Appendix VI - analytical solution for the point stable population structure.

The deduction for any point (x,y) started by assuming as the new starting time (t_s) one where the whole population was already at its stable structure and asymptotic growth rate (λ).Then, the fertility vector had the solution:

$$F_{(x_0, y_0, t+t_s)} = \lambda^t . F_{(x_0, y_0, t_s)} = \lambda^t . \begin{bmatrix} tet_{(x_0, y_0, t_s+1)} \\ 0 \\ 0 \\ 0 \\ carp_{(x_0, y_0, t_s+1)} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(eqn.1)}$$

where λ was estimated as $(N_{t+ts}/N_{ts})^{1/t}$. Since the fertility vector Fv could be written as a function of its initial structure, it was possible to estimate N_{t+ts} iterating from N_{ts} equation 5 in chapter 5. Neglecting the x_0 and y_0 subscripts for space location and standardizing the population to its initial size by multiplying by $1/\lambda^t$:

$$\frac{N_{1+t_s}}{\lambda} = \frac{T.N_{t_s} + Fv_{t_s}}{\lambda} = T.N_{t_s}.\lambda^{-1} + Fv_{t_s}\lambda^{-1}$$

$$\frac{N_{2+t_s}}{\lambda^2} = \frac{T.\left(T.N_{t_s} + Fv_{t_s}\right) + F_{t_s}.\lambda}{\lambda^2} = T^2.N_{t_s}.\lambda^{-2} + T.Fv_{t_s}\lambda^{-2} + Fv_{t_s}.\lambda^{-1}$$

$$\frac{N_{t+t_s}}{\lambda^t} = T^t.N_{t_s}.\lambda^{-t} + T^{t-1}.Fv_{t_s}.\lambda^{-t} + T^{t-2}.Fv_{t_s}.\lambda^{1-t} + \ldots + T^{t-t}.Fv_{t_s}.\lambda^{(t-1)-t} \qquad \text{(eqn.2)}$$

$$\frac{N_{t+t_s}}{\lambda^t} = \underbrace{T^t.N_{t_s}.\lambda^{-t}}_{A} + \underbrace{\sum_{i=1}^{t}\left(T^{i-1}.\lambda^{-i}\right).Fv_{t_s}}_{B}$$

The term A was relative to the fate of the individuals already in the population at the new starting time (t_s) while B was the terms relative to the production and fate of new individuals. The sum in B is a geometric succession:

$$S = \sum a.r^{i-1} \quad , \quad a = \lambda^{-1} \quad \text{and} \quad r = T.\lambda^{-1} \qquad \text{(eqn.3)}$$

It was convergent because the population vector with exponential growth was standardized to the initial population size. Yet, if the population was decaying B would always be convergent even in the absence of standardization. The analytical solution of a geometric series was transposed from the one dimensional case to the matrix dimensional case. This is possible provided the matrix is square, non-singular and therefore invertible and the unidimensional operands are replaced by the matrix operands. The solution to the population vector became:

$$\frac{N_{t+t_s}}{\lambda^t} = \underbrace{T^t.N_{t_s}.\lambda^{-t}}_{A} + \underbrace{\lambda^{-1}\left(I - \left(T.\lambda^{-1}\right)^t \right).inv\left(I - T.\lambda^{-1}\right).Fv_{t_s}}_{B} \qquad \text{(eqn.4)}$$

where I is the identity matrix. Stable structure meant the proportionality among stages estimated from equation 4 was kept as time tended to infinity. So, solving equation 4 when t tends to infinity, being T a matrix which only described the fate of individuals already alive and knowing that if any individual could die he eventually did die, it was verified that:

$$\lim_{t \to +\infty} T^t.N_{t_s}.\lambda^{-t} = 0_{(m,1)} \quad \text{and} \quad \lim_{t \to +\infty} \left(T.\lambda^{-1}\right)^t = 0_{(m,m)} \qquad \text{(eqn.5)}$$

where m is the number of stages which in this case were 8. So the stable population structure in point (x_0,y_0) was given by equation 6. It was not a fully analytical solution because Fv_{ts} still had to be numerically determined.

$$N_{(x_0,y_0,t_s)} = \lim_{t \to +\infty} \frac{N_{(x_0,y_0,t+t_s)}}{\lambda^t} = \lambda^{-1}.inv\left(I - T_{(x_0,y_0)}.\lambda^{-1}\right).Fv_{(x_0,y_0,t_s)} \qquad \text{(eqn.6)}$$

The developments in equation 5 did fail upon the simulation of some survival dominated life strategies: when the population was slowly decaying there could be cases where the survival of individuals from a certain stage surpassed the population's growth rate, in which case the correspondent matrix and vector entries rather tended to infinity. Then the developments were only valid until equation 4. However, slowly decaying populations violate the assumption of a whole population stable structure because they imply on the long run the existence of fractionary individuals. So, these cases were misplaced for this section and did not unvalidate the solution for the stable structure given by equation 6.