}{2}} \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \|\bar{\mathbf{f}}(t)\|_2^2 dt \right]^{\frac{\gamma}{2}} + C(\gamma, K, T) \right\}. \tag{4.59} \end{aligned}$$

An application of Gronwall’s inequality in (4.59) yields

$$\begin{aligned} &\mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{\mathbf{u}}(t)\|_2^2 \right]^{\frac{\gamma}{2}} \\ &\leq C_1 \left\{ \left(\frac{1}{\eta_1} + \kappa \right)^{\frac{\gamma}{2}} \left(\int_{\mathbf{V}} \|\bar{\mathbf{z}}\|_{\mathbf{V}}^2 d\Lambda_0(\bar{\mathbf{z}}) \right)^{\frac{\gamma}{2}} + \left(\int_{\mathbf{L}^2(\mathcal{O}_T)} \|\bar{\mathbf{g}}\|_{\mathbf{L}^2(\mathcal{O}_T)}^2 d\Lambda_{\bar{\mathbf{f}}}(\bar{\mathbf{g}}) \right)^{\frac{\gamma}{2}} + C(\gamma, K, T) \right\}, \end{aligned}$$

where $C_1 := C_1(\kappa, \gamma, \eta_1, K, T)$. Collecting the obtained information in (4.59), we obtain the required result. □

4.7 Non-Stationary flows

In this subsection, we prove the main result of this work, that is, the *existence of martingale solutions* to the stochastic problem (1.1)-(1.4).

Proof of Theorem 2.6 The proof of the theorem is lengthy, so we have divided it into several steps. We start by approximating the original problem by an auxiliary problem under the conditions of the previous sections. By Theorems 4.1 and 4.2, we have the existence of solutions to the approximate system (4.2). Later, we obtain the uniform estimates, followed by

weak convergence of subsequences as a direct application of the Banach-Alaoglu theorem. In the second part, we establish some compactness arguments for the solution to the approximate system (4.2). We pass to the limit in the viscous term with the help of the monotone operator theory.

Step (1): (A priori estimate and weak convergence). We consider the following system:

$$\begin{cases} d(\mathbb{I} - \kappa \Delta)\mathbf{u}_n(t) = [\operatorname{div}(\mathbf{A}(\mathbf{u}_n(t))) - \operatorname{div}(\mathbf{u}_n(t) \otimes \mathbf{u}_n(t)) - \frac{1}{n}|\mathbf{u}_n(t)|^{q-2}\mathbf{u}_n(t) \\ \quad + \nabla\pi(t) + \mathbf{f}_n(t)]dt + \Phi(t, \mathbf{u}_n(t))d\mathbf{W}(t), \\ \mathbf{u}_n(0) = \mathbf{u}_0^n. \end{cases} \tag{4.60}$$

Using Theorems 4.1 and 4.2, $\alpha = \frac{1}{n}$, we have the existence of a martingale solution

$$((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), \mathbf{u}_n, \mathbf{u}_0^n, \mathbf{f}_n, \mathbf{W})$$

to (4.60) with $\mathbf{u}_n \in \mathcal{V}_{p,q}$, $\Lambda_0 = \mathbb{P} \circ (\mathbf{u}_0^n)^{-1}$ and $\Lambda_{\mathbf{f}_n} = \mathbb{P} \circ (\mathbf{f}_n)^{-1}$. For the sake of writing, we have removed the underline bars. In view of the aforementioned results, we can write \mathbb{P} -a.s.,

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}_n(t) \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_n(t) : \nabla \boldsymbol{\phi} \, dx + \nu \int_0^t \int_{\mathcal{O}} \mathbf{A}(\mathbf{u}_n) : \mathbf{D}(\boldsymbol{\phi}) \, dx ds \\ & + \frac{1}{n} \int_0^t \int_{\mathcal{O}} \mathbf{a}(\mathbf{u}_n) \cdot \boldsymbol{\phi} \, dx ds \\ & = \int_{\mathcal{O}} \mathbf{u}_0^n \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0^n : \nabla \boldsymbol{\phi} \, dx + \int_0^t \int_{\mathcal{O}} \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \boldsymbol{\phi} \, dx ds + \int_0^t \int_{\mathcal{O}} \mathbf{f}_n \cdot \boldsymbol{\phi} \, dx ds \\ & + \int_0^t \int_{\mathcal{O}} \Phi(\mathbf{u}_n) d\mathbf{W}(s) \cdot \boldsymbol{\phi} \, dx, \text{ for all } \boldsymbol{\phi} \in \mathcal{V}. \end{aligned}$$

We can choose the probability space independently of n . The same holds for the Wiener process $\mathbf{W}(\cdot)$. On the other hand, Theorem 4.1 gives us the uniform estimates for \mathbf{u}_n in the space

$$L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathbf{V})) \cap L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; \mathbf{W}_0^{1,p}(\mathcal{O})^d)).$$

By Corollary 4.3 and assumptions (2.9)-(2.10) on Λ_0 and $\Lambda_{\mathbf{f}_n}$, with $\gamma \geq 2$, we find

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in (0, T)} \left\{ \|\mathbf{u}_n(t)\|_2^2 + \kappa \|\nabla \mathbf{u}_n(t)\|_2^2 \right\}^{\frac{\gamma}{2}} \right] + C(p, \mathcal{O}) \nu \mathbb{E} \left[\int_0^T \|\nabla \mathbf{u}_n(t)\|_\rho^p dt \right]^{\frac{\gamma}{2}} \\ & + \frac{C}{n} \mathbb{E} \left[\int_0^T \|\mathbf{u}_n(t)\|_q^q dt \right]^{\frac{\gamma}{2}} \leq C_1(\kappa, \gamma, \eta_1, K, T). \end{aligned} \tag{4.61}$$

Claim 4 For any $p \geq 1$ and

$$\gamma \geq \max \left\{ \frac{pd}{d-2}, 2 + \frac{2p}{d-2} \right\}$$

there holds

$$\mathbb{E} \left[\int_0^T \|\mathbf{u}_n(t)\|_\rho^\rho dt \right] \leq C, \quad \rho := \frac{pd}{d-2} \text{ if } d \neq 2, \text{ and any } \rho \in [1, \infty) \text{ if } d = 2. \tag{4.62}$$

Proof of Claim 4 Assume that $d \neq 2$ (for $d = 2$ is easier). For $1 \leq p \leq 2$, we can use Sobolev’s and Hölder’s inequalities, together with (4.61), so that for $\rho \geq \max\left\{\frac{pd}{d-2}, 2\right\}$, one has

$$\begin{aligned} \mathbb{E}\left[\int_0^T \|\mathbf{u}_n(t)\|_{\frac{pd}{d-2}}^{\frac{pd}{d-2}} dt\right] &\leq C\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}_n(t)\|_2^{\frac{pd}{d-2}} dt\right] = C\mathbb{E}\left[\int_0^T \left[\left(\|\nabla\mathbf{u}_n(t)\|_2^2\right)^{\frac{\gamma}{2}}\right]^{\frac{pd}{\gamma(d-2)}} dt\right] \\ &\leq C(T)\left\{\mathbb{E}\left[\sup_{t \in [0, T]} \left\{\|\mathbf{u}_n(t)\|_2^2 + \kappa\|\nabla\mathbf{u}_n(t)\|_2^2\right\}^{\frac{\gamma}{2}}\right]\right\}^{\frac{pd}{\gamma(d-2)}} \leq C. \end{aligned}$$

If $p > 2$, we can use the embedding

$$L^\infty(0, T; \mathbf{V}) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d) \hookrightarrow L^{\frac{pd}{d-2}}(0, T; L^{\frac{pd}{d-2}}(\mathcal{O})^d),$$

together with the Sobolev and Hölder’s inequalities, and with (4.61), to show that

$$\begin{aligned} \mathbb{E}\left[\int_0^T \|\mathbf{u}_n(t)\|_{\frac{pd}{d-2}}^{\frac{pd}{d-2}} dt\right] &\leq C\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}_n(t)\|_2^{\frac{2p}{d-2}} \|\nabla\mathbf{u}_n(t)\|_p^p dt\right], \quad p \geq 2 \\ &= C\mathbb{E}\left[\int_0^T \left[\left(\|\nabla\mathbf{u}_n(t)\|_2^2\right)^{\frac{\gamma}{2}}\right]^{\frac{2p}{\gamma(d-2)}} \|\nabla\mathbf{u}_n(t)\|_p^p dt\right] \\ &\leq C\left\{\mathbb{E}\left[\sup_{t \in [0, T]} \left\{\|\nabla\mathbf{u}_n(t)\|_2^2\right\}^{\frac{\gamma}{2}}\right]\right\}^{\frac{2p}{\gamma(d-2)}} \\ &\quad \left\{\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}_n(t)\|_p^p dt\right]^{\frac{\gamma(p-2)}{\gamma(d-2)-2p}}\right\}^{\frac{\gamma(d-2)-2p}{\gamma(d-2)}}, \quad \gamma \geq \frac{2p}{d-2} \\ &\leq C(T)\left\{\mathbb{E}\left[\sup_{t \in [0, T]} \left\{\|\mathbf{u}_n(t)\|_2^2 + \kappa\|\nabla\mathbf{u}_n(t)\|_2^2\right\}^{\frac{\gamma}{2}}\right]\right\}^{\frac{2p}{\gamma(d-2)}} \left\{\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}_n(t)\|_p^p dt\right]^{\frac{\gamma}{2}}\right\}^{\frac{2}{\gamma}} \leq C. \end{aligned}$$

The penultimate inequality holds provided $\gamma \geq 2 + \frac{2p}{d-2}$. □

Claim 5 For some $\gamma \geq \frac{2d}{d-2}$, there holds

$$\mathbb{E}\left[\int_0^T \|\mathbf{u}_n(t) \otimes \mathbf{u}_n(t)\|_{q_0}^{q_0} dt + \int_0^T \|\operatorname{div}(\mathbf{u}_n(t) \otimes \mathbf{u}_n(t))\|_{q_0}^{q_0} dt\right] \leq C, \tag{4.63}$$

for $1 \leq q_0 \leq \frac{d}{d-1}$.

Proof of Claim 5 Assume also here that $d \neq 2$. By Hölder’s and Sobolev’s inequalities, along with (4.61), one immediately has

$$\begin{aligned} \mathbb{E}\left[\int_0^T \|\mathbf{u}_n(t) \otimes \mathbf{u}_n(t)\|_{q_0}^{q_0} dt\right] &\leq \mathbb{E}\left[\int_0^T \|\mathbf{u}_n(t)\|_{2q_0}^{2q_0} dt\right] \leq C\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}_n(t)\|_2^{2q_0} dt\right] \\ &\leq CT\mathbb{E}\left[\sup_{t \in [0, T]} \|\nabla\mathbf{u}_n(t)\|_2^{2q_0}\right] \leq C, \end{aligned}$$

for $q_0 \leq \frac{d}{d-2}$ and $\gamma \geq 2q_0$.

In turn, using Hölder’s and Sobolev’s inequalities, together with (4.61), one has

$$\begin{aligned} \mathbb{E} \left[\int_0^T \|\operatorname{div}(\mathbf{u}_n(t) \otimes \mathbf{u}_n(t))\|_{q_0}^{q_0} dt \right] &\leq \mathbb{E} \left[\int_0^T \|\mathbf{u}_n(t)\|\|\nabla \mathbf{u}_n\|_{q_0}^{q_0} dt \right] \\ &\leq \mathbb{E} \left[\int_0^T \|\mathbf{u}_n(t)\|_{\frac{2q_0}{2-q_0}}^{q_0} \|\nabla \mathbf{u}_n(t)\|_2^{q_0} dt \right] \\ &\leq C \mathbb{E} \left[\int_0^T \|\nabla \mathbf{u}_n(t)\|_2^{2q_0} dt \right] \quad \text{for } 1 \leq q_0 \leq \frac{d}{d-1} \\ &\leq CT \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \mathbf{u}_n(t)\|_2^{2q_0} \right] \leq C, \quad \text{for } \gamma \geq 2q_0. \end{aligned}$$

Combining the above estimates, we prove (4.63) for

$$1 \leq q_0 \leq \frac{d}{d-1} \quad \text{and} \quad \gamma \geq \frac{2d}{d-2}. \tag{4.64}$$

The case of $d = 2$ is easy. The first part follows immediately from (4.62) and the second part is an easy consequence of Sobolev’s inequality. \square

Applying the Banach-Alaoglu theorem, we can pass the limit along a subsequence (still denoting by the same index) as follows:

$$\mathbf{u}_n \xrightarrow{n \rightarrow \infty} \mathbf{u}, \quad \text{in } L^{\frac{\gamma p}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; W_0^{1,p}(\mathcal{O})^d)), \tag{4.65}$$

$$\mathbf{u}_n \xrightarrow{n \rightarrow \infty} \mathbf{u}, \quad \text{in } L^\gamma(\Omega, \mathcal{F}, \mathbb{P}; L^\rho(0, T; \mathbf{V})), \quad \forall \rho \geq 1, \tag{4.66}$$

$$\frac{1}{n} |\mathbf{u}_n|^{q-2} \mathbf{u}_n \xrightarrow{n \rightarrow \infty} 0, \quad \text{in } L^{\frac{\gamma q'}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^{q'}(0, T; L^{q'}(\mathcal{O})^d)), \tag{4.67}$$

$$\mathbf{u}_n \otimes \mathbf{u}_n \xrightarrow{n \rightarrow \infty} \mathbf{w}, \quad \text{in } L^{q_0}(\Omega, \mathcal{F}, \mathbb{P}; L^{q_0}(0, T; W^{1,q_0}(\mathcal{O})^{d \times d})), \tag{4.68}$$

$$\mathbf{A}(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \mathbf{S}, \quad \text{in } L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; L^{p'}(\mathcal{O})^{d \times d})), \tag{4.69}$$

$$\mathbf{A}(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \mathbf{S}, \quad \text{in } L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; W^{-1,p'}(\mathcal{O})^{d \times d})), \tag{4.70}$$

$$\Phi(\mathbf{u}_n) \xrightarrow{n \rightarrow \infty} \widehat{\Phi}, \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\rho(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \quad \forall \rho \geq 1. \tag{4.71}$$

Moreover, we have

$$\begin{aligned} \mathbf{u} &\in L^\gamma(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathbf{V})), \\ \widehat{\Phi} &\in L^\gamma(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))). \end{aligned}$$

Now, for the reconstruction of the pressure term, we set

$$\begin{aligned} \mathcal{H}_1^n &:= \mathbf{A}(\mathbf{u}_n), \\ \mathcal{H}_2^n &:= \mathbf{u}_n \otimes \mathbf{u}_n + \nabla \Delta^{-1} \mathbf{f}_n + \nabla \Delta^{-1} \left(\frac{1}{n} |\mathbf{u}_n|^{q-2} \mathbf{u}_n \right), \\ \Phi^n &:= \Phi(\mathbf{u}_n). \end{aligned}$$

Applying Theorem 3.1, Corollaries 3.5 and 3.6, we get functions π_h^n, π_1^n and π_2^n adapted to $\{\mathcal{F}_t\}_{t \in [0, T]}$ and Φ_π^n progressively measurable such that \mathbb{P} -a.s.,

$$\int_{\mathcal{O}} (\mathbf{u}_n - \nabla \pi_h^n)(t) \cdot \boldsymbol{\phi} \, dx + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_n(t) : \nabla \boldsymbol{\phi} \, dx$$

$$\begin{aligned}
 &= \int_{\mathcal{O}} \mathbf{u}_0^n \cdot \boldsymbol{\phi} \, d\mathbf{x} + \kappa \int_{\mathcal{O}} \nabla \mathbf{u}_0^n : \nabla \boldsymbol{\phi} \, d\mathbf{x} - \nu \int_0^t \int_{\mathcal{O}} (\mathcal{H}_1^n - \pi_1^n \mathbf{I}) : \nabla \boldsymbol{\phi} \, d\mathbf{x} \, ds \\
 &+ \int_0^t \int_{\mathcal{O}} \operatorname{div} (\mathcal{H}_2^n - \pi_2^n \mathbf{I}) : \boldsymbol{\phi} \, d\mathbf{x} \, ds + \int_0^t \int_{\mathcal{O}} \Phi^n \, dW(s) \cdot \boldsymbol{\phi} \, d\mathbf{x} + \int_0^t \int_{\mathcal{O}} \Phi_{\pi}^n \, dW(s) \cdot \boldsymbol{\phi} \, d\mathbf{x}.
 \end{aligned}
 \tag{4.72}$$

Hence, the following functions are uniformly bounded, with respect to n , in the below mentioned spaces

$$\mathcal{H}_1^n \in L^{\frac{\nu p'}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; L^{p'}(\mathcal{O})^{d \times d})), \tag{4.73}$$

$$\mathcal{H}_2^n \in L^{q_0}(\Omega, \mathcal{F}, \mathbb{P}; L^{q_0}(0, T; W^{1, q_0}(\mathcal{O})^{d \times d})), \tag{4.74}$$

$$\Phi^n \in L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\infty}(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \tag{4.75}$$

where we have used the continuity of $\nabla \Delta^{-1}$ from $L^{q_0}(\mathcal{O})^d$ to $W^{1, q_0}(\mathcal{O})^{d \times d}$. We know that the corresponding pressure functions are also uniformly bounded in the scalar spaces. That is

$$\pi_h^n \in L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\infty}(0, T; L^2(\mathcal{O}))), \tag{4.76}$$

$$\pi_1^n \in L^{\frac{\nu p'}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; L^{p'}(\mathcal{O}))), \tag{4.77}$$

$$\pi_2^n \in L^{q_0}(\Omega, \mathcal{F}, \mathbb{P}; L^{q_0}(0, T; W^{1, q_0}(\mathcal{O}))), \tag{4.78}$$

$$\Phi_{\pi}^n \in L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\infty}(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \tag{4.79}$$

where we have applied Corollaries 3.5 and 3.6. Using the regularity theory for the harmonic functions and Corollary 3.6 to the harmonic pressure term, we find that

$$\pi_h^n \in L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\rho}(0, T; W^{k, \infty}(\mathcal{O}))), \quad \forall k \in \mathbb{N}, \quad \forall \rho \geq 1. \tag{4.80}$$

Passing $n \rightarrow \infty$ along subsequences, we obtain the following convergence results:

$$\pi_h^n \xrightarrow{n \rightarrow \infty} \pi_h, \quad \text{in } L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\rho}(0, T; W^{k, \rho}(\mathcal{O}))), \quad \forall \rho \geq 1, \tag{4.81}$$

$$\pi_1^n \xrightarrow{n \rightarrow \infty} \pi_1, \quad \text{in } L^{\frac{\nu p'}{2}}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(0, T; L^{p'}(\mathcal{O}))), \tag{4.82}$$

$$\pi_2^n \xrightarrow{n \rightarrow \infty} \pi_2, \quad \text{in } L^{q_0}(\Omega, \mathcal{F}, \mathbb{P}; L^{q_0}(0, T; W^{1, q_0}(\mathcal{O}))), \tag{4.83}$$

$$\Phi_{\pi}^n \xrightarrow{n \rightarrow \infty} \Phi_{\pi}, \quad \text{in } L^{\gamma}(\Omega, \mathcal{F}, \mathbb{P}; L^{\rho}(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \quad \forall \rho \geq 1. \tag{4.84}$$

Now, our goal is to show that in (4.68) and (4.71), we have $\mathbf{w} = \mathbf{u} \otimes \mathbf{u}$ and $\widehat{\Phi} = \Phi(\mathbf{u})$. To prove this, we use the compactness arguments and a version of Skorokhod’s theorem (see [36, Theorem 2]) which help us in the construction of a new probability space, similar to the proof of Theorem 4.2. In the final part, we will prove that $\mathbf{S} = \mathbf{A}(\mathbf{u})$.

Step (2): (Compactness). In this step, we prove the compactness of the sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$. Here we follow the approach used in [34, Section 4] (see also [14, Section 5]). As the pressure needs to be included in the compact method, we have to work with weak convergence results. However, in this case, we cannot apply the classical Skorokhod theorem due to the presence of the non-metric space (see (4.87) below). Even though there is a generalization of this result that includes weak topologies in Banach spaces: the Jakubowski-Skorokhod theorem (see [36, Theorem 2]), which can be applied to quasi-Polish spaces (see [17]).

From (4.60)-(4.62), we compute that

$$\mathbb{E} \left[\left\| (I - \kappa \Delta) \mathbf{u}_n(t) - \int_0^t \Phi(\mathbf{u}_n) dW(s) \right\|_{W^{1,q_0}(0,T;W_\sigma^{-1,q_0}(\mathcal{O}^d))} \right] \leq C.$$

For the stochastic integral term we use (4.31) (see [34, Lemma 4.6]), for some $\mu > 0$

$$\mathbb{E} \left[\left\| \int_0^t \Phi(\mathbf{u}_n) dW(s) \right\|_{C^\mu([0,T];L^2(\mathcal{O}^d))} \right] \leq C,$$

which is a consequence of (4.61), and the assumption (2.11). Using the above information and similar arguments to (4.33) and (4.34), we find

$$\mathbb{E} \left[\|\mathbf{u}_n\|_{W^{\eta,q_0}(0,T;W_\sigma^{1,q_0}(\mathcal{O}^d))} \right] \leq C, \quad \text{for some } \eta > 0. \tag{4.85}$$

Using the embedding $W_\sigma^{1,q_0}(\mathcal{O}^d) \hookrightarrow L_\sigma^{q_0}(\mathcal{O}^d)$, for any $q_0 \in [1, \infty)$, we obtain

$$\mathbb{E} \left[\|\mathbf{u}_n\|_{W^{\eta,q_0}(0,T;L_\sigma^{q_0}(\mathcal{O}^d))} \right] \leq C. \tag{4.86}$$

Again, by [29, Theorem 2.1], for $p > \frac{2d}{d+2}$, we obtain the following compact embedding:

$$W^{\eta,q_0}(0,T;L_\sigma^{q_0}(\mathcal{O}^d)) \cap L^p(0,T;W_{0,\sigma}^{1,p}(\mathcal{O}^d)) \hookrightarrow L^\rho(0,T;L_\sigma^\rho(\mathcal{O}^d)),$$

for all $\rho < \min \left\{ \frac{p(d+2)}{d}, \frac{2d}{d-2} \right\}$. Note that $\rho < \min \left\{ \frac{p(d+2)}{d}, p^*, 2^* \right\}$ and for $p > \frac{2d}{d+2}$ we know that $\frac{p(d+2)}{d} < p^*$.

We use the above compact embedding for the compactness of $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$. One can argue for the harmonic pressure term $\{\pi_h^n\}_{n \in \mathbb{N}}$ in a similar manner. Reasoning as in [65, Eqn. (4.24)], we can combine this with the regularity theory for harmonic functions and with the Lebesgue dominated convergence theorem to prove that the following compact embedding holds:

$$L^\infty(0,T;L^2(\mathcal{O})) \cap \{\Delta u(t) = 0, \text{ for a.e. } t\} \hookrightarrow L^\rho(0,T;L^\rho(\mathcal{O})).$$

Now, we define a path space

$$\begin{aligned} \mathfrak{X} := & L^\rho(0,T;L_\sigma^\rho(\mathcal{O}^d)) \otimes L^\rho(0,T;L^\rho(\mathcal{O})) \otimes L_w^{p'}(0,T;L^{p'}(\mathcal{O})) \otimes L_w^{q_0}(0,T;W^{1,q_0}(\mathcal{O})) \\ & \otimes L_w^\rho(0,T;L_2(\mathbf{U},L^2(\mathcal{O}^d))) \otimes C([0,T];\mathbf{U}_0) \otimes \mathbf{V} \otimes L^2(\mathcal{O}_T), \end{aligned} \tag{4.87}$$

where w denotes the weak topology. The following notations are to be used next:

- (1) $\mathcal{Q}_{\mathbf{u}_n}$ denotes the law of \mathbf{u}_n on $L^\rho(0,T;L_\sigma^\rho(\mathcal{O}^d))$,
- (2) $\mathcal{Q}_{\pi_h^n}$ denotes the law of π_h^n on $L^\rho(0,T;L^\rho(\mathcal{O}))$,
- (3) $\mathcal{Q}_{\pi_1^n}$ denotes the law of π_1^n on $L_w^{p'}(0,T;L^{p'}(\mathcal{O}))$,
- (4) $\mathcal{Q}_{\pi_2^n}$ denotes the law of π_2^n on $L_w^{q_0}(0,T;W^{1,q_0}(\mathcal{O}))$,
- (5) $\mathcal{Q}_{\Phi_\pi^n}$ denotes the law of Φ_π^n on $L_w^\rho(0,T;L_2(\mathbf{U},L^2(\mathcal{O}^d)))$,
- (6) \mathcal{Q}_{W_n} denotes the law of W_n on $C([0,T];\mathbf{U}_0)$,
- (7) \mathcal{Q}_n denotes the joint law of $\mathbf{u}_n, \pi_h^n, \pi_1^n, \pi_2^n, \Phi_\pi^n, W_n, \mathbf{u}_0^n, \mathbf{f}_n$ on \mathfrak{X} .

Now, our main goal is to prove the tightness of the measure \mathcal{Q}_n . For that, we first consider the ball B_N in the space $W^{\eta,q_0}(0,T;L_\sigma^{q_0}(\mathcal{O}^d)) \cap L^p(0,T;W_{0,\sigma}^{1,p}(\mathcal{O}^d))$ and the complement of this ball is denoted by B_N^c . Thus, we have

$$\mathcal{Q}_{\mathbf{u}_n}(B_N^c) = \mathbb{P} \left(\|\mathbf{u}_n\|_{W_\sigma^{\eta,q_0}(0,T;L_\sigma^{q_0}(\mathcal{O}^d))} + \|\mathbf{u}_n\|_{L^p(0,T;W_{0,\sigma}^{1,p}(\mathcal{O}^d))} \geq N \right)$$

$$\leq \frac{1}{N} \mathbb{E} \left[\|\mathbf{u}_n\|_{W_\sigma^{\eta, q_0}(0, T; L_\sigma^{q_0}(\mathcal{O})^d)} + \|\mathbf{u}_n\|_{L^p(0, T; W_{0, \sigma}^{1, p}(\mathcal{O})^d)} \right] \leq \frac{C}{N},$$

where we have used (4.61) and (4.86). Thus, for any fixed $\xi > 0$, we can find $N(\xi)$ such that

$$\varrho_{\mathbf{u}_n}(B_{N(\xi)}) \geq 1 - \frac{\xi}{8}.$$

Using (4.80), we can also prove that the law of π_h^n is tight, that is, there exists a compact subset $K_\pi \subset L^p(0, T; L^p(\mathcal{O}))$ such that $\varrho_{\pi_h^n}(K_\pi) \geq 1 - \frac{\xi}{8}$. As we know that our spaces are reflexive, therefore we can find compact sets for π_1^n, π_2^n and Φ_π^n with measures $\geq 1 - \frac{\xi}{8}$. And the law ϱ_W is tight as it is the same with the law of Brownian motion W which is a Radon measure on the space $C([0, T]; \mathbf{U}_0)$. So, there exists another compact set $K_\xi \subset C([0, T]; \mathbf{U}_0)$ with $\varrho_{W_n}(K_\xi) \geq 1 - \frac{\xi}{8}$. Using the similar arguments, we can find compact subsets of \mathbf{V} and $L^2(\mathcal{O}_T)$ with their measures $(\Lambda_0$ and $\Lambda_{f_n}) \geq 1 - \frac{\xi}{8}$. Therefore, there exists a compact subset $\mathfrak{Y}_\xi \subset \mathfrak{Y}$ with $\varrho_n(\mathfrak{Y}_\xi) \geq 1 - \xi$. Hence, $\{\varrho_n\}_{n \in \mathbb{N}}$ is tight in the same space. Applying the Jakubowski version of Skorokhod Theorem (see [36, Theorem 2]), we ensure the existence of a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, and a sequence of random variables $(\bar{\mathbf{u}}_n, \bar{\pi}_h^n, \bar{\pi}_1^n, \bar{\pi}_2^n, \bar{\Phi}_\pi^n, \bar{W}_n, \bar{\mathbf{u}}_0^n, \bar{f}_n)$ and a random variable $(\bar{\mathbf{u}}, \bar{\pi}_h, \bar{\pi}_1, \bar{\pi}_2, \bar{\Phi}_\pi, \bar{W}, \bar{\mathbf{u}}_0, \bar{f})$ defined on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ taking values in the space \mathfrak{Y} (see (4.87)) such that the following holds:

- (1) The laws of $(\bar{\mathbf{u}}_n, \bar{\pi}_h^n, \bar{\pi}_1^n, \bar{\pi}_2^n, \bar{\Phi}_\pi^n, \bar{W}_n, \bar{\mathbf{u}}_0^n, \bar{f}_n)$ and $(\bar{\mathbf{u}}, \bar{\pi}_h, \bar{\pi}_1, \bar{\pi}_2, \bar{\Phi}_\pi, \bar{W}, \bar{\mathbf{u}}_0, \bar{f})$ are the same under $\bar{\mathbb{P}}$ and coincide with ϱ_n and $\varrho := \lim_{n \rightarrow \infty} \varrho_n$, respectively;
- (2) The following weak convergence holds

$$\begin{aligned} \bar{\pi}_1^n &\xrightarrow[n \rightarrow \infty]{} \bar{\pi}_1, && \text{in } L^{p'}(0, T; L^{p'}(\mathcal{O})), \\ \bar{\pi}_2^n &\xrightarrow[n \rightarrow \infty]{} \bar{\pi}_2, && \text{in } L^{q_0}(0, T; W^{1, q_0}(\mathcal{O})), \\ \bar{\Phi}_\pi^n &\xrightarrow[n \rightarrow \infty]{} \bar{\Phi}_\pi, && \text{in } L^p(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d)), \end{aligned}$$

$\bar{\mathbb{P}}$ -a.s.;

- (3) The following strong convergence holds

$$\begin{aligned} \bar{\mathbf{u}}_n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}, && \text{in } L^p(0, T; L_\sigma^p(\mathcal{O})^d), \\ \bar{\pi}_h^n &\xrightarrow[n \rightarrow \infty]{} \bar{\pi}_h, && \text{in } L^p(0, T; L^p(\mathcal{O})), \\ \bar{W}_n &\xrightarrow[n \rightarrow \infty]{} \bar{W}, && \text{in } C([0, T]; \mathbf{U}_0), \\ \bar{\mathbf{u}}_0^n &\xrightarrow[n \rightarrow \infty]{} \bar{\mathbf{u}}_0, && \text{in } \mathbf{V}, \\ \bar{f}_n &\xrightarrow[n \rightarrow \infty]{} \bar{f}, && \text{in } L^2(0, T; L^2(\mathcal{O})^d), \end{aligned}$$

$\bar{\mathbb{P}}$ -a.s.;

- (4) For all $\beta < \infty$, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{W}_n(t)\|_{\mathbf{U}_0}^\beta \right] = \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{W}(t)\|_{\mathbf{U}_0}^\beta \right].$$

Using the equivalency in distributions, we find the following weak convergence results:

$$\bar{\pi}_1^n \xrightarrow[n \rightarrow \infty]{} \bar{\pi}_1, \quad \text{in } L^{p'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{p'}(0, T; L^{p'}(\mathcal{O}))),$$

$$\begin{aligned} \bar{\pi}_2^n &\xrightarrow{n \rightarrow \infty} \bar{\pi}_2, & \text{in } L^{q_0}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q_0}(0, T; W^{1, q_0}(\mathcal{O}))), \\ \bar{\Phi}_\pi^n &\xrightarrow{n \rightarrow \infty} \bar{\Phi}_\pi, & \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))). \end{aligned}$$

Using Vitali’s convergence theorem along subsequences, we obtain the following strong convergence results:

$$\bar{W}_n \xrightarrow{n \rightarrow \infty} \bar{W}, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; C([0, T]; \mathbf{U}_0)), \tag{4.88}$$

$$\bar{u}_n \xrightarrow{n \rightarrow \infty} \bar{u}, \quad \text{in } L^\rho(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; L^\rho_\sigma(\mathcal{O})^d)), \tag{4.89}$$

$$\nabla^j \bar{\pi}_h^n \xrightarrow{n \rightarrow \infty} \nabla^j \bar{\pi}_h, \quad \text{in } L^\rho(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; L^\rho(\mathcal{O}))), \tag{4.90}$$

$$\bar{u}_0^n \xrightarrow{n \rightarrow \infty} \bar{u}_0, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; \mathbf{V}), \tag{4.91}$$

$$\bar{f}_n \xrightarrow{n \rightarrow \infty} \bar{f}, \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^2(0, T; L^2(\mathcal{O})^d)), \tag{4.92}$$

for all $\rho < \min \left\{ \frac{p(d+2)}{d}, \frac{2d}{d-2} \right\}$. In the harmonic pressure term, we have used the regularity theory for harmonic maps. As a consequence, for any $\beta < \infty$, we have

$$\bar{u}_n \otimes \bar{u}_n \xrightarrow{n \rightarrow \infty} \bar{u} \otimes \bar{u}, \quad \text{in } L^{q_0}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q_0}(0, T; W^{1, q_0}(\mathcal{O})^{d \times d})), \tag{4.93}$$

$$\Phi(\bar{u}_n) \xrightarrow{n \rightarrow \infty} \Phi(\bar{u}), \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\beta(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \tag{4.94}$$

$$\Phi_\pi(\bar{u}_n) \xrightarrow{n \rightarrow \infty} \Phi_\pi(\bar{u}), \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\beta(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))). \tag{4.95}$$

Now, we define the $\bar{\mathbb{P}}$ -augmented canonical filtration, which is denoted by $\{\bar{\mathcal{F}}_t\}_{t \geq 0}$, of the process $(\bar{u}, \bar{\pi}_h, \bar{\pi}_1, \bar{\pi}_2, \bar{\Phi}, \bar{W}, \bar{f})$, that is,

$$\bar{\mathcal{F}}_t = \sigma(\sigma(\mathbf{h}_t \bar{u}, \mathbf{h}_t \bar{\pi}_h, \mathbf{h}_t \bar{\pi}_1, \mathbf{h}_t \bar{\pi}_2, \mathbf{h}_t \bar{\Phi}_\pi, \mathbf{h}_t \bar{W}, \mathbf{h}_t \bar{f}) \cup \{M \in \bar{\mathcal{F}}; \bar{\mathbb{P}}(M) = 0\}), \quad t \in [0, T].$$

Proceeding as in the proof of Theorem 4.2, choosing in the present case (non-divergence free) test functions from $C_0^\infty(\mathcal{O})^d$, we can show that the following equation holds on the new probability space. That is, we have \mathbb{P} -a.s.,

$$\begin{aligned} &\int_{\mathcal{O}} (\bar{u}_n - \nabla \bar{\pi}_h^n)(t) \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{u}_n(t) : \nabla \phi \, dx \\ &= \int_{\mathcal{O}} \bar{u}_0^n \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{u}_0^n : \nabla \phi \, dx - \nu \int_0^t \int_{\mathcal{O}} (\bar{\mathcal{H}}_1^n - \bar{\pi}_1^n \mathbf{I}) : \nabla \phi \, dx ds \\ &\quad + \int_0^t \int_{\mathcal{O}} \operatorname{div}(\bar{\mathcal{H}}_2^n - \bar{\pi}_2^n \mathbf{I}) \cdot \phi \, dx ds + \int_0^t \int_{\mathcal{O}} \Phi(\bar{u}_n) d\bar{W}_n(s) \cdot \phi \, dx \end{aligned} \tag{4.96}$$

$$+ \int_0^t \int_{\mathcal{O}} \bar{\Phi}_\pi^n d\bar{W}_n(s) \cdot \phi \, dx, \tag{4.97}$$

for all $t \in [0, T]$, and $\phi \in C_0^\infty(\mathcal{O})^d$, where we have set

$$\bar{\mathcal{H}}_1^n := \mathbf{A}(\bar{u}_n),$$

$$\bar{\mathcal{H}}_2^n := \bar{u}_n \otimes \bar{u}_n + \nabla \Delta^{-1} \bar{f}_n + \nabla \Delta^{-1} \left(\frac{1}{n} |\bar{u}_n|^{q-2} \bar{u}_n \right).$$

Passing $n \rightarrow \infty$, using the above convergence results, and [26, Lemma 2.1] for the convergence of the stochastic integral term, we arrive at $\bar{\mathbb{P}}$ -a.s.,

$$\int_{\mathcal{O}} (\bar{u} - \nabla \bar{\pi}_h)(t) \cdot \phi \, dx + \kappa \int_{\mathcal{O}} \nabla \bar{u}(t) : \nabla \phi \, dx$$

$$\begin{aligned}
 &= \int_{\mathcal{O}} \bar{\mathbf{u}}_0 \cdot \boldsymbol{\phi} \, d\mathbf{x} + \kappa \int_{\mathcal{O}} \nabla \bar{\mathbf{u}}_0 : \nabla \boldsymbol{\phi} \, d\mathbf{x} - \nu \int_0^t \int_{\mathcal{O}} (\bar{\mathcal{H}}_1 - \bar{\pi}_1 \mathbf{I}) : \nabla \boldsymbol{\phi} \, d\mathbf{x} ds \\
 &+ \int_0^t \int_{\mathcal{O}} \operatorname{div} (\bar{\mathcal{H}}_2 - \bar{\pi}_2 \mathbf{I}) \cdot \boldsymbol{\phi} \, d\mathbf{x} ds + \int_0^t \int_{\mathcal{O}} \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}}(s) \cdot \boldsymbol{\phi} \, d\mathbf{x} + \int_0^t \int_{\mathcal{O}} \bar{\Phi}_\pi d\bar{\mathbf{W}}(s) \cdot \boldsymbol{\phi} \, d\mathbf{x},
 \end{aligned} \tag{4.98}$$

for all $\boldsymbol{\phi} \in C_0^\infty(\mathcal{O})^d$, where $\bar{\mathcal{H}}_1 := \bar{\mathbf{S}}$ and $\bar{\mathcal{H}}_2 := \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \nabla \Delta^{-1} \bar{\mathbf{f}}$.

Now, our aim is to show that $\bar{\mathbf{S}} = \mathbf{A}(\bar{\mathbf{u}})$. Let us set

$$\begin{aligned}
 \bar{\mathcal{G}}_1^n &:= \mathbf{A}(\bar{\mathbf{u}}_n) - \bar{\mathbf{S}}, \\
 \bar{\mathcal{G}}_2^n &:= \bar{\mathbf{u}}_n \otimes \bar{\mathbf{u}}_n - \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \nabla \Delta^{-1} (\bar{\mathbf{f}}_n - \bar{\mathbf{f}}) + \nabla \Delta^{-1} \left(\frac{1}{n} |\bar{\mathbf{u}}_n|^{q-2} \bar{\mathbf{u}}_n \right), \\
 \bar{\theta}_h^n &:= \bar{\pi}_h^n - \bar{\pi}_h, \quad \bar{\theta}_1^n := \bar{\pi}_1^n - \bar{\pi}_1, \quad \bar{\theta}_2^n := \bar{\pi}_2^n - \bar{\pi}_2.
 \end{aligned}$$

The following convergence results hold true:

$$\bar{\mathbf{u}}_n - \bar{\mathbf{u}} \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^{\frac{\gamma p}{2}}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^p(0, T; W_0^{1,p}(\mathcal{O})^d)), \tag{4.99}$$

$$\bar{\mathbf{u}}_n - \bar{\mathbf{u}} \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; \mathbf{V})), \quad \forall \rho < \infty, \tag{4.100}$$

$$\bar{\mathcal{G}}_1^n \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^{\frac{\gamma p'}{2}}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{p'}(0, T; L^{p'}(\mathcal{O})^{d \times d})), \tag{4.101}$$

$$\bar{\mathcal{G}}_2^n \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^{q_0}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q_0}(0, T; W^{1,q_0}(\mathcal{O})^{d \times d})), \tag{4.102}$$

$$\Phi(\bar{\mathbf{u}}_n) - \Phi(\bar{\mathbf{u}}) \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \quad \forall \rho < \infty. \tag{4.103}$$

For the pressure terms, we have the following convergence results:

$$\bar{\theta}_h^n \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; W^{j,\rho}(\mathcal{O}))), \quad \forall \rho < \infty, \tag{4.104}$$

$$\bar{\theta}_1^n \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^{\frac{\gamma p'}{2}}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{p'}(0, T; L^{p'}(\mathcal{O}))), \tag{4.105}$$

$$\bar{\theta}_2^n \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^{q_0}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q_0}(0, T; W^{1,q_0}(\mathcal{O}))), \tag{4.106}$$

$$\Phi_\pi(\bar{\mathbf{u}}_n) - \Phi_\pi(\bar{\mathbf{u}}) \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{in } L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\rho(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \quad \forall \rho < \infty. \tag{4.107}$$

Moreover, we have

$$\bar{\theta}_h^n \in L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\infty(0, T; L^2(\mathcal{O}))), \tag{4.108}$$

$$\Phi(\bar{\mathbf{u}}_n), \Phi_\pi(\bar{\mathbf{u}}_n) \in L^\gamma(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^\infty(0, T; \mathcal{L}_2(\mathbf{U}, L^2(\mathcal{O})^d))), \tag{4.109}$$

uniformly in n . We can rewrite the difference between the approximate equation (4.96) and limit equation (4.98) as $\bar{\mathbb{P}}$ -a.s.,

$$\begin{aligned}
 &\int_{\mathcal{O}} (\bar{\mathbf{u}}_n - \bar{\mathbf{u}} - \nabla \bar{\theta}_h^n)(t) \cdot \boldsymbol{\phi} \, d\mathbf{x} + \kappa \int_{\mathcal{O}} (\nabla(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}))(t) : \nabla \boldsymbol{\phi} \, d\mathbf{x} \\
 &= \int_{\mathcal{O}} (\bar{\mathbf{u}}_0^n - \bar{\mathbf{u}}_0) \cdot \boldsymbol{\phi} \, d\mathbf{x} + \kappa \int_{\mathcal{O}} (\nabla(\bar{\mathbf{u}}_0^n - \bar{\mathbf{u}}_0)) : \nabla \boldsymbol{\phi} \, d\mathbf{x} - \nu \int_0^t \int_{\mathcal{O}} (\bar{\mathcal{G}}_1^n - \bar{\theta}_1^n \mathbf{I}) : \nabla \boldsymbol{\phi} \, d\mathbf{x} ds \\
 &+ \int_0^t \int_{\mathcal{O}} \operatorname{div} (\bar{\mathcal{G}}_2^n - \bar{\theta}_2^n \mathbf{I}) \cdot \boldsymbol{\phi} \, d\mathbf{x} ds + \int_0^t \int_{\mathcal{O}} (\Phi(\bar{\mathbf{u}}_n) d\bar{\mathbf{W}}_n(s) - \Phi(\bar{\mathbf{u}}) d\bar{\mathbf{W}}(s)) \cdot \boldsymbol{\phi} \, d\mathbf{x}
 \end{aligned}$$

$$+ \int_0^t \int_{\mathcal{O}} (\Phi_{\pi}(\bar{u}_n) d\bar{W}_n - \Phi_{\pi}(\bar{u}) d\bar{W}(s)) \cdot \phi \, dx, \tag{4.110}$$

for all $\phi \in C_0^\infty(\mathcal{O})^d$.

We introduce the sequence $\bar{v}_n := \bar{u}_n - \nabla \bar{\pi}_h^n$, and the sequence $\bar{v}_{n,1} = \bar{v}_n - \bar{v}$, where $\bar{v} = \bar{u} - \nabla \bar{\pi}_h$, for which we have

$$\bar{v}_{n,1} \xrightarrow{n \rightarrow \infty} 0, \text{ in } L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^p(0, T; W_0^{1,p}(\mathcal{O})^d)), \tag{4.111}$$

$$\bar{v}_{n,1} \xrightarrow{n \rightarrow \infty} 0, \text{ in } L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^p(0, T; L_\sigma^p(\mathcal{O})^d)). \tag{4.112}$$

Thus, for all $\phi \in C_0^\infty(\mathcal{O})^d$, we arrive at $\bar{\mathbb{P}}$ -a.s.,

$$\begin{aligned} & \int_{\mathcal{O}} (I - \kappa \Delta) \bar{v}_{n,1}(t) \cdot \phi \, dx \\ &= \int_{\mathcal{O}} \bar{v}_0^{n,1} \cdot \phi \, dx + \nu \int_0^t \int_{\mathcal{O}} \operatorname{div} (\bar{\mathbb{G}}_1^n - \bar{\theta}_1^n I) \cdot \phi \, dx ds \\ &+ \int_0^t \int_{\mathcal{O}} \operatorname{div} (\bar{\mathbb{G}}_2^n - \bar{\theta}_2^n I) \cdot \phi \, dx ds + \int_0^t \int_{\mathcal{O}} (\Phi(\bar{u}_n) d\bar{W}_n(s) - \Phi(\bar{u}) d\bar{W}(s)) \cdot \phi \, dx \\ &+ \int_0^t \int_{\mathcal{O}} (\Phi_{\pi}(\bar{u}_n) d\bar{W}_n - \Phi_{\pi}(\bar{u}) d\bar{W}(s)) \cdot \phi \, dx. \end{aligned} \tag{4.113}$$

Let us prove that if we test the above expression with $\bar{v}_{n,1}$ (validity of Itô’s formula), then the right hand side of (4.113) is well-defined. In view of the regularity of $\bar{v}_{n,1}$, we only need to work on the third term of the right hand side of (4.113). Assume that $d \neq 2$, as the case of $d = 2$ is easier. By Sobolev’s embedding, we have

$$\mathbf{V} \hookrightarrow L^{q'_0}(\mathcal{O})^d, \text{ for } q'_0 \leq \frac{2d}{d-2} \text{ or } q_0 \geq \frac{2d}{d+2}.$$

From (4.64), we obtain

$$\frac{2d}{d+2} \leq q_0 \leq \frac{d}{d-1}, \text{ for } d \leq 4.$$

Combining the above facts, we conclude that

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} \operatorname{div} (\bar{\mathbb{G}}_2^n - \bar{\theta}_2^n I) \cdot \bar{v}_{n,1} \, dx ds \right] \\ & \leq \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} \|\operatorname{div} (\bar{\mathbb{G}}_2^n - \bar{\theta}_2^n I)\|_{q_0} \|\bar{v}_{n,1}\|_{q'_0} \, dx ds \right] \\ & \leq \mathbb{E} \left[\left(\int_0^t \int_{\mathcal{O}} \|\bar{v}_{n,1}\|_{q'_0}^{q'_0} \, dx ds \right)^{\frac{1}{q'_0}} \left(\int_0^t \int_{\mathcal{O}} \|\operatorname{div} (\bar{\mathbb{G}}_2^n - \bar{\theta}_2^n I)\|_{q_0}^{q_0} \, dx ds \right)^{\frac{1}{q_0}} \right] \\ & \leq T^{\frac{1}{q'_0}} \left\{ \mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla \bar{v}_{n,1}(s)\|_2^{q'_0} \right] \right\}^{\frac{1}{q'_0}} \left\{ \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} \|\operatorname{div} (\bar{\mathbb{G}}_2^n - \bar{\theta}_2^n I)\|_{q_0}^{q_0} \, dx ds \right] \right\}^{\frac{1}{q_0}} \\ & < \infty, \end{aligned} \tag{4.114}$$

where we have used Hölder’s inequality.

Claim 6

$$\bar{\mathbf{S}} = \mathbf{A}(\bar{u}). \tag{4.115}$$

Proof of Claim 6 Using density arguments, we can choose our test functions $\phi \in W_0^{1,p}(\mathcal{O})^d \cap W_0^{1,2}(\mathcal{O})^d \cap L^{q_0'}(\mathcal{O})^d$. In view of the regularity of $\bar{v}_{n,1}(\cdot)$, we can apply the infinite-dimensional Itô formula to the process $\|(I - \kappa \Delta)^{\frac{1}{2}} \bar{v}_{n,1}(\cdot)\|_{\mathbb{H}}^2$, to find (see Subsection 4.8 for a justification)

$$\begin{aligned} & \|\bar{v}_{n,1}(t)\|_2^2 + \kappa \|\nabla \bar{v}_{n,1}(t)\|_2^2 \\ &= \|\bar{v}_0^{n,1}\|_2^2 + \kappa \|\nabla \bar{v}_0^{n,1}\|_2^2 + 2\nu \int_0^t \int_{\mathcal{O}} \operatorname{div}(\bar{G}_1^n - \bar{\theta}_1^n \mathbf{1}) \cdot \bar{v}_{n,1} \, d\mathbf{x} \, ds \\ & \quad + 2 \int_0^t \int_{\mathcal{O}} \operatorname{div}(\bar{G}_2^n - \bar{\theta}_2^n) \cdot \bar{v}_{n,1} \, d\mathbf{x} \, ds + 2 \int_0^t \int_{\mathcal{O}} (\Phi(\bar{u}_n) d\bar{W}_n(s) - \Phi(\bar{u}) d\bar{W}(s)) \cdot \bar{v}_{n,1} \, d\mathbf{x} \\ & \quad + 2 \int_0^t \int_{\mathcal{O}} (\Phi_{\pi}(\bar{u}_n) d\bar{W}_n(s) - \Phi_{\pi}(\bar{u}) d\bar{W}(s)) \cdot \bar{v}_{n,1} \, d\mathbf{x} \\ & \quad + \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} \Phi(\bar{u}_n) d\bar{W}_n - \int_0^{\cdot} \Phi(\bar{u}) d\bar{W} \right\rangle_s \, ds \\ & \quad + \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} \Phi_{\pi}(\bar{u}_n) d\bar{W}_n - \int_0^{\cdot} \Phi_{\pi}(\bar{u}) d\bar{W} \right\rangle_s \, ds \\ & =: \sum_{j=1}^8 I_j^n. \end{aligned} \tag{4.116}$$

Using the strong convergence (4.91) and Theorem 3.1 (2), we obtain $\mathbb{E}[I_1^n], \mathbb{E}[I_2^n] \xrightarrow{n \rightarrow \infty} 0$. Now, our aim is to show that the expected values of the terms I_j^n for $j = 4, \dots, 8$ goes to 0. We know that the expectations of the terms I_5^n and I_6^n are equal to 0, being local martingale terms.

The strong convergence (4.112) gives

$$\bar{v}_{n,1} \xrightarrow{n \rightarrow \infty} 0, \quad \text{in } L^{q_0'}(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^{q_0'}(0, T; L_{\sigma}^{q_0'}(\mathcal{O})^d)), \quad \text{for } \frac{pd}{pd - d + 2} \leq q_0 \leq \frac{d}{d - 1}.$$

Using (4.102), (4.106) and the above consequence of strong convergence (4.112), we conclude that $\mathbb{E}[I_4^n] \xrightarrow{n \rightarrow \infty} 0$ (cf. (4.114)).

Let us consider the term I_7^n , and estimate it using the Cauchy-Schwarz inequality as

$$\begin{aligned} I_7^n &= \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} (\Phi(\bar{u}_n) - \Phi(\bar{u})) dW_n \right\rangle_s \, d\mathbf{x} + \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} \Phi(\bar{u}) d(\bar{W}_n - \bar{W}) \right\rangle_s \, d\mathbf{x} \\ & \quad + \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} (\Phi(\bar{u}_n) - \Phi(\bar{u})) d\bar{W}_n, \int_0^{\cdot} \Phi(\bar{u}) d(\bar{W}_n - \bar{W}) \right\rangle_s \, d\mathbf{x} \\ & \leq C \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} (\Phi(\bar{u}_n) - \Phi(\bar{u})) d\bar{W}_n \right\rangle_s \, d\mathbf{x} + C \int_0^t \int_{\mathcal{O}} d \left\langle \int_0^{\cdot} \Phi(\bar{u}) d(\bar{W}_n - \bar{W}) \right\rangle_s \, d\mathbf{x} \\ & =: C(I_{71}^n + I_{72}^n). \end{aligned}$$

Using (2.11) and (4.89), we find

$$\mathbb{E}[I_{71}^n] \leq C \mathbb{E} \left[\int_0^t \|\Phi(\bar{u}_n) - \Phi(\bar{u})\|_{\mathcal{L}_2}^2 \, ds \right] \leq C \mathbb{E} \left[\int_0^t \|\bar{u}_n - \bar{u}\|_2^2 \, ds \right] \xrightarrow{n \rightarrow \infty} 0.$$

For the term I_{72}^n , we use the fact that $\bar{\mathbf{u}} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; L^2(0, T; \mathbf{V}))$, (2.12) and (4.88), to get

$$\begin{aligned} \mathbb{E}[I_{72}^n] &= \mathbb{E} \left[\int_0^T \sum_i \left(\int_{\mathcal{O}} |\phi_i(\bar{\mathbf{u}})|^2 \text{Var}(\bar{\beta}_i^n(1) - \bar{\beta}_i(1)) \, d\mathbf{x} \right) dt \right] \\ &\leq \mathbb{E} \left[\int_0^T \left(\int_{\mathcal{O}} \sup_i i^2 |\phi_i(\bar{\mathbf{u}})|^2 \, d\mathbf{x} \right) dt \right] \sum_i \frac{1}{i^2} \text{Var}(\bar{\beta}_i^n(1) - \bar{\beta}_i(1)) \\ &\leq C \mathbb{E} \left[\int_0^T \left(\int_{\mathcal{O}} (1 + |\bar{\mathbf{u}}|^2) \, d\mathbf{x} \right) dt \right] \sum_i \frac{1}{i^2} \text{Var}(\bar{\beta}_i^n(1) - \bar{\beta}_i(1)) \\ &\leq C \mathbb{E} \left[\int_0^T \left(\int_{\mathcal{O}} (1 + |\bar{\mathbf{u}}|^2) \, d\mathbf{x} \right) dt \right] \mathbb{E}[\|\bar{\mathbf{W}}_n(1) - \bar{\mathbf{W}}(1)\|_{\mathbf{U}_0}^2] \\ &\leq C \mathbb{E} \left[\int_0^T (1 + \|\bar{\mathbf{u}}\|_2^2) \, dt \right] \mathbb{E}[\|\bar{\mathbf{W}}_n - \bar{\mathbf{W}}\|_{C([0, T]; \mathbf{U}_0)}^2] \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

We know that Φ_π inherits the properties of Φ . Therefore, I_8^n can be estimated in a similar manner.

Taking expectation in (4.116), we find

$$\begin{aligned} &2\nu \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} \bar{\mathbb{G}}_1^n : \mathbf{D}(\bar{\mathbf{v}}_{n,1}) \, d\mathbf{x} \, ds \right] \\ &= -\mathbb{E}[\|\bar{\mathbf{v}}_{n,1}(t)\|_2^2 + \kappa \|\nabla \bar{\mathbf{v}}_{n,1}\|_2^2] + \sum_{j=1}^2 \mathbb{E}[I_j^n] + 2\nu \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} \nabla \bar{\theta}_1^n \cdot \bar{\mathbf{v}}_{n,1} \, d\mathbf{x} \, ds \right] + \sum_{j=4}^8 \mathbb{E}[I_j^n]. \end{aligned}$$

Combining the above estimates, we arrive at

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \bar{\mathbf{S}}) : (\mathbf{D}(\bar{\mathbf{u}}_n) - \mathbf{D}(\bar{\mathbf{u}})) \, d\mathbf{x} \, ds \right] \leq 0, \tag{4.117}$$

where we have used the fact that $\text{div } \bar{\mathbf{v}}_{n,1} = 0$. Let us now consider

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \mathbf{A}(\bar{\mathbf{u}})) : (\mathbf{D}(\bar{\mathbf{u}}_n) - \mathbf{D}(\bar{\mathbf{u}})) \, d\mathbf{x} \, ds \right] \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \bar{\mathbf{S}}) : (\mathbf{D}(\bar{\mathbf{u}}_n) - \mathbf{D}(\bar{\mathbf{u}})) \, d\mathbf{x} \, ds \right] \\ &\quad + \limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} (\bar{\mathbf{S}} - \mathbf{A}(\bar{\mathbf{u}})) : (\mathbf{D}(\bar{\mathbf{u}}_n) - \mathbf{D}(\bar{\mathbf{u}})) \, d\mathbf{x} \, ds \right] \leq 0, \end{aligned} \tag{4.118}$$

where we have used (4.117) and also the weak convergence (4.99). Using the monotonicity of the operator $\mathbf{A}(\cdot)$ (see (2.4)), we also obtain

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} (\mathbf{A}(\bar{\mathbf{u}}_n) - \mathbf{A}(\bar{\mathbf{u}})) : (\mathbf{D}(\bar{\mathbf{u}}_n) - \mathbf{D}(\bar{\mathbf{u}})) \, d\mathbf{x} \, ds \right] \geq 0. \tag{4.119}$$

Combining (4.118), (4.119), (2.2) and (2.3), one can deduce that (cf. [27, Eqn. (1.6)] or [30])

$$\mathbf{D}(\bar{\mathbf{u}}_n) \xrightarrow{n \rightarrow \infty} \mathbf{D}(\bar{\mathbf{u}}), \quad \bar{\mathbb{P}} \otimes \lambda^{d+1} - \text{a.e.}$$

The above convergence justifies the limit procedure in the energy estimate, that is, $\bar{\mathbf{S}} = \mathbf{A}(\bar{\mathbf{u}})$, which completes the proof of Claim 6 (for more details, see [65, Appendix A]). \square

This completes the proof of Theorem 2.6. □

Remark 4.4 Since $\bar{u} \in L^\infty(0, T; \mathbf{V}) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d)$ is regular, we note that in order to obtain the Claim 6, neither Lipschitz truncation ([25]) nor L^∞ -truncation ([14, 65]) are required.

4.8 Pathwise Uniqueness of Solution

We rewrite the equation (1.1) as follows:

$$d(I - \kappa \Delta)^{\frac{1}{2}} \mathbf{u} = (I - \kappa \Delta)^{-\frac{1}{2}} \{ \operatorname{div} (v \mathbf{A}(\mathbf{u}) - (\mathbf{u} \otimes \mathbf{u}) - \pi \mathbf{I} - \mathbf{F}) \} dt + (I - \kappa \Delta)^{-\frac{1}{2}} \Phi(\mathbf{u}) dW(t), \tag{4.120}$$

where $\mathbf{F} \in L^2(0, T; W^{1,2}(\mathcal{O})^{d \times d})$ such that $\operatorname{div} \mathbf{F} = -f$.

The existence of a solution \mathbf{u} is already established in Theorem 2.6 and $\mathbf{u} \in L^\infty(0, T; \mathbf{V}) \cap L^p(0, T; W_0^{1,p}(\mathcal{O})^d)$, \mathbb{P} -a.s., for $p > \frac{2d}{d+2}$. Let us set $\mathbf{v}(\cdot) := (I - \kappa \Delta)^{\frac{1}{2}} \mathbf{u}(\cdot)$. In order to establish the energy equality (Itô’s formula), in view of [33, Theorem 2.1, Eqn. (1.2)], we only need to show that

$$\mathbf{v}^* = (I - \kappa \Delta)^{-\frac{1}{2}} \{ \operatorname{div} (v \mathbf{A}(\mathbf{u}) - (\mathbf{u} \otimes \mathbf{u}) - \pi \mathbf{I} - \mathbf{F}) \} \in L^1(0, T; L^2(\mathcal{O})^d).$$

In order to verify $\mathbf{v}^* \in L^1(0, T; L^2(\mathcal{O})^d)$, we consider

$$\begin{aligned} \int_0^t \|\mathbf{v}^*(s)\|_2 ds &\leq \int_0^t \|(I - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div} \mathbf{F}(s)\|_2 ds + \int_0^t \|(I - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div} \pi \mathbf{I}\|_2 ds \\ &\quad + v \int_0^t \|(I - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div} \mathbf{A}(\mathbf{u}(s))\|_2 ds \\ &\quad + \int_0^t \|(I - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div}(\mathbf{u}(s) \otimes \mathbf{u}(s))\|_2 ds \\ &=: \sum_{i=1}^4 I_i. \end{aligned}$$

We consider the term I_1 and estimate it using Hölder’s inequality, to find

$$I_1 \leq C \int_0^t \|\mathbf{F}(s)\|_2 ds \leq C \|\mathbf{F}\|_{L^2(0, T; W^{1,2}(\mathcal{O})^{d \times d})} < \infty.$$

Similarly, we can estimate the term I_2 as

$$I_2 \leq C \int_0^t \|\pi(s)\|_{L^2} ds \leq C \|\pi\|_{L^2(0, T; L^2(\mathcal{O}))} < \infty.$$

Now, we consider the term I_3 and estimate it in the following way:

$$\begin{aligned} I_3 &\leq C v \int_0^t \|\mathbf{D}(\mathbf{u}(s))\|_2^{p-2} \|\mathbf{D}(\mathbf{u}(s))\|_2 ds \leq C \int_0^t \|\nabla \mathbf{u}(s)\|_{2(p-1)}^{p-1} ds \\ &\leq C \sup_{t \in [0, T]} \|\nabla \mathbf{u}(t)\|_2^{p-1} < \infty, \end{aligned}$$

for $p \leq 2$.

We estimate the final term I_4 , with the help of the embedding $W_0^{1,2}(\mathcal{O})^d \hookrightarrow L^4(\mathcal{O})^d$ for $2 \leq d \leq 4$, and Hölder’s inequality as

$$I_4 \leq C \int_0^t \|\mathbf{u}(s) \otimes \mathbf{u}(s)\|_2 ds \leq C \int_0^t \|\mathbf{u}(s)\|_4^2 ds \leq C \sup_{t \in [0, T]} \|\nabla \mathbf{u}(s)\|_2^2 < \infty.$$

Combining the estimates of $I_1 - I_4$, we deduce that $\mathbf{v}^* \in L^1(0, T; L^2(\mathcal{O})^d)$ and hence we can use Itô’s formula (see [33, Theorem 2.1]) to obtain the following energy equality, \mathbb{P} -a.s.,

$$\begin{aligned} & \|\mathbf{u}(t)\|_2^2 + \kappa \|\nabla \mathbf{u}(t)\|_2^2 + 2\nu \int_0^t \|\mathbf{D}(\mathbf{u}(s))\|_p^p ds \\ &= \|\mathbf{u}_0\|_2^2 + \kappa \|\nabla \mathbf{u}_0\|_2^2 + 2 \int_0^t (\mathbf{f}(s), \mathbf{u}(s)) ds + \int_0^t \|\Phi(\mathbf{u}(s))\|_2^2 ds \\ &+ 2 \int_0^t (\Phi(\mathbf{u}(s)) dW(s), \mathbf{u}(s)), \end{aligned} \tag{4.121}$$

for all $t \in [0, T]$, where $p \in (\frac{2d}{d+2}, 2]$.

For $p \geq 2$, we consider the Gelfand triplet $L^p(\mathcal{O})^d \subset \mathbf{H} \subset L^{p'}(\mathcal{O})^d$. In view of [33, Theorem 2.1, Eqn. (1.2)], we only need to show the following:

$$\nu \int_0^t \|(\mathbf{I} - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div} \mathbf{A}(\mathbf{u}(s))\|_{p'}^{p'} ds < \infty, \tag{4.122}$$

and the remaining terms can be estimated with the help of the embedding $L^{p'}(0, T; L^2(\mathcal{O})^d) \subset L^{p'}(0, T; L^{p'}(\mathcal{O})^d)$, for $p \geq 2$. Let us verify (4.122) in the following way:

$$\nu \int_0^t \|(\mathbf{I} - \kappa \Delta)^{-\frac{1}{2}} \operatorname{div} \mathbf{A}(\mathbf{u}(s))\|_{p'}^{p'} ds \leq C \int_0^t \|\mathbf{A}(\mathbf{u}(s))\|_{p'}^{p'} ds \leq C \int_0^t \|\nabla \mathbf{u}(s)\|_p^p ds < \infty.$$

Combining the estimates (4.122), I_1, I_2 and I_4 imply the required energy equality (4.121) for the case $p \in [2, \infty)$.

Now, we discuss the pathwise uniqueness of the solution to the system (1.1)-(1.4).

Theorem 4.5 (Uniqueness) *Under the assumptions of Theorem 2.6, solution of the system (1.1)-(1.4) is pathwise unique.*

Proof Let $\mathbf{u}_1(\cdot), \mathbf{u}_2(\cdot)$ be any two solutions of the system (4.120) with the initial data \mathbf{u}_0^1 and \mathbf{u}_0^2 , respectively. For $M > 0$, let us define

$$\tau_M^1 = \inf_{t \in [0, T]} \{t : \|\nabla \mathbf{u}_1(t)\|_2 \geq M\}, \text{ and } \tau_M^2 = \inf_{t \in [0, T]} \{t : \|\nabla \mathbf{u}_2(t)\|_2 \geq M\}.$$

Set $\tau_M := \tau_M^1 \wedge \tau_M^2$. Let us define $\mathbf{w}(\cdot) := \mathbf{u}_1(\cdot) - \mathbf{u}_2(\cdot)$ and $\tilde{\Phi}(\cdot) := \Phi(\mathbf{u}_1(\cdot)) - \Phi(\mathbf{u}_2(\cdot))$. Then, $\mathbf{w}(\cdot)$ satisfies the following system:

$$\begin{cases} d(\mathbf{I} - \kappa \Delta)^{\frac{1}{2}} \mathbf{w} = (\mathbf{I} - \kappa \Delta)^{-\frac{1}{2}} [\operatorname{div} (\mathbf{A}(\mathbf{u}_1) - \mathbf{A}(\mathbf{u}_2)) - \operatorname{div} ((\mathbf{u}_1 \otimes \mathbf{u}_1) - (\mathbf{u}_2 \otimes \mathbf{u}_2)) \\ \quad - \operatorname{div} ((\pi_1 - \pi_2)\mathbf{I})] dt + (\mathbf{I} - \kappa \Delta)^{-\frac{1}{2}} \tilde{\Phi} dW, \\ \mathbf{w}(0) = \mathbf{u}_0^1 - \mathbf{u}_0^2, \end{cases} \tag{4.123}$$

for all $t \in [0, T]$ in $L^{p'}$.

Let us define

$$\varphi(t) := \exp\left(-C_1 \int_0^t \|\nabla \mathbf{u}_2(s)\|_2 ds\right),$$

where C_1 is the constant appearing in (4.126) below. Applying infinite-dimensional Itô's formula (see [33, Theorem 2.1]) to the process $\varphi(\cdot)\|(\mathbf{I} - \kappa \Delta)^{\frac{1}{2}} \mathbf{w}(\cdot)\|_2^2$, we find \mathbb{P} -a.s.,

$$\begin{aligned} & \varphi(t \wedge \tau_M) \{ \|\mathbf{w}(t \wedge \tau_M)\|_2^2 + \kappa \|\nabla \mathbf{w}(t \wedge \tau_M)\|_2^2 \} \\ &= \|\mathbf{w}(0)\|_2^2 + \kappa \|\nabla \mathbf{w}(0)\|_2^2 - 2\nu \int_0^{t \wedge \tau_M} \varphi(s) \langle \mathbf{A}(\mathbf{u}_1(s)) - \mathbf{A}(\mathbf{u}_2(s)), \mathbf{D}(\mathbf{w}) \rangle ds \\ &+ 2 \int_0^{t \wedge \tau_M} \varphi(s) \left(\langle \operatorname{div} \{ (\mathbf{u}_1(s) \otimes \mathbf{u}_1(s)) - (\mathbf{u}_2(s) \otimes \mathbf{u}_2(s)) \}, \mathbf{w}(s) \rangle \right. \\ &\left. - C_1 \|\nabla \mathbf{u}_2(s)\|_2 \|\mathbf{w}(s)\|_2^2 \right) ds \end{aligned} \tag{4.124}$$

$$+ \int_0^{t \wedge \tau_M} \varphi(s) \|\tilde{\Phi}(s)\|_{\mathcal{L}_2}^2 ds + 2 \int_0^{t \wedge \tau_M} \varphi(s) (\tilde{\Phi}(s) d\mathbf{W}(s), \mathbf{w}(s)). \tag{4.125}$$

Using the following facts:

$$|\langle \operatorname{div} \{ (\mathbf{u}_1 \otimes \mathbf{u}_1) - (\mathbf{u}_2 \otimes \mathbf{u}_2) \}, \mathbf{w} \rangle| \leq \|\nabla \mathbf{u}_2\|_2 \|\mathbf{w}\|_4^2 \leq C_1 \|\nabla \mathbf{u}_2\|_2 \|\nabla \mathbf{w}\|_2^2, \tag{4.126}$$

for $2 \leq d \leq 4$, and

$$\langle \mathbf{A}(\mathbf{u}_1) - \mathbf{A}(\mathbf{u}_2), \mathbf{D}(\mathbf{w}) \rangle \geq 0,$$

and $\operatorname{div} \mathbf{w} = 0$, we arrive at \mathbb{P} -a.s.,

$$\begin{aligned} & \varphi(t \wedge \tau_M) \{ \|\mathbf{w}(t \wedge \tau_M)\|_2^2 + \kappa \|\nabla \mathbf{w}(t \wedge \tau_M)\|_2^2 \} \\ & \leq \|\mathbf{w}(0)\|_2^2 + \kappa \|\nabla \mathbf{w}(0)\|_2^2 + \int_0^{t \wedge \tau_M} \varphi(s) \|\tilde{\Phi}(s)\|_{\mathcal{L}_2}^2 ds + 2 \int_0^{t \wedge \tau_M} \varphi(s) (\tilde{\Phi}(s) d\mathbf{W}(s), \mathbf{w}(s)). \end{aligned}$$

Taking expectation, using Hypothesis 2.11, Poincaré's inequality and the fact that the final term is a martingale, we deduce

$$\begin{aligned} & \mathbb{E} \left[\varphi(t \wedge \tau_M) \{ \|\mathbf{w}(t \wedge \tau_M)\|_2^2 + \kappa \|\nabla \mathbf{w}(t \wedge \tau_M)\|_2^2 \} \right] \\ & \leq \left(\frac{1}{\eta_1} + \kappa \right) \|\nabla \mathbf{w}(0)\|_2^2 + \frac{C(K)}{\eta_1} \mathbb{E} \left[\int_0^{t \wedge \tau_M} \varphi(s) \|\nabla \mathbf{w}(s)\|_2^2 ds \right]. \end{aligned}$$

An application of Gronwall's inequality yields

$$\mathbb{E} \left[\varphi(t \wedge \tau_M) \|\nabla \mathbf{w}(t \wedge \tau_M)\|_2^2 \right] \leq C(\eta_1, \kappa, K) \|\nabla \mathbf{w}(0)\|_2^2,$$

where we have used that fact that $\int_0^t \|\nabla \mathbf{u}_2(s)\|_2 ds < \infty$, \mathbb{P} -a.s.

Thus the initial data $\mathbf{u}_0^1 = \mathbf{u}_0^2 = \mathbf{u}_0$ leads to $\mathbf{w}(t \wedge \tau_M) = 0$, \mathbb{P} -a.s. But using the fact that $\tau_M \rightarrow T$, \mathbb{P} -a.s., implies $\mathbf{w}(t) = 0$ and hence $\mathbf{u}_1(t) = \mathbf{u}_2(t)$, \mathbb{P} -a.s., for all $t \in [0, T]$, which completes the proof. \square

Proof of Theorem 2.7 We have already established the existence of a probabilistically weak solution and pathwise uniqueness in Theorems 2.6 and 4.5, respectively. Therefore, combining Theorems 2.6, 4.5 and an application of Yamada-Watanabe theorem (see [62, Theorem 2.1]) leads to the proof of this theorem. \square

Acknowledgements The first author would like to thank Ministry of Education, Government of India - MHRD for financial assistance. H. B. de Oliveira would like to thank CIDMA under the Portuguese Foundation for Science and Technology Multi-Annual Financing Program for R&D Units (FCT, <https://ror.org/00snfq58>). M. T. Mohan would like to thank the Department of Science and Technology (DST) Science & Engineering Research Board (SERB), India for a MATRICS grant (MTR/2021/000066). The authors would like to express their sincere gratitude to the reviewer for his/her insightful comments and suggestions.

Author Contributions All authors have contributed equally.

Funding DST-SERB, India, MTR/2021/000066 (M. T. Mohan).
<https://ror.org/00snfq58> (H. B. de Oliveira).

Data Availability Not applicable.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethical Approval Not applicable.

References

1. Amrouche, C., Berselli, C.L., Lewandowski, R., Nguyen, D.D.: Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis. *Nonlinear Anal.* **196**, 111790 (2020)
2. Anh, C.T., Thanh, N.V.: On the existence and long-time behavior of solutions to stochastic three-dimensional Navier-Stokes-Voigt equations. *Stochastics* **91**, 485–513 (2019)
3. Antontsev, S.N., de Oliveira, H.B., Khompysh, K.: Generalized Kelvin-Voigt equations for nonhomogeneous and incompressible fluids. *Commun. Math. Sci.* **17**, 1915–1948 (2019)
4. Antontsev, S.N., de Oliveira, H.B., Khompysh, K.: Kelvin-Voigt equations perturbed by anisotropic relaxation, diffusion and damping. *J. Math. Anal. Appl.* **473**, 1122–1154 (2019)
5. Antontsev, S.N., de Oliveira, H.B., Khompysh, K.: Kelvin-Voigt equations with anisotropic diffusion, relaxation and damping: blow-up and large time behavior. *Asymptot. Anal.* **121**, 125–157 (2021)
6. Antontsev, S.N., de Oliveira, H.B., Khompysh, K.: Existence and large time behavior for generalized Kelvin-Voigt equations governing nonhomogeneous and incompressible fluids. *J. Phys. Conf. Ser.* **1268**, 012008 (2019)
7. Antontsev, S.N., de Oliveira, H.B., Khompysh, K.: The classical Kelvin-Voigt problem for incompressible fluids with unknown non-constant density: existence, uniqueness and regularity. *Nonlinearity* **34**, 3083–3111 (2021)
8. Antontsev, S.N., Khompysh, K.: Kelvin-Voigt equation with p-Laplacian and damping term: existence, relaxation and blow-up. *J. Math. Anal. Appl.* **446**, 1255–1273 (2017)
9. Antontsev, S.N., Khompysh, K.: Generalized Kelvin-Voigt equations with p-Laplacian and source/absorption terms. *J. Math. Anal. Appl.* **456**, 99–116 (2017)
10. Barnes, H.A.: *A Handbook of Elementary Rheology*. University of Wales, Cambrian Printers, Aberystwyth (2000)
11. Bensoussan, A., Temam, R.: Équations stochastiques du type Navier-Stokes. *J. Functional Analysis* **13**, 195–222 (1973)
12. Berselli, C.L., Bisconti, L.: On the structural stability of the Euler-Voigt and Navier-Stokes-Voigt models. *Nonlinear Anal.* **75**, 117–130 (2012)
13. Berselli, C.L., Spirito, S.: Suitable weak solutions to the 3D Navier-Stokes equations are constructed with the Voigt approximation. *J. Differential Equations* **262**, 3285–3316 (2017)
14. Breit, D.: Existence theory for stochastic power law fluids. *J. Math. Fluid Mech.* **17**, 295–326 (2015)
15. Breit, D., Diening, L., Schwarzacher, S.: Solenoidal Lipschitz truncation for parabolic PDEs. *Math. Models Methods Appl. Sci.* **23**, 2671–2700 (2013)
16. Breit, D., Gmeineder, F.: Electro-rheological fluids under random influences: martingale and strong solutions. *Stoch. Partial Differ. Equ. Anal. Comput.* **7**, 699–745 (2019)
17. Breit, D., Hofmanová, M.: Stochastic Navier-Stokes equations for compressible fluids. *Indiana Univ. Math. J.* **65**, 1183–1250 (2016)

18. Bogovskii, M. E.: Solutions of some problems of vector analysis, associated with the operators div and grad. In: Theory of cubature formulas and the application of functional Analysis to Problems of Mathematical Physics (Russian), Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, **149**, 5–40 (1980)
19. Burkholder, D.L.: The best constant in the Davis inequality for the expectation of the martingale square function. *Trans. Amer. Math. Soc.* **354**, 91–105 (2002)
20. Cao, Y., Lunasin, M.E., Titi, E.S.: Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. *Commun. Math. Sci.* **4**, 823–848 (2006)
21. Chen, J., Chen, Z.M.: Stochastic non-Newtonian fluid motion equations of a nonlinear bipolar viscous fluid. *J. Math. Anal. Appl.* **369**, 486–509 (2010)
22. Clément, P., Sweers, G.: Uniform anti-maximum principles for polyharmonic equations. *Proc. Am. Math. Soc.* **129**, 467–474 (2000)
23. Clément, P., Sweers, G.: Uniform anti-maximum principles. *J. Differential Equations* **164**, 118–154 (2000)
24. Da Prato, G., Zabczyk, J.: *Stochastic Equations in Infinite Dimensions*, 2nd edn. Cambridge University Press, Cambridge (2014)
25. de Oliveira, H.B.: Existence of weak solutions for the generalized Navier-Stokes equations with damping. *NoDEA Nonlinear Differential Equations Appl.* **20**, 797–824 (2013)
26. Debussche, A., Glatt-Holtz, N., Temam, R.: Local Martingale and pathwise solutions for an abstract fluids model. *Phys. D Nonlinear Phenom.* **240**, 1123–1144 (2011)
27. Diening, L., Málek, J., Steinhauer, M.: On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM Control Optim. Calc. Var.* **14**, 211–232 (2008)
28. Flandoli, F.: An introduction to 3D stochastic fluid dynamics, *SPDE in Hydrodynamic: Recent Progress and Prospects*, 51–150, Lecture Notes in Math., **1942**, Springer, Berlin, (2008)
29. Flandoli, F., Gątarek, D.: Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields* **102**, 367–391 (1995)
30. Frehse, J., Málek, J., Josef, S., Steinhauer, M.: On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.* **34**, 1064–1083 (2003)
31. Galdi, G.P.: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems*, Springer, New York (2011)
32. Glowinski, R., Marrocco, R.: Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.* **9**, 41–76 (1975)
33. Gyöngy, I., Šiška, D.: Itô formula for processes taking values in intersection of finitely many Banach spaces. *Stoch. Partial Differ. Equ. Anal. Comput.* **5**, 428–455 (2017)
34. Hoffmanová, M.: Degenerate parabolic stochastic partial differential equations. *Stochastic Process. Appl.* **123**, 4294–4336 (2013)
35. Ikeda, N., Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, 2nd edn., North-Holland Mathematical Library 24, North-Holland, Amsterdam (1989)
36. Jakubowski, A.: The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen* **42**, 209–216 (1997)
37. Joseph, D. D.: *Fluid Dynamics of Viscoelastic Liquids*, Appl. Math. Sci., 84 Springer-Verlag, New York, (1990). xviii+755 pp
38. Kalantarov, K.V., Titi, E.S.: Global attractors and determining modes for the 3D Navier-Stokes-Voigt equations. *Chinese Ann. Math. Ser. B* **30**, 697–714 (2009)
39. Kondratiev, V.A., Oleinik, O.A.: On Korn’s inequalities. *C. R. Acad. Sci. Paris Sér. I Math.* **308**, 483–487 (1989)
40. Kotsiolis, A. A., Oskolkov, A. P., Shadiev, R.: Global a priori estimates on the half-line $t \geq 0$, asymptotic stability and time periodicity of “small” solutions of equations for the motion of Oldroyd fluids and Kelvin-Voigt fluids, LOMI Preprints, **R-10-89**, Akad. Nauk SSSR, Mat. Inst. Leningrad. Otdel., Leningrad, (1989). 65 pp
41. Kumar, P., Kinra, K., Mohan, M.T.: A local in time existence and uniqueness result of an inverse problem for the Kelvin-Voigt fluids. *Inverse Prob.* **37**(8), 085005 (2021)
42. Ladyžhenskaya, O.A.: Certain nonlinear problems of the theory of continuous media, *Proc. Internat. Congr. Math. (Moscow, 1966)*, 560–573, Izdat. “Mir”, Moscow
43. Ladyžhenskaya, O.A.: On some gaps in two of my papers on the Navier-Stokes equations and the way of closing them. *J. Math. Sci.* **115**, 2789–2791 (2003)
44. Layton, W., Lewandowski, R.: On a well-posed turbulence model. *Discrete Contin. Dyn. Syst. Ser. B* **6**, 111–128 (2006)
45. Lewandowski, R., Berselli, C.L.: On the Bardina’s model in the whole space. *J. Math. Fluid Mech.* **20**, 1335–1351 (2018)

46. Levant, B., Ramos, F., Titi, E.S.: On the statistical properties of the 3D incompressible Navier-Stokes-Voigt model. *Commun. Math. Sci.* **8**, 277–293 (2010)
47. Lions, J.-L.: *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris (1969)
48. Liu, H., Sun, C.: Large deviations for the 3D stochastic Navier-Stokes-Voigt equations. *Appl. Anal.* **97**, 919–937 (2018)
49. Liu, W., Röckner, M.: *Stochastic Partial Differential Equations: An Introduction*. Springer, Berlin (2015)
50. Málek, J., Nečas, J., Rokyta, M., Růžička, M.: *Weak measured-valued solutions to evolutionary PDEs*. Chapman & Hall, London (1996)
51. Mikulevicius, R., Rozovskii, B.L.: Stochastic Navier-Stokes equations for turbulent flows. *SIAM J. Math. Anal.* **35**, 1250–1310 (2004)
52. Mohan, M.T.: On the three dimensional Kelvin-Voigt fluids: global solvability, exponential stability and exact controllability of Galerkin approximations. *Evol. Equ. Control Theory* **9**(2), 301–339 (2020)
53. Müller, R.: Das schwache Dirichletproblem in L_q für den Bipotentialoperator in beschränkten Gebieten und in Aubenge-bieten. *Bayreuth. Math. Schr.* **49**, 115–211 (1995). (**Dissertation, Universität Bayreuth, Bayreuth (1994)**)
54. Obukhovskii, V., Zecca, P., Zvyagin, V.: Optimal feedback control in the problem of the motion of a viscoelastic fluid. *Topol. Methods Nonlinear Anal.* **23**, 323–337 (2004)
55. Ondreját, M.: Stochastic nonlinear wave equations in local Sobolev spaces. *Electron. J. Probab.* **15**, 1041–1091 (2010)
56. Oskolkov, A.P.: The uniqueness and global solvability of boundary-value problems for the equations of motion for aqueous solutions of polymers. *J. Math. Sci.* **8**, 427–455 (1977)
57. Oskolkov, A.P.: Nonlocal problems for the equations of Kelvin-Voigt fluids and their ϵ -approximations. *J. Math. Sci.* **87**, 3393–3408 (1997)
58. Pokorný, M.: *Navier-Stokes Equations*. Charles University Textnotes, Prague (2022)
59. Prévôt, C., Röckner, M.: *A Concise Course on Stochastic Partial Differential Equations*. Springer, Berlin Heidelberg (2007)
60. Ramos, F., Titi, E.S.: Invariant measures for the 3D Navier-Stokes-Voigt equations and their Navier-Stokes limit. *Discrete Contin. Dyn. Syst.* **28**, 375–403 (2010)
61. Soher, H.: *The Navier-Stokes Equations: An elementary functional analytic approach*. Birkhäuser, Basel (2001)
62. Röckner, M., Schmuland, B., Zhang, X.: Yamada-Watanabe theorem for stochastic evolution equations in infinite dimensions. *Condens. Matter Phys.* **11**, 247–259 (2008)
63. Temam, R.: *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam (1984)
64. Terasawa, Y., Yoshida, N.: Stochastic power law fluids: existence and uniqueness of weak solutions. *Ann. Appl. Probab.* **21**, 1827–1859 (2011)
65. Wolf, J.: Existence of weak solutions to the equations of nonstationary motion of non-Newtonian fluids with shear-dependent viscosity. *J. Math. Fluid Mech.* **9**, 104–138 (2007)
66. Yoshida, N.: Stochastic shear thickening fluids: Strong convergence of the Galerkin approximation and the energy equality. *Ann. Appl. Probab.* **22**, 1215–1242 (2012)
67. Zhang, Q.: Stability of pullback random attractors for stochastic 3D Navier-Stokes-Voigt equations with delays. *Acta Appl. Math.* **184**, 4 (2023)
68. Zvyagin, V.G., Turbin, M.V.: The study of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids. *J. Math. Sci.* **168**, 157–308 (2010)
69. Zvyagin, V.G., Orlov, V.: On weak solutions of the equations of motion of a viscoelastic medium with variable boundary. *Bound. Value Probl.*, **3**, 215–245 (2005)

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