

RESEARCH ARTICLE

Simple transitive 2-representations of Soergel bimodules for finite Coxeter types

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Abstract

In this paper, we show that Soergel bimodules for finite Coxeter types have only finitely many equivalence classes of simple transitive 2-representations and we complete their classification in all types but H_3 and H_4 .

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1 | INTRODUCTION

The series of papers [40–45] laid the foundations for a systematic theory of finitary 2-representations of finitary 2-categories, which is a categorical analog of the theory of finite-dimensional representations of finite-dimensional algebras. This theory is a part of modern 2-representation theory which originated in seminal papers like [6, 16, 22, 50]. The main difference to the ‘alternative’ theory of module categories over tensor categories, see [15] and references therein, is not that 2-categories can have more than one object: every finitary 2-category gives rise to a multifinitary monoidal category with the same finitary 2-representation theory and vice versa, see [15, Remark 4.3.7; 36, Subsection 2.4]. The main difference is that finitary 2-representations and 2-categories are not assumed to be abelian, but additive and idempotent complete. This is important for the main character of this paper, the one-object \mathbb{C} -linear 2-category $\mathcal{S} = \mathcal{S}(W, S)$ of Soergel bimodules for a Coxeter system (W, S) of finite type over the coinvariant algebra of W , which is finitary but not abelian. (Our choice of ground field in this paper is crucial, as \mathcal{S} behaves very differently over a field of positive characteristic.) One can, of course, consider an abelianization \mathcal{C} of a finitary 2-category \mathcal{C} , but such abelianizations do not admit adjunctions in general, which is a serious obstruction. Recall that a finitary 2-category is (quasi) fiat if it has a duality structure satisfying conditions akin to those for (rigid or) pivotal monoidal categories (this is the categorical analog of an algebra having an involution). Tensor categories are rigid by assumption and many of the structural results on their module categories do not hold without that assumption, see [15], but \mathcal{C} is not equivalent to a multitensor category for a general quasi-fiat 2-category \mathcal{C} . For this reason, there are many differences between the representation theories of quasi-fiat 2-categories and multitensor categories, for example, cell structures have played a key role in the first one but no role at all in the second. Even the formulation of analogous results in the two theories, and the respective proofs, can have important differences, for example, the role of coalgebras in quasi-fiat 2-categories is analogous to that of algebras in multitensor categories, but technically more involved, compare, for example, [15, Subsections 7.8–7.10; 35, Section 4]. However, the two representation theories coincide when \mathcal{C} is fusion or pivotal fusion, that is, locally semisimple (meaning that morphism categories are semisimple) and quasi-fiat or fiat, which is very helpful for the purpose of this paper, as we will explain below.

A key result in finitary 2-representation theory is the (weak) Jordan–Hölder theorem [44, Theorem 8] for finitary 2-representations, in which the role of the simples is played by the so-called simple transitive 2-representations. Classifying the simple transitive 2-representations of a given finitary 2-category \mathcal{C} is therefore a fundamental problem in 2-representation theory, which we will refer to as the *Classification Problem*. When \mathcal{C} is fusion, there are only finitely many equivalence classes of simple transitive 2-representations by (a consequence of) Ocneanu rigidity, see, for example, [15, Proposition 3.4.6 and Corollary 9.1.6]. For certain fusion 2-categories those equivalence classes have even been classified explicitly, see Section 8. However, when \mathcal{C} is not locally semisimple, that finiteness result need not be true, see, for example, [17, Subsection 4.3], and in most cases the Classification Problem is unsolved.

In this paper, we address the Classification Problem for the fiat 2-category \mathcal{S} of Soergel bimodules. This 2-category, which is naturally graded and categorifies the corresponding Hecke algebra $H = H(W, S)$, was introduced by Soergel [54, 56] to give an alternative proof of the famous Kazhdan–Lusztig conjectures of positive integrality in the case of Weyl groups and provides a framework in which to generalize and approach these conjectures for arbitrary Coxeter groups, see, for example, [55]. A general and completely algebraic proof using Soergel bimodules was eventually found by Elias and Williamson [14]. Since their introduction, Soergel bimodules and their (graded) 2-representations have played a fundamental role in representation theory, both in characteristic zero, see, for example, [24, 27] (and the above-mentioned papers), and, more recently, in positive characteristic, see, for example, [29, 52]. It is therefore interesting to study its (graded) 2-representation theory, initially in characteristic zero. However, only very partial results on the classification of its simple transitive 2-representations were known. To be precise, the Classification Problem was completely solved for finite Coxeter type A in [44] both in the graded and the ungraded setting, for finite dihedral type in [23, 38] in the graded setting and almost completely in the ungraded setting, and for a small number of other special cases in other Coxeter types, see, for example, [23, 37], both in the graded and the ungraded setting. In all these cases, the classification turned out to be finite. However, in general it was not even known if the number of equivalence classes of simple transitive 2-representations of \mathcal{S} is finite, left aside any classification. In this paper, we show that there are indeed only finitely many equivalence classes of simple transitive 2-representations of \mathcal{S} for any finite Coxeter type, and classify them completely for all finite Coxeter types except H_3 and H_4 . All our results are proved in the graded setting, but they hold in the ungraded setting as well, see Subsection 7.4.

Let us explain our results in some more detail. First of all, the Classification Problem for \mathcal{S} can be reduced to that of certain subquotients. To do that, one uses the cell structure of \mathcal{S} , which is the categorical analog of the Kazhdan–Lusztig cell structure of H . By [5, Subsection 3.2], for every (graded or ungraded) simple transitive 2-representation \mathbf{M} of \mathcal{S} there is a unique two-sided cell, called *apex*, that is not annihilated by \mathbf{M} and is maximal in the two-sided order with respect to that property. The Classification Problem for \mathcal{S} can therefore be addressed apex by apex, and for a fixed two-sided cell \mathcal{J} we denote the 2-category of (graded) simple transitive 2-representations of \mathcal{S} with apex \mathcal{J} by $\mathcal{S}\text{-(g)stmod}_{\mathcal{J}}$. Next, we can reduce the Classification Problem even further by (strong) \mathcal{H} -reduction, see [37, Theorem 15] or [36, Theorem 4.32]. For any diagonal \mathcal{H} -cell $\mathcal{H} \subseteq \mathcal{J}$, which, by definition, the intersection of a left cell in \mathcal{J} and the dual right cell, there is a subquotient $\mathcal{S}_{\mathcal{H}}$ of \mathcal{S} whose only left, right and two-sided cells are the trivial cell and \mathcal{H} . By [36, Theorem 4.32], there is a biequivalence

$$\mathcal{S}\text{-(g)stmod}_{\mathcal{J}} \simeq \mathcal{S}_{\mathcal{H}}\text{-(g)stmod}_{\mathcal{H}}.$$

This is a major reduction, because $\mathcal{S}_{\mathcal{H}}$ is much smaller than \mathcal{S} and one can pick any diagonal \mathcal{H} -cell in \mathcal{J} (which is helpful in practice, because not all diagonal \mathcal{H} -cells of \mathcal{J} necessarily have the same structure, see Section 8).

By the results in this paper, the benefits turn out to be even bigger. Based on Elias and Williamson’s results in [14], Lusztig defined in [31, Subsection 18.15] a locally semisimple bicategory $\mathcal{A}_{\mathcal{J}}$, for any two-sided cell \mathcal{J} of any Coxeter group. This bicategory categorifies the summand of the *asymptotic Hecke algebra* corresponding to the cell \mathcal{J} . The bicategory $\mathcal{A}_{\mathcal{J}}$ is pivotal fusion by [12] and contains a one-object pivotal fusion full subcategory $\mathcal{A}_{\mathcal{H}}$ for any diagonal \mathcal{H} -cell $\mathcal{H} \subset \mathcal{J}$, which we call the *asymptotic bicategory* associated to \mathcal{H} . Being fusion, the bicategory $\mathcal{A}_{\mathcal{H}}$ has only one cell, corresponding to \mathcal{H} , which is left, right, two-sided and diagonal simultaneously, so all

its simple transitive 2-representations have apex \mathcal{H} . The main insight of this paper is that these asymptotic bicategories completely determine the (graded) simple transitive 2-representations of \mathcal{S} . To be precise, our main result (Theorem 7.1) is the existence of a biequivalence of graded 2-categories

$$\mathcal{S}_{\mathcal{H}}\text{-gstmod}_{\mathcal{H}} \simeq \mathcal{A}_{\mathcal{H}}\text{-stmod}'$$

for every diagonal \mathcal{H} -cell \mathcal{H} of \mathcal{S} for any finite Coxeter type. Here $\mathcal{A}_{\mathcal{H}}\text{-stmod}'$ is the graded 2-category with translation obtained from $\mathcal{A}_{\mathcal{H}}\text{-stmod}$ by a well-known and straightforward construction, which we will recall in Subsection 2.6. As a matter of fact, $\mathcal{A}_{\mathcal{H}}\text{-stmod}'$ and $\mathcal{A}_{\mathcal{H}}\text{-stmod}$ are biequivalent as ungraded 2-categories, but not as graded 2-categories. In the ungraded case, therefore, the above biequivalence becomes

$$\mathcal{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}} \simeq \mathcal{A}_{\mathcal{H}}\text{-stmod},$$

see Subsection 7.4. By [15, Corollary 9.1.6 and Proposition 3.4.6] and strong \mathcal{H} -reduction, this implies that the number of equivalence classes of (graded) simple transitive 2-representations of \mathcal{S} is finite for any finite Coxeter type, cf. Corollary 7.3 and the end of Subsection 7.4. Moreover, for all finite Coxeter types but H_3 and H_4 , all two-sided cells contain a diagonal \mathcal{H} -cell \mathcal{H} for which both $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{A}_{\mathcal{H}}\text{-stmod}$ are known explicitly, as summarized in Section 8. For some two-sided cells in Coxeter types H_3 and H_4 this is unfortunately not true, see Section 8. Therefore, we get a complete solution of the Classification Problem for \mathcal{S} for all finite Coxeter types but H_3 and H_4 , and even for those two Coxeter types we get a complete solution for more than half the number of apexes, see Section 8. Finally, Theorem 7.1 also implies (see Theorem 7.10) that $\mathcal{S}\text{-(g)stmod}_{\mathcal{J}}$ is locally (graded) semisimple, meaning that its morphism categories are all (graded) semisimple, for any two-sided cell \mathcal{J} and any finite Coxeter type.

Let us briefly sketch the key ingredients of the proof of Theorem 7.1. Let $\mathbf{C}_{\mathcal{H}}$ be the cell 2-representation of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} , which categorifies the Kazhdan–Lusztig cell module of the Hecke algebra associated to \mathcal{H} . The main ingredient in the proof of Theorem 7.1 is the existence of a graded biequivalence

$$\mathcal{E}\text{nd}_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \mathcal{A}_{\mathcal{H}}^{\prime, \text{op}},$$

cf. Corollary 6.12. (For the experts, we remark that the proof of this proposition uses that the Duflo involution $\mathbf{C}_d \in \mathcal{H}$ is a separable Frobenius algebra in $\mathcal{S}_{\mathcal{H}}$, as we show in Subsection 4.4 and Proposition 6.3.) In the ungraded setting, the above biequivalence becomes

$$\mathcal{E}\text{nd}_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \mathcal{A}_{\mathcal{H}}^{\text{op}}.$$

Another important ingredient in the proof of our main result is the *double centralizer theorem*, see [36, Theorem 5.2], which implies that there is a biequivalence

$$\mathcal{S}_{\mathcal{H}}\text{-(g)stmod}_{\mathcal{H}} \simeq \left(\mathcal{E}\text{nd}_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \right)^{\text{op}}\text{-(g)stmod},$$

as we show in Proposition 7.8. (Note that the bicategory $\mathcal{B}_{\mathcal{H}}$ in that proposition, which will be defined in (4.6), is graded biequivalent to $(\mathcal{E}\text{nd}_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}))^{\text{op}}$ by Corollary 4.8.)

Along the way, we also show several results that are interesting in their own right, for example, the aforementioned fact that $C_d \in \mathcal{H}$ is a separable Frobenius algebra in $\mathcal{S}_{\mathcal{H}}$. To achieve this, we prove in Proposition 4.3 that the underlying algebra B of $\mathbf{C}_{\mathcal{H}}$ is a finite-dimensional positively graded weakly symmetric Frobenius algebra of graded length $2\mathbf{a}$, where \mathbf{a} is the value of Lusztig's \mathbf{a} -function on \mathcal{H} . In certain cases B is known to be symmetric, see Remark 4.4. However, we do not know if B is symmetric in general.

2 | RECOLLECTIONS

2.1 | Categorical conventions

Categories \mathcal{C} and 2-categories, 2-semicategories and bicategories \mathcal{C} in this paper are assumed to be essentially small. We also view a monoidal category as a 2-category with one (possibly unspecified) object; a perspective which we will use throughout, for example, for Soergel bimodules. We will also use the following notation:

- objects in categories (which are not morphism categories in 2-categories) are denoted by letters such as $X \in \mathcal{C}$, and morphisms by $f \in \mathcal{C}$;
- objects in 2-categories are denoted by $i \in \mathcal{C}$, 1-morphisms by $F \in \mathcal{C}$ and 2-morphisms by Greek letters such as $\alpha \in \mathcal{C}$;
- for any $i, j \in \mathcal{C}$, we denote by $\mathcal{C}(i, j)$ the corresponding morphism category;
- identity 1-morphisms are denoted by $\mathbb{1}_i$ and identity 2-morphisms by id_F , where the subscripts are sometimes omitted;
- we write $FG = F \circ G$ for composition of 1-morphisms, and \circ_v and \circ_h denote vertical and horizontal compositions of 2-morphisms, respectively.

We will also use bicategories, silently adapting definitions and results to the weaker setting if necessary, using [36]. We will stress when we do not work with genuine 2-categories, 2-functors, and so on. The reader is referred to, for example, [28, 34] or [2] for these and related notions.

2.2 | Finitary and fiat 2-categories, and their 2-representations

Let \mathbb{k} be an algebraically closed field.

A category \mathcal{C} is called *finitary* (over \mathbb{k}) if it is equivalent to the category of finitely generated, injective (or projective) modules over some associative, finite-dimensional \mathbb{k} -algebra. These categories assemble into a 2-category $\mathfrak{A}_{\mathbb{k}}^f$ having additive, \mathbb{k} -linear functors and natural transformations as 1- and 2-morphisms, respectively. Similarly, a 2-category \mathcal{C} is *finitary* (over \mathbb{k}) if it has finitely many objects, all identity 1-morphisms $\mathbb{1}_i$ are indecomposable and each morphism category $\mathcal{C}(i, j)$ is finitary over \mathbb{k} with all compositions being (bi)additive and \mathbb{k} -(bi)linear. We further say that \mathcal{C} is *fiat* if it has a weak antiinvolution \star , reversing the direction of both 1- and 2-morphisms, and adjunction 2-morphisms associated to \star . If \star is just a weak antiequivalence of finite order, then \mathcal{C} is called *quasi-fiat*. Finally, a finitary 2-category \mathcal{C} is called *locally semisimple* if its morphism categories are all semisimple. Let us note that our use of the term ‘locally semisimple’ is similar to such standard categorical terminology as ‘locally small’, with ‘locally’ referring to 1-morphism categories.

Remark 2.1. For completeness, dropping the assumption of the identity 1-morphisms being indecomposable gives what we call multifinitary, multifiat or quasi-multifiat 2-categories. Most of the theory goes through for these as well, see [36], but we will not need that level of generality in this paper.

Example 2.2. For a finite group G , the (strictified) one-object 2-category $\mathcal{R}\text{ep}(G, \mathbb{k})$ of finite-dimensional representations of G over \mathbb{k} is fiat if and only if the algebra $\mathbb{k}[G]$ has finite representation type. This is true, for example, if $\text{char}(\mathbb{k}) \nmid \#G$, in which case $\mathcal{R}\text{ep}(G, \mathbb{k})$ is locally semisimple.

Let $\mathbb{k} = \mathbb{C}$. Another example of a fiat 2-category is $\mathcal{S} = \mathcal{S}_{\mathbb{C}}(W, S)$, the one-object 2-category of Soergel bimodules over the coinvariant algebra of a finite Coxeter group, see Section 3.

Example 2.3. For any finite-dimensional algebra B we have an associated one-object 2-category \mathcal{C}_B , called the 2-category of projective functors, whose 1-morphisms are direct sums of functors with summands isomorphic to the identity functor or to tensoring with projective B - B -bimodules. (Despite the name of \mathcal{C}_B , the identity functor is not a projective functor, but is needed to make \mathcal{C}_B a genuine 2-category instead of a 2-semicategory.) Assume that B is basic and connected and that $1 = e_1 + \dots + e_n$ is a splitting of the identity into orthogonal primitive idempotents, then $\{Be_i \otimes_{\mathbb{k}} e_j B \mid i, j = 1, \dots, n\}$ is a complete and irredundant set of indecomposable projective B - B -bimodules. By, for example, [40, Lemma 45 and its proof], for every $i, j = 1, \dots, n$,

$$(Be_i \otimes_{\mathbb{k}} e_j B)^{\star} \cong (e_j B)^{\star} \otimes_{\mathbb{k}} e_i B.$$

The left injective B -modules $(e_j B)^{\star}$, for $j = 1, \dots, n$, are projective if and only if B is Frobenius (since B is assumed to be basic, it is self-injective if and only if it is Frobenius). Therefore, \mathcal{C}_B is quasi-fiat if and only if B is Frobenius.

Assume that \mathcal{C}_B is quasi-fiat. Then $(e_j B)^{\star} \cong Be_{\sigma(j)}$, for $j = 1, \dots, n$, where σ is the Nakayama permutation. Applying \star twice yields

$$(Be_i \otimes_{\mathbb{k}} e_j B)^{\star\star} \cong Be_{\sigma(i)} \otimes_{\mathbb{k}} e_{\sigma(j)} B,$$

for all $i, j = 1, \dots, n$. We thus see that \mathcal{C}_B is fiat if and only if B weakly symmetric. For more details, see [40, Subsection 7.3].

A locally semisimple quasi-fiat 2-category is called a *fusion 2-category* and a locally semisimple fiat 2-category can be equipped with the structure of a *pivotal fusion 2-category*. Note that one-object (pivotal) fusion 2-categories, also known as (pivotal) fusion categories, form an important class of tensor categories which has been intensively studied, see, for example, [15].

Example 2.4. The 2-category $\mathcal{R}\text{ep}(G, \mathbb{k})$ is pivotal fusion unless $\text{char}(\mathbb{k}) \mid \#G$. By contrast, the 2-category \mathcal{S} is not (pivotal) fusion, since it is not locally semisimple.

Remark 2.5. Throughout this paper, we will consistently use the above 2-categorical terminology. Let us, for convenience, list the correspondence between our terminology and the one used in [15] in the abelian setting for one-object 2-categories/monoidal categories:

- quasi-fiat \leftrightarrow finitary and rigid;

- fiat \leftrightarrow finitary and pivotal;
- locally semisimple \leftrightarrow semisimple;
- (pivotal) fusion 2-category \leftrightarrow (pivotal) fusion category.

Note that the notion of fiat only requires the existence of a natural isomorphism between the identity and $\star\star$, whereas the notion of pivotal requires a choice of such a natural isomorphism. In practice, this subtle difference is not important for us because the abstract results hold for fiat 2-categories, whereas in the examples the pivotal structure is known and fixed.

For a finitary 2-category \mathcal{C} , a *finitary 2-representation* \mathbf{M} is an additive, \mathbb{k} -linear 2-functor from \mathcal{C} to $\mathfrak{A}_{\mathbb{k}}^f$. Finitary 2-representations of \mathcal{C} form a 2-category, as we will explain in Subsection 2.5; in particular, there is an appropriate notion of equivalence. The *underlying category* of \mathbf{M} is defined as

$$\mathcal{M} := \coprod_{i \in \mathcal{C}} \mathbf{M}(i).$$

The *rank* of \mathbf{M} is by definition the number of isomorphism classes of indecomposable objects in \mathcal{M} . Moreover, we will often use the action notation $FX := \mathbf{M}(F)(X)$ for 2-representations.

Example 2.6. If \mathcal{C} is finitary, then the so-called *principal* or *Yoneda* 2-representation $\mathbf{P}_i := \mathcal{C}(i, _)$ is finitary, for all $i \in \mathcal{C}$.

A finitary 2-representation \mathbf{M} is called *transitive* if, for any $i \in \mathcal{C}$ and any non-zero object $X \in \mathbf{M}(i)$, the additive closure (in the sense of being closed under direct sums, direct summands and isomorphisms)

$$\text{add}\{FX \mid j \in \mathcal{C}, F \in \mathcal{C}(i, j)\}$$

coincides with \mathcal{M} . Recall that an *ideal* of \mathbf{M} is by definition a \mathcal{C} -stable ideal of \mathcal{M} . A *transitive* 2-representation \mathbf{M} is said to be *simple transitive* provided that it has no non-trivial ideals. Moreover, every transitive 2-representation has a unique simple transitive quotient.

The importance of simple transitive 2-representations is explained, in particular, by the existence of a weak version of the Jordan–Hölder theorem. Namely, for any finitary 2-representation \mathbf{M} of \mathcal{C} , there is a finite filtration

$$0 = \mathbf{M}_0 \subset \mathbf{M}_1 \subset \cdots \subset \mathbf{M}_m = \mathbf{M},$$

where every 2-representation \mathbf{M}_k generates an ideal \mathbf{I}_k in \mathbf{M}_{k+1} such that $\mathbf{M}_{k+1}/\mathbf{I}_k$ is transitive, and thus, has a unique associated simple transitive quotient \mathbf{L}_{k+1} . Up to equivalence and ordering, the set $\{\mathbf{L}_k \mid 1 \leq k \leq m\}$ is an invariant of \mathbf{M} .

The above motivates the *Classification Problem*, that is, the classification of simple transitive 2-representations for a fixed finitary 2-category.

Example 2.7. For the 2-category $\mathcal{R}\text{ep}(G, \mathbb{C})$ the Classification Problem has a well-known solution in terms of subgroups of G and their group cohomology, see Section 8 for details.

By contrast, solving the Classification Problem for \mathcal{S} is the main objective of this paper.

See [40, Subsections 2.2–2.4; 42, Subsection 2.3; 44, Subsection 3.5; 45, Subsection 2.5] for details.

2.3 | Cells and cell 2-representations

For every finitary 2-category \mathcal{C} , one has the notion of *cells*: For any pair of indecomposable 1-morphisms F and G , we define $F \geq_L G$ if F is isomorphic to a direct summand of HG , for some 1-morphism H . This produces the *left* preorder \geq_L , for which the equivalence classes are called *left cells*. Similarly one obtains *right cells* and *two-sided cells*. By [5, Subsection 3.2], for any transitive 2-representation \mathbf{M} there is a unique two-sided cell \mathcal{J} , an invariant of \mathbf{M} called the *apex*, which is not annihilated by \mathbf{M} and is maximal, in the two-sided order, with respect to this property.

Example 2.8. Any one-object (pivotal) fusion 2-category \mathcal{C} , also known as (pivotal) fusion category, has only one cell, which is left, right and two-sided. This follows from the fact that XX^* and X^*X both contain the identity 1-morphism as a direct summand, for all indecomposable $X \in \mathcal{C}$.

Example 2.9. When $\text{char}(\mathbb{k}) \nmid \#G$, the 2-category $\mathcal{R}\text{ep}(G, \mathbb{k})$ is pivotal fusion and has, therefore, only one cell, which is left, right and two-sided. When $\text{char}(\mathbb{k}) \mid \#G$, the 2-category $\mathcal{R}\text{ep}(G, \mathbb{k})$ has more than one cell, for example, the projective modules form a two-sided cell.

As we will recall in Subsection 3.1, the cells of \mathcal{S} are given by the Kazhdan–Lusztig cells in the sense of [20].

Each left cell \mathcal{L} of \mathcal{C} can be used to define a *cell 2-representation* $\mathbf{C}_{\mathcal{L}}$ as follows. First we note that all 1-morphisms in \mathcal{L} have the same domain, say \mathfrak{i} . Define a 2-subrepresentation $\mathbf{M}^{\geq \mathcal{L}}$ of $\mathbf{P}_{\mathfrak{i}}$ using the induced action of \mathcal{C} on

$$\text{add}(\{F \mid F \geq_L \mathcal{L}\}).$$

The 2-representation $\mathbf{M}^{\geq \mathcal{L}}$ has a unique maximal ideal \mathbf{I} and we define

$$\mathbf{C}_{\mathcal{L}} := \mathbf{M}^{\geq \mathcal{L}} / \mathbf{I},$$

which is a simple transitive 2-representation by construction. If \mathcal{C} is quasi-fiat, then the apex of $\mathbf{C}_{\mathcal{L}}$ is the two-sided cell containing \mathcal{L} .

Example 2.10. The (unique) cell 2-representation of $\mathcal{R}\text{ep}(G, \mathbb{C})$ coincides with its unique principal 2-representation.

The cell 2-representations of \mathcal{S} categorify the Kazhdan–Lusztig cell representations, see Subsection 3.1.

Finally, recall that a 2-category is called *\mathcal{J} -simple*, where \mathcal{J} is a two-sided cell, if any non-zero 2-ideal contains the identity 2-morphisms of all 1-morphisms in \mathcal{J} . If \mathcal{C} is fiat, then \mathcal{C} has an associated *\mathcal{J} -simple* subquotient 2-category $\mathcal{C}_{\mathcal{J}}$, whose only two-sided cells are \mathcal{J} and the two-sided cells containing $\mathbb{1}_{\mathfrak{i}}$, such that \mathfrak{i} is the source of some 1-morphism in \mathcal{J} . Note that these cells may all coincide, for example, if \mathcal{C} is fusion.

For further details, we refer to [5, Subsection 3.2; 36, Subsections 2.5 and 2.6; 37, Subsection 4.2; 40, Subsection 4.5; 44, Section 3].

2.4 | Coalgebra and algebra 1-morphisms

Finitary 2-categories can be injectively or projectively abelianized. The *injective abelianization* is denoted $\underline{\mathcal{C}}$ and the *projective abelianization* is denoted $\overline{\mathcal{C}}$. Moreover, \mathcal{C} embeds into $\underline{\mathcal{C}}$ or into $\overline{\mathcal{C}}$, and the isomorphism closure of the image of this embedding is the 2-full 2-subcategory of injective or projective 1-morphisms, respectively. In particular, in these abelianizations, each indecomposable 1-morphism $F \in \mathcal{C}$ has an associated simple socle or head, respectively.

These abelianizations are rather technical and not all properties of \mathcal{C} carry over to the abelianizations. In particular, the abelianizations of fiat 2-categories are usually not even finitary and the involution $*$ only gives rise to an antiequivalence between $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$.

The same abelianizations, *mutatis mutandis*, exist for finitary 2-representations, where we use the same notation.

See [35, Section 3] for details.

Example 2.11. When $\text{char}(\mathbb{k}) \nmid \#G$, we have $\underline{\mathcal{R}\text{ep}}(G, \mathbb{k}) \simeq \overline{\mathcal{R}\text{ep}}(G, \mathbb{k}) \simeq \mathcal{R}\text{ep}(G, \mathbb{k})$, because $\mathcal{R}\text{ep}(G, \mathbb{k})$ is locally semisimple. In contrast, \mathcal{S} is not abelian and neither $\underline{\mathcal{S}}$ nor $\overline{\mathcal{S}}$ are equivalent to it.

Abelianizations play a key role in the construction and study of 2-representations: Recall that a *coalgebra* $C := (C, \delta_C, \epsilon_C)$ in $\underline{\mathcal{C}}$ is a 1-morphism in some $\underline{\mathcal{C}}(i, i)$ equipped with 2-morphisms $\delta_C : C \rightarrow CC$, called comultiplication, and $\epsilon_C : C \rightarrow \mathbb{1}_i$, called counit, satisfying the usual conditions. Dually, one can define algebras as well. A *right C-comodule* $M = (M, \delta_{M,C})$ is a 1-morphism in some $\underline{\mathcal{C}}(i, j)$ together with a right coaction $\delta_{M,C} : M \rightarrow MC$, again satisfying the usual coherence conditions. For each object $j \in \underline{\mathcal{C}}$, these assemble into a category

$$\text{comod}_{\underline{\mathcal{C}}}(C)_j := \{(M, \delta_{M,C}) \mid M \in \underline{\mathcal{C}}(i, j), M \text{ is a right } C\text{-comodule}\},$$

whose morphisms are called *right C-comodule morphisms* and satisfy the usual intertwining condition. All of these notions can be defined, *mutatis mutandis*, for left coactions as well, and we put the coalgebras on the left-hand side to stress that we have a left coaction. Let

$$\text{inj}_{\underline{\mathcal{C}}}(C)_j := \{(M, \delta_{M,C}) \mid M \in \text{comod}_{\underline{\mathcal{C}}}(C)_j \text{ injective}\}$$

denote the full subcategory of $\text{comod}_{\underline{\mathcal{C}}}(C)_j$ consisting of all injective objects. We set

$$\text{comod}_{\underline{\mathcal{C}}}(C) = \coprod_{j \in \underline{\mathcal{C}}} \text{comod}_{\underline{\mathcal{C}}}(C)_j \quad \text{and} \quad \text{inj}_{\underline{\mathcal{C}}}(C) = \coprod_{j \in \underline{\mathcal{C}}} \text{inj}_{\underline{\mathcal{C}}}(C)_j.$$

Note that left composition defines a natural $\underline{\mathcal{C}}$ -action on $\text{comod}_{\underline{\mathcal{C}}}(C)$, which restricts to a natural left $\underline{\mathcal{C}}$ -action on $\text{inj}_{\underline{\mathcal{C}}}(C)$. We use a bold font whenever we want to consider these as 2-representations, for example, we write $\mathbf{inj}_{\underline{\mathcal{C}}}(C)$ instead of $\text{inj}_{\underline{\mathcal{C}}}(C)$.

Example 2.12. The identity 1-morphism $\mathbb{1}_i$, for any $i \in \underline{\mathcal{C}}$, has a natural structure of a coalgebra, with both the comultiplication and the counit being given by the identity 2-morphism. The 2-representation $\mathbf{comod}_{\underline{\mathcal{C}}}(\mathbb{1}_i)$ of $\underline{\mathcal{C}}$ is equivalent to $\underline{\mathbf{P}}_i$ and the 2-representation $\mathbf{inj}_{\underline{\mathcal{C}}}(\mathbb{1}_i)$ of $\underline{\mathcal{C}}$ is equivalent to \mathbf{P}_i .

Another natural source of coalgebras in $\underline{\mathcal{C}}$ is provided by the internal cohom construction: Let \mathbf{M} be a finitary 2-representation of \mathcal{C} and consider objects $X \in \mathbf{M}(i)$, $Y \in \mathbf{M}(j)$. Their *internal cohom* $[X, Y] \in \underline{\mathcal{C}}(i, j)$ is uniquely determined, up to isomorphism, by the isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}(j)}(Y, GX) \cong \mathrm{Hom}_{\underline{\mathcal{C}}(i, j)}([X, Y], G)$$

for all $G \in \underline{\mathcal{C}}(i, j)$. If it is not immediately obvious in which 2-category or bicategory the internal cohom is taken, we will use a subscript $[X, Y]_{\underline{\mathcal{C}}}$ to clarify. The 1-morphism $C^X := [X, X]$ has a natural structure of a coalgebra in $\underline{\mathcal{C}}$. Moreover, if \mathcal{C} is fiat and X generates \mathbf{M} in the sense that $\mathrm{add}\{FX \mid F \in \mathcal{C}\} = \mathcal{M}$, then $\mathbf{inj}_{\underline{\mathcal{C}}}(C^X) \simeq \mathbf{M}$. In this way, we can realize any finitary 2-representation of a fiat 2-category \mathcal{C} , up to equivalence, as $\mathbf{inj}_{\underline{\mathcal{C}}}(C)$ for some coalgebra C in $\underline{\mathcal{C}}$ (or in the injective abelianization of the additive closure of \mathcal{C} , if no generator in a single $\mathbf{M}(i)$ exists), and simple transitive 2-representations of \mathcal{C} (which always have a generator) correspond to cosimple coalgebras in $\underline{\mathcal{C}}$. If \mathbf{M} is a simple transitive 2-representation of \mathcal{C} with apex J and \mathcal{C} is J -simple, then C^X is in $\underline{\mathcal{C}}$ for every $X \in \mathcal{M}$.

Finally, we recall that coalgebras in \mathcal{C} have an associated Morita–Takeuchi theory. In particular, C and C' are Morita–Takeuchi equivalent if and only if $\mathbf{inj}_{\underline{\mathcal{C}}}(C) \simeq \mathbf{inj}_{\underline{\mathcal{C}}}(C')$. See also, for example, [15, Chapter 7; 35, Sections 4 and 5; 37, Subsection 3.6].

For later use, we recall one technical result and record three others. The first result, Theorem 2.13, is the content of theorem [23, Theorem 2], which we will need repeatedly throughout the paper.

Theorem 2.13. *Let \mathbf{M} be a simple transitive 2-representation of a fiat 2-category \mathcal{C} with apex J . For every 1-morphism $F \in \mathcal{J}$, the endofunctor $\overline{\mathbf{M}}(F)$ is projective (in the category of right exact linear endofunctors).*

In particular, suppose that \mathcal{C} has only one object i and that B is the basic connected underlying algebra of $\mathbf{M}(i)$, that is, $\mathcal{M} \simeq B\text{-proj}$ and $\overline{\mathcal{M}} \simeq B\text{-mod}$. Theorem 2.13 then implies that, for every 1-morphism $F \in \mathcal{J}$, the action of $\overline{\mathbf{M}}(F)$ on $B\text{-mod}$ is given by tensoring over B with a projective B - B -bimodule B_F . Note that $B_F \otimes_B M \in B\text{-proj}$ for any $M \in B\text{-mod}$.

Corollary 2.14. *The algebra B is Frobenius.*

Proof. Let $1 = e_1 + \dots + e_n$ be a splitting of the unit into orthogonal primitive idempotents in B . Then $\{Be_i \otimes_k e_j B \mid i, j = 1, \dots, n\}$ is a complete and irredundant set of indecomposable projective B - B -bimodules.

Choose $j \in \{1, \dots, n\}$. By Theorem 2.13 and transitivity of \mathbf{M} , for every $i \in \{1, \dots, n\}$ with $i \neq j$, there is an $F \in \mathrm{add}(\mathcal{J})$ such that Be_i is a direct summand of $B_F \otimes_B Be_j$, so B_F contains a direct summand of the form

$$Be_i \otimes_k e_k B,$$

for some $k \in \{1, \dots, n\}$. As in Example 2.3, the decomposition of $B_F \cong B_{F^{\star\star}}$ thus contains a direct summand of the form

$$(e_i B)^{\star} \otimes_k e_l B,$$

for some $l \in \{1, \dots, n\}$. This shows that $(e_i B)^{\star}$ is a projective left B -module, by Theorem 2.13.

Since this holds for all $i \in \{1, \dots, n\}$, every left injective B -module is also projective, that is, B is self-injective. Since every self-injective basic algebra is Frobenius, the result follows. \square

Corollary 2.14 shows that Theorem 2.13 can also be formulated in terms of the injective abelianization $\underline{\mathbf{M}}$, so $\mathcal{M} \simeq B\text{-inj} = B\text{-proj}$ and $\underline{\mathcal{M}} \cong B\text{-mod}$, and the action of $\underline{\mathbf{M}}(F)$ on $B\text{-mod}$ is also given by B_F , which is injective-projective as an B - B -bimodule. This consequence of Theorem 2.13 and Corollary 2.14 is crucial and will be used repeatedly throughout the paper.

Remark 2.15. The 2-category \mathcal{C}_B in Example 2.3 is fiat if and only if B is weakly symmetric, whereas simple transitivity of \mathbf{M} only implies that $B = B^M$ is Frobenius. The reason is that, for any $F \in \mathcal{J}$, the B - B -bimodule B_F is a direct sum of indecomposable B - B -bimodules. When \mathcal{C} is fiat, then $B_F^{**} \cong B_F$, but $**$ can still permute the indecomposable direct summands, so the Nakayama permutation of B need not be trivial. In Proposition 5.8, however, we will show that B^M is weakly symmetric if \mathbf{M} is a (graded) simple transitive 2-representation of Soergel bimodules of finite Coxeter type.

Proposition 2.16. *Suppose that \mathcal{C} is a one-object pivotal fusion 2-category and \mathbf{M} a finitary 2-representation of \mathcal{C} . If \mathbf{M} is simple transitive, then \mathcal{M} is semisimple.*

Proof. Suppose that \mathbf{M} is simple transitive. As explained in Example 2.8, \mathcal{C} has only one cell, which is thus the apex of \mathbf{M} . By Theorem 2.13, this implies that $\overline{\mathbf{M}}(\mathbb{1})$ is a projective endofunctor of $\overline{\mathcal{M}}$, where $\mathbb{1}$ is the unique identity 1-morphism of \mathcal{C} . This implies that every simple $L \in \overline{\mathcal{M}}$ is projective, since $L = \mathbf{M}(\mathbb{1})(L)$. Therefore, \mathcal{M} is semisimple. \square

Corollary 2.17. *Suppose that \mathcal{C} is a one-object pivotal fusion 2-category and C a cosimple coalgebra in \mathcal{C} . Then $\text{inj}_{\mathcal{C}}(C) = \text{comod}_{\mathcal{C}}(C)$ is semisimple.*

Proof. If C is a cosimple coalgebra in \mathcal{C} , then $\text{inj}_{\mathcal{C}}(C)$ is simple transitive by [37, Corollary 12]. Therefore, its underlying category is semisimple by Proposition 2.16. \square

2.5 | 2-Categories of 2-representations

The classification problem suggests studying the following 2-categories. Given two 2-categories \mathcal{C} and \mathcal{D} , let $[\mathcal{C}, \mathcal{D}]$ denote the 2-category of 2-functors together with 2-natural transformations and modifications.

If \mathcal{C} is finitary, then let $\mathcal{C}\text{-afmod} := [\mathcal{C}, \mathfrak{A}_{\mathbb{k}}^f]$ denote the additive, \mathbb{k} -linear 2-category of its finitary 2-representations. We let $\mathcal{C}\text{-stmod}$ denote the 1,2-full 2-subcategory of $\mathcal{C}\text{-afmod}$ consisting of all simple transitive 2-representations. Further, we indicate by the subscript \mathcal{J} , where \mathcal{J} is a given two-sided cell of \mathcal{C} , the 1,2-full 2-subcategories consisting of the finitary 2-representations of \mathcal{C} whose weak Jordan–Hölder constituents all have apex \mathcal{J} . Finally, we use the superscript ex to indicate the 2-full 2-subcategories having the same objects, but 1-morphisms being exact, in the sense of their component functors extending to exact functors in the (injective) abelianization.

Example 2.18. Suppose that \mathcal{C} is fiat and \mathcal{L} a left cell inside a two-sided cell \mathcal{J} of \mathcal{C} . Then $\mathbf{C}_{\mathcal{L}} \in \mathcal{C}\text{-stmod}_{\mathcal{J}}^{\text{ex}} \subset \mathcal{C}\text{-stmod}_{\mathcal{J}} \subset \mathcal{C}\text{-stmod} \subset \mathcal{C}\text{-afmod}$.

As discussed in detail in [36, Subsection 2.6], there are various relations between the above 2-categories. We recall two of the most crucial ones here in the form in which we need them.

The first relates to all morphisms between simple transitive 2-representations with the same apex being exact:

Proposition 2.19. *Suppose that \mathcal{C} is fiat. For any two-sided cell \mathcal{J} of \mathcal{C} , the 2-categories $\mathcal{C}\text{-stmod}_{\mathcal{J}}^{\text{ex}}$ and $\mathcal{C}\text{-stmod}_{\mathcal{J}}$ are equal.*

The second one is the so-called *strong \mathcal{H} -reduction* [36, Theorem 4.32]: Suppose that \mathcal{C} is fiat and let \mathcal{J} be a two-sided cell of \mathcal{C} . Then \star preserves \mathcal{J} and the \mathcal{J} -simple subquotient $\mathcal{C}_{\mathcal{J}}$, see the end of Subsection 2.3, is also fiat and, by [36, Theorem 4.28], there is a biequivalence

$$\mathcal{C}_{\mathcal{J}}\text{-stmod}_{\mathcal{J}} \simeq \mathcal{C}\text{-stmod}_{\mathcal{J}}.$$

The point of this biequivalence is that the structure of $\mathcal{C}_{\mathcal{J}}$ is simpler than that of the whole \mathcal{C} in general.

However, there are even simpler 2-categories, which still contain enough information to solve the Classification Problem. We define *\mathcal{H} -cells* as the intersection of left and right cells. Suppose that \mathcal{C} is fiat. Then, for every left cell \mathcal{L} , we define the associated *diagonal \mathcal{H} -cell*

$$\mathcal{H}(\mathcal{L}) := \mathcal{L} \cap \mathcal{L}^*.$$

By construction, $\mathcal{H} = \mathcal{H}(\mathcal{L})$ lies in the same two-sided cell \mathcal{J} as \mathcal{L} . Further, recall that each \mathcal{L} contains a unique distinguished 1-morphism $D = D(\mathcal{L})$ called *Duflo involution* and that, in fact, $D \in \mathcal{H}(\mathcal{L})$.

Given a left cell \mathcal{L} in some two-sided cell \mathcal{J} , we define $\mathcal{C}_{\mathcal{H}}$ to be the 2-full 2-subcategory of $\mathcal{C}_{\mathcal{J}}$ generated by all 1-morphisms in $\mathcal{H} := \mathcal{H}(\mathcal{L})$ together with the identity 1-morphism $1_{\mathfrak{i}}$, where \mathfrak{i} is the unique domain and codomain of all 1-morphisms in \mathcal{H} . The 2-category $\mathcal{C}_{\mathcal{H}}$ is fiat, has \mathcal{H} as its maximal two-sided cell and is \mathcal{H} -simple. The importance of $\mathcal{C}_{\mathcal{H}}$ is expressed by the following result, called *strong \mathcal{H} -reduction*.

Theorem 2.20. *Let \mathcal{C} be a fiat 2-category and fix a two-sided cell \mathcal{J} of \mathcal{C} . For any diagonal \mathcal{H} -cell $\mathcal{H} \subset \mathcal{J}$, there is a biequivalence*

$$\mathcal{C}\text{-stmod}_{\mathcal{J}} \simeq \mathcal{C}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

Example 2.21. The main purpose of this paper is to study \mathcal{S} -stmod. Theorem 2.20 tells us that, by varying over all two-sided cells \mathcal{J} and choosing an arbitrary but fixed diagonal \mathcal{H} -cell in each \mathcal{J} , this problem can be reduced to studying $\mathcal{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}$. The main result of this paper, which is Theorem 7.1, shows that the latter problem can be reduced even further.

The main reference for the above is [36, Subsections 2.6, 4.7 and 4.8].

2.6 | Grading conventions

Let $\mathbb{k}\text{-fgmod}$ denote the category of finite-dimensional (\mathbb{Z} -)graded \mathbb{k} -vector spaces. An object in $\mathbb{k}\text{-fgmod}$ has the form $V = \bigoplus_{t \in \mathbb{Z}} V_t$, where V_t denotes the linear subspace of elements of V which are homogeneous of degree t . Morphisms in $\mathbb{k}\text{-fgmod}$ are \mathbb{k} -linear maps (not necessarily homo-

geneous, but each morphism is a linear combination of homogeneous morphisms). The group \mathbb{Z} acts on $\mathbb{k}\text{-fgmod}$ by grading shift $\langle - \rangle$ via the rule $(V\langle t \rangle)_s = V_{s+t}$, for all $s, t \in \mathbb{Z}$. From now on, if $p(v) = n_{-k}v^{-k} + \dots + n_l v^l \in \mathbb{N}_0[v, v^{-1}]$ and V is a graded vector space, then we use the notation

$$V^{\oplus p} := V\langle k \rangle^{\oplus n_{-k}} \oplus \dots \oplus V\langle -l \rangle^{\oplus n_l}.$$

With this notation, we have, for example, $V\langle -t \rangle = V^{\oplus t}$. Further, if V is a finite-dimensional graded vector space, then its graded dimension $\text{grdim}(V) \in \mathbb{N}_0[v, v^{-1}]$ is uniquely defined by the property that the graded vector spaces V and $\mathbb{k}^{\oplus \text{grdim}(V)}$ are isomorphic by a homogeneous isomorphism of degree 0, where \mathbb{k} is concentrated in degree 0. A finite-dimensional algebra A is called *positively graded* if it is non-negatively graded, that is, $\text{grdim}(A) \in \mathbb{N}_0[v]$, and its degree 0 component A_0 is semisimple. A finite-dimensional positively graded algebra A is called a *graded Frobenius algebra of length k* if $\text{Hom}_{\mathbb{k}}(A, \mathbb{k}) \cong A^{\oplus v^{-k}}$ as graded left A -modules, where again the isomorphism is homogeneous of degree 0.

A *graded \mathbb{k} -linear category* C is a category enriched over $\mathbb{k}\text{-fgmod}$ (in this paper, most categories will have finite-dimensional morphism spaces, but there are some exceptions). This means that

$$\text{Hom}_C(X, Y) = \bigoplus_{t \in \mathbb{Z}} \text{Hom}_C(X, Y)_t$$

and the composite of two homogeneous morphisms of degrees t_1 and t_2 is homogeneous of degree $t_1 + t_2$. We let

$$\text{hom}_C(X, Y) := \text{Hom}_C(X, Y)_0.$$

By definition, $C^{(0)}$ is the (non-full) subcategory of C given by taking the degree 0 morphisms between all objects of C , that is,

$$\text{Hom}_{C^{(0)}}(X, Y) := \text{hom}_C(X, Y).$$

The grading on $C^{(0)}$ is, of course, trivial.

We call a functor $F : C \rightarrow D$ between graded \mathbb{k} -linear categories a *graded functor*, provided the assignment

$$\text{Hom}_C(X, Y) \rightarrow \text{Hom}_D(F(X), F(Y))$$

on morphisms is homogeneous of degree 0.

Following [7], a \mathbb{k} -linear category C is a \mathbb{Z} -category if it is equipped with a group homomorphism from \mathbb{Z} to the group of autoequivalences of C . It is a *free \mathbb{Z} -category* if the action of \mathbb{Z} thus obtained is free. For $t \in \mathbb{Z}$, we again write $X\langle t \rangle = X^{\oplus v^{-t}}$ for the action of t on C .

We say that a graded \mathbb{k} -linear category C *admits translation* if it is a free \mathbb{Z} -category and for every $t \in \mathbb{Z}$ there is a homogeneous natural isomorphism $\phi_t : \langle 0 \rangle \xrightarrow{\cong} \langle t \rangle$ of degree t , such that $\phi_0 = \text{Id}_{\mathcal{C}}$ and $\phi_s \phi_t = \phi_{s+t}$, for $s, t \in \mathbb{Z}$. In particular, we have

$$\text{Hom}_C(X^{\oplus v^{-s}}, Y^{\oplus v^{-t}}) \cong \text{Hom}_C(X, Y)^{\oplus v^{s-t}}$$

for all objects $X, Y \in C$ and $s, t \in \mathbb{Z}$. If C comes with a fixed choice of isomorphisms ϕ_t , we say that C is a category *with translation*.

A graded functor $F : C \rightarrow D$ between two graded categories with translation automatically preserves translations (up to natural isomorphism), as follows from the existence of the homogeneous degree 0 natural isomorphism

$$F(X^{\oplus v^t}) \xrightarrow{F(\phi_t^C)} F(X) \xrightarrow{\phi_{-t}^D} F(X)^{\oplus v^t}$$

for any $X \in C$ and $t \in \mathbb{Z}$.

Suppose that C admits translation. Then in $C^{(0)}$ the objects X and $X^{\oplus v^t}$ are isomorphic if and only if $t = 0$, so $C^{(0)}$ does not admit translation, although it is a free \mathbb{Z} -category. Note also that $C^{(0)}$ is never finitary, because $C^{(0)}$ has infinitely many isomorphism classes of indecomposable objects. We say that a free \mathbb{Z} -category \mathcal{X} is \mathbb{Z} -finitary if there is a finite set of indecomposable objects $X_i \in \mathcal{X}$, $i \in I$, such that every indecomposable object in \mathcal{X} is isomorphic to $X_i^{\oplus v^t}$, for some $i \in I$ and $t \in \mathbb{Z}$, and every object in \mathcal{X} has an essentially unique decomposition into indecomposables objects. We will call C *graded finitary* provided that $C^{(0)}$ is \mathbb{Z} -finitary.

Given a graded \mathbb{k} -linear category C , we can define a new graded \mathbb{k} -linear category C' with objects (X, t) , where $X \in C$ and $t \in \mathbb{Z}$, and

$$\mathrm{Hom}_{C'}((X, s), (Y, t)) := \mathrm{Hom}_C(X, Y)^{\oplus v^{s-t}},$$

with the evident composition. Then C' has a translation, with ϕ_t given by the identity on C shifted by t degrees, and the natural embedding of C into C' sending X to $(X, 0)$ is an equivalence of ungraded categories.

Remark 2.22. Note that the above embedding is degree-preserving but it need not be an equivalence of graded categories in general, for example, if $C = C^{(0)}$, then the functorial inverse of the embedding would have to be given by $(X, t) \mapsto X$, for all $X \in C$ and $t \in \mathbb{Z}$. However, that functor is not degree-preserving, as it maps any homogeneous morphism of any degree in C' to a homogeneous morphism of degree 0 in C . Nonetheless, if C admits translation, then it is equivalent to C' as a graded category, because $(X, t) \mapsto X^{\oplus v^{-t}}$ defines a functorial inverse of the above embedding.

Observe that, given a \mathbb{k} -linear category C admitting translation, C is equal to the skew-category $C^{(0)}[\mathbb{Z}]$ in the sense of [7, Definition 2.3]. Recall that for any \mathbb{k} -linear free \mathbb{Z} -category D , the objects of $D[\mathbb{Z}]$ coincide with those of D and the morphism spaces are defined by

$$\mathrm{Hom}_{D[\mathbb{Z}]}(X, Y) := \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}_D(X, Y^{\oplus v^{-t}}),$$

with the composition being induced by the one in D after shifting. The skew-category $D[\mathbb{Z}]$ is naturally graded and comes with the natural translation induced by the identity on D . By construction, the isomorphism classes of its objects correspond bijectively to the \mathbb{Z} -orbits of objects in D .

For future use, we record the following easy lemma.

Lemma 2.23. *Let $C = C^{(0)}$ be a trivially graded \mathbb{k} -linear category, and let C' be defined as above. Then $(C')^{(0)}$ is equivalent, as a \mathbb{k} -linear free \mathbb{Z} -category, to $\bigoplus_{t \in \mathbb{Z}} C(t)$, where $C(t)$ is the full subcategory of $(C')^{(0)}$ with objects (X, t) for $X \in C$.*

Moreover, let $D := (C')^{(0)}[\mathbb{Z}]$. Then C' and D are equivalent as graded \mathbb{k} -linear categories.

Proof. The first statement is clear and the second is a special case of [7, Proposition 2.6]. \square

Another notion that we will need is that of graded semisimplicity. A graded finitary category C is called *graded semisimple* if $C^{(0)}$ is semisimple. Note that in the latter case there is a finite number of simple object $L_i \in C^{(0)}$, $i \in I$, and every simple object in $C^{(0)}$ is isomorphic to $L_i^{\oplus v^{-t}}$, for some $i \in I$ and $t \in \mathbb{Z}$.

Graded \mathbb{k} -linear categories admitting translation, together with graded functors and all natural transformations form a 2-category, which we will denote by $\mathfrak{A}_{\mathbb{k}}^g$. We denote by $\mathfrak{A}_{\mathbb{k}}^{gf}$ the 1,2-full 2-subcategory whose objects are graded finitary categories admitting translation.

As our notation emphasizes, all non-trivially graded \mathbb{k} -linear categories in this paper will be assumed to admit translation.

References for the above are, for example, [7; 18, Section 1].

2.7 | Graded 2-categories of 2-representations

Many 2-categories in this paper will be graded, meaning their morphism categories are in $\mathfrak{A}_{\mathbb{k}}^g$ and horizontal composition is a graded bifunctor.

By a *graded finitary 2-category*, we mean a graded \mathbb{k} -linear 2-category which has finitely many objects, on which identity 1-morphisms are indecomposable, and morphism categories in $\mathfrak{A}_{\mathbb{k}}^{gf}$ with horizontal composition being a graded \mathbb{k} -bilinear bifunctor.

If \mathcal{C} is a graded \mathbb{k} -linear 2-category, not necessarily admitting translation, we define the 2-category with translation \mathcal{C}' , with the same objects as \mathcal{C} and morphism spaces defined by $\mathcal{C}'(i, j) := \mathcal{C}(i, j)'$ and the evident horizontal composition.

For a graded 2-category \mathcal{C} , we denote by $\mathcal{C}^{(0)}$ the 2-subcategory whose morphism categories are given by $\mathcal{C}^{(0)}(i, j) := \mathcal{C}(i, j)^{(0)}$. As in the case of categories, even if \mathcal{C} is graded finitary, the 2-subcategory $\mathcal{C}^{(0)}$ will never be finitary, since F and $F^{\oplus t}$ are not isomorphic in $\mathcal{C}^{(0)}$ unless $t = 0$. However, $\mathcal{C}^{(0)}$ is a \mathbb{Z} -finitary 2-category, meaning that its morphism categories are \mathbb{Z} -finitary categories and horizontal composition is compatible with the free \mathbb{Z} -actions. Moreover, the skew 2-category $\mathcal{C}^{(0)}[\mathbb{Z}]$, whose morphism categories are given by $\mathcal{C}^{(0)}[\mathbb{Z}](i, j) := \mathcal{C}^{(0)}(i, j)[\mathbb{Z}]$, is biequivalent to \mathcal{C} (since $\mathcal{C}^{(0)}$ is a free \mathbb{Z} -category).

A graded finitary 2-category \mathcal{C} is called *locally graded semisimple* if all its morphism categories are graded semisimple. Equivalently, this means that $\mathcal{C}^{(0)}$ is locally semisimple.

A *graded finitary 2-representation* \mathbf{M} of a graded finitary 2-category \mathcal{C} is a graded \mathbb{k} -linear 2-functor $\mathcal{C} \rightarrow \mathfrak{A}_{\mathbb{k}}^{gf}$, where by a graded 2-functor we mean that all its component functors are graded.

Similarly, a morphism of graded finitary 2-representations is a collection of graded functors together with homogeneous natural isomorphisms of degree 0 providing the compatibility with the 2-actions.

Graded finitary 2-representations of a graded finitary 2-category \mathcal{C} , together with morphisms of graded finitary 2-representations and modifications, form a graded 2-category, since spaces of modifications are naturally enriched over $\mathbb{k}\text{-fgmod}$. This 2-category is denoted by $\mathcal{C}\text{-gafmod}$.

The concepts of transitive, simple transitive and cell 2-representations are defined as in the ungraded case. In particular, we can define the 1,2-full 2-subcategory $\mathcal{C}\text{-gstmod}$, whose objects are graded simple transitive 2-representations of \mathcal{C} . As in the ungraded case, a subscript \mathcal{J} indicates that we are restricting our attention to simple transitive 2-representations with apex \mathcal{J} .

For a graded finitary 2-representation \mathbf{M} of a graded finitary 2-category \mathcal{C} , we define a graded finitary 2-representation \mathbf{M}' of \mathcal{C}' by $\mathbf{M}'(\mathfrak{i}) := \mathbf{M}(\mathfrak{i})'$, for every $\mathfrak{i} \in \mathcal{C}$, with the obvious extension of the action.

Similarly, for a graded finitary 2-representation \mathbf{M} of a graded finitary 2-category \mathcal{C} , we define a 2-representation $\mathbf{M}^{(0)}$ of $\mathcal{C}^{(0)}$ by $\mathbf{M}^{(0)}(\mathfrak{i}) := \mathbf{M}(\mathfrak{i})^{(0)}$ for every $\mathfrak{i} \in \mathcal{C}$. Again, $\mathbf{M}^{(0)}$ is a \mathbb{Z} -finitary 2-representation, meaning that $\mathbf{M}^{(0)}(\mathfrak{i})$ is a \mathbb{Z} -finitary category, for every $\mathfrak{i} \in \mathcal{C}$, and all other data of the 2-representation are compatible with the free \mathbb{Z} -actions. Conversely, given a \mathbb{Z} -finitary 2-representation \mathbf{N} of $\mathcal{C}^{(0)}$, we define a graded finitary 2-representation $\mathbf{N}[\mathbb{Z}]$ of \mathcal{C} by setting $\mathbf{N}[\mathbb{Z}](\mathfrak{i}) := \mathbf{N}(\mathfrak{i})[\mathbb{Z}]$ and extending the other data of the 2-representation in the obvious way. Finally, we say that a graded finitary 2-representation \mathbf{M} is *graded semisimple* if $\mathbf{M}^{(0)}$ is semisimple.

Following [39, Subsection 6.4], given a graded finitary 2-representation \mathbf{M} , we can associate a coalgebra C in $\underline{\mathcal{C}}^{(0)}$ to $\mathbf{M}^{(0)}$, and $\mathbf{inj}_{\underline{\mathcal{C}}}(\underline{C}) \simeq \mathbf{M}$. In other words, we can associate to \mathbf{M} a coalgebra C whose comultiplication and counit are homogeneous of degree 0. We, therefore, call it a *graded coalgebra*. Now, suppose that \mathbf{M} is simple transitive, then C is cosimple in $\underline{\mathcal{C}}$, see Subsection 2.4. This implies that C is also cosimple in $\underline{\mathcal{C}}^{(0)}$, of course, whence $\mathbf{M}^{(0)} \simeq \mathbf{inj}_{\underline{\mathcal{C}}^{(0)}}(\underline{C})$ is also simple transitive (again, see Subsection 2.4). In all this, however, the definition of the graded injective abelianization $\underline{\mathcal{C}}$ requires some care. It is defined as

$$\underline{\mathcal{C}} := (\underline{\mathcal{C}}^{(0)})'.$$

This means that the 1-morphisms in $\underline{\mathcal{C}}$ are complexes of the form

$$X \xrightarrow{\alpha} Y,$$

where X, Y are 1-morphisms in \mathcal{C} and $\alpha \in \text{hom}_{\mathcal{C}}(X, Y)$, and 2-morphisms in $\underline{\mathcal{C}}$ are defined in the usual way without the restriction of homogeneity. With this definition, $\underline{\mathcal{C}}$ is indeed a graded abelian 2-category and, for any graded coalgebra C in $\underline{\mathcal{C}}(\mathfrak{i}, \mathfrak{i})$, the category $\mathbf{inj}_{\underline{\mathcal{C}}}(\underline{C})$ is indeed a graded subcategory of $\coprod_{j \in \mathcal{C}} \underline{\mathcal{C}}(\mathfrak{i}, j)$.

3 | SOERGEL BIMODULES AND THE ASYMPTOTIC BICATEGORY

3.1 | Soergel bimodules

From now on we set $\mathbb{k} = \mathbb{C}$ (or any other algebraically closed field of characteristic zero), and all categories and 2-categories, and so on, are over \mathbb{C} unless indicated otherwise.

Let (W, S) be a Coxeter system of finite type. We fix a reflection faithful representation of W and let C be the corresponding coinvariant algebra with the usual (\mathbb{Z}) -grading. We denote by $\mathcal{S} = \mathcal{S}_C(W, S)$ the 2-category of *Soergel bimodules* over C , by which we mean the 2-category with one object \emptyset , which we identify with the category of finite-dimensional graded C -modules, whose 1-morphisms are endofunctors of the latter category isomorphic to tensoring with Soergel bimodules, and whose 2-morphisms are natural transformations, which we can identify with bimodule homomorphisms. The 2-category \mathcal{S} is graded fiat (this follows from, for example, [14], see also [40, Subsection 7.1] for the case of Weyl groups).

By [56] (see also [13]), there is a $\mathbb{Z}[v, v^{-1}]$ -linear isomorphism between the split Grothendieck ring of $\mathcal{S}^{(0)}$ and the Hecke algebra $H := H_{\mathbb{Z}[v, v^{-1}]}(W, S)$ of W such that, for each $w \in W$, there is a unique (up to homogeneous isomorphism) indecomposable Soergel bimodule B_w in $\mathcal{S}^{(0)}$ whose Grothendieck class is sent to the Kazhdan–Lusztig basis element b_w corresponding to w , and

$[X^{\oplus v}] = v[X]$ for any 1-morphism X in $\mathcal{S}^{(0)}$. We call this fact the *Soergel–Elias–Williamson categorification theorem* and we call the isomorphism the *character isomorphism*. As a consequence, the cell structure of \mathcal{S} is given by the Kazhdan–Lusztig combinatorics. In particular, *Lusztig’s conjectures* [31, Conjecture 14.2], which we will use several times, hold in our case, see, for example, [31, Subsection 15.1] or [9, Corollary 1.4].

We set

$$C_w := B_w^{\oplus v^{a(w)}} \quad \text{and} \quad c_w := v^{a(w)} b_w,$$

where we recall that, to each $w \in W$, Lusztig [30] assigns a number $\mathbf{a}(w)$, called its *\mathbf{a} -value*, such that the function $\mathbf{a} : W \rightarrow \mathbb{N}_0$ is constant on two-sided cells. In $\overline{\mathcal{S}}$ the projective 1-morphism B_w has a simple head, and dually in $\underline{\mathcal{S}}$ the injective 1-morphism B_w has a simple socle.

Example 3.1. For $s \in S$ we have $B_s B_s \cong B_s^{\oplus (v^{-1}+v)}$ in $\mathcal{S}^{(0)}$, but $C_s C_s \cong C_s^{\oplus (1+v^2)}$. The heads of B_s and C_s , seen as projective 1-morphisms in \mathcal{S} , are concentrated in degrees -1 and 0 , respectively.

For $x, y, z \in W$, we let $h_{x,y,z} \in \mathbb{N}_0[v, v^{-1}]$ and $\gamma_{x,y,z^{-1}} \in \mathbb{N}_0$ be given by

$$B_x B_y \cong \bigoplus_{z \in W} B_z^{\oplus h_{x,y,z}} \text{ in } \mathcal{S}^{(0)}, \quad h_{x,y,z} = v^{a(z)} \gamma_{x,y,z^{-1}} \pmod{v^{a(z)-1} \mathbb{N}_0[v^{-1}]}, \quad (3.1)$$

where \mathbf{a} is Lusztig’s \mathbf{a} -function. Since the h are bar invariant, that is, invariant with respect to the symmetry $v \leftrightarrow v^{-1}$, we also have

$$h_{x,y,z} = v^{-a(z)} \gamma_{x,y,z^{-1}} \pmod{v^{-a(z)+1} \mathbb{N}_0[v]}. \quad (3.2)$$

By the Soergel–Elias–Williamson categorification theorem, the h are also the structure constants of H with respect to the basis $\{b_w \mid w \in W\}$ and the γ are the structure constants of Lusztig’s *asymptotic Hecke algebra*, denoted A , with respect to the basis $\{a_w \mid w \in W\}$. (Lusztig [32] called the asymptotic Hecke algebra the *\mathcal{J} -ring*, denoting its basis elements by t_w in [32].)

Recall from Subsection 2.5 that, for every two-sided cell \mathcal{J} , there is a \mathcal{J} -simple subquotient $\mathcal{S}_{\mathcal{J}}$ of \mathcal{S} and, for every left cell \mathcal{L} in \mathcal{J} , there is a diagonal \mathcal{H} -cell $\mathcal{H} = \mathcal{H}(\mathcal{L}) = \mathcal{L} \cap \mathcal{L}^*$ and a 2-full 2-subcategory $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{J}}$ which is \mathcal{H} -simple. Recall also that every left cell \mathcal{L} contains a unique Duflo involution $d = d_{\mathcal{L}}$ and that $d_{\mathcal{L}} \in \mathcal{H}(\mathcal{L})$. Let $\mathcal{D}(\mathcal{J})$ denote the set of Duflo involutions in \mathcal{J} . Both $\mathcal{S}_{\mathcal{J}}$ and $\mathcal{S}_{\mathcal{H}}$ have asymptotic counterparts too, which we denote by $\mathcal{A}_{\mathcal{J}}$ and $\mathcal{A}_{\mathcal{H}}$, respectively, and will recall in Subsection 3.2. For every two-sided cell \mathcal{J} of W , the split Grothendieck group of $\mathcal{A}_{\mathcal{J}}$ is a subalgebra of A , denoted $A_{\mathcal{J}}$. The unit of $A_{\mathcal{J}}$ is equal to $\sum_{d \in \mathcal{D}(\mathcal{J})} a_d$ and there is a decomposition of algebras

$$A \cong \prod_{\mathcal{J} \subset W} A_{\mathcal{J}},$$

where the direct product runs over all two-sided cells of W . Note that $A_{\mathcal{J}}$ is an idempotent subalgebra of A . Similarly, for every diagonal \mathcal{H} -cell $\mathcal{H} \subset \mathcal{J}$, the split Grothendieck group of $\mathcal{A}_{\mathcal{H}}$ is an idempotent subalgebra of $A_{\mathcal{J}}$, denoted by $A_{\mathcal{H}}$. The unit of $A_{\mathcal{H}}$ is equal to $a_{d_{\mathcal{H}}}$, but the direct product of these $A_{\mathcal{H}}$ is strictly smaller than $A_{\mathcal{J}}$ in general. In fact, $A_{\mathcal{J}}$ decomposes as

$$A_{\mathcal{J}} \cong \bigoplus_{\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{J}} A_{\mathcal{L}_1^* \cap \mathcal{L}_2}$$

as a vector space, where the direct sum runs over all pairs of left cells in \mathcal{J} and $A_{\mathcal{L}_1^* \cap \mathcal{L}_2}$ is the direct summand of $A_{\mathcal{J}}$ generated by $\{a_w \mid w \in \mathcal{L}_1^* \cap \mathcal{L}_2\}$, which is an $A_{\mathcal{H}(\mathcal{L}_1)}\text{-}A_{\mathcal{H}(\mathcal{L}_2)}$ -bimodule. Alternatively, we can see $A_{\mathcal{J}}$ as a finitary category, whose objects are indexed by the left cells $\mathcal{L} \subset \mathcal{J}$ and whose morphism categories are defined by

$$\text{Hom}_{A_{\mathcal{J}}}(\mathcal{L}_1, \mathcal{L}_2) := A_{\mathcal{L}_1^* \cap \mathcal{L}_2}.$$

The identity morphism on \mathcal{L} is given by $a_{d_{\mathcal{L}}}$.

From Soergel’s hom formula, see [14, Theorem 3.6], we obtain

$$\dim\left(\text{hom}_{\mathcal{S}}(B_v, B_w^{\oplus v^k})\right) = \delta_{v,w} \delta_{0,k}, \quad v, w \in W, k \in \mathbb{Z}_{\geq 0}. \tag{3.3}$$

By slight abuse of terminology, we will also refer to (3.3) as *Soergel’s hom formula*. Consequently, for any subset $U \subset W$, the 2-endomorphism algebra of $\bigoplus_{w \in U} B_w$ in \mathcal{S} is positively graded. This property is inherited by $\mathcal{S}_{\mathcal{H}}$.

Let $\mathbf{a}(\mathcal{H}) := \mathbf{a}(x)$ for any $x \in \mathcal{H}$. Finally, the following lemmas are evident and we state them for later use.

Lemma 3.2. *In $\mathcal{S}_{\mathcal{H}}^{(0)}$, for all $x, y, z \in \mathcal{H}$, we have that*

$$C_x C_y \cong \bigoplus_{z \in \mathcal{H}} C_z^{\oplus v^{\mathbf{a}(\mathcal{H})} h_{x,y,z}}.$$

In particular, if $C_z^{\oplus v^k}$ is isomorphic to a direct summand of $C_x C_y$, then $k \geq 0$.

Lemma 3.2 has the following ‘negative’ counterpart.

Lemma 3.3. *For $x \in \mathcal{H}$, set $\tilde{C}_x := C_x^{\oplus v^{-2\mathbf{a}(\mathcal{H})}} = B_x^{\oplus v^{-\mathbf{a}(\mathcal{H})}}$. Then, in $\mathcal{S}_{\mathcal{H}}^{(0)}$, for all $x, y, z \in \mathcal{H}$, we have that*

$$\tilde{C}_x \tilde{C}_y \cong \bigoplus_{z \in \mathcal{H}} \tilde{C}_z^{\oplus v^{-\mathbf{a}(\mathcal{H})} h_{x,y,z}}.$$

In particular, if $\tilde{C}_z^{\oplus v^k}$ is isomorphic to a direct summand of $\tilde{C}_x \tilde{C}_y$, then $k \leq 0$.

3.2 | The asymptotic bicategory

We define two 2-semicategories inside $\mathcal{S}_{\mathcal{H}}^{(0)}$:

$$\mathcal{X} := \text{add}\left(\{C_w^{\oplus v^k} \mid w \in \mathcal{H}, k \geq 0\}\right)^{(0)},$$

$$\tilde{\mathcal{X}} := \text{add}\left(\{C_w^{\oplus v^k} \mid w \in \mathcal{H}, k > 0\}\right)^{(0)}.$$

The 2-semicategory \mathcal{X} is, in fact, a lax monoidal category with lax identity 1-morphism C_d and strict associators. Let us explain this in detail. From [42, Subsection 7.6] and positivity of the grading on Soergel bimodules, it follows that there is a unique, up to a non-zero scalar, 2-morphism $\epsilon_d : C_d \rightarrow 1_\emptyset$ in $\mathcal{S}^{(0)}$. The lax structure of the identity 1-morphism of \mathcal{X} on C_d is now defined, for $X \in \mathcal{X}$, by the two 2-morphisms in $\mathcal{S}^{(0)}$

$$\ell_X : C_d X \xrightarrow{\epsilon_d \circ_h \text{id}_X} X, \quad r_X : X C_d \xrightarrow{\text{id}_X \circ_h \epsilon_d} X,$$

with the unitality condition expressed, for $X, Y \in \mathcal{X}$, by the diagram

$$\begin{array}{ccc} (XC_d)Y & \xlongequal{\quad} & X(C_d Y) \\ \swarrow r_X \circ_h \text{id}_Y & & \searrow \text{id}_X \circ_h \ell_Y \\ & XY & \end{array} \tag{3.4}$$

which commutes by associativity and the interchange law.

The bicategory \mathcal{A}_H is defined as the quotient $\mathcal{X}/(\tilde{\mathcal{X}})$, where $(\tilde{\mathcal{X}})$ denotes the 2-ideal of \mathcal{X} generated by $\tilde{\mathcal{X}}$. We denote by $\Pi : \mathcal{X} \rightarrow \mathcal{X}/(\tilde{\mathcal{X}})$ the natural projection.

Note that \mathcal{A}_H is only a bicategory as it does not contain any strict identity 1-morphism, but, at the same time, the composition in \mathcal{A}_H is strictly associative. Up to isomorphism, the identity 1-morphism in \mathcal{A}_H is the image $A_d := \Pi(C_d)$ of the lax identity 1-morphism C_d of \mathcal{X} . Finally, note that there is no non-trivial \mathbb{Z} -action on \mathcal{A}_H and the grading is trivial, that is, all 2-morphisms are homogeneous of degree 0.

By the Soergel–Elias–Williamson categorification theorem and (3.1), \mathcal{A}_H categorifies the asymptotic Hecke algebra A_H associated to H in the sense that $A_w := \Pi(C_w)$ categorifies the basis element a_w for $w \in H$. The algebra A_H is a fusion algebra and the following categorifies this fact.

Proposition 3.4. *\mathcal{A}_H is a (one-object) pivotal fusion bicategory. Moreover, if W is a finite Weyl group, then \mathcal{A}_H is biequivalent to C_ζ^H from [4, Section 5; 51].*

Note that this implies that $\mathcal{A}_H, \overline{\mathcal{A}_H}$ and $\overline{\overline{\mathcal{A}_H}}$ are canonically isomorphic.

Proof. The bicategory \mathcal{A}_H is fiat by [12, Theorem 5.2] (see also [4, Subsection 4.3; 31, Subsection 18.19; 33, Subsection 9.3]). Moreover, due to the definition of \mathcal{A}_H and Soergel’s hom-formula (see (3.3)), \mathcal{A}_H is locally semisimple. Finally, the connection to [4, Section 5; 51] follows from Soergel’s identification of \mathcal{S} and the category of semisimple perverse sheaves on the associated flag variety, see [54, Erweiterungssatz 5]. □

Remark 3.5. For completeness, we note that there are analogs of the above for two-sided cells \mathcal{J} instead of diagonal \mathcal{H} -cells. In fact, for finite and affine Weyl groups, Lusztig and Bezrukavnikov–Finkelberg–Ostrik (see [4] and the references therein) showed that the pivotal fusion bicategories $\mathcal{A}_{\mathcal{J}}$ corresponding to \mathcal{J} can be described geometrically as bicategories of vector bundles on a square of a finite set equivariant with respect to an algebraic group.

A small remark about terminology: When \mathcal{J} is finite, for example, for finite Coxeter type, one can also see $\mathcal{A}_{\mathcal{J}}$ as a pivotal multifusion category with decomposable identity object isomorphic to $\bigoplus_{d \in \mathcal{J}} C_d$, where the finite direct sum is over all Duflo involutions in \mathcal{J} . In fact, this is Lusztig’s

point of view, see [31, Subsection 18.15]. When \mathcal{J} has infinitely many Duflo involutions (which, of course, can only happen when \mathcal{J} is infinite), there is no identity object, as Lusztig indeed remarks. However, for all \mathcal{J} , finite or infinite, one can consider $\mathcal{A}_{\mathcal{J}}$ to be a pivotal fusion bicategory, whose (possibly infinitely many) objects are indexed by the left cells of \mathcal{J} , which correspond bijectively to the Duflo involutions in \mathcal{J} .

Finally, if \mathcal{J} is finite, by strong \mathcal{H} -reduction (which we recalled in Theorem 2.20), Proposition 3.4 and the fact that one-object fusion bicategories have only one two-sided cell, there is a biequivalence $\mathcal{A}_{\mathcal{J}}\text{-stmod} \simeq \mathcal{A}_{\mathcal{H}}\text{-stmod}$ for any diagonal \mathcal{H} -cell $\mathcal{H} \subseteq \mathcal{J}$. As we will summarize in Section 8, in each two-sided cell \mathcal{J} of a (finite) Weyl group one can pick a nice diagonal \mathcal{H} -cell \mathcal{H} , that is, one such that $\mathcal{A}_{\mathcal{H}}$ is a well-known pivotal fusion bicategory for which the Classification Problem has been solved. For (finite) non-Weyl groups this is no longer true in general. However, even for those groups, roughly half the number of the two-sided cells contains a nice diagonal \mathcal{H} -cell. This is why we restrict our attention to $\mathcal{A}_{\mathcal{H}}$ in this paper.

3.3 | Going up

This subsection investigates a certain oplax pseudofunctor from $\mathcal{A}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ which we loosely call ‘going up’.

By the definition of $\mathcal{A}_{\mathcal{H}}$, the projection Π is the identity on $\text{add}(\{C_w \mid w \in \mathcal{H}\})^{(0)}$ and zero on (\mathcal{X}) , so it is a genuine (in contrast to lax) 2-semifunctor. It sends the lax identity C_d of \mathcal{X} to the identity A_d of $\mathcal{A}_{\mathcal{H}}$ (which is a weak identity, in the sense that there are non-trivial left and right unitors) and, for any $X, Y \in \mathcal{X}$, we have $\Pi(X)\Pi(Y) = \Pi(XY)$, by definition.

Indeed, for any $X = \Pi(F) \in \mathcal{A}_{\mathcal{H}}$, we can define the 2-morphism

$$\lambda_X : A_d X = \Pi(C_d)X = \Pi(C_d)\Pi(F) = \Pi(C_d F) \xrightarrow{\Pi(\ell_F)} \Pi(F) = X.$$

Then λ is a natural transformation from $A_d \circ _-$ to the identity 2-functor on $\mathcal{A}_{\mathcal{H}}$ and we have $\lambda_{\Pi(F)} = \Pi(\ell_F)$. Similarly, one can define a natural transformation ρ from $_-\circ A_d$ to the identity 2-functor on $\mathcal{A}_{\mathcal{H}}$ via $\rho_{\Pi(F)} = \Pi(r_F)$. Here λ and ρ are the left and right unitors for the bicategory $\mathcal{A}_{\mathcal{H}}$. In details, by applying Π to (3.4), we obtain the commutative diagram

$$\begin{array}{ccc} (XA_d)Y & \xlongequal{\quad\quad\quad} & X(A_d Y) \\ \parallel & & \parallel \\ (X\Pi(C_d))Y & \xrightarrow{\rho_X \circ \text{id}_Y} & XY \xleftarrow{\text{id}_X \circ \lambda_Y} & X(\Pi(C_d)Y), \end{array}$$

for all $X, Y \in \mathcal{A}_{\mathcal{H}}$.

Observing that $\lambda_{\Pi(F)} = \Pi(\ell_F)$ and $\rho_{\Pi(F)} = \Pi(r_F)$, we obtain the commutative diagrams

$$\begin{array}{ccc} A_d \Pi(F) & \xlongequal{\quad\quad\quad} & \Pi(C_d)\Pi(F) & \quad & \Pi(F)A_d & \xlongequal{\quad\quad\quad} & \Pi(F)\Pi(C_d) \\ \lambda_{\Pi(F)} \downarrow & & \parallel & & \rho_{\Pi(F)} \downarrow & & \parallel \\ \Pi(F) & \xleftarrow{\quad\quad\quad \Pi(\ell_F) \quad\quad\quad} & \Pi(C_d F), & & \Pi(F) & \xleftarrow{\quad\quad\quad \Pi(r_F) \quad\quad\quad} & \Pi(FC_d), \end{array}$$

which, together with the fact that Π preserves strict associativity, imply that Π is a lax pseudo-functor.

Positivity of the grading on Soergel bimodules shows that the functor underlying Π has a left adjoint

$$\Theta : \mathcal{A}_{\mathcal{H}} \rightarrow \mathcal{X}, \quad \text{Hom}_{\mathcal{X}}(\Theta(F), G) \cong \text{Hom}_{\mathcal{A}_{\mathcal{H}}}(F, \Pi(G)), \quad (3.5)$$

which is unique up to natural 2-isomorphisms. Up to isomorphism, Θ is determined by $\Theta(A_w) \cong C_w$, in particular, Θ is an embedding.

Since Π is lax, the doctrinal adjunction [21] (see also [53, Formula (3.5)]) implies that Θ is an oplax pseudofunctor. Note that, for each $X \in \mathcal{X}$, the object $\Pi(X)$ is isomorphic to $\Pi(Y)$, where Y is the subobject of X generated in degree 0. As $Y \cong \Theta(F)$, for some $F \in \mathcal{A}_{\mathcal{H}}$, the adjunction morphisms guarantee that $\Pi\Theta \cong \text{id}_{\mathcal{A}_{\mathcal{H}}}$.

We can further use $\epsilon_d : C_d \rightarrow \mathbb{1}_{\emptyset}$ to extend Θ to an oplax pseudofunctor from $\mathcal{A}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ via the embeddings $\mathcal{X} \hookrightarrow \mathcal{S}_{\mathcal{H}}^{(0)} \hookrightarrow \mathcal{S}_{\mathcal{H}}$.

We use the notation $\mathbb{1}_{\mathcal{A}_{\mathcal{H}}} := A_d$, where d is the Duflo involution in \mathcal{H} . Note that $\mathbb{1}_{\mathcal{A}_{\mathcal{H}}}$ is only a weak identity 1-morphism. Then $\Theta(\mathbb{1}_{\mathcal{A}_{\mathcal{H}}}) = C_d$.

Lemma 3.6. *For any coalgebra A in $\mathcal{A}_{\mathcal{H}}$, the oplax 2-functor Θ induces the structure of a coalgebra in $\mathcal{S}_{\mathcal{H}}^{(0)}$ on $\Theta(A)$. Moreover, a (left or right) A -comodule X in $\mathcal{A}_{\mathcal{H}}$ is sent to a (left or right) $\Theta(A)$ -comodule $\Theta(X)$ in $\mathcal{S}_{\mathcal{H}}^{(0)}$, which can, of course, be viewed as a comodule in $\mathcal{S}_{\mathcal{H}}$.*

Proof. *Mutatis mutandis* as in [19, Proposition 5.5]. □

Proposition 3.7. *If A as in Lemma 3.6 is cosimple in $\mathcal{A}_{\mathcal{H}}$, then $\Theta(A)$ is cosimple in $\mathcal{S}_{\mathcal{H}}$.*

Note that Proposition 3.7 rules out the existence of ungraded subcoalgebras of $\Theta(A)$ as well.

Proof. By construction, $\Theta(A)$ belongs to $\text{add}(\mathcal{H})$. Let B be a cosimple subcoalgebra of $\Theta(A)$ in $\mathcal{S}_{\mathcal{H}}$. By [37, Corollary 12], the corresponding 2-representation $\mathbf{inj}_{\mathcal{S}_{\mathcal{H}}}(B)$ is simple transitive. By [36, Example 4.20], we have $B \cong [B, B]_{\mathcal{S}_{\mathcal{H}}}$, which, by [36, Theorem 4.19], implies that B is a direct summand of $\Theta(A)$ in $\mathcal{S}_{\mathcal{H}}$. Hence, B is of the form $\Theta(\tilde{A})$, for some 1-morphism \tilde{A} in $\mathcal{A}_{\mathcal{H}}$. It is easy to check that \tilde{A} is a subcoalgebra of A in $\mathcal{A}_{\mathcal{H}}$ and is thus isomorphic to A . The claim follows. □

Example 3.8. Being an identity 1-morphism, A_d is a cosimple coalgebra in $\mathcal{A}_{\mathcal{H}}$. By Lemma 3.6 and Proposition 3.7, this implies that C_d has the structure of a cosimple coalgebra in $\mathcal{S}_{\mathcal{H}}$ whose structure 2-morphisms are homogeneous of degree 0. By duality, this implies that \tilde{C}_d has the structure of a simple algebra in $\mathcal{S}_{\mathcal{H}}$, again with structure 2-morphisms homogeneous of degree 0.

In Subsection 4.4, we will additionally see that C_d has the structure of a Frobenius algebra in $\mathcal{S}_{\mathcal{H}}$.

We denote by \square the cotensor product, see [57, Section 0].

Lemma 3.9. *Let A be a coalgebra in $\mathcal{A}_{\mathcal{H}}$, X a right A -comodule and Y a left A -comodule. Then there is a 2-isomorphism in $\mathcal{S}_{\mathcal{H}}^{(0)}$*

$$\Theta(X \square_A Y) \cong \Theta(X) \square_{\Theta(A)} \Theta(Y)$$

that is natural in X and Y .

Proof. Consider the commutative diagram in $\mathcal{S}_H^{(0)}$

$$\begin{array}{ccccc}
 \Theta(X \square_A Y) & \hookrightarrow & \Theta(XY) & \longrightarrow & \Theta(XA Y) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Theta(X) \square_{\Theta(A)} \Theta(Y) & \hookrightarrow & \Theta(X)\Theta(Y) & \xrightarrow{\varphi} & \Theta(X)\Theta(A)\Theta(Y),
 \end{array}$$

where the top row is given by applying Θ to the definition of $X \square_A Y$, the bottom row is the definition of $\Theta(X) \square_{\Theta(A)} \Theta(Y)$ and the two solid inclusions are given by the oplax structure of Θ . The dashed inclusion is induced by commutativity of the solid square and the universality property of the cotensor product. As all solid arrows are natural in X and Y , the same holds for the dashed inclusion.

Note that the vertical arrows in the solid square restrict to 2-isomorphisms in degree 0. Consequently, the dashed arrow is also 2-isomorphism in degree 0. To prove the lemma, we need to show that φ is injective when restricted to summands of $\Theta(X)\Theta(Y)$ generated in positive degrees (as 1-morphisms in $\mathcal{S}_H^{(0)}$). We will verify this by passing to the cell 2-representation \mathbf{C}_H .

Let B be the underlying algebra of \mathbf{C}_H . Then B is naturally a positively graded algebra. The bimodules representing $\Theta(A)$, $\Theta(X)$ and $\Theta(Y)$ are all projective by Theorem 2.13 and generated in degree 0, by construction. Let M be an indecomposable summand of $\mathbf{C}_H(\Theta(X)\Theta(Y))$ not generated in degree 0. Being a summand in the tensor product over B of two projective B - B -bimodules generated in degree 0, we can view M as a summand of $(B \otimes_C B) \otimes_B (B \otimes_C B) \cong B \otimes_C B \otimes_C B$. Writing any non-zero element u of M in a fixed homogeneous basis of $B \otimes_C B \otimes_C B$, the element u will have a non-zero coefficient at some $b \otimes c \otimes d$, where b, c and d are elements from a fixed homogeneous basis of B and c has positive degree. Under φ , such $b \otimes c \otimes d$ is sent to

$$q := b \otimes e \otimes c \otimes d - b \otimes c \otimes f \otimes d,$$

where e and f are non-zero elements of degree 0 in B . The element q is clearly non-zero in $\mathbf{C}_H(\Theta(X)\Theta(A)\Theta(Y))$. All other basis elements appearing with non-zero coefficient in u will be mapped to elements with zero coefficients at $b \otimes e \otimes c \otimes d$ and $b \otimes c \otimes f \otimes d$. The claim follows. □

Proposition 3.10. *Let A and B be Morita–Takeuchi equivalent coalgebras in \mathcal{A}_H . Then $\Theta(A)$ and $\Theta(B)$ are Morita–Takeuchi equivalent in $\mathcal{S}_H^{(0)}$.*

Proof. By [35, Theorem 5.1], there exist a B - A -bicomodule X and an A - B -bicomodule Y such that

$${}_B X \square_A Y_B \cong {}_B B_B \quad \text{and} \quad {}_A Y \square_B X_A \cong {}_A A_A.$$

Applying Lemmas 3.6 and 3.9, we obtain

$$\begin{aligned}
 \Theta(B) \Theta(X) \square_{\Theta(A)} \Theta(Y)_{\Theta(B)} &\cong \Theta(B) \Theta(B)_{\Theta(B)}, \\
 \Theta(A) \Theta(Y) \square_{\Theta(B)} \Theta(X)_{\Theta(A)} &\cong \Theta(A) \Theta(A)_{\Theta(A)}.
 \end{aligned}$$

The claim of the proposition now follows from [35, Theorem 5.1]. □

Proposition 3.11. *If A as in Lemma 3.6 is cosimple and $X \in \text{inj}_{\mathcal{A}_H}(A)$, then $\Theta(X)$ is in $\text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$.*

Note that Lemma 3.6 shows that the right coaction of $\Theta(A)$ on $\Theta(X)$ is, in fact, homogeneous of degree 0.

Proof. By additivity, it suffices to prove that $\Theta(A_w A)$ is in $\text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$, for any $w \in \mathcal{H}$.

Set $B = [A_w A, A_w A]_{\mathcal{A}_H}$. By [36, Corollary 4.17], the Morita–Takeuchi equivalence between A and B is given by the A - B -bicomodule AA_w^* and the B - A -bicomodule $A_w A$. By Lemma 3.9 and Proposition 3.10, the $\Theta(A)$ - $\Theta(B)$ -bicomodule $\Theta(AA_w^*)$ and the $\Theta(B)$ - $\Theta(A)$ -bicomodule $\Theta(A_w A)$ provide a Morita–Takeuchi equivalence between $\Theta(A)$ and $\Theta(B)$ in $\mathcal{S}_H^{(0)}$. In particular, $\Theta(A_w A)$ is in $\text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$. \square

Corollary 3.12. *If A as in Lemma 3.6 is cosimple, then the ranks of $\text{inj}_{\mathcal{A}_H}(A)$ and $\text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$ are equal.*

Proof. The fact that $\Pi\Theta \cong \text{id}_{\mathcal{A}_H}$ and Proposition 3.11 imply that application of Θ induces an injection from the set of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{A}_H}(A)$ to the set of isomorphism classes of indecomposable objects in $\text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$. Let us now argue that this is also surjective.

Let $M \in \text{inj}_{\underline{\mathcal{S}}_H}(\Theta(A))$ be indecomposable. Up to an overall grading shift, we may assume that there is a decomposition in $\mathcal{S}_H^{(0)}$ of the form

$$M \cong \bigoplus_{w \in \mathcal{H}} C_w^{\oplus p_w},$$

where the p_w belong to $\mathbb{N}_0[v]$ and are such that $p_w(0)$ is non-zero for at least one $w \in \mathcal{H}$. Here $p_w(0)$ means the evaluation of $p_w(v)$ at $v = 0$. Let

$$M^0 := \bigoplus_{w \in \mathcal{H}} C_w^{\oplus p_w(0)}.$$

By positivity of the grading on \mathcal{S}_H and Lemma 3.2, M^0 is a $\Theta(A)$ -subcomodule of M in $\mathcal{S}_H^{(0)}$.

Note that $\Pi(\Theta(A)) \cong A$ by construction. Similarly, $\Pi(M^0\Theta(A)) \cong \Pi(M^0)\Pi(\Theta(A))$. Hence, there is an induced A -comodule structure on $\Pi(M^0)$. By Corollary 2.17, $\Pi(M^0)$ is injective as an A -comodule since A is cosimple and \mathcal{A}_H is fusion. Now, $\Theta(\Pi(M^0)) \cong M^0$ is an injective $\Theta(A)$ -comodule in $\mathcal{S}_H^{(0)}$ by Proposition 3.11. Consequently, M^0 is isomorphic to M in $\mathcal{S}_H^{(0)}$. \square

3.4 | Going down

In this subsection we describe, see Lemma 3.13, and investigate what we loosely call ‘going down’. As the name suggests, this is, in some sense, the opposite of going up.

Let \mathbf{M} be a graded simple transitive 2-representation of \mathcal{S}_H with apex \mathcal{H} . Let C be a graded coalgebra in \mathcal{S}_H such that \mathbf{M} is equivalent to $\text{inj}_{\underline{\mathcal{S}}_H}(C)$. Let X_1, \dots, X_n be a complete and irredundant list of representatives of isomorphism classes (up to grading shift) of indecomposable objects

in $\text{inj}_{\mathcal{S}_H}(C)$, normalized such that as 1-morphisms in \mathcal{S}_H they are concentrated in non-negative degrees with non-zero degree zero part. Set $X := X_1 \oplus X_2 \oplus \dots \oplus X_n$.

Lemma 3.13. *The quotient*

$$\text{add}(\{X^{\oplus v^k} \mid k \geq 0\})^{(0)} / \left(\text{add}(\{X^{\oplus v^k} \mid k > 0\})^{(0)} \right)$$

carries an induced action of \mathcal{A}_H . Here $(\text{add}(\{X^{\oplus v^k} \mid k > 0\})^{(0)})$ is the ideal generated by $\text{add}(\{X^{\oplus v^k} \mid k > 0\})^{(0)}$.

Proof. Lemma 3.2 implies that, if $X_m^{\oplus v^k}$ is isomorphic to a direct summand of $C_w X_j$ in $\mathcal{S}_H^{(0)}$, where $w \in H$, then $k \geq 0$. Therefore $\text{add}(\{X^{\oplus v^k} \mid k \geq 0\})^{(0)}$ is stable under the action of $\text{add}(\{C_w^{\oplus v^l} \mid w \in W, l \geq 0\})^{(0)}$. Moreover, $(\text{add}(\{C_w^{\oplus v^l} \mid w \in W, l > 0\})^{(0)})$ maps $\text{add}(\{X^{\oplus v^k} \mid k \geq 0\})^{(0)}$ to $(\text{add}(\{X^{\oplus v^k} \mid k > 0\})^{(0)})$. The claim follows. \square

We state and prove the following lemma for \mathcal{S}_H or \mathcal{A}_H , although we only need it in the latter case, since it is a useful result in its own right.

Lemma 3.14. *Let $\mathcal{C} = \mathcal{S}_H$ or $\mathcal{C} = \mathcal{A}_H$. Given any graded simple transitive 2-representation \mathbf{M} of \mathcal{C} with apex H , there exists a coalgebra C in \mathcal{C} such that $\text{inj}_{\mathcal{C}}(C)$ is equivalent to \mathbf{M} and C is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{C}}(C)$ under the forgetful functor to \mathcal{C} .*

Proof. Let C' be any coalgebra in \mathcal{C} and Y a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{C}}(C')$. Then the coalgebra $C := C^Y$ in \mathcal{C} is Morita–Takeuchi equivalent to C' , and the equivalence between $\text{inj}_{\mathcal{C}}(C')$ and $\text{inj}_{\mathcal{C}}(C)$ identifies Y with C , so C is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{C}}(C)$ under the forgetful functor to \mathcal{C} , as required. \square

Proposition 3.15. *There is an injection $\hat{\Theta}$ from the set of equivalence classes of simple transitive 2-representations of \mathcal{A}_H to the set of equivalence classes of graded simple transitive 2-representations of \mathcal{S}_H with apex H .*

Proof. Let \mathbf{N} be a simple transitive 2-representation of \mathcal{A}_H . Let A be a coalgebra in \mathcal{A}_H such that \mathbf{N} is equivalent to $\text{inj}_{\mathcal{A}_H}(A)$. Then $\text{inj}_{\mathcal{S}_H}(\Theta(A))$ is a graded simple transitive 2-representation of \mathcal{S}_H (with apex H) due to Proposition 3.7 and [37, Corollary 12]. By Proposition 3.10, this yields a well-defined map $\hat{\Theta}$.

Now assume A is chosen such that A is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{A}_H}(A)$ under the forgetful functor to \mathcal{A}_H , see Lemma 3.14. Set $C := \Theta(A)$. Then, by Proposition 3.11 and Corollary 3.12, the object X defined in the going down procedure is isomorphic to C . The 2-representation of \mathcal{A}_H obtained by going down is now, clearly, equivalent to $\text{inj}_{\mathcal{A}_H}(A)$. Therefore, the map defined in the previous paragraph is injective, as claimed. \square

As we will see later, Theorem 7.1 actually implies that $\hat{\Theta}$ is a bijection.

4 | THE ROLE OF THE DUFLO INVOLUTION

Throughout this section, we set $\mathbf{a} := \mathbf{a}(\mathcal{H})$ for our fixed diagonal \mathcal{H} -cell, and let d be the Duflo involution in \mathcal{H} .

4.1 | Cell 2-representations and Duflo involutions

Let $w \in \mathcal{H}$ and \tilde{L}_w be the corresponding simple object in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$, concentrated in degree 0. By [42, Section 7] (see also [40, Subsection 4.5]), $\tilde{\mathbf{C}}_w \tilde{L}_d$ is an indecomposable injective object in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ with simple socle \tilde{L}_w concentrated in degree 0.

Dually, let L_w be the simple object in $\overline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$, corresponding to w , concentrated in degree 0. Then $\mathbf{C}_w L_d$ is an indecomposable projective object in $\overline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ with simple head L_w concentrated in degree 0.

Lemma 4.1. *For any $x, w \in \mathcal{H}$,*

- (i) *the injective objects $\tilde{\mathbf{C}}_x \tilde{L}_w$ and $\mathbf{C}_x \tilde{L}_w$ in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ are concentrated between the degrees $-2\mathbf{a}$ and 0 and between the degrees 0 and $2\mathbf{a}$, respectively;*
- (ii) *the projective objects $\tilde{\mathbf{C}}_x L_w$ and $\mathbf{C}_x L_w$ in $\overline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ are concentrated between the degrees $-2\mathbf{a}$ and 0 and between the degrees 0 and $2\mathbf{a}$, respectively.*

Proof. Let us prove the first statement in (i). As the 2-category of Soergel bimodules is positively graded, the fact that $\tilde{\mathbf{C}}_x \tilde{\mathbf{C}}_w \tilde{L}_d$ is concentrated in non-positive degrees follows from Lemma 3.3. This in turn implies that $\tilde{\mathbf{C}}_x \tilde{L}_w$ is concentrated in non-positive degrees as well.

By adjunction, we have

$$\mathrm{hom}_{\underline{\mathbf{C}}_{\mathcal{H}}}(\tilde{\mathbf{C}}_x \tilde{L}_w, \tilde{L}_y^{\oplus v^k}) \cong \mathrm{hom}_{\underline{\mathbf{C}}_{\mathcal{H}}}(\tilde{L}_w, \tilde{\mathbf{C}}_{x^{-1}} \tilde{L}_y^{\oplus v^{(2\mathbf{a}+k)}}).$$

As the right-hand side of the above isomorphism is zero for $2\mathbf{a} + k < 0$, we deduce that $\mathrm{hom}_{\underline{\mathbf{C}}_{\mathcal{H}}}(\tilde{\mathbf{C}}_x \tilde{L}_w, \tilde{L}_y^{\oplus v^k})$ is also zero if $k < -2\mathbf{a}$.

The second statement in (i) follows from the first one and the fact that $\tilde{\mathbf{C}}_x = \mathbf{C}_x^{\oplus v^{-2\mathbf{a}}}$.

The dual statements in (ii) are proved in exactly the same way, using Lemma 3.2 instead of Lemma 3.3. \square

Let $\mathbf{P} := \mathbf{P}_{\emptyset}$ be the principal 2-representation of $\mathcal{S}_{\mathcal{H}}$ and $\underline{\mathbf{P}}$ its injective abelianization, see Example 2.6. Denote by I_w and I_e the corresponding injective object in $\underline{\mathbf{P}}(\emptyset)$ with respect to $\tilde{\mathbf{C}}_w$ and $\tilde{\mathbf{C}}_e = \mathbb{1}_{\emptyset}$, respectively, see [35, Subsection 3.1] for details. Note that I_w has simple socle \tilde{L}_w concentrated in degree 0 and I_e has simple socle \tilde{L}_e concentrated in degree \mathbf{a} .

Lemma 4.2. *For any $x \in \mathcal{H}$, the following hold.*

- (i) *The injective object $\tilde{\mathbf{C}}_x \tilde{L}_d$ in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ has simple head \tilde{L}_x concentrated in degree $-2\mathbf{a}$.*
- (ii) *The projective object $\mathbf{C}_x L_d$ in $\overline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$ has simple socle L_x concentrated in degree $2\mathbf{a}$.*

Proof. We only prove statement (i) since the dual statement (ii) follows by similar arguments. As $\mathcal{S}_{\mathcal{H}}$ is fiat, the underlying algebra of $\underline{\mathbf{C}}_{\mathcal{H}}$ is self-injective, by Corollary 2.14, which implies that the

indecomposable injective object $\tilde{C}_x \tilde{L}_d$ is also projective and thus, has a simple head. Therefore, it suffices to prove that $\text{hom}_{\underline{\mathbf{C}}_{\mathcal{H}}}(\tilde{C}_x \tilde{L}_d^{\oplus v^{2a}}, \tilde{L}_x)$ is non-zero. By adjunction, this is equivalent to proving that

$$\text{hom}_{\underline{\mathbf{C}}_{\mathcal{H}}}(\tilde{L}_d, \tilde{C}_{x^{-1}} \tilde{L}_x) \neq 0, \tag{4.1}$$

which holds if and only if $\tilde{C}_d \tilde{L}_d$ appears as a direct summand of $\tilde{C}_{x^{-1}} \tilde{L}_x$ in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$.

Note that the latter holds if and only if the same is true in $\underline{\mathbf{P}}(\emptyset)$. (Recall the two equivalent constructions of cell 2-representations, see [40, Subsection 4.5].) In the principal 2-representation, we can use the following fact. Since $\tilde{C}_x I_e \cong I_x$, we have

$$\text{hom}_{\underline{\mathbf{P}}}(\tilde{C}_x \tilde{L}_d, I_e^{\oplus v^k}) \cong \text{hom}_{\underline{\mathbf{P}}}(\tilde{L}_d, I_{x^{-1}}^{\oplus v^{(2a+k)}}),$$

whose right-hand side is zero unless $x = d$ and $k = -2a$. In other words, the only $x \in \mathcal{H}$ such that $\tilde{C}_x \tilde{L}_d$ has a composition factor isomorphic to \tilde{L}_e , up to a shift, is $x = d$. Therefore, to prove (4.1), it is enough to show that $\text{hom}_{\underline{\mathbf{P}}}(\tilde{C}_{x^{-1}} \tilde{L}_x, I_e^{\oplus v^{-2a}})$ is non-zero. By adjunction, we have

$$\text{hom}_{\underline{\mathbf{P}}}(\tilde{C}_{x^{-1}} \tilde{L}_x, I_e^{\oplus v^{-2a}}) \cong \text{hom}_{\underline{\mathbf{P}}}(\tilde{L}_x, I_x),$$

where the right-hand side is non-zero. Hence, $\tilde{L}_x^{\oplus v^{-2a}}$ appears in the head of $\tilde{C}_x \tilde{L}_d$ in $\underline{\mathbf{C}}_{\mathcal{H}}(\emptyset)$. \square

Assume that $B = B^{\underline{\mathbf{C}}_{\mathcal{H}}}$ is the underlying basic algebra of $\underline{\mathbf{C}}_{\mathcal{H}}$. Then the indecomposable objects in $\mathbf{C}_{\mathcal{H}}(\emptyset)$ are identified with indecomposable injective B -modules.

Proposition 4.3. *The algebra B is a finite-dimensional positively graded weakly symmetric Frobenius algebra of graded length $2a$.*

Proof. By construction, the algebra B is non-negatively graded and its degree 0 part, which is isomorphic to $C^{\#\mathcal{H}}$, is semisimple, so B is positively graded.

Since $\mathcal{S}_{\mathcal{H}}$ is fiat, the algebra B is Frobenius, see Corollary 2.14.

The fact that B is weakly symmetric follows from Lemma 4.2(i). Together with Lemma 4.1(ii), this implies B is of graded length $2a$. \square

Remark 4.4. In some cases, we know that B is symmetric.

- (i) Let $\mathcal{L}(\mathcal{H})$ be the left cell for \mathcal{H} . From [46, Theorem 4.6] we know that B is symmetric if W is a finite Weyl group and

$$\text{there exists } I \subset S \text{ such that } w_0 w_0^I \in \mathcal{L}(\mathcal{H}).$$

- (ii) For any finite Coxeter group W , B is symmetric if \mathcal{H} is a diagonal \mathcal{H} -cell in the subregular two-sided cell of \mathcal{S} , see [23, Corollary 5, Proposition 14 and the comment below it, and Remarks 29, 32 and 38; 38, Theorem I].

However, we do not know if this holds in general.

Since \mathcal{S}_H is fiat, the action of C_w on the category of B-modules is exact. Further, by Theorem 2.13, we know that the action of C_w via \mathbf{C}_H is given by tensoring with a projective-injective bimodule. It follows from Lemma 4.1(i) that the bimodule representing C_w is isomorphic to a direct sum of bimodules of the form $B e_u \otimes_C e_v B$, possibly with multiplicities but without grading shifts, where e_u, e_v are some primitive idempotents of B. By Proposition 4.3 and the fact that \mathbf{C}_H is a faithful 2-functor which is degree-preserving on 2-morphisms, this implies that the 1-morphism C_w in \mathcal{S}_H has graded length at most $4a$.

Recall from Example 3.8 that C_d is a cosimple coalgebra 1-morphism in \mathcal{S}_H . By [37, Corollary 12], $\mathbf{M} := \mathbf{inj}_{\mathcal{S}_H}(C_d)$ is a graded simple transitive 2-representation of \mathcal{S}_H with apex H . We denote by B^M the basic algebra underlying \mathbf{M} .

Proposition 4.5. *The algebra B^M is a positively graded Frobenius algebra of graded length $2a$.*

Proof. The algebra B^M is graded, by definition, and Frobenius by Corollary 2.14.

By Lemma 3.14, we can choose a coalgebra A in \mathcal{A}_H such that $\mathbf{inj}_{\mathcal{A}_H}(A)$ is equivalent to $\mathbf{inj}_{\mathcal{A}_H}(A_d)$ and A is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in $\mathbf{inj}_{\mathcal{A}_H}(A)$ under the forgetful functor to \mathcal{A}_H . This implies that $A \cong \bigoplus_{w \in H} A_w$ as 1-morphisms in \mathcal{A}_H , because A_d is the identity 1-morphism of \mathcal{A}_H .

Define $C := \Theta(A)$ in $\mathcal{S}_H^{(0)}$. Since $\Theta(A_w) \cong C_w$, for all $w \in H$, there is an isomorphism $C \cong \bigoplus_{w \in H} C_w$ in $\mathcal{S}_H^{(0)}$ and, by Proposition 3.10, there is an equivalence $\mathbf{inj}_{\mathcal{S}_H}(C) \simeq \mathbf{M}$. From the definition of C , for all $k \in \mathbb{Z}$, we have

$$\mathrm{hom}_{\mathcal{S}_H}(C, \mathbb{1}_{\mathcal{S}_H}^{\oplus v^k}) \cong \mathrm{hom}_{\mathbf{M}}(C, \mathbb{1}_{\mathcal{S}_H}^{\oplus v^k} C). \tag{4.2}$$

If $k > 0$, then Soergel’s hom formula [14, Theorem 3.6] (here we need the full formula, not just the restricted version which we recalled in (3.3)) implies that the left-hand side of (4.2) is zero. If $k = 0$, then Soergel’s hom formula implies that the left-hand side of (4.2) has dimension one, because

$$\dim \left(\mathrm{hom}_{\mathcal{S}_H}(C_w, \mathbb{1}_{\mathcal{S}_H}) \right) = \begin{cases} 1 & \text{if } w = d, \\ 0 & \text{else.} \end{cases}$$

This implies that B^M is positively graded.

Since $C \cong \bigoplus_{w \in H} C_w$ in $\mathcal{S}_H^{(0)}$, the left-hand side of (4.2) is zero if $k < -2a$ and hence, the graded length of B^M is at most $2a$. As the algebra underlying the cell 2-representation has graded length $2a$, see Proposition 4.3, we know that there exists a C_w such that $\mathrm{hom}_{\mathcal{S}_H}(C_w, \mathbb{1}_{\mathcal{S}_H}^{\oplus v^{-2a}}) \neq 0$, which implies that the left-hand side of (4.2) is non-zero for $k = -2a$. Thus, the graded length of B^M is exactly $2a$. This completes the proof. \square

Lemma 4.6. *For all $w \in H$, the 1-morphism C_w in \mathcal{S}_H is of graded length $4a$.*

Proof. Recall from [36, Example 4.20] that $C_d \cong [C_d, C_d]$ in $\mathcal{S}_H^{(0)}$, where the internal cohom-construction is with respect to $\mathbf{M} = \mathbf{inj}_{\mathcal{S}_H}(C_d)$. We first claim that

$$\mathrm{hom}_{\mathcal{S}_H}(C_d, C^{\oplus v^{-4a}}) \cong \mathrm{hom}_{\mathbf{M}}(C_d, CC_d^{\oplus v^{-4a}}) \tag{4.3}$$

is non-zero, where C is as in the proof of Proposition 4.5. By Lemma 3.2, we have

$$C_w C_d \cong \bigoplus_{z \in \mathcal{H}} C_z^{\oplus v^a h_{w,d,z}}$$

in $\mathcal{S}_H^{(0)}$. Noting that (3.1), (3.2) and [31, Conjecture 14.2.P2, P5 and P7] imply that

$$\gamma_{w,d,z^{-1}} = \gamma_{z^{-1},w,d} = \delta_{z,w},$$

we deduce that

$$v^a h_{w,d,z} \in \begin{cases} 1 + \dots + v^{2a} & \text{if } z = w, \\ v\mathbb{N}_0[v] \cap v^{2a-1}\mathbb{N}_0[v^{-1}] & \text{if } z \neq w. \end{cases}$$

Therefore, the object $C^{\oplus v^{-2a}}$ is isomorphic to a direct summand of $CC_d^{\oplus v^{-4a}}$ in $\mathbf{M}^{(0)}(\emptyset)$. The head of the indecomposable injective object C_d in \mathbf{M} is isomorphic to a direct summand of the socle of $C^{\oplus v^{-2a}}$ in $\mathbf{M}^{(0)}(\emptyset)$, see Proposition 4.5 and its proof. Hence, the right-hand side of (4.3) is non-zero, which implies that the left-hand side is non-zero. This shows that C has a direct summand in $\mathcal{S}_H^{(0)}$ isomorphic to C_v with graded length at least $4a$. Therefore, C_v must have graded length exactly $4a$, as we already know that it is at most $4a$. By adjunction, we have

$$0 \neq \text{hom}_{\mathcal{S}_H}(C_d, C_v^{\oplus v^{-4a}}) \cong \text{hom}_{\mathcal{S}_H}(\tilde{C}_{v^{-1}}, \tilde{C}_d^{\oplus v^{-4a}}) \cong \text{hom}_{\mathcal{S}_H}(C_{v^{-1}}, C_d^{\oplus v^{-4a}}), \tag{4.4}$$

yielding that the graded length of C_d is at least $4a$. Thus, as above, it must be equal to $4a$. Note that for any $w \in \mathcal{H}$ we have $v^a h_{w,w^{-1},d} \in 1 + \dots + v^{2a}$. Therefore, each $\tilde{C}_{w^{-1}} C_w^{\oplus v^{-4a}}$ contains a direct summand $C_d^{\oplus v^{-4a}}$ in $\mathcal{S}_H^{(0)}$. By (4.4), we have

$$0 \neq \text{hom}_{\mathcal{S}_H}(C_{v^{-1}}, \tilde{C}_{w^{-1}} C_w^{\oplus v^{-4a}}) \cong \text{hom}_{\mathcal{S}_H}(C_w C_{v^{-1}}, C_w^{\oplus v^{-4a}}).$$

By Lemma 3.2, the direct summands of $C_w C_{v^{-1}}$ in $\mathcal{S}_H^{(0)}$ have non-negative shift. Again as before, this shows that the graded length of C_w is at least $4a$, which implies that it must be equal to $4a$. \square

Recall from Example 3.8 that \tilde{C}_d is a simple algebra 1-morphism in $\overline{\mathcal{S}_H}$.

Proposition 4.7.

- (i) The 2-representation $\mathbf{inj}_{\overline{\mathcal{S}_H}}(C_d)$ is equivalent to the cell 2-representation \mathbf{C}_H .
- (ii) The 2-representation $\mathbf{proj}_{\overline{\mathcal{S}_H}}(\tilde{C}_d)$ is also equivalent to the cell 2-representation \mathbf{C}_H .

Proof. Again, we will prove the statement (i) and the dual statement (ii) follows verbatim. Consider C_d as an object of $\mathbf{C}_H(\emptyset)$ and set $C := [C_d, C_d]$ (note that here the internal cohom-construction is with respect to \mathbf{C}_H and not \mathbf{M} as in the proof of Lemma 4.6). As a 1-morphism in $\mathcal{S}_H^{(0)}$ we have

$$C \cong \bigoplus_{w \in \mathcal{H}} C_w^{\oplus p_w},$$

with $p_w \in \mathbb{N}_0[v, v^{-1}]$. Furthermore,

$$\mathrm{hom}_{\mathcal{S}_H}(C, C_w^{\oplus v^k}) = \mathrm{hom}_{\mathcal{S}_H}([C_d, C_d], C_w^{\oplus v^k}) \cong \mathrm{hom}_{\mathbf{C}_H}(C_d, C_w^{\oplus v^k} C_d). \quad (4.5)$$

By positivity of the grading on \mathcal{S}_H and Lemma 3.2, we see that $p_w \in \mathbb{N}_0[v^{-1}]$.

If $k < -4\mathbf{a}$, then the right-hand side of (4.5) is zero, because C_d is an indecomposable injective object of graded length $2\mathbf{a}$ by Lemma 4.1, and the action of C_w increases the graded length by at most $2\mathbf{a}$ by Lemma 3.2. On the left-hand side, the indecomposable injective object C_w has graded length $4\mathbf{a}$, see Lemma 4.6, which implies that C lives in non-negative degrees, that is, $p_w = p_w(0)$.

For $k = 0$, the right-hand side of (4.5) is one-dimensional if $w = d$, and zero otherwise. This implies $C \cong C_d$ in $\mathcal{S}_H^{(0)}$. Since the degree 0 maps $C_d \rightarrow \mathbb{1}_{\mathcal{S}_H}$ and $C_d \rightarrow C_d C_d$ are unique up to scalar, it follows that $C \cong \Theta(\mathbb{1}_{\mathcal{S}_H})$ as coalgebra 1-morphisms in $\mathcal{S}_H^{(0)}$. \square

Let

$$\mathcal{B}_H := (C_d)\mathrm{biinj}_{\underline{\mathrm{add}}(\mathcal{H})}(C_d) \quad (4.6)$$

denote the graded one-object bicategory of biinjective C_d - C_d -bicomodules in $\underline{\mathrm{add}}(\mathcal{H})$, where horizontal composition is given by the cotensor product $-\square_{C_d}-$ and the identity 1-morphism is C_d . By definition, a biinjective C_d - C_d -bicomodule is injective as a left and as a right C_d -comodule, but not necessarily as a C_d - C_d -bicomodule, for example, C_d is biinjective but not injective as a C_d - C_d -bicomodule when $d \neq e, w_0$. The bicategory \mathcal{B}_H will play an important role in this article, cf. Proposition 6.11 and Theorem 7.1. For now, it is important to record the following corollary.

Corollary 4.8. *There is a graded biequivalence*

$$\mathcal{B}_H^{\mathrm{op}} \simeq \mathcal{E}\mathrm{nd}_{\mathcal{S}_H}(\mathbf{C}_H).$$

Proof. By Proposition 4.7(i), there is a graded biequivalence

$$\mathcal{E}\mathrm{nd}_{\mathcal{S}_H}(\mathbf{C}_H) \simeq \mathcal{E}\mathrm{nd}_{\mathcal{S}_H}(\underline{\mathrm{inj}}_{\mathcal{S}_H}(C_d)).$$

Using this identification, the assignment

$$\mathcal{B}_H^{\mathrm{op}} \rightarrow \mathcal{E}\mathrm{nd}_{\mathcal{S}_H}(\mathbf{C}_H), \quad X \mapsto -\square_{C_d} X$$

defines a biequivalence, as follows from Proposition 2.19 and [36, Theorems 4.26 and 4.31]. \square

4.2 | The categorified bar involution

Proposition 4.9. *There exists a functorial involution $\vee: \mathcal{S} \rightarrow \mathcal{S}$, which is covariant on 1-morphisms and contravariant and degree-preserving on 2-morphisms, such that we have*

$$(B_w^{\oplus v^k})^\vee \cong B_w^{\oplus v^{-k}} \quad (4.7)$$

in $\mathcal{S}^{(0)}$ for all $w \in W$ and $k \in \mathbb{Z}$.

Proof. Let \mathcal{D} be the diagrammatic (one-object) Soergel 2-category as in [13], whose 1-morphisms are of the form $\underline{w}\langle k \rangle$, where \underline{w} is a word in the simple reflections of W and $k \in \mathbb{Z}$ indicates a formal degree shift, and whose 2-morphisms are Soergel diagrams, defined by generators and relations. By [13, Theorem 6.30], we can identify \mathcal{S} with $\text{add}(\mathcal{D})$. (Strictly speaking, we have to quotient $\text{add}(\mathcal{D})$ by the 2-ideal generated by the totally invariant polynomials with no constant term in the base ring R in that paper, because \mathcal{S} was defined over the coinvariant algebra. This is a technical detail, which we will suppress from now on.) Now, $\vee : \mathcal{D} \rightarrow \mathcal{D}$ is defined by $(\underline{w}\langle -k \rangle)^\vee := \underline{w}\langle k \rangle$, for all words \underline{w} and $k \in \mathbb{Z}$, and by flipping the Soergel diagrams upside-down. By definition, \vee is covariant on 1-morphisms, contravariant and degree-preserving on 2-morphisms. For any word \underline{w} in the simple reflections of W , let $\text{BS}(\underline{w})$ be the corresponding Bott–Samelson bimodule in \mathcal{S} . Then $(\text{BS}(\underline{w})^{\oplus v^k})^\vee \cong \text{BS}(\underline{w})^{\oplus v^{-k}}$ under the identification $\mathcal{S} \simeq \text{add}(\mathcal{D})$.

Extend \vee to $\mathcal{S} \simeq \text{add}(\mathcal{D})$. To show (4.7), we use induction on the length $\ell(w)$ of w , the case $\ell(w) = 0$ being immediate. Assume that $\ell(w) > 0$ and that (4.7) holds for all $v \in W$ with $\ell(v) < \ell(w)$ and all $k \in \mathbb{Z}$. By [55, Satz 6.14] (see also [13, Theorem 3.16] and the text around it) and [13, Corollary 6.27], we have

$$\text{BS}(\underline{w}) \cong B_w \oplus \bigoplus_{v < w} B_v^{\oplus p_{w,v}} \text{ in } \mathcal{S}^{(0)}$$

for every $w \in W$, where \underline{w} is an arbitrary reduced expression for w , the $p_{w,v} \in \mathbb{N}_0[v, v^{-1}]$ are invariant under the bar involution (which follows from [31, Chapter 4] and the Soergel–Elias–Williamson categorification theorem) and $<$ is the Bruhat order. As remarked above, we have $\text{BS}(\underline{w})^\vee \cong \text{BS}(\underline{w})$ in $\mathcal{S}^{(0)}$. By induction, we also have $(B_v^{\oplus p_{w,v}})^\vee \cong B_v^{\oplus p_{w,v}}$ in $\mathcal{S}^{(0)}$ for all $v < w$, using the bar invariance of $p_{w,v}$. Since $\mathcal{S}^{(0)}$ is Krull–Schmidt, we deduce that $B_w^\vee \cong B_w$ in $\mathcal{S}^{(0)}$. This implies that (4.7) holds for all $k \in \mathbb{Z}$. \square

Recall that the bar involution on the Hecke algebra is uniquely determined by the fact that it is \mathbb{Z} -linear, sends $v^k \mapsto v^{-k}$ for all $k \in \mathbb{Z}$ and fixes the Kazhdan–Lusztig basis elements, see, for example, [31, Chapters 4 and 5]. By Proposition 4.9 and [13, Corollary 6.27 and Theorem 6.30], the duality \vee thus categorifies the bar involution on the Hecke algebra. We will therefore refer to it as the *categorified bar involution*.

Remark 4.10. The origin of the categorified bar involution is [55]. In the diagrammatic setting \vee appears in [13, Definition 6.22], where it is denoted ι and gives an antiinvolution on double light leaves.

Proposition 4.11. *The categorified bar involution on \mathcal{S} induces a functorial involution on $\mathcal{S}_{\mathcal{H}}$, also denoted \vee , which is covariant on 1-morphisms and contravariant and degree-preserving on 2-morphisms, such that $(B_x^{\oplus v^k})^\vee \cong B_x^{\oplus v^{-k}}$ in $\mathcal{S}_{\mathcal{H}}^{(0)}$, for all $x \in \mathcal{H}$ and $k \in \mathbb{Z}$. This functorial involution extends to an equivalence between $\underline{\mathcal{S}}_{\mathcal{H}}$ and $\overline{\mathcal{S}}_{\mathcal{H}}$ which sends injective 1-morphisms in the first 2-category to projective 1-morphisms in the second.*

Proof. Since \vee is covariant on 1-morphisms and $(B_w^{\oplus v^k})^\vee \cong B_w^{\oplus v^{-k}}$ in $\mathcal{S}^{(0)}$, for $w \in W$ and $k \in \mathbb{Z}$, it preserves left, right and two-sided cells.

As \vee sends identity 2-morphisms to identity 2-morphisms, it also preserves the maximal 2-ideal in \mathcal{S} that does not contain the identity 2-morphisms on the 1-morphisms in $\text{add}(\mathcal{J})$, whence

it descends to the \mathcal{J} -simple quotient $\mathcal{S}_{\leq \mathcal{J}}$. Since \vee also preserves the two-sided cell \mathcal{J} and its diagonal \mathcal{H} -cells, it restricts to the 2-full 2-subcategories $\mathcal{S}_{\mathcal{J}}$ and $\mathcal{S}_{\mathcal{H}}$, which proves the first claim.

Finally, since \vee is contravariant on 2-morphisms, it extends to an equivalence between $\overline{\mathcal{S}_{\mathcal{H}}}$ and $\underline{\mathcal{S}_{\mathcal{H}}}$. \square

Corollary 4.12. *The functorial involution \vee induces a functorial involution on $\mathbf{C}_{\mathcal{H}}$, also denoted \vee , which is contravariant and degree-preserving on morphisms and satisfies $(\mathbb{B}_x^{\oplus v^k})^{\vee} \cong \mathbb{B}_x^{\oplus v^{-k}}$ in $\mathbf{C}_{\mathcal{H}}^{(0)}(\emptyset)$, for all $x \in \mathcal{H}$ and $k \in \mathbb{Z}$. This functorial involution extends to an equivalence between $\mathbf{C}_{\mathcal{H}}$ and $\overline{\mathbf{C}_{\mathcal{H}}}$ which sends injective objects in the category underlying the first to projective objects in the category underlying the second.*

Proof. As already remarked, the functorial involution \vee also preserves left cells. The rest now follows as in the proof of Proposition 4.11. \square

Remark 4.13. The existence of \vee implies that any statement in Subsection 4.1 has a dual counterpart. In particular, the equivalence $\underline{\mathbf{C}_{\mathcal{H}}} \simeq \overline{\mathbf{C}_{\mathcal{H}}}$ induces a degree-preserving isomorphism of algebras $\mathbb{B}^{\vee} \cong \mathbb{B}^{\text{op}}$, see Proposition 4.3.

4.3 | Explicit bimodules for the cell 2-representation

In the following, we will use projective abelianizations instead of injective ones. As we are in the fiat setup, the difference does not play an essential role on an abstract level, but with this choice we can conveniently describe the action of C_w by projective bimodules and their composition by tensoring over the underlying algebra, instead of injective bicomodules and cotensoring over the underlying coalgebra.

Denote by

$$\mathbb{B} := \text{End}_{\overline{\mathbf{C}_{\mathcal{H}}}} \left(\bigoplus_{w \in \mathcal{H}} C_w \right)^{\text{op}}$$

the algebra underlying the cell 2-representation. Fix a complete set of orthogonal primitive idempotents $e_w \in \mathbb{B}$, for $w \in \mathcal{H}$, corresponding to the indecomposable projective objects $C_w \in \mathbf{C}_{\mathcal{H}}(\emptyset)$. Set $Q_w := \mathbb{B}e_w$ and let L_w (recall the notation in Subsection 4.1) be the simple head of Q_w in $\overline{\mathbf{C}_{\mathcal{H}}}$. Note that L_w is one-dimensional and isomorphic to $\mathbb{C}e_w$ as a vector space. Note that $C_w L_d \cong Q_w$ for $w \in \mathcal{H}$, see Subsection 2.3 and [42, Section 7]. Lemma 4.2(ii) implies that the socle of Q_w is isomorphic to $L_w^{\oplus v^{2a}}$.

For every pair $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \text{grdim}(\text{Hom}_{\mathbb{B}}(Q_x, Q_y)) &= \text{grdim}(\text{Hom}_{\mathbb{B}}(C_x L_d, C_y L_d)) \\ &= \text{grdim}(\text{Hom}_{\mathbb{B}}(C_{y^{-1}} C_x L_d^{\oplus v^{-2a}}, L_d)) \\ &= \sum_{z \in \mathcal{H}} h_{y^{-1}, x, z} \text{grdim}(\text{Hom}_{\mathbb{B}}(C_z L_d^{\oplus v^{-a}}, L_d)) \\ &= \sum_{z \in \mathcal{H}} h_{y^{-1}, x, z} \text{grdim}(\text{Hom}_{\mathbb{B}}(Q_z^{\oplus v^{-a}}, L_d)) \\ &= v^a h_{y^{-1}, x, d}. \end{aligned} \tag{4.8}$$

By [31, Subsection 13.6], we know that

$$v^a h_{y^{-1},x,d} \in \begin{cases} 1 + \dots + v^{2a} & \text{if } x = y; \\ v\mathbb{N}_0[v] \cap v^{2a-1}\mathbb{N}_0[v^{-1}] & \text{if } x \neq y. \end{cases}$$

Recall that, by definition of the Kazhdan–Lusztig basis, the $h_{y^{-1},x,d}$ are invariant under the bar involution.

Remark 4.14. By [31, 13.1(e)], we have $h_{v^{-1},x,d} = h_{x^{-1},v,d}$, which corresponds to the fact that

$$\text{grdim}(\text{Hom}_{\mathbb{B}}(Q_x, Q_v)) = \text{grdim}(\text{Hom}_{\mathbb{B}}(Q_v, Q_x))$$

for all $x, v \in \mathcal{H}$.

For the formulation of the next proposition, we recall that the $\gamma_{w,v,u^{-1}}$ are defined in Subsection 3.1.

Proposition 4.15. *For any $w \in \mathcal{H}$, the action of C_w on the category of finite-dimensional graded B-modules is isomorphic to tensoring (over B) with the graded projective B-B-bimodule*

$$\bigoplus_{u,v \in \mathcal{H}} (\text{Be}_u \otimes_{\mathbb{C}} e_v \text{B})^{\oplus \gamma_{w,v,u^{-1}}}.$$

Proof. By Theorem 2.13, we know that the action of C_w is given by tensoring with a B-B-bimodule of the form

$$\bigoplus_{u,v \in \mathcal{H}} (\text{Be}_u \otimes_{\mathbb{C}} e_v \text{B})^{\oplus c_{w,v,u}},$$

for certain $c_{w,v,u} \in \mathbb{N}_0[v, v^{-1}]$. We also know that, for any $x \in \mathcal{H}$, we must have

$$C_w Q_x \cong C_w C_x L_d \cong \bigoplus_{u \in \mathcal{H}} Q_u^{\oplus v^a h_{w,x,u}}.$$

On the other hand, using (4.8), we obtain

$$\begin{aligned} \bigoplus_{u,v \in \mathcal{H}} (\text{Be}_u \otimes_{\mathbb{C}} e_v \text{B})^{\oplus c_{w,v,u}} \otimes_{\mathbb{B}} \text{Be}_x &\cong \bigoplus_{u,v \in \mathcal{H}} (\text{Be}_u \otimes_{\mathbb{C}} e_v \text{Be}_x)^{\oplus c_{w,v,u}} \\ &\cong \bigoplus_{u,v \in \mathcal{H}} \text{Be}_u^{\oplus \text{grdim}(e_v \text{Be}_x) c_{w,v,u}} \\ &\cong \bigoplus_{u,v \in \mathcal{H}} \text{Be}_u^{\oplus v^a h_{x^{-1},v,d} c_{w,v,u}}, \end{aligned}$$

and hence, deduce that the $c_{w,v,u}$ have to satisfy

$$\sum_{v \in \mathcal{H}} h_{x^{-1},v,d} c_{w,v,u} = h_{w,x,u}, \tag{4.9}$$

for all $w, x, u, v \in \mathcal{H}$.

For every fixed pair $w, u \in \mathcal{H}$, this is a system of $\#\mathcal{H}$ linear equations, indexed by $x \in \mathcal{H}$, in $\#\mathcal{H}$ variables, indexed by $v \in \mathcal{H}$. We claim that $c_{w,v,u} = \gamma_{w,v,u^{-1}}$, for $v \in \mathcal{H}$, is the unique solution of (4.9).

Let us first show that $c_{w,v,u} = \gamma_{w,v,u^{-1}}$ is a solution of (4.9), that is, that we have

$$\sum_{v \in \mathcal{H}} h_{x^{-1},v,d} \gamma_{w,v,u^{-1}} = h_{w,x,u}.$$

This equation is similar to one in [31, Subsection 18.8] and can be proved in the same way, using:

- the equation at the beginning of the proof of Theorem 18.9(b) in [31], that is,

$$\sum_{z \in W} h_{x_1, x_2, z} \gamma_{z, x_3, y^{-1}} = \sum_{z \in W} h_{x_1, z, y} \gamma_{x_2, x_3, z^{-1}}; \quad (4.10)$$

- the symmetries in [31, 13.1(e)];
- [31, Proposition 13.9(b) and Conjecture 14.2.P7], that is,

$$h_{a,b,c} = h_{b^{-1}, a^{-1}, c^{-1}}, \quad \gamma_{a,b,c} = \gamma_{b^{-1}, a^{-1}, c^{-1}}, \quad \gamma_{a,b,c} = \gamma_{c,a,b}; \quad (4.11)$$

- [31, Conjectures 14.2.P2 and 14.2.P5], that is,

$$\gamma_{v,u,d} = \delta_{v,u^{-1}}. \quad (4.12)$$

By (4.10), (4.11) and (4.12), we have

$$\begin{aligned} \sum_{v \in \mathcal{H}} h_{x^{-1},v,d} \gamma_{w,v,u^{-1}} &= \sum_{v \in \mathcal{H}} h_{x^{-1},v,d} \gamma_{v^{-1}, w^{-1}, u} \\ &= \sum_{v \in \mathcal{H}} h_{x^{-1},v,d} \gamma_{w^{-1}, u, v^{-1}} \\ &= \sum_{v \in \mathcal{H}} h_{x^{-1}, w^{-1}, v} \gamma_{v, u, d} \\ &= h_{x^{-1}, w^{-1}, u^{-1}} \\ &= h_{w, x, u}. \end{aligned}$$

Finally, note

$$h_{x^{-1},v,d} = v^{-a} \gamma_{x^{-1},v,d} \pmod{v^{-a+1} \mathbb{N}_0[v]},$$

so the determinant of the matrix

$$(h_{x^{-1},v,d})_{x,v \in \mathcal{H}}$$

belongs to $v^{-a(\#\mathcal{H})}(1 + v\mathbb{N}_0[v])$, and the matrix is hence invertible over $\mathbb{C}(v)$. Our system of linear equations in (4.9) therefore has a unique solution and the statement of the proposition follows. \square

4.4 | The Frobenius structure on the Duflo involution

In this subsection, we describe the structure of a Frobenius algebra on the Duflo involution in $\mathcal{S}_{\mathcal{H}}$ explicitly. As $C_d \cong \Theta(A_d)$ is a graded coalgebra, the comultiplication and counit 2-morphisms are homogeneous of degree 0. By duality, $\check{C}_d = C_d^{\oplus v^{-2a}}$ is a graded algebra. This implies that the multiplication and unit 2-morphisms of C_d are homogeneous of degree $-2a$ and $2a$, respectively. It remains to show that the multiplication and the comultiplication of C_d satisfy the Frobenius conditions.

Recall the algebra B constructed in Subsection 4.3. Proposition 4.15, combined with (4.11) and (4.12), implies that C_d acts via the B - B -bimodule

$$\bigoplus_{u \in \mathcal{H}} Be_u \otimes_{\mathbb{C}} e_u B.$$

The comultiplication on this bimodule is given by

$$\delta_d : \bigoplus_{u \in \mathcal{H}} Be_u \otimes_{\mathbb{C}} e_u B \rightarrow \bigoplus_{u, v \in \mathcal{H}} Be_u \otimes_{\mathbb{C}} e_u Be_v \otimes_{\mathbb{C}} e_v B, \quad e_u \otimes e_u \mapsto \delta_d(u) e_u \otimes e_u \otimes e_u$$

and the counit by

$$\epsilon_d : \bigoplus_{u \in \mathcal{H}} Be_u \otimes_{\mathbb{C}} e_u B \rightarrow B, \quad ae_u \otimes e_u b \mapsto \delta_d(u)^{-1} ae_u b$$

for certain $\delta_d(u) \in \mathbb{C}^{\times}$. In general we do not have a presentation of $\mathcal{S}_{\mathcal{H}}$ in terms of generating 2-morphisms and relations, so we cannot compute the $\delta_d(u)$ explicitly. In the specific case of dihedral Soergel bimodules such a presentation does exist and the $\delta_d(u)$ were computed explicitly in [38, Subsection 4.2]. In general, all we can say is that if the bimodule map representing δ_d involves $\delta_d(u)$, then the bimodule map representing ϵ_d involves $\delta_d(u)^{-1}$ by counitality.

To describe the algebra structure, consider the Frobenius trace $\text{tr} : B \rightarrow \mathbb{C}$ and note that $\text{tr}(e_u be_v) = 0$ for a homogeneous element $e_u be_v$ unless $u = v$ (because B is weakly symmetric) and the degree of $e_u be_v$ is $2a$. Furthermore, for any $u, v \in \mathcal{H}$, let $\{b_{i,v,u} \mid i = 1, \dots, m_{u,v}\}$ and $\{b^{i,u,v} \mid i = 1, \dots, m_{u,v}\}$ be dual bases of $e_v Be_u$ and $e_u Be_v$, respectively, with respect to the Frobenius trace tr , that is,

$$\text{tr}(b_{i,v,u} b^{j,u,v}) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Here $m_{u,v} = h_{u^{-1},v,d}(1) = h_{v^{-1},u,d}(1)$, with h defined as in (3.1). Following the same convention, the dual of e_u is denoted by e^u , for $u \in \mathcal{H}$. Let σ be the Nakayama automorphism of B defined by $\text{tr}(ab) = \text{tr}(b\sigma(a))$, for any $a, b \in B$. Note that, since B is weakly symmetric, we have $\sigma(e_u) = e_u$ and $\sigma(e^u) = e^u$ for all $u \in \mathcal{H}$. Without loss of generality, we therefore assume that $e_u = b_{1,u,u}$ and $e^u = b^{1,u,u}$. The algebra structure is then given by the multiplication

$$\mu_d : \bigoplus_{u,v \in \mathcal{H}} (Be_u \otimes_{\mathbb{C}} e_u Be_v \otimes_{\mathbb{C}} e_v B)^{\oplus v^{-2a}} \rightarrow \bigoplus_{u \in \mathcal{H}} Be_u \otimes_{\mathbb{C}} e_u B,$$

$$e_u \otimes e_u ae_v \otimes e_v \mapsto \delta_{u,v} \mu_d(u) \text{tr}(a) e_u \otimes e_u$$

and the unit

$$\iota_d : B \rightarrow \bigoplus_{u \in H} (Be_u \otimes_{\mathbb{C}} e_u B)^{\oplus v^{-2a}}, \quad e_u \mapsto \sum_{v \in H} \mu_d(v)^{-1} \sum_{i=1}^{m_{u,v}} b^{i,u,v} \otimes b_{i,v,u},$$

for certain $\mu_d(u) \in \mathbb{C}^\times$. As for the comultiplication and counit 2-morphisms, we cannot determine the $\mu_d(u)$ explicitly in general. For the specific case of dihedral Soergel bimodules, those scalars were computed explicitly in [38, Subsection 4.2]. In the definition of μ_d , note that $\text{tr}_B(a) = 0$ unless $u = v$.

We already know that C_d is a coalgebra and an algebra. One can immediately check that δ_d and μ_d satisfy the Frobenius conditions. Since the cell 2-representation is faithful on 2-morphisms, this proves that C_d is a Frobenius algebra in \mathcal{S}_H .

The result in this subsection is weaker than the Klein–Elias–Hogancamp conjecture [11, Conjecture 4.40; 26, Subsection 5.2], which conjectures the existence of a Frobenius structure on C_d in \mathcal{S} itself. Indeed, we do not know how to ‘lift’ the Frobenius structure on C_d from the \mathcal{H} -simple quotient \mathcal{S}_H to the whole of \mathcal{S} . The problem is that $C_d C_d$, as noted in [11, Conjecture 4.40], may contain indecomposable direct summands in $\mathcal{S}^{(0)}$ isomorphic to $C_w^{\oplus v^t}$, with $w >_J H$ and either $t < \mathbf{a} - \mathbf{a}(w)$ or $3\mathbf{a} - \mathbf{a}(w) < t$. Note that, in terms of C_z , Soergel’s hom formula in (3.3) becomes:

$$\dim(\text{hom}_{\mathcal{S}}(C_v, C_w^{\oplus v^{(k+\mathbf{a}(v)-\mathbf{a}(w))}})) = \delta_{v,w} \delta_{0,k}, \quad v, w \in W, k \in \mathbb{Z}_{\geq 0}.$$

In particular, this implies that the dimension of

$$\text{hom}_{\mathcal{S}}(C_d, C_d C_d) \quad \text{or} \quad \text{hom}_{\mathcal{S}}(C_d C_d, C_d^{\oplus v^{2a}})$$

need not be one, in general. Let us give one simple example.

Example 4.16. For rank 2 or lower $C_d C_d$ never contains such direct summands, but for higher ranks it frequently does. For an explicit and minimal example, let W be of type A_3 with simple reflections s_1, s_2, s_3 , where we write $i = s_i$ for short, and Coxeter diagram

$$1 \text{ --- } 2 \text{ --- } 3 .$$

Set $d = 12321$. Then $\mathbf{a} = 3$. Consider also the longest element $w_0 = 121321$ of W (whose \mathbf{a} -value is 6), which is strictly greater than d in the two-sided order. We have

$$C_d C_d \cong C_d^{\oplus 1 \oplus 3v^2 \oplus 3v^4 \oplus v^6} \oplus C_{w_0}^{\oplus v^{-4} \oplus 4v^{-2} \oplus 6 \oplus 4v^2 \oplus v^4} \text{ in } \mathcal{S}^{(0)},$$

where the minimal shift of C_{w_0} is strictly smaller than $\mathbf{a} - \mathbf{a}(w_0) = -3$ and the maximal shift is strictly bigger than $3\mathbf{a} - \mathbf{a}(w_0) = 3$.

5 | LIFTED SIMPLE TRANSITIVE 2-REPRESENTATIONS

5.1 | The underlying algebra

Suppose $(A, \delta_A, \epsilon_A)$ is a cosimple coalgebra in \mathcal{A}_H . By [37, Corollary 12] and Corollary 2.17, $\mathbf{N} := \mathbf{inj}_{\mathcal{A}_H}(A)$ is a simple transitive birepresentation of \mathcal{A}_H and $\mathcal{N} = \text{inj}_{\mathcal{A}_H}(A) = \text{comod}_{\mathcal{A}_H}(A)$ is semisimple. By Lemma 3.6 and Proposition 3.7, $\Theta(A)$ is also a cosimple coalgebra, which implies

that $\mathbf{M} := \mathbf{inj}_{\mathcal{S}_H}(\Theta(A))$ is a graded simple transitive 2-representation of \mathcal{S}_H with apex \mathcal{H} , using [37, Corollary 12] again.

Since \mathcal{S}_H is locally semisimple, see Proposition 3.4, A must contain a direct summand isomorphic to A_d , which is the identity 1-morphism in \mathcal{S}_H , and $\epsilon_A : A \rightarrow A_d$ is the projection, which is a morphism of coalgebras. Hence, we obtain a faithful morphism of birepresentations of \mathcal{S}_H

$$\Phi_{\mathcal{S}_H} : \mathbf{inj}_{\mathcal{S}_H}(A) = \mathbf{N} \rightarrow \mathbf{inj}_{\mathcal{S}_H}(A_d) \simeq \mathbf{A}_H,$$

which is the identity on morphisms and sends $(N, \delta_{N,A})$ in \mathbf{N} to $(N, (\text{id}_N \circ_h \epsilon_A) \circ_v \delta_{N,A})$. Here \mathbf{A}_H denotes the cell birepresentation of \mathcal{S}_H .

Since Θ is \mathbb{C} -linear, the above implies that $\Theta(A)$ contains a direct summand in $\mathcal{S}_H^{(0)}$ isomorphic to $\Theta(A_d) = C_d$ and the counit $\epsilon_{\Theta(A)} : \Theta(A) \rightarrow \mathbb{1}_{\mathcal{S}_H}$ is the composite of the projection $\pi_d := \Theta(\epsilon_A) : \Theta(A) \rightarrow C_d$ and $\epsilon_d : C_d \rightarrow \mathbb{1}_{\mathcal{S}_H}$. In particular, we obtain a faithful, degree-preserving morphism of 2-representations of \mathcal{S}_H

$$\Phi_{\mathcal{S}_H} : \mathbf{inj}_{\mathcal{S}_H}(\Theta(A)) = \mathbf{M} \rightarrow \mathbf{inj}_{\mathcal{S}_H}(C_d) \simeq \mathbf{C}_H,$$

which is the identity on morphisms and sends the comodule $(M, \delta_{M,\Theta(A)})$ in $\mathbf{M}(\emptyset)$ to the comodule $(M, (\text{id}_M \circ_h \pi_d) \circ_v \delta_{M,\Theta(A)})$.

Altogether, this yields a commuting square

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{\Phi_{\mathcal{S}_H}} & \mathbf{C}_H \\ \uparrow \Theta & & \uparrow \Theta \\ \mathbf{N} & \xrightarrow{\Phi_{\mathcal{S}_H}} & \mathbf{A}_H. \end{array}$$

Let N_1, \dots, N_r be a complete set of pairwise non-isomorphic simple (=indecomposable) objects in $\mathbf{N}(\emptyset)$. For every $i = 1, \dots, r$, we have

$$\Phi_{\mathcal{S}_H}(N_i) \cong \bigoplus_{w \in \mathcal{H}} A_w^{\oplus p_{i,w}},$$

for certain $p_{i,w} \in \mathbb{N}_0$.

Let $M_i := \Theta(N_i)$, for $i = 1, \dots, r$. Then M_1, \dots, M_r is a complete and irredundant set of indecomposable objects of $\mathbf{M}(\emptyset)$ up to isomorphism and grading shift, and

$$\Phi_{\mathcal{S}_H}(M_i) \cong \bigoplus_{w \in \mathcal{H}} C_w^{\oplus p_{i,w}} \text{ in } \mathbf{C}_H^{(0)}(\emptyset), \tag{5.1}$$

for every $i = 1, \dots, r$.

Lemma 5.1. For $w \in \mathcal{H}$ and $i, j = 1, \dots, r$, define $\tilde{h}_{w,i,j} \in \mathbb{N}[v, v^{-1}]$ by

$$C_w M_i \cong \bigoplus_{j=1}^r M_j^{\oplus \tilde{h}_{w,i,j}} \text{ in } \mathbf{M}^{(0)}(\emptyset). \tag{5.2}$$

Then

$$\tilde{h}_{w,i,j} \in v^{2a} \mathbb{N}_0[v^{-1}] \cap \mathbb{N}_0[v].$$

Proof. The proof follows from comparing two decompositions in $\mathbf{C}_{\mathcal{H}}^{(0)}(\emptyset)$. On one hand, by (5.1) and (5.2), we have

$$\Phi_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_w M_i) \cong \bigoplus_{j=1}^r \Phi_{\mathcal{S}_{\mathcal{H}}}\left(M_j^{\oplus \tilde{h}_{w,i,j}}\right) \cong \bigoplus_{j=1}^r \bigoplus_{v \in \mathcal{H}} \mathbf{C}_v^{\oplus \tilde{h}_{w,i,j} p_{j,v}}. \quad (5.3)$$

On the other hand, by (5.1) and the fact that $\Phi_{\mathcal{S}_{\mathcal{H}}}$ is a morphism of 2-representations, we have

$$\Phi_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_w M_i) \cong \mathbf{C}_w \Phi_{\mathcal{S}_{\mathcal{H}}}(M_i) \cong \bigoplus_{u \in \mathcal{H}} \mathbf{C}_w \mathbf{C}_u^{\oplus p_{i,u}} \cong \bigoplus_{u,v \in \mathcal{H}} \mathbf{C}_v^{\oplus v^a p_{i,u} h_{w,u,v}}. \quad (5.4)$$

Comparing (5.3) and (5.4) for a fixed v , we obtain the equation

$$\sum_{j=1}^r \tilde{h}_{w,i,j} p_{j,v} = v^a \sum_{u \in \mathcal{H}} p_{i,u} h_{w,u,v}.$$

The facts that $p_{i,u}, p_{j,v} \in \mathbb{N}_0$ and $v^a h_{w,u,v} \in v^{2a} \mathbb{N}_0[v^{-1}] \cap \mathbb{N}_0[v]$ now imply the result, as for every $j = 1, \dots, r$ there exists at least one $v \in \mathcal{H}$ such that $p_{j,v} \neq 0$. \square

Define

$$\mathbf{B}^{\mathbf{M}} := \text{End}_{\mathbf{M}}\left(\bigoplus_{i=1}^r M_i\right)^{\text{op}}.$$

Then $\mathbf{M}(\emptyset)$ is equivalent to the category of finite-dimensional graded injective $\mathbf{B}^{\mathbf{M}}$ -modules.

Proposition 5.2. *The algebra $\mathbf{B}^{\mathbf{M}}$ is a finite-dimensional positively graded Frobenius algebra of graded length $2a$.*

Proof. The case of \mathbf{M} being the cell 2-representation $\mathbf{C}_{\mathcal{H}}$ is discussed in Proposition 4.3 (or in Proposition 4.5 because of Proposition 4.7). In the case when \mathbf{M} is not necessarily the cell 2-representation, the proposition follows from similar arguments as in the proof of Proposition 4.5. \square

Remark 5.3. In contrast to the situation in Subsection 4.1, we do not know *a priori* that $\mathbf{B}^{\mathbf{M}}$ is weakly symmetric. Therefore, we have to include a possible Nakayama permutation in Subsection 5.3. Only at the end of that section, we will be able to show that it is trivial.

5.2 | A characterization of 2-representations in the image of $\hat{\Theta}$

We note the following characterization of the graded simple transitive 2-representations in the image of the map $\hat{\Theta}$ from Proposition 3.15. As we will see in Theorem 7.1, these exhaust the graded simple transitive 2-representation of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} , up to equivalence.

Theorem 5.4. *Let \mathbf{M} be a graded simple transitive 2-representation of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} . Then \mathbf{M} is in the image of $\hat{\Theta}$ if and only if the following conditions are satisfied:*

- (i) *there is a choice $\{M_i \mid i \in I\}$ of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\emptyset)$ such that the endomorphism algebra \mathbf{B} of $M := \bigoplus_{i \in I} M_i$ in $\mathbf{M}(\emptyset)$ is positively graded, and, additionally;*
- (ii) *for every $w \in \mathcal{H}$ and $i \in I$, the object $C_w M_i \in \mathbf{M}^{(0)}(\emptyset)$ decomposes into a direct sum whose summands, up to degree-preserving isomorphism, are of the form $M_j^{\oplus v^l}$, where $j \in I$ and $0 \leq l \leq 2\mathbf{a}$;*
- (iii) *the graded length of \mathbf{B} is not greater than $2\mathbf{a}$.*

Proof. For the ‘only if’ part observe that, by Proposition 3.11, we can pick a choice of representatives of isomorphism classes of indecomposable objects in \mathbf{M} , which are in the image of Θ . The condition in (ii) then follows from Lemma 5.1. Conditions (i) and (iii) hold by Proposition 5.2.

The ‘if’ direction follows the proof of Proposition 4.7 closely. Set $C := [M, M]$. Then

$$\mathrm{hom}_{\mathcal{S}_{\mathcal{H}}}(C, C_w^{\oplus v^k}) \cong \mathrm{hom}_{\mathbf{M}}(M, C_w^{\oplus v^k} M). \quad (5.5)$$

Conditions (i) and (ii) imply that the right-hand side is zero if $k > 0$. Hence, writing

$$C \cong \bigoplus_{w \in \mathcal{H}} C_w^{\oplus p_w} \text{ in } \mathcal{S}_{\mathcal{H}}^{(0)},$$

we obtain $p_w \in \mathbb{N}_0[v^{-1}]$.

Next we want to establish an analog of Lemma 4.1. Namely, we claim that, for any simple object L in $\mathbf{M}(\emptyset)$ concentrated in degree 0, and for any $w \in \mathcal{H}$, the injective object $C_w L$ is concentrated between the degrees 0 and $2\mathbf{a}$. Similarly to the proof of Lemma 4.1, the fact that $C_w L$ is concentrated in positive degrees follows from Conditions (i) and (ii). The fact that $C_w L$ is concentrated in degrees below $2\mathbf{a}$ follows from conditions (ii) and (iii).

Now, if $k < -4\mathbf{a}$, then the right-hand side of (5.5) is zero since, by condition (iii), M is a projective-injective object of graded length at most $2\mathbf{a}$ and the action of C_w is given by projective functors which increase the graded length by at most $2\mathbf{a}$ (see the previous paragraph). Given the graded length of C_w in $\mathcal{S}_{\mathcal{H}}$, see Lemma 4.6, this again shows that $p_w = p_w(0) \in \mathbb{N}_0$ for all $w \in \mathcal{H}$, so C is in the image of Θ , as claimed. \square

5.3 | Explicit bimodules for the 2-action

The degree-zero part of $\mathbf{B}^{\mathbf{M}}$ is isomorphic to $\bigoplus_{i=1}^r \mathbb{C}e_i$, where e_1, \dots, e_r is a complete and irredundant set of primitive, orthogonal idempotents corresponding to M_1, \dots, M_r , respectively. Due to (5.1), every M_i is concentrated between degrees 0 and $2\mathbf{a}$, whence

$$M_i \cong \mathrm{Hom}_{\mathbb{C}}(e_i \mathbf{B}^{\mathbf{M}}, \mathbb{C})^{\oplus v^{2\mathbf{a}}} \cong \mathbf{B}^{\mathbf{M}} e_{\sigma(i)},$$

where σ is the Nakayama permutation of $\mathbf{B}^{\mathbf{M}}$. By Theorem 2.13, the action of C_w on the category of finite-dimensional graded injective $\mathbf{B}^{\mathbf{M}}$ -modules, for $w \in \mathcal{H}$, is given by tensoring over $\mathbf{B}^{\mathbf{M}}$ with

a $B^{\mathbf{M}}\text{-}B^{\mathbf{M}}$ -bimodule of the form

$$\bigoplus_{i,j=1}^r (B^{\mathbf{M}}e_{\sigma(j)} \otimes_{\mathbb{C}} e_{\sigma(i)}B^{\mathbf{M}})^{\oplus \tilde{\gamma}_{w,i,j}},$$

for certain $\tilde{\gamma}_{w,i,j} \in \mathbb{N}_0[v, v^{-1}]$.

Proposition 5.5. *We have*

$$\tilde{\gamma}_{w,k,j} = \tilde{h}_{w,k,j}(0) \in \mathbb{N}_0,$$

for all $w \in \mathcal{H}$ and $j, k = 1, \dots, r$.

Proof. For $w \in \mathcal{H}$ and $1 \leq k \leq r$, we obtain two different expressions for $C_w B^{\mathbf{M}}e_{\sigma(k)}$ in $B^{\mathbf{M}}\text{-fgproj}^{(0)}$, the category of finite-dimensional graded projective $B^{\mathbf{M}}$ -modules. On one hand, by (5.2) and the fact that $M_l \cong B^{\mathbf{M}}e_{\sigma(l)}$, we have

$$C_w B^{\mathbf{M}}e_{\sigma(k)} \cong \bigoplus_{j=1}^r B^{\mathbf{M}}e_{\sigma(j)}^{\oplus \tilde{h}_{w,k,j}}. \quad (5.6)$$

On the other hand, we have

$$C_w B^{\mathbf{M}}e_{\sigma(k)} \cong \bigoplus_{i,j=1}^r B^{\mathbf{M}}e_{\sigma(j)}^{\oplus \tilde{\gamma}_{w,i,j} \text{grdim}(e_{\sigma(i)}B^{\mathbf{M}}e_{\sigma(k)})}. \quad (5.7)$$

Comparing the terms in (5.6) and (5.7) for a fixed j , shows that

$$\sum_{i=1}^r \tilde{\gamma}_{w,i,j} \text{grdim}(e_{\sigma(i)}B^{\mathbf{M}}e_{\sigma(k)}) = \tilde{h}_{w,k,j}. \quad (5.8)$$

Suppose that $\tilde{\gamma}_{w,\sigma^{-1}(k),j}$ has a non-zero term belonging to $v\mathbb{N}_0[v]$ for some w, i, j . By (5.8) and the fact that $\text{grdim}(e_k B^{\mathbf{M}}e_{\sigma(k)})$ has highest term v^{2a} , see Proposition 5.2, this implies that $\tilde{h}_{w,k,j}$ has a non-zero term belonging to $v^{2a+1}\mathbb{N}_0[v]$. However, this contradicts Lemma 5.1.

Since $\text{grdim}(e_{\sigma(i)}B^{\mathbf{M}}e_{\sigma(k)}) \in \delta_{i,k} + v\mathbb{N}_0[v]$, the equation in (5.8) implies that $\tilde{\gamma}_{w,k,j}$ cannot have non-zero terms belonging to $v^{-1}\mathbb{N}_0[v^{-1}]$ either, whence

$$\tilde{\gamma}_{w,k,j} = \tilde{h}_{w,k,j}(0) \in \mathbb{N}_0,$$

for all $w \in \mathcal{H}$ and $j, k = 1, \dots, r$. □

In particular, note that the fact that the constant term in $\text{grdim}(e_{\sigma(i)}B^{\mathbf{M}}e_{\sigma(k)})$ is 1 if $i = k$, and 0 otherwise, implies that

$$C_d B^{\mathbf{M}}e_{\sigma(k)} \cong \bigoplus_{j=1}^r B^{\mathbf{M}}e_{\sigma(j)}^{\oplus \tilde{\gamma}_{d,k,j}} \oplus R,$$

where all summands of R have coefficients in $v\mathbb{N}_0[v]$. Since the first summand descends to the action of A_d , which is the identity 1-morphism in \mathcal{A}_H , on $\mathbf{N}(\emptyset)$, by Lemma 3.13, we see that $\tilde{\gamma}_{d,k,j} = \delta_{k,j}$. By Proposition 5.5 and Equation (5.8), this shows that the action of $C_d = \Theta(A_d)$ is given by tensoring with the bimodule

$$\bigoplus_{i=1}^r \mathbf{B}^M e_i \otimes_{\mathbb{C}} e_i \mathbf{B}^M \tag{5.9}$$

and that

$$\text{grdim}(\text{Hom}_{\mathbf{B}^M}(\mathbf{B}^M e_i, \mathbf{B}^M e_k)) = \text{grdim}(e_i \mathbf{B}^M e_k) = \tilde{h}_{d,\sigma^{-1}(k),\sigma^{-1}(i)}. \tag{5.10}$$

In [31, Theorem 18.9], Lusztig defined a homomorphism $\phi : H \rightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}[v, v^{-1}]$ of $\mathbb{Z}[v, v^{-1}]$ -algebras. Its restriction to \mathcal{H} is given by

$$\phi_{\mathcal{H}}(c_w) = \sum_{u \in \mathcal{H}} v^a h_{w,d,u} a_u,$$

where $c_w := [C_w]$ in the split Grothendieck group $[\mathcal{S}_H^{(0)}]_{\oplus}$ (which should not be confused with Lusztig's c_w) and $a_u := [A_u]$ in $[\mathcal{A}_H]_{\oplus}$. Let $\phi_H^!$ denote the pullback of ϕ .

Proposition 5.6. *We have*

$$[\mathbf{M}]_{\oplus} \cong \phi_H^!([\mathbf{N}]_{\oplus}).$$

Proof. By (5.9), there is a degree-preserving isomorphism

$$C_d C e_{\sigma(i)} \cong \mathbf{B}^M e_{\sigma(i)}.$$

Using this, we obtain two expressions for $C_w \mathbf{B}^M e_{\sigma(i)}$ in $\mathbf{B}^M\text{-fgproj}^{(0)}$. On one hand,

$$C_w \mathbf{B}^M e_{\sigma(i)} \cong \bigoplus_{j=1}^r \mathbf{B}^M e_{\sigma(j)}^{\oplus \tilde{h}_{w,i,j}}. \tag{5.11}$$

On the other hand,

$$\begin{aligned} C_w \mathbf{B}^M e_{\sigma(i)} &\cong C_w C_d C e_{\sigma(i)} \cong \bigoplus_{u \in \mathcal{H}} C_u^{\oplus v^a h_{w,d,u}} C e_{\sigma(i)} \\ &\cong \bigoplus_{u \in \mathcal{H}} \bigoplus_{j=1}^r \mathbf{B}^M e_{\sigma(j)}^{\oplus v^a h_{w,d,u} \tilde{\gamma}_{u,i,j}}. \end{aligned} \tag{5.12}$$

Comparing (5.11) and (5.12) for a fixed j yields

$$\tilde{h}_{w,i,j} = \sum_{u \in \mathcal{H}} v^a h_{w,d,u} \tilde{\gamma}_{u,i,j}, \tag{5.13}$$

which is precisely what we had to prove. □

Corollary 5.7. $v^{-\mathbf{a}}\tilde{h}_{w,i,j}$ is bar invariant.

Proof. Equation (5.13) implies that $v^{-\mathbf{a}}\tilde{h}_{w,i,j}$ is bar invariant, since the $h_{w,d,u}$ and $\tilde{\gamma}_{u,i,j}$ are bar invariant. This completes the proof. \square

Proposition 5.8. The algebra $B^{\mathbf{M}}$ is weakly symmetric.

Proof. Recalling (3.1), (3.2) and (4.12), we know that

$$v^{\mathbf{a}}h_{d,d,u} \in \begin{cases} 1 + \dots + v^{2\mathbf{a}} & \text{if } u = d; \\ v\mathbb{N}_0[v] \cap v^{2\mathbf{a}-1}\mathbb{N}_0[v^{-1}] & \text{if } u \neq d. \end{cases}$$

By (5.13) and the equality $\tilde{\gamma}_{d,i,j} = \delta_{i,j}$, (5.10) then shows that

$$\text{grdim}(e_i B^{\mathbf{M}} e_i) = \tilde{h}_{d,\sigma^{-1}(i),\sigma^{-1}(i)} \in 1 + \dots + v^{2\mathbf{a}}.$$

By Proposition 5.2, the highest non-zero term of $\text{grdim}(e_i B^{\mathbf{M}} e_j)$ is $v^{2\mathbf{a}}$ if and only if $j = \sigma(i)$. Therefore, we obtain $\sigma(i) = i$ for all i and the claim follows. \square

6 | SEPARABILITY AND SEMISIMPLICITY

Recall the definition of $\mathcal{B}_{\mathcal{H}}$ from (4.6). The main point of this section is to prove that the coalgebra C_d in $\mathcal{S}_{\mathcal{H}}$ is separable, see Propositions 6.3 and 6.5, which in turn enables us to show that $\mathcal{B}_{\mathcal{H}}$ is locally graded semisimple, see Proposition 6.11.

6.1 | Some (diagrammatic) preliminaries

Let $B = B^{\mathcal{C}_{\mathcal{H}}}$ be the basic underlying algebra of the cell 2-representation $\mathbf{C}_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$, see Subsection 4.1. This in particular means that there is an equivalence of 2-representations of $\mathcal{S}_{\mathcal{H}}$ on

$$\mathbf{C}_{\mathcal{H}}(\emptyset) \simeq B\text{-fgproj}$$

such that the 2-action on $B\text{-fgproj}$, the category of finite-dimensional graded projective B -modules, is given by a graded pseudofunctor from $\mathcal{S}_{\mathcal{H}}$ to \mathcal{C}_B (projective B - B -bimodules, see Example 2.3) which is 2-faithful, that is, faithful on 2-morphisms. Recall that $B \cong \bigoplus_{i=0}^{2\mathbf{a}} B_i$ is a finite-dimensional positively graded weakly symmetric Frobenius algebra of graded length $2\mathbf{a}$, where $\mathbf{a} = \mathbf{a}(\mathcal{H})$ is the value of Lusztig's \mathbf{a} -function on \mathcal{H} , see Proposition 4.3. Note that therefore the objects in $B\text{-fgproj}$ are projective-injective and thus, $B\text{-fgproj} = B\text{-fginj}$, the latter denoting the category of finite-dimensional graded injective B -modules.

By Subsection 4.4, C_d is a graded Frobenius algebra in $\mathcal{S}_{\mathcal{H}}$ with homogeneous comultiplication $\delta_d : C_d \rightarrow C_d C_d$ and counit $\epsilon_d : C_d \rightarrow C_e$ of degree 0, and homogeneous multiplication $\mu_d : C_d C_d \rightarrow C_d$ and unit $\iota_d : C_e \rightarrow C_d$ of degree $-2\mathbf{a}$ and $2\mathbf{a}$, respectively. Recall that $\gamma_{d,u,v} = \delta_{u,v^{-1}}$, so Proposition 4.15 shows that C_d acts on $B\text{-fgproj}$ by tensoring over B with the projective

B-B-bimodule

$$\bigoplus_{x \in H} B e_x \otimes e_x B,$$

and, consequently, the action of $C_d C_d$ is given by tensoring with

$$\bigoplus_{x,y \in H} B e_x \otimes e_x B e_y \otimes e_y B.$$

The B-B-bimodule maps corresponding to $\delta_d, \mu_d, \epsilon_d$ and ι_d , which we defined in Subsection 4.4, are graphically illustrated as follows (for convenience, we recall the formulas defining them):

$$\delta_d : e_x \otimes e_x \mapsto \delta_d(x) e_x \otimes e_x \otimes e_x \leftrightarrow \begin{array}{c} d \quad d \\ \text{---} \\ \text{---} \\ | \\ d \end{array} \text{ (degree 0),} \tag{6.1}$$

$$\mu_d : e_x \otimes a \otimes e_y \mapsto \delta_{x,y} \mu_d(x) \text{tr}(a) e_x \otimes e_x \leftrightarrow \begin{array}{c} d \\ | \\ \text{---} \\ d \quad d \end{array} \text{ (degree } -2\mathbf{a}), \tag{6.2}$$

$$\epsilon_d : a e_x \otimes e_x a' \mapsto \delta_d(x)^{-1} a a' \leftrightarrow \begin{array}{c} \bullet \\ | \\ d \end{array} \text{ (degree 0),} \tag{6.3}$$

$$\iota_d : e_x \mapsto \sum_{y \in H} \mu_d(y)^{-1} \sum_{i=1}^{m_{x,y}} b^{i,x,y} \otimes b_{i,y,x} \leftrightarrow \begin{array}{c} d \\ \bullet \\ | \end{array} \text{ (degree } 2\mathbf{a}). \tag{6.4}$$

Here the source (bottom) and target (top) of the diagrams in (6.1), (6.2), (6.3) and (6.4) are composites of unshifted copies of C_d .

Remark 6.1. Unlike the degrees in, for example, [11, Conjecture 4.40]), the ones above are ‘unbalanced’, that is, $\text{deg}(\mu_d) \neq \text{deg}(\delta_d)$ and $\text{deg}(\iota_d) \neq \text{deg}(\epsilon_d)$, because we are working with C_d instead of B_d .

To avoid overloading the diagrams with shifts, we will also sometimes use a diagram of a 2-morphism $f : X \rightarrow Y$ of degree d to depict the 2-morphism $f \langle d \rangle : X \rightarrow Y^{\oplus v^{-d}}$. Note that the dual C_d^* of C_d is isomorphic to $C_d^{\oplus v^{-2\mathbf{a}}}$ in $\mathcal{S}_H^{(0)}$, so we can see, for example, $\mu_d \langle -2\mathbf{a} \rangle$ also as a 2-morphism of degree 0 from $C_d C_d^*$ or $C_d^* C_d$, which are isomorphic in $\mathcal{S}_H^{(0)}$, to C_d . The coevaluation and evaluation of C_d in $\mathcal{S}_H^{(0)}$ are given by

$$\begin{aligned} \text{coev}_d &= \delta_d \circ_v \iota_d \langle 2\mathbf{a} \rangle : C_e \rightarrow C_d^{\oplus v^{-2\mathbf{a}}} C_d, \\ \text{ev}_d &= \epsilon_d \circ_v \mu_d \langle -2\mathbf{a} \rangle : C_d C_d^{\oplus v^{-2\mathbf{a}}} \rightarrow C_e, \end{aligned} \tag{6.5}$$

which we draw as

$$\begin{aligned} \text{coev}_d &\leftrightarrow \text{cup} := \text{cup with stem} \quad (\text{degree } 2a), \\ \text{ev}_d &\leftrightarrow \text{cap} := \text{cap with stem} \quad (\text{degree } -2a). \end{aligned}$$

Note that ${}^*C_d \cong C_d^*$ in $\mathcal{S}_H^{(0)}$ and we could also define $\text{coev}'_d : C_e \rightarrow C_d C_d^*$ and $\text{ev}'_d : C_d^* C_d \rightarrow C_e$ in the evident way. Since $C_d^{**} \cong C_d$, the diagrams for the evaluation and the coevaluations of C_d and C_d^* are the same, which justifies the lack of arrows on the diagrams.

Recall that being a Frobenius algebra in \mathcal{S}_H means that the (co)multiplication and the (co)unit satisfy the diagrammatic equations

$$\begin{aligned} \text{cup with stem} &= \text{cup with stem}, & \text{cap with stem} &= \text{cap with stem}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} \text{cup with dot} &= \text{cup with dot}, & \text{cap with dot} &= \text{cap with dot}, \end{aligned} \tag{6.7}$$

$$\text{multiplication} = \text{multiplication} = \text{multiplication}. \tag{6.8}$$

Recall further that (6.6), (6.7) and (6.8) imply

$$\begin{aligned} \text{cup with stem} &= \text{cup with stem}, \\ \text{cup with stem} &= \text{cup with stem}, & \text{cup with stem} &= \text{cup with stem}, \\ \text{cup with dot} &= \text{cup with dot}, & \text{cup with dot} &= \text{cup with dot}. \end{aligned} \tag{6.9}$$

By (6.9) we only need to consider isotopy classes of diagrams, and we simplify our diagrams by drawing them in a more topological fashion. However, some diagrams are not the same, as the calculus is not necessarily cyclic for general B-B-bimodule maps. In particular, for any

$$\alpha \in \text{hom}_{\mathcal{S}_H}(C_d \dots C_d, C_d \dots C_d^{\oplus v^{-t+s}}) \rightsquigarrow \begin{array}{c} d \dots d \\ | \\ \textcircled{\alpha} \\ | \\ d \dots d \end{array} \quad (\text{degree } t - s), \tag{6.10}$$

where the dots indicate multiple strands, we define the *right mate* α^* and the *left mate* ${}^*\alpha$ of α via

$$\begin{array}{c} d \dots d \\ | \\ \textcircled{\alpha}^* \\ | \\ d \dots d \end{array} := \begin{array}{c} d \dots d \\ | \\ \textcircled{\alpha} \\ | \\ d \dots d \end{array}, \quad \begin{array}{c} d \dots d \\ | \\ {}^*\alpha \\ | \\ d \dots d \end{array} := \begin{array}{c} d \dots d \\ | \\ \textcircled{\alpha} \\ | \\ d \dots d \end{array},$$

which have the same degree as the diagram in (6.10) itself. To be more concrete, let us compute the mates of $\alpha \in \text{hom}_{\mathcal{S}_H}(C_d, C_d^{\oplus v^{-t+s}})$ in the cell 2-representation. Suppose that the B-B-bimodule map α is given by

$$\alpha : e_x \otimes e_x \mapsto \sum_{y \in H} \sum_{i=1}^{n_{x,y}} \alpha_{i,x,y}^{(1)} \otimes \alpha_{i,y,x}^{(2)}, \tag{6.11}$$

for $n_{x,y} \in \mathbb{N}_0$ and some homogeneous $\alpha_{i,x,y}^{(1)} \in e_x B e_y$ and $\alpha_{i,y,x}^{(2)} \in e_y B e_x$ such that $\text{deg}(\alpha_{i,x,y}^{(1)}) + \text{deg}(\alpha_{i,y,x}^{(2)}) = t - s$. By (6.5), the B-B-bimodule maps corresponding to $\text{coev}_d \langle -2\mathbf{a} \rangle$ and $\text{ev}_d \langle 2\mathbf{a} \rangle$ are given by

$$\begin{aligned} \delta_d \circ \text{coev}_d : e_x \mapsto \sum_{y \in H} \sum_{i=1}^{m_{x,y}} \mu_d(y)^{-1} \delta_d(y) b^{i,x,y} \otimes e_y \otimes b_{i,y,x}, \\ \epsilon_d \circ \text{ev}_d : e_x \otimes a \otimes e_y \mapsto \delta_{x,y} \mu_d(x) \delta_d(x)^{-1} \text{tr}(a) e_x, \end{aligned}$$

for $x, y \in H$. Then α^* and ${}^*\alpha$ are given by

$$\begin{aligned} \alpha^* : e_x \otimes e_x \mapsto \sum_{y \in H} \sum_{i=1}^{n_{y,x}} \mu_d(x) \delta_d(x)^{-1} \mu_d(y)^{-1} \delta_d(y) \sigma(\alpha_{i,x,y}^{(2)}) \otimes \alpha_{i,y,x}^{(1)}, \\ {}^*\alpha : e_x \otimes e_x \mapsto \sum_{y \in H} \sum_{i=1}^{n_{y,x}} \mu_d(x) \delta_d(x)^{-1} \mu_d(y)^{-1} \delta_d(y) \alpha_{i,x,y}^{(2)} \otimes \sigma^{-1}(\alpha_{i,y,x}^{(1)}), \end{aligned}$$

where σ is the Nakayama automorphism. In other words, the B-B-bimodule map associated to α^* can be obtained from the one associated to ${}^*\alpha$ by applying $\sigma \otimes \sigma$. To indicate this

diagrammatically, we write

$$\begin{array}{c} d \\ | \\ \textcircled{\alpha}^* \\ | \\ d \end{array} = \sigma \left(\begin{array}{c} d \\ | \\ \textcircled{\alpha}^* \\ | \\ d \end{array} \right). \tag{6.12}$$

Note that the equations in (6.9) imply that the left and right mates of the structural 2-morphisms of C_d agree, that is, $\mu_d = \delta_d^* = {}^* \delta_d$ and $\iota_d = \epsilon_d^* = {}^* \epsilon_d$.

6.2 | Dot diagrams

Next, recall that $C_{\mathcal{H}} \simeq \mathbf{inj}_{\mathcal{S}_{\mathcal{H}}}(C_d)$. For short, we denote the morphism spaces in the latter category by Hom_{C_d} (for the enriched version) and hom_{C_d} (for morphism spaces of degree 0). A specific object in this category is C_d , since the coalgebra structure of C_d gives it the structure of an injective right comodule in $\mathcal{S}_{\mathcal{H}}$ over itself.

Lemma 6.2. *Let α be a homogeneous endomorphism of C_d in $\mathcal{S}_{\mathcal{H}}$ of degree t , for some $t \in \mathbb{Z}$. Then we have $\alpha \in \text{hom}_{C_d}(C_d, C_d^{\oplus v^{-t}})$ if and only if the corresponding B-B-bimodule map is of the form*

$$e_x \otimes e_x \mapsto \alpha_x \otimes e_x, \quad \forall x \in \mathcal{H},$$

where α_x is some homogeneous element in $e_x B e_x$ of degree t .

Proof. Let $x \in \mathcal{H}$ and suppose that the B-B-bimodule map associated to α is given by (6.11). Now compare the B-B-bimodule maps associated to $\delta_d \circ_v \alpha$ and $(\alpha \circ_{\mathcal{H}} \text{id}_{C_d}) \circ_v \delta_d$, respectively,

$$\begin{aligned}
 e_x \otimes e_x &\mapsto \sum_{y \in \mathcal{H}} \sum_{i=1}^{n_{x,y}} \delta_d(y) \alpha_{i,x,y}^{(1)} \otimes e_y \otimes \alpha_{i,y,x}^{(2)}, \\
 e_x \otimes e_x &\mapsto \sum_{y \in \mathcal{H}} \sum_{i=1}^{n_{x,y}} \delta_d(x) \alpha_{i,x,y}^{(1)} \otimes \alpha_{i,y,x}^{(2)} \otimes e_x.
 \end{aligned}$$

By 2-faithfulness of the 2-functor $\mathcal{S}_{\mathcal{H}} \rightarrow \mathcal{C}_{\mathcal{B}}$ defining the cell 2-representation (see the beginning of this section), we see that the equality $\delta_d \circ_v \alpha = (\alpha \circ_{\mathcal{H}} \text{id}_{C_d}) \circ_v \delta_d$ holds if and only if

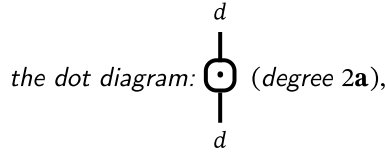
$$\alpha_{i,y,x}^{(2)} \in \begin{cases} \mathbb{C} e_x & \text{if } x = y; \\ \{0\} & \text{otherwise,} \end{cases}$$

which proves the lemma. □

Before the next statement, which is a version of separability, recall the following special cases of Soergel’s hom formula (3.3):

$$\text{hom}_{C_d}(C_d, C_d) \cong \mathbb{C} \cong \text{hom}_{C_d}(C_d, C_d^{\oplus v^{-2a}}). \tag{6.13}$$

Proposition 6.3. *There is a unique 2-morphism in $\text{hom}_{C_d}(C_d, C_d^{\oplus v^{-2a}})$, which we call the dot diagram and which we depict as*



such that



$$\text{dot diagram} = \text{vertical line } d \text{ at both ends} \tag{6.14}$$

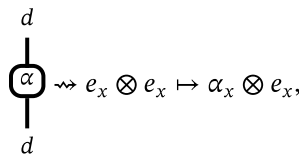
Proof. Choose any non-zero 2-morphism $\alpha \in \text{hom}_{C_d}(C_d, C_d^{\oplus v^{-2a}})$, which we depict as in (6.10). By (6.13), we have



$$\text{dot diagram with } \alpha = \lambda \cdot \text{vertical line } d \text{ at both ends} \tag{6.15}$$

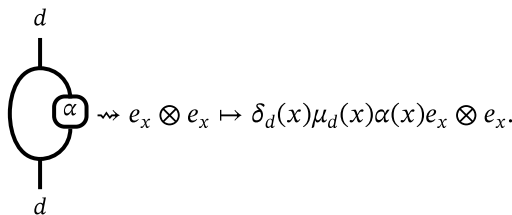
for some scalar $\lambda \in \mathbb{C}$. We claim that $\lambda \neq 0$.

To prove this claim, we only need to compute the B-B-bimodule map corresponding to the morphism on the left-hand side in (6.15). By Lemma 6.2, we have



$$\text{dot diagram with } \alpha \rightsquigarrow e_x \otimes e_x \mapsto \alpha_x \otimes e_x,$$

where $\alpha_x = \alpha(x)e^x$ for some scalar $\alpha(x) \in \mathbb{C}$ depending on $x \in \mathcal{H}$. Hence, (6.1) and (6.2) imply that



$$\text{dot diagram with } \alpha \rightsquigarrow e_x \otimes e_x \mapsto \delta_d(x) \mu_d(x) \alpha(x) e_x \otimes e_x.$$

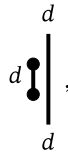
We see that $\lambda = \delta_d(x)\mu_d(x)\alpha(x)$ for all $x \in \mathcal{H}$. In particular, the scalar $\delta_d(x)\mu_d(x)\alpha(x)$ does not depend on the choice of x . Since α is not zero, there exists an $x \in \mathcal{H}$ such that $\alpha(x) \neq 0$, which implies that $\lambda \neq 0$ because $\delta_d(x)\mu_d(x) \neq 0$ for all $x \in \mathcal{H}$. Note that this also means that $\alpha(x) \neq 0$ for all $x \in \mathcal{H}$.

Now define

$$\begin{array}{c} d \\ | \\ \circ \\ | \\ d \end{array} := \lambda^{-1} \cdot \begin{array}{c} d \\ | \\ \circ \\ | \\ d \end{array}.$$

Clearly, this is the unique 2-morphism in $\text{hom}_{C_d}(C_d, C_d^{\oplus v^{-2a}}) \cong \mathbb{C}$ satisfying (6.14). □

Remark 6.4. One might be tempted to use the diagrammatic calculus to define the dot diagram as a multiple of



but we do not know whether this 2-morphism in $\text{hom}_{C_d}(C_d, C_d^{\oplus v^{-2a}})$ is non-zero in general.

For later use, note that

$$\begin{array}{c} d \\ | \\ \circ \\ | \\ d \end{array} \rightsquigarrow e_x \otimes e_x \mapsto \nu_d(x)^{-1} e^x \otimes e_x, \tag{6.16}$$

for all $x \in \mathcal{H}$, where $\nu_d(x) := \delta_d(x)\mu_d(x) \in \mathbb{C}^\times$. Note also that the fact that the dot diagram is a homomorphism of right C_d -comodules in $\mathcal{S}_{\mathcal{H}}$ translates to

$$\begin{array}{c} d \quad d \\ | \quad | \\ \circ \\ | \\ d \end{array} = \begin{array}{c} d \quad d \\ | \quad | \\ \circ \\ | \\ d \end{array}. \tag{6.17}$$

Since σ fixes e_x and e^x , the equality in (6.12) and the B-B-bimodule map in (6.16) show that the right and left mates of the dot diagram agree, that is,

$$\begin{array}{c} d \\ | \\ \circ \\ | \\ d \end{array}^* = \begin{array}{c} d \\ | \\ \circ \\ | \\ d \end{array}.$$

Either of these diagrams we therefore call the *dual dot diagram*, and the following is an almost immediate consequence of Proposition 6.3, which can be proved diagrammatically using (6.9) and (6.17).

Proposition 6.5. *The dual dot diagram is a homomorphism of left C_d -comodules in \mathcal{S}_H and satisfies*

$$\begin{array}{c} d \\ | \\ \text{---} \circ \text{---} \\ | \\ d \end{array} = \begin{array}{c} d \\ | \\ \text{---} \\ | \\ d \end{array} . \tag{6.18}$$

Diagrammatically, this means that the dual dot diagram satisfies a sliding relation analogous to (6.17). Furthermore, as in (6.16), the image of the dual dot diagram in \mathcal{C}_B is given by

$$\begin{array}{c} d \\ | \\ \text{---} \circ \text{---} \\ | \\ d \end{array} \rightsquigarrow e_x \otimes e_x \mapsto \nu_d(x)^{-1} e_x \otimes e^x$$

for all $x \in H$.

A straightforward calculation in \mathcal{C}_B proves the following diagrammatic equations, which will be very useful in the following.

Lemma 6.6. *We have*

$$\begin{array}{c} d \quad d \\ | \quad | \\ \text{---} \circ \text{---} \\ | \quad | \\ d \quad d \end{array} = \begin{array}{c} d \\ | \\ \text{---} \circ \text{---} \\ | \\ d \end{array} , \quad \begin{array}{c} d \quad d \\ | \quad | \\ \text{---} \circ \text{---} \\ | \quad | \\ d \quad d \end{array} = \begin{array}{c} d \\ | \\ \text{---} \circ \text{---} \\ | \\ d \end{array} . \tag{6.19}$$

6.3 | Graded semisimple bicomodules

We are now ready to determine the structure of \mathcal{B}_H , in particular, that it is locally graded semisimple.

Lemma 6.7. *The 1-morphism $C_x^{\oplus \nu^t}$ has a unique C_d - C_d -bicomodule structure in $\mathcal{S}_H^{(0)}$, for every $x \in H$ and $t \in \mathbb{Z}$*

Proof. For every $x \in \mathcal{H}$, we have $C_x \cong \Theta(A_x)$, see (3.5) and the text below it. Since A_d is the identity 1-morphism in \mathcal{A}_H , the 1-morphism A_x is naturally an A_d - A_d -bicomodule. This, together with the fact that Θ is an oplax pseudofunctor, implies that C_x is a C_d - C_d -bicomodule in $\mathcal{S}_H^{(0)}$.

Of course, this implies that $C_x^{\oplus vt}$ is a C_d - C_d -bicomodule in $\mathcal{S}_H^{(0)}$ for all $t \in \mathbb{Z}$, with the same (degree 0) structure 2-morphisms. We denote the left and right C_d -coactions on $C_x^{\oplus vt}$ by $\delta_{C_d, C_x^{\oplus vt}}$ and $\delta_{C_x^{\oplus vt}, C_d}$, respectively. Note that the bicomodule structure on each $C_x^{\oplus vt}$ in $\mathcal{S}_H^{(0)}$ is unique because Soergel’s hom formula (3.3) gives

$$\dim \left(\text{hom}_{\mathcal{S}_H} (C_x^{\oplus vt}, C_x^{\oplus vt} C_d) \right) = 1 = \dim \left(\text{hom}_{\mathcal{S}_H} (C_x^{\oplus vt}, C_d C_x^{\oplus vt}) \right),$$

and the choice of $\delta_{C_d, C_x^{\oplus vt}}$ and $\delta_{C_x^{\oplus vt}, C_d}$ is fixed by counitality. □

Lemma 6.8. *For any $x, y \in \mathcal{H}$ and $t \in \mathbb{Z}$, we have*

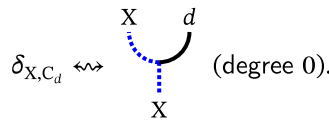
$$\dim \left(\text{hom}_{\mathcal{B}_H} (C_x, C_y^{\oplus vt}) \right) = \delta_{x,y} \delta_{t,0}.$$

Proof. By arguments analogous to those in the proof of Lemma 6.2, but this time applied to the left and the right C_d -coactions, we see that

$$\dim \left(\text{hom}_{\mathcal{B}_H} (C_x, C_y^{\oplus vt}) \right) = 0 \text{ unless } t = 0.$$

The result for $t = 0$ follows from (3.3). □

Since C_d is a graded Frobenius algebra in \mathcal{S}_H (cf. Subsection 4.4), any right, left or bicomodule of C_d has a compatible right, left or bimodule structure over C_d up to a degree shift of $-2\mathbf{a}$. This is well-known, but for convenience and later use, we briefly repeat the main arguments and constructions. The shortest way to do this is diagrammatically. Let X be a right C_d -comodule in $\mathcal{S}_H^{(0)}$ with right coaction $\delta_{X, C_d} : X \rightarrow XC_d$. We depict δ_{X, C_d} as



The coassociativity and counitality of δ_{X, C_d} are expressed by the diagrammatic equations

(6.20)

We can then define the right C_d -action $\mu_{X,C_d} : XC_d \rightarrow X$ of degree $-2\mathbf{a}$ as

$$\mu_{X,C_d} \rightsquigarrow \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad d \quad := \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad d \quad \text{(degree } -2\mathbf{a}\text{)}. \quad (6.21)$$

Note also that (6.9) implies that

$$\begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad . \quad (6.22)$$

Similarly, it is easy to see that

$$\begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad , \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad , \quad (6.23)$$

$$\begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad \begin{array}{c} d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad ,$$

that is, μ_{X,C_d} is associative and unital and there is a Frobenius-type compatibility between μ_{X,C_d} and δ_{X,C_d} . Thus, (6.20), (6.21), (6.22) and (6.23) imply that the diagrammatic calculus is again topological in nature, which we will use to simplify our diagrammatic arguments.

By symmetry, similar results hold for left coactions and left actions of C_d and one can check that any C_d - C_d -bicomodule in \mathcal{S}_H has a compatible C_d - C_d -bicomodule structure and vice versa (which diagrammatically translates into a height exchange, see (6.24)). To simplify our diagrams for bimodules and bicomodules, we also allow four valent vertices, for example,

$$\begin{array}{c} d \quad X \quad d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad := \quad \begin{array}{c} d \quad X \quad d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad = \quad \begin{array}{c} d \quad X \quad d \\ \text{---} \\ \text{---} \\ \text{---} \\ X \end{array} \quad , \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ d \quad X \quad d \end{array} \quad := \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ d \quad X \quad d \end{array} \quad = \quad \begin{array}{c} X \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \\ d \quad X \quad d \end{array} \quad . \quad (6.24)$$

The following lemma is the analog of Lemma 6.6.

Lemma 6.9. *We have*

$$\begin{array}{c}
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array} \\
 = \\
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array}
 \end{array}
 , \quad
 \begin{array}{c}
 \begin{array}{c} d & X \\ \vdots & | \\ \circ & \\ \vdots & | \\ d & X \end{array} \\
 = \\
 \begin{array}{c} d & X \\ \vdots & | \\ \circ & \\ \vdots & | \\ d & X \end{array}
 \end{array}
 . \tag{6.25}$$

Proof. The trick is to reduce the statement to the one from Lemma 6.6. We show how to do this for the first diagrammatic equation.

$$\begin{array}{c}
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array} \\
 = \\
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array} \\
 = \\
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array} \\
 \stackrel{(6.19)}{=} \\
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array} \\
 = \\
 \begin{array}{c} X & d \\ \vdots & | \\ \circ & \\ \vdots & | \\ X & d \end{array}
 \end{array}
 .$$

The proof of the second diagrammatic equation is analogous. □

The following lemma is the analog of (6.14) and (6.18).

Lemma 6.10. *We have*

$$\begin{array}{c}
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 = \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 = \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array}
 \end{array}
 . \tag{6.26}$$

Proof. Let us prove the first equality.

$$\begin{array}{c}
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 \stackrel{(6.20)}{=} \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 = \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 \stackrel{(6.14)}{=} \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array} \\
 \stackrel{(6.20)}{=} \\
 \begin{array}{c} X \\ \vdots \\ \circ \\ \vdots \\ X \end{array}
 \end{array}
 .$$

The second equation can be proved by symmetry. □

For any $t \in \mathbb{Z}$, let $\mathcal{A}_H\langle t \rangle$ be the category defined as

$$\mathcal{A}_H\langle t \rangle := \text{add}\left(\{C_w^{\oplus vk} \mid w \in H, k \geq t\}\right)^{(0)} / \left(\text{add}\left(\{C_w^{\oplus vk} \mid w \in H, k > t\}\right)^{(0)}\right).$$

By definition, we will write

$$A_w^{\oplus v^{-t}} = \Pi(C_w^{\oplus v^{-t}}) \in \mathcal{A}_H\langle t \rangle,$$

for any $w \in H$. Note that horizontal composition in \mathcal{S}_H descends to $\circ_h : \mathcal{A}_H\langle s \rangle \times \mathcal{A}_H\langle t \rangle \rightarrow \mathcal{A}_H\langle s+t \rangle$, so the action of \mathcal{A}_H on $\mathcal{A}_H\langle t \rangle$ gives rise to a semisimple simple transitive 2-representation of \mathcal{A}_H . In particular, $\bigoplus_{t \in \mathbb{Z}} \mathcal{A}_H\langle t \rangle$ is a \mathbb{Z} -finitary locally semisimple bicategory, as defined in Subsection 2.7.

Recall also from Subsection 2.7 the definition of the graded finitary 2-category \mathcal{C}' with translation for a given graded \mathbb{k} -linear 2-category \mathcal{C} .

Proposition 6.11. *We have an equivalence of \mathbb{Z} -finitary locally semisimple bicategories*

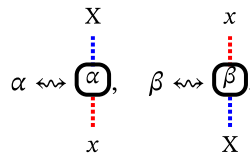
$$\mathcal{B}_H^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_H\langle t \rangle.$$

Moreover, this implies an equivalence of locally graded semisimple bicategories

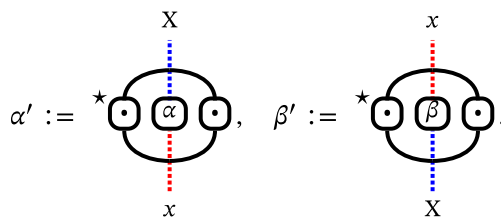
$$\mathcal{B}_H \simeq \mathcal{A}'_H. \tag{6.27}$$

Proof. Let X be a 1-morphism in $\mathcal{B}_H^{(0)}$. Suppose that $C_x^{\oplus vt}$ is a direct summand of X in $\mathcal{S}_H^{(0)}$ for some $x \in H$ and $t \in \mathbb{Z}$. We first claim that $C_x^{\oplus vt}$ is also a direct summand of X in $\mathcal{B}_H^{(0)}$.

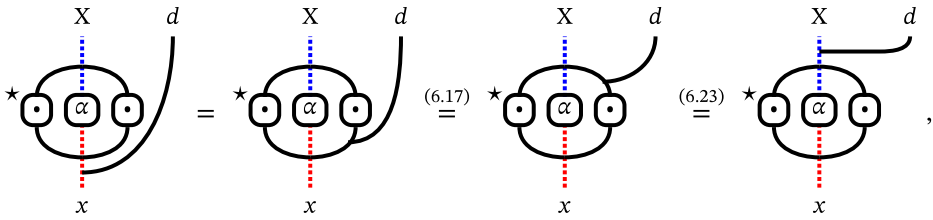
To this end, let $\alpha \in \text{hom}_{\mathcal{S}_H}(C_x^{\oplus vt}, X)$ and $\beta \in \text{hom}_{\mathcal{S}_H}(X, C_x^{\oplus vt})$ be the embedding and the projection, respectively, that is, $\beta \circ_v \alpha = \text{id}_{C_x^{\oplus vt}}$. Depict them using our usual conventions, with dotted strands of a different color for C_x (which we again denote by x in the diagrams) and X :



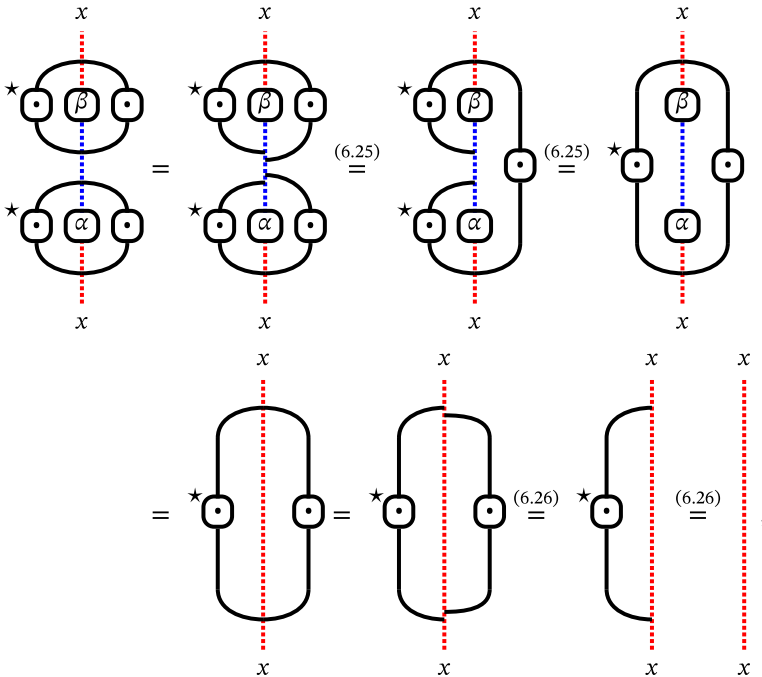
Define $\alpha' \in \text{hom}_{\mathcal{B}_H}(C_x^{\oplus vt}, X)$ and $\beta' \in \text{hom}_{\mathcal{B}_H}(X, C_x^{\oplus vt})$ as



Note that α' and β' are indeed 2-morphisms in $\mathcal{B}_H^{(0)}$, for example,



which shows that α' commutes with the right C_d -coaction. Next, by (6.25), (6.26), the Frobenius properties and $\beta \circ_V \alpha = \text{id}_{C_x^{\oplus_V t}}$, we have $\beta' \circ_V \alpha' = \text{id}_{C_x^{\oplus_V t}}$. Indeed,



which proves our claim. Hence, the decomposition of X into indecomposables is the same in $\mathcal{B}_H^{(0)}$ as in $\mathcal{S}_H^{(0)}$.

Together with Lemma 6.8, this implies that

$$\mathcal{B}_H^{(0)}(\emptyset, \emptyset) \simeq \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_H(\emptyset, \emptyset)\langle t \rangle$$

as \mathbb{Z} -finitary categories.

To show that this is an equivalence of bicategories, recall the oplax pseudofunctor $\Theta : \mathcal{A}_H \rightarrow \mathcal{S}_H^{(0)}$ from (3.5). Since each $X \in \mathcal{A}_H$ has a canonical A_d - A_d -bicomodule structure, $\Theta(X)$ has an induced $C_d \cong \Theta(A_d)$ -bicomodule structure in $\mathcal{S}_H^{(0)}$ by the oplax condition of Θ . By Lemma 3.9, there are homogeneous 2-isomorphisms of degree 0

$$\Theta(XY) \cong \Theta(X \square_{A_d} Y) \cong \Theta(X) \square_{\Theta(A_d)} \Theta(Y) \cong \Theta(X) \square_{C_d} \Theta(Y)$$

that are natural in X and Y . This shows that Θ gives rise to a \mathbb{C} -linear pseudofunctor

$$\Theta : \mathcal{A}_{\mathcal{H}} \rightarrow \mathcal{B}_{\mathcal{H}}^{(0)},$$

for which we use the same notation.

Observe that Θ extends uniquely to a \mathbb{C} -linear pseudofunctor

$$\Theta' : \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_{\mathcal{H}} \langle t \rangle \rightarrow \mathcal{B}_{\mathcal{H}}^{(0)}$$

which is compatible with the \mathbb{Z} -actions, and again 2-full by Lemma 6.8 and 2-faithful by semisimplicity. By the claim at the beginning of the proof and Lemma 6.7, each $X \in \mathcal{S}_{\mathcal{H}}^{(0)}$ has therefore a unique C_d - C_d -bicomodule structure and is isomorphic to $\Theta'(Z)$ for some $Z \in \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_{\mathcal{H}} \langle t \rangle$. This shows that Θ' is essentially surjective on 1-morphisms. Hence,

$$\mathcal{B}_{\mathcal{H}}^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_{\mathcal{H}} \langle t \rangle$$

as free \mathbb{Z} -bicategories.

Since $\bigoplus_{t \in \mathbb{Z}} \mathcal{A}_{\mathcal{H}} \langle t \rangle \simeq (\mathcal{A}'_{\mathcal{H}})^{(0)}$ by Lemma 2.23, this implies that there are biequivalences of free \mathbb{Z} -bicategories

$$(\mathcal{A}'_{\mathcal{H}})^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_{\mathcal{H}} \langle t \rangle \simeq \mathcal{B}_{\mathcal{H}}^{(0)}.$$

Moreover, by the paragraph above Lemma 2.23, $\mathcal{B}_{\mathcal{H}} \simeq \mathcal{B}_{\mathcal{H}}^{(0)}[\mathbb{Z}]$ and, therefore, we obtain biequivalences of locally graded bicategories

$$\mathcal{B}_{\mathcal{H}} \simeq \mathcal{B}_{\mathcal{H}}^{(0)}[\mathbb{Z}] \simeq (\mathcal{A}'_{\mathcal{H}})^{(0)}[\mathbb{Z}] \simeq \mathcal{A}'_{\mathcal{H}},$$

where the third biequivalence follows from Lemma 2.23 and the second from the previous chain of biequivalences. □

The following corollary follows immediately from Corollary 4.8 and Proposition 6.11.

Corollary 6.12. *There is a biequivalence of graded bicategories*

$$\mathcal{E}nd_{\mathcal{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \mathcal{A}'_{\mathcal{H}}{}^{l, \text{op}}.$$

Remark 6.13. If we forget the gradings, then the biequivalence (6.27) becomes a biequivalence of (ungraded) locally semisimple bicategories

$$\mathcal{B}_{\mathcal{H}} \simeq \mathcal{A}_{\mathcal{H}}.$$

This induces the structure of a pivotal fusion bicategory on $\mathcal{B}_{\mathcal{H}}$, by Proposition 3.4. Abusing the terminology slightly, we therefore say that $\mathcal{B}_{\mathcal{H}}^{(0)}$ is *pivotal \mathbb{Z} -fusion*, in particular, it is a semisimple pivotal monoidal category.

Note that the dual of $C_w^{\oplus v^t}$ in $\mathcal{B}_{\mathcal{H}}^{(0)}$ is isomorphic to $C_{w^{-1}}^{\oplus v^{-t}}$, for any $w \in \mathcal{H}$ and $t \in \mathbb{Z}$. This in contrast to the dual of $C_w^{\oplus v^t}$ in $\mathcal{S}_{\mathcal{H}}^{(0)}$, which is isomorphic to $C_{w^{-1}}^{\oplus v^{-t-2a}}$.

Remark 6.14. The proof of Proposition 6.11 also shows that

$$\mathcal{B}_H \simeq (\mathbf{C}_d)\text{comod}_{\text{add}(\mathcal{H})}(\mathbf{C}_d).$$

By the graded version of Theorem 2.20, there is a pair of graded biequivalences

$$\mathcal{S}_H\text{-gstmod}_H \simeq \mathcal{S}_J\text{-gstmod}_J \simeq \mathcal{S}\text{-gstmod}_J,$$

where J is the two-sided cell containing H . Among other things, the first biequivalence uses the fact that the additive closure of \mathcal{H} in \mathcal{S}_H is equal to the additive closure of \mathcal{H} in \mathcal{S}_J . The composite of both biequivalences sends $\mathbf{C}_H \simeq \mathbf{inj}_{\mathcal{S}_H}(\mathbf{C}_d)$ to $\mathbf{C}_L \simeq \mathbf{inj}_{\mathcal{S}}(\mathbf{C}_d)$, where L is the left cell such that $H = L \cap L^*$. It also provides a graded equivalence between $\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)$ and $\mathcal{E}nd_{\mathcal{S}}(\mathbf{C}_L)$, both being equivalent to $\mathcal{B}_H^{\text{op}} \simeq \mathcal{A}_H^{\prime,\text{op}}$.

We can now explain a generalization of Proposition 6.11. Let $\mathcal{S}\text{-gcell}_J$ be the 2-category whose

- objects are left cells L in J , identified with the cell 2-representation \mathbf{C}_L of \mathcal{S} ;
- morphism categories $\mathcal{S}\text{-gcell}_J(L_1, L_2) := \text{Hom}_{\mathcal{S}}(\mathbf{C}_{L_1}, \mathbf{C}_{L_2})$ with horizontal composition given by composition of morphisms of 2-representations.

Recalling that every left cell L contains a unique Duflo involution d_L and the equivalence $\mathbf{C}_L \simeq \mathbf{inj}_{\mathcal{S}}(\mathbf{C}_{d_L})$, $\mathcal{S}\text{-gcell}_J$ is biequivalent to the bicategory $\mathcal{B}_J^{\text{op}}$, where \mathcal{B}_J has the same objects but with morphism categories

$$\mathcal{B}_J(L_1, L_2) = (\mathbf{C}_{d_{L_1}})\mathbf{biinj}_{\text{add}(J)}(\mathbf{C}_{d_{L_2}}),$$

where horizontal composition is given by taking cotensor products over the relevant Duflo involution and the identity 1-morphism on L is \mathbf{C}_{d_L} .

As already recalled in Remark 3.5, we can define the bicategory \mathcal{A}_J in analogy to \mathcal{A}_H using the perverse filtration of $\text{add}(J)$.

Theorem 6.15. *The 2-category $\mathcal{S}\text{-gcell}_J$ is locally graded semisimple.*

Proof. As in Subsection 4.4 and Proposition 6.3, the proof follows from the fact that the Duflo involution \mathbf{C}_{d_L} is a separable Frobenius algebra in \mathcal{S}_J for every left cell L in J . Recall that \mathcal{A}_H is a 2-full one-object subbicategory (that is, a full monoidal subcategory) of the bicategory \mathcal{A}_J . As we already mentioned in Remark 3.5, the latter can be regarded as a pivotal multifusion bicategory with one object (whose identity 1-morphism is decomposable in general) or, as we will do in this paper, as a pivotal fusion bicategory whose objects are the left cells $L \subseteq J$ and whose morphism categories $\mathcal{A}_J(L_1, L_2)$ consist of 1-morphisms in $\mathcal{A}_{L_1^* \cap L_2}$, for the \mathcal{H} -cell $L_1^* \cap L_2$.

Analogously to Proposition 6.11, one can show that \mathcal{B}_J is locally graded semisimple and that there is a biequivalence of \mathbb{Z} -finitary bicategories

$$\mathcal{B}_J^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathcal{A}_J\langle t \rangle,$$

which implies that there is a graded biequivalence

$$\mathcal{B}_J \simeq \mathcal{A}'_J.$$

Since there is also a graded biequivalence $\mathcal{S}\text{-gcell}_J \simeq \mathcal{B}_J^{\text{op}}$, the result follows. □

Remark 6.16. The previous theorem shows that $\mathcal{S}\text{-gcell}_{\mathcal{J}}$ is graded biequivalent to $\mathcal{A}_{\mathcal{J}}^{\prime, \text{op}}$ and hence locally graded semisimple. However, the graded cell 2-representations of \mathcal{S} with apex \mathcal{J} , which form the objects of $\mathcal{S}\text{-gcell}_{\mathcal{J}}$, are not semisimple themselves in general, as follows from Proposition 4.3.

At the same time, we can analogously define the graded 2-category $\mathcal{A}_{\mathcal{J}}^{\prime}\text{-gcell}_{\mathcal{J}}$ as having objects \mathcal{L} for left cells $\mathcal{L} \subset \mathcal{J}$, which we now identify with the corresponding graded cell 2-representations of $\mathcal{A}_{\mathcal{J}}^{\prime}$ (which are graded semisimple), and with morphism categories given by the corresponding morphisms of 2-representations. Again, this is graded biequivalent to the bicategory with the same objects, and the graded morphism category from \mathcal{L}_1 to \mathcal{L}_2 given by $(A_{d_{\mathcal{L}_1}})\text{biinj}_{\mathcal{A}_{\mathcal{J}}^{\prime}}(A_{d_{\mathcal{L}_2}})$. Since the $A_{d_{\mathcal{L}_1}}$ are idempotents with the trivial coalgebra structure, this implies that $\mathcal{A}_{\mathcal{J}}^{\prime}\text{-gcell}_{\mathcal{J}}$ is graded biequivalent to $(\mathcal{A}_{\mathcal{J}}^{\prime})^{\text{op}}$ and hence there is a graded biequivalence

$$\mathcal{S}\text{-gcell}_{\mathcal{J}} \simeq \mathcal{A}_{\mathcal{J}}^{\prime}\text{-gcell}_{\mathcal{J}}. \quad (6.28)$$

In Theorem 7.10, we will prove that $\mathcal{S}\text{-gstmod}_{\mathcal{J}}$ is locally graded semisimple, which is a generalization of Theorem 6.15. The reason for including the latter theorem here is that the proof above, which does not depend on the results in Section 7, shows that both 2-categories in (6.28) are graded biequivalent to $\mathcal{A}_{\mathcal{J}}^{\prime, \text{op}}$, which is a nice result in its own right. The analogous result for $\mathcal{S}\text{-gstmod}_{\mathcal{J}}$ is more involved and, since we do not need it in this paper, we have omitted it.

7 | THE MAIN THEOREM

In this section, we will prove our main theorem:

Theorem 7.1. *There are biequivalences of graded 2-categories*

$$\mathcal{S}_{\mathcal{H}}\text{-gstmod}_{\mathcal{H}} \simeq \mathcal{B}_{\mathcal{H}}\text{-gstmod} \simeq \mathcal{A}_{\mathcal{H}}\text{-stmod}'.$$

Remark 7.2. To avoid confusion about the gradings, recall that $\mathcal{B}_{\mathcal{H}} \simeq \mathcal{A}_{\mathcal{H}}^{\prime}$ by Proposition 6.11. The last biequivalence in Theorem 7.1 then means that, for every $\mathbf{M} \in \mathcal{A}_{\mathcal{H}}^{\prime}\text{-gstmod}$, there is an $\mathbf{N} \in \mathcal{A}_{\mathcal{H}}\text{-stmod}$ such that $\mathbf{M} \simeq \mathbf{N}'$ in $\mathcal{A}_{\mathcal{H}}^{\prime}\text{-gstmod}$, and this correspondence establishes a bijection between the equivalence classes of objects in $\mathcal{A}_{\mathcal{H}}^{\prime}\text{-gstmod}$ and $\mathcal{A}_{\mathcal{H}}\text{-stmod}$. Moreover, for every pair $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{A}_{\mathcal{H}}^{\prime}\text{-gstmod}$, there is an equivalence of graded categories $\text{Hom}_{\mathcal{A}_{\mathcal{H}}^{\prime}}(\mathbf{M}_1, \mathbf{M}_2) \simeq \text{Hom}_{\mathcal{A}_{\mathcal{H}}}(\mathbf{N}_1, \mathbf{N}_2)'$.

Note that this does *not* mean that $\mathcal{A}_{\mathcal{H}}^{\prime}\text{-gstmod}$ and $\mathcal{A}_{\mathcal{H}}\text{-stmod}$ are biequivalent as graded 2-categories, because the morphism categories of the former 2-category are non-trivially graded, whereas those of the latter 2-category are trivially graded (meaning that all 2-morphisms live in degree 0), cf. Remark 2.22.

The proof of Theorem 7.1 is the content of Propositions 7.5 and 7.8 in the next two subsections, each of which proves one of the two biequivalences above, but let us first state and prove a very important corollary.

Corollary 7.3. *For any finite Coxeter type, there are finitely many equivalence classes of graded simple transitive 2-representations of \mathcal{S} .*

Proof. Let W be any finite Coxeter group and $\mathcal{S} = \mathcal{S}(W)$. By the composite of the biequivalences in Theorem 7.1, there is a bijection between the set of equivalence classes of graded simple transitive 2-representations of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} and the set of equivalence classes of simple transitive birepresentations of $\mathcal{A}_{\mathcal{H}}$, for any diagonal \mathcal{H} -cell of \mathcal{S} . Since every $\mathcal{A}_{\mathcal{H}}$ is pivotal fusion, the corollary now follows from [15, Corollary 9.1.6 and Proposition 3.4.6], strong \mathcal{H} -reduction, see [36, Theorem 4.32; 37, Theorem 15]), and the fact that \mathcal{S} has only finitely many two-sided cells. \square

7.1 | The right biequivalence in Theorem 7.1

Lemma 7.4. *Every graded simple transitive 2-representation of $\mathcal{B}_{\mathcal{H}}$ is graded semisimple.*

Proof. Let \mathbf{M} be a graded simple transitive 2-representation of $\mathcal{B}_{\mathcal{H}}$. We want to show that the radical of \mathcal{M} is trivial, which implies the claim. Recall that the radical \mathcal{R} is the unique maximal nilpotent ideal of \mathcal{M} , which can be defined as the union of the radicals of all full subcategories of \mathcal{M} with finitely many objects. Since such a subcategory can also be seen as a finite-dimensional algebra, its radical can be defined as in [8, Section 3.1]. By [8, Theorem 3.1.1], this also implies that \mathcal{M}/\mathcal{R} is semisimple. We first consider the $\mathcal{B}_{\mathcal{H}}^{(0)}$ 2-representation $\mathbf{M}^{(0)}$, and note that this does not contain any proper ideals, see the last paragraph of Subsection 2.7.

We then use Proposition 6.11 to restrict $\mathbf{M}^{(0)}$ to a 2-representation of $\mathcal{A}_{\mathcal{H}}$, and let \mathbf{N} denote the transitive 2-subrepresentation of $\mathcal{A}_{\mathcal{H}}$ generated by a non-zero indecomposable object X in $\mathbf{M}^{(0)}(\emptyset)$.

Recall that $\mathcal{A}_{\mathcal{H}}$ is pivotal fusion and therefore only has one two-sided cell, namely \mathcal{H} , which contains the identity 1-morphism A_d . Thus, Theorem 2.13 implies that A_d acts projectively, so any simple transitive 2-representation of $\mathcal{A}_{\mathcal{H}}$ is semisimple. This in turn implies that the radical $\text{rad}(\mathbf{N}(\emptyset))$ is the unique maximal ideal of \mathbf{N} . By Proposition 6.11, the direct sums

$$\bigoplus_{t \in \mathbb{Z}} \text{rad}(\mathbf{N}(\emptyset)(t)), \quad \bigoplus_{s \neq t \in \mathbb{Z}} \text{Hom}(\mathbf{N}(\emptyset), \mathbf{N}(\emptyset)(t-s))$$

are both $\mathcal{B}_{\mathcal{H}}^{(0)}$ -invariant, and hence so is their sum, which equals $\text{rad}(\mathbf{M}^{(0)}(\emptyset))$. Since $\mathbf{M}^{(0)}$ does not contain any proper ideals, the latter radical is zero. Thus, $\mathbf{M}^{(0)}$ is semisimple, and hence \mathbf{M} is graded semisimple. \square

Proposition 7.5. *There is a biequivalence of graded 2-categories*

$$\mathcal{B}_{\mathcal{H}}\text{-gstmod} \simeq \mathcal{A}_{\mathcal{H}}\text{-stmod}'.$$

Proof. Given a simple transitive 2-representation \mathbf{N} of $\mathcal{A}_{\mathcal{H}}$, we can choose a simple object $X \in \mathbf{N}(\emptyset)$ to obtain $\mathbf{N} \simeq \mathbf{inj}_{\mathcal{A}_{\mathcal{H}}}([X, X])$. By Proposition 6.11 and [35, Theorem 9; 37, Corollary 12], $\mathbf{M} := \mathbf{inj}_{\mathcal{B}_{\mathcal{H}}}([X, X]) \simeq \mathbf{N}'$ is a graded simple transitive 2-representation of $\mathcal{B}_{\mathcal{H}}$. (To simplify notation, we use the identification $[X, X] = \Theta([X, X])$.) This provides an assignment $\mathcal{A}_{\mathcal{H}}\text{-stmod}' \rightarrow \mathcal{B}_{\mathcal{H}}\text{-gstmod}$ on objects, and we claim that it is essentially surjective.

Indeed, let \mathbf{M} be a graded simple transitive 2-representation of $\mathcal{B}_{\mathcal{H}}$. As before, we consider the $\mathcal{B}_{\mathcal{H}}^{(0)}$ 2-representation $\mathbf{M}^{(0)}$, which again does not contain any proper ideals, see Subsection 2.7, and restrict it to $\mathcal{A}_{\mathcal{H}}$. Letting X be a non-zero indecomposable object in $\mathbf{M}^{(0)}(\emptyset)$, we let \mathbf{N} denote the transitive 2-subrepresentation of $\mathcal{A}_{\mathcal{H}}$ generated by X . By virtue of $\mathcal{A}_{\mathcal{H}}$ being pivotal

fusion, \mathbf{N} is simple transitive and, moreover, the subcategory $\mathbf{N}(\emptyset)$ of $\mathbf{M}^{(0)}(\emptyset)$ is also semisimple by Lemma 7.4.

Also by Lemma 7.4, we have

$$\mathbf{M}^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathbf{N}\langle t \rangle \quad (7.1)$$

as 2-representations of $\mathcal{B}_{\mathcal{H}}^{(0)}$. By [35, Theorem 4.7; 37, Corollary 12] (which in the context of finitary semisimple categories are the dual versions of [15, Corollary 7.10.5 and Exercise 7.10.6]) and local semisimplicity of $\mathcal{A}_{\mathcal{H}}$, we have

$$\mathbf{N} \simeq \mathbf{inj}_{\mathcal{A}_{\mathcal{H}}}([X, X]), \quad (7.2)$$

where $[X, X] \in \mathcal{A}_{\mathcal{H}}$ is the cosimple coalgebra given by the internal cohom construction, see Subsection 2.4. Proposition 6.11 implies that it is also cosimple as a coalgebra in $\mathcal{B}_{\mathcal{H}}^{(0)}$ and from (7.1) and (7.2) it follows that

$$\mathbf{M}^{(0)} \simeq \bigoplus_{t \in \mathbb{Z}} \mathbf{inj}_{\mathcal{A}_{\mathcal{H}}}([X, X])\langle t \rangle \simeq \mathbf{inj}_{\mathcal{B}_{\mathcal{H}}^{(0)}}([X, X])$$

as 2-representations of $\mathcal{B}_{\mathcal{H}}^{(0)}$. Moreover, $\mathcal{B}_{\mathcal{H}} \simeq (\mathcal{B}_{\mathcal{H}}^{(0)})'$ and, by semisimplicity, we have

$$\mathbf{M} \simeq \mathbf{N}' \simeq \mathbf{inj}_{\mathcal{B}_{\mathcal{H}}}([X, X])$$

as graded finitary 2-representations, and our assignment is essentially surjective

We further claim that any morphism Φ of 2-representations of $\mathcal{A}_{\mathcal{H}}$ between \mathbf{N}_1 and \mathbf{N}_2 extends in a unique way to a morphism of graded 2-representations of $\mathcal{B}_{\mathcal{H}}$ between $\mathbf{M}_1 = \mathbf{N}'_1$ and $\mathbf{M}_2 = \mathbf{N}'_2$. To see this, note that any object of $\mathbf{M}_1(\emptyset)^{(0)}$ is a direct sum of objects of the form $X^{\oplus v}$ for some $X \in \mathbf{N}_1(\emptyset)$. Since $X^{\oplus v} = \mathbf{M}_1(1_{\emptyset}^{\oplus v})X$, we obtain that

$$\Phi(X^{\oplus v}) = \Phi(\mathbf{M}_1(1_{\emptyset}^{\oplus v})X) \cong \mathbf{M}_2(1_{\emptyset}^{\oplus v})\Phi(X) = \Phi(X)^{\oplus v} \text{ in } \mathbf{M}_2(\emptyset)^{(0)}.$$

Similarly, all components of a morphism f in $\mathbf{M}_1(\emptyset)$ are scalar multiples of $\text{id}_X^{\oplus v}$ for indecomposable X , again by semisimplicity. Since $\text{id}_X^{\oplus v} = \mathbf{M}_1(\text{id}_{1_{\emptyset}^{\oplus v}})_X$, we obtain

$$\Phi(\text{id}_X^{\oplus v}) = \Phi(\mathbf{M}_1(\text{id}_{1_{\emptyset}^{\oplus v}})_X) = \mathbf{M}_2(\text{id}_{1_{\emptyset}^{\oplus v}})_{\Phi(X)} = \mathbf{M}_2(\text{id}_{1_{\emptyset}^{\oplus v}})_{\Phi(X)}^{\oplus v} = \text{id}_{\Phi(X)}^{\oplus v},$$

which completes the proof of our claim. Recall that 1-morphisms in $\mathcal{A}_{\mathcal{H}}\text{-stmod}'$ are of the form (Φ, s) , where Φ is a morphism of 2-representations of $\mathcal{A}_{\mathcal{H}}$ between \mathbf{N}_1 and \mathbf{N}_2 and $s \in \mathbb{Z}$. Extending Φ as above, we send (Φ, s) to $\Phi\langle s \rangle$. Modifications extend similarly, and we obtain a graded pseudofunctor $\mathcal{A}_{\mathcal{H}}\text{-stmod}' \rightarrow \mathcal{B}_{\mathcal{H}}\text{-gstmod}$.

To check that the above graded functor

$$\text{Hom}_{\mathcal{A}_{\mathcal{H}}}(\mathbf{N}_1, \mathbf{N}_2)' \rightarrow \text{Hom}_{\mathcal{B}_{\mathcal{H}}}(\mathbf{M}_1, \mathbf{M}_2)$$

is an equivalence, let $\Phi : \mathbf{M}_1 \rightarrow \mathbf{M}_2$ be a morphism of graded 2-representations for $\mathcal{B}_{\mathcal{H}}$. Suppose we have defined $\mathbf{N}_1 \simeq \mathbf{inj}_{\mathcal{A}_{\mathcal{H}}}([X_1, X_1])$, $\mathbf{N}_2 \simeq \mathbf{inj}_{\mathcal{A}_{\mathcal{H}}}([X_2, X_2])$ via a choice of objects $X_i \in \mathbf{M}_i(\emptyset)$. If $\Phi(X_1) \in \mathbf{N}_2(\emptyset)\langle t \rangle$, it is easy to see that $\Phi\langle -t \rangle$ restricts to a morphism of 2-representations

$\Psi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$. Modifications restrict naturally. It is immediate that these assignments provide a quasi-inverse to the functor $\text{Hom}_{\mathcal{A}_H}(\mathbf{N}_1, \mathbf{N}_2)' \rightarrow \text{Hom}_{\mathcal{B}_H}(\mathbf{M}_1, \mathbf{M}_2)$. \square

7.2 | The left biequivalence in Theorem 7.1

A functor between two finitary categories is called *injective* if its extension to the injective abelianizations of those categories is an injective object in the category of left exact functors between the said abelianizations. Further, a morphism of finitary 2-representations is injective if its underlying functors are injective functors. For a finitary 2-category \mathcal{C} and two finitary 2-representations \mathbf{M}, \mathbf{N} of \mathcal{C} , we denote by $\text{Hom}_{\mathcal{C}}^{\text{inj}}(\mathbf{M}, \mathbf{N})$ the category of injective morphisms of 2-representations. Moreover, considering injective endomorphisms of \mathbf{M} , we obtain the 2-semicategory $\mathcal{E}\text{nd}_{\mathcal{C}}^{\text{inj}}(\mathbf{M})$. We further denote by $\text{End}_{\mathcal{C}}^{\text{inj},r}(\mathbf{M})$ the additive closure of the injective endomorphisms and the identity morphism of \mathbf{M} . This category also defines the 2-category $\mathcal{E}\text{nd}_{\mathcal{C}}^{\text{inj},r}(\mathbf{M})$ whose single object is \mathbf{M} .

For future use, we record the following lemma.

Lemma 7.6. *Let $F : B \rightarrow C$ be an injective functor and assume that both B and C are finitary Frobenius categories. Then F is isomorphic to tensoring with a projective bimodule and the collection of such injective functor is closed under adjunctions.*

Proof. Assume without loss of generality that $\underline{B} \simeq B\text{-mod}$ and $\underline{C} \simeq C\text{-mod}$ for self-injective algebras B and C , respectively. Any left exact functor from $B\text{-mod}$ to $C\text{-mod}$ is isomorphic to $\text{Hom}_B(X, _)$ for some B - C -bimodules X . The morphism space between such functors $\text{Hom}_B(X', _)$ and $\text{Hom}_B(X, _)$ is given by $\text{Hom}_{B\text{-}C}(X, X')$, thus $\text{Hom}_B(X, _)$ is injective if and only if X is a projective bimodule. In this case, using $\text{Hom}_B(B \otimes_k C, _) \cong C^* \otimes_k B \otimes_B _$, together with additivity of Hom and self-injectivity of C , we see that injective functors correspond to tensoring with projective bimodules, and that these are, moreover, closed under taking adjoints. \square

Before we can prove the left biequivalence in Theorem 7.1, we need to recall the appropriate application of the double centralizer theorem [36, Theorem 5.2] to the case of the cell 2-representation \mathbf{C}_H of \mathcal{S}_H . The 2-actions of \mathcal{S}_H and $\mathcal{E}\text{nd}_{\mathcal{S}_H}(\mathbf{C}_H)$ on \mathbf{C}_H commute by definition, this implies that there is the canonical (graded) 2-functor

$$\text{can} : \mathcal{S}_H \rightarrow \mathcal{E}\text{nd}_{\mathcal{E}\text{nd}_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{ex}}(\mathbf{C}_H).$$

The following theorem is a special, but graded, case of [36, Theorem 5.2].

Theorem 7.7. *The canonical 2-functor can is fully faithful on 2-morphisms and essentially surjective on 1-morphisms when restricted to $\text{add}(\mathcal{H})$ and corestricted to $\mathcal{E}\text{nd}_{\mathcal{E}\text{nd}_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(\mathbf{C}_H)$.*

We are now ready to prove the existence of the left graded biequivalence in Theorem 7.1.

Proposition 7.8. *There is a biequivalence of graded 2-categories*

$$\mathcal{S}_H\text{-gstmod}_H \simeq \mathcal{B}_H\text{-gstmod}.$$

Proof. Consider the following graded fiat 2-category \mathcal{S}_H with two objects, denoted c and b . We identify the object c with the graded category $C := \mathbf{C}_H(\emptyset)$, that is, the underlying graded category of the cell 2-representation of \mathcal{S}_H with apex H , and the object b with the graded category $B := \mathcal{B}_H(\emptyset, \emptyset)$, that is, the graded category underlying the principal birepresentation of \mathcal{B}_H . Recall that, due to the graded equivalence $\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H) \simeq \mathcal{B}_H^{\text{op}}$ in Corollary 4.8, the 2-category $\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)$ acts, by graded endofunctors and homogeneous degree-0 natural transformations, on both C and B .

The graded morphism categories of \mathcal{S}_H are defined as

$$\mathcal{S}_H(b, b) := \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}(B, B),$$

$$\mathcal{S}_H(b, c) := \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(B, C),$$

$$\mathcal{S}_H(c, b) := \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(C, B),$$

$$\mathcal{S}_H(c, c) := \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}, r}(C, C).$$

Note that

$$\text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}(B, B) = \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(B, B) = \text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}, r}(B, B)$$

since B is graded semisimple by Proposition 6.11. The graded 2-categorical structure on \mathcal{S}_H is given by composition in the same way as in, for example, [42, Subsection 2.3], and the fiat structure is defined by taking adjoints, which preserves injective functors, since the underlying categories are Frobenius (see Lemma 7.6).

Next, note that the regular birepresentation of \mathcal{B}_H defines a natural graded embedding

$$\mathbf{B}_H : \mathcal{B}_H \rightarrow \mathcal{S}_H(b, b), \quad (7.3)$$

as it clearly commutes with the right 2-action of \mathcal{B}_H on B . Locally graded semisimplicity of \mathcal{B}_H implies that this graded embedding is a graded biequivalence.

By Theorem 7.7, the natural graded embedding of \mathcal{S}_H into $\mathcal{S}_H(c, c)$ is fully faithful on 2-morphisms and essentially surjective on 1-morphisms when restricted to $\text{add}(H)$ and corestricted to $\text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(C, C)$. In particular, the endomorphism 2-category of the object c has two H -cells: that containing the identity and that consisting of indecomposable objects in $\text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(C, C)$, which we can identify with H . Theorem 7.7 then implies that there is a biequivalence of graded 2-categories

$$\mathcal{S}_H\text{-gstmod}_H \simeq \mathcal{S}_H(c, c)\text{-gstmod}_H.$$

The statement now follows by strong H -reduction, see Theorem 2.20. To see this, note that \mathcal{S}_H has only one non-trivial two-sided cell (provided H is not trivial, of course), which we denote by J and is formed by the isomorphism classes of all indecomposable 1-morphisms that are not isomorphic to 1_c . This two-sided cell contains two diagonal H -cells, the one formed by the isomorphism classes of the indecomposable 1-morphisms in $\mathcal{S}_H(b, b)$ and the other formed by the isomorphism classes of the indecomposable 1-morphisms in $\text{Hom}_{\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H)}^{\text{inj}}(C, C)$, which can both be identi-

fied with \mathcal{H} due to Proposition 6.11 and Theorem 7.7. By the above and Theorem 2.20, there are biequivalences of graded 2-categories

$$\mathcal{S}_{\mathcal{H}}\text{-gstmod}_{\mathcal{H}} \simeq \mathcal{I}_{\mathcal{H}}\text{-gstmod}_{\mathcal{J}} \simeq \mathcal{B}_{\mathcal{H}}\text{-gstmod}_{\mathcal{H}} = \mathcal{B}_{\mathcal{H}}\text{-gstmod},$$

where the middle biequivalence is induced by $\mathbf{B}_{\mathcal{H}}$ in (7.3) and the last equality uses that $\mathcal{B}_{\mathcal{H}}$ only has one \mathcal{H} -cell, which follows from Remark 2.8 and Proposition 6.11. This completes the proof. \square

Remark 7.9. The construction of the 2-category $\mathcal{I}_{\mathcal{H}}$ in the proof of Proposition 7.8 is in the spirit of what is known as a *Morita context* in the tensor category literature, see [47, Remark 3.18 and section 4].

7.3 | Local graded semisimplicity

In this subsection, we generalize Theorem 6.15.

Theorem 7.10. *The 2-category $\mathcal{S}\text{-gstmod}_{\mathcal{J}}$ is locally graded semisimple.*

Proof. By Theorem 2.20 and Theorem 7.1, it suffices to prove that $\mathcal{A}_{\mathcal{H}}\text{-stmod}$ is locally semisimple. Noting that $\mathcal{A}_{\mathcal{H}}$ is pivotal fusion and that, therefore, its simple transitive 2-representations coincide with its so-called *indecomposable exact module categories*, this is a consequence of [16, Theorems 2.15 and 2.16]. \square

7.4 | The ungraded case

In this section, let $\mathcal{S} = \mathcal{S}(W, S)$ be the ungraded category of Soergel bimodules for a finite Coxeter system (W, S) . This category is defined just as its graded counterpart, but ignores the grading. The indecomposables in \mathcal{S} are still indexed by W and will be denoted by C_x as well. This follows from the fact that for any object $X \in \mathcal{S}$, the endomorphism ring $\text{End}_{\mathcal{S}}(X)$ is local in the graded setting if and only if it is local in the ungraded setting. The character isomorphism in the Soergel–Elias–Williamson categorification theorem then has a \mathbb{Z} -linear counterpart in the ungraded setting which sends $[C_x] \in [\mathcal{S}]_{\oplus}$ to $c_x \in \mathbb{Z}[W]$, where c_x is the image of the Kazhdan–Lusztig basis element under the \mathbb{Z} -linear map $H(W) \rightarrow \mathbb{Z}[W]$ which sends the standard basis element T_w to w , for $w \in W$, and v to 1.

Recall that $\mathcal{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}$ denotes the 2-category of all simple transitive 2-representations of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} without any grading assumptions. Any simple transitive graded 2-representation of the graded $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} remains simple transitive in the ungraded setting, but there is no *a priori* reason why the forgetful 2-functor

$$F : \mathcal{S}_{\mathcal{H}}\text{-gstmod}_{\mathcal{H}} \rightarrow \mathcal{S}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}$$

should be essentially surjective. In principle, there could be ungradable simple transitive 2-representations of $\mathcal{S}_{\mathcal{H}}$ with apex \mathcal{H} . This was actually an open question until now, for example, for finite dihedral Coxeter type it was unknown if the bicolored ADE classification of the graded simple transitive 2-representations of \mathcal{S} with subregular apex in [23, 38] remained valid

in the ungraded case. We can now answer that question affirmatively, because F is indeed a biequivalence for any finite Coxeter type.

This follows from the fact that in the ungraded case the double centralizer property from Theorem 7.7 still holds (note that [36, Theorem 5.2] was formulated and proved in the ungraded setting) and that there is a biequivalence

$$\mathcal{E}nd_{\mathcal{S}_H}(\mathbf{C}_H) \simeq \mathcal{A}_H^{\text{op}}. \quad (7.4)$$

The proof of Theorem 7.7 does not use the grading in an essential way, so that theorem remains true in the ungraded setup. The arguments which prove the existence of the biequivalence in (7.4) in the ungraded case, are exactly the same as the ones which prove Proposition 6.11. The only difference is that there is no translation in the ungraded setting, so the trivial \mathbb{Z} -cover of \mathcal{A}_H ‘collapses’ to \mathcal{A}_H itself. Just as in the proof of Theorem 7.1, the double centralizer theorem and the biequivalence in (7.4) imply that there is a biequivalence

$$\mathcal{S}_H\text{-stmod}_H \simeq \mathcal{A}_H\text{-stmod}, \quad (7.5)$$

by Theorem 2.20 (which also holds both in the graded and the ungraded setting). This shows that the forgetful 2-functor F is indeed a biequivalence.

As in Corollary 7.3, the biequivalence in (7.5) implies that, for any finite Coxeter type, there are finitely many equivalence classes of simple transitive 2-representations of \mathcal{S} .

8 | CLASSIFICATION RESULTS

The asymptotic bicategory and its 2-representations

Recall that, by Proposition 3.4, \mathcal{A}_H is a pivotal fusion bicategory and thus, H is its only cell and all of its simple transitive 2-representations are semisimple. Further, the notion of a simple transitive 2-representation agrees with the notion of an indecomposable exact module category, which is often used as the terminology in [4, 15, 25, 49–51] below.

Up to a handful of exceptions, the asymptotic bicategory \mathcal{A}_H comes in three flavors and for all of them a classification of simple transitive 2-representations is known, as we will summarize now (giving more details below). Recall that $\mathcal{R}ep(G) = \mathcal{R}ep(G, \mathbb{C})$ is the pivotal fusion 2-category of finite-dimensional G -modules. Let $\mathcal{V}ect(G) = \mathcal{V}ect_{\mathbb{C}}(G)$ denote the pivotal fusion 2-category of finite-dimensional G -graded \mathbb{C} -vector spaces ($\mathcal{V}ect = \mathcal{V}ect(1)$ are plain finite-dimensional vector spaces), and $\mathcal{S}\mathcal{O}(3)_k$ the pivotal fusion 2-category of complex, finite-dimensional representations of quantum \mathfrak{so}_3 semisimplified at level k and without twists, see, for example, [15, Examples 2.3.4 and 8.18.5].

- (a) *Finite Weyl type (excluding G_2): generic case.* Up to three exceptions in types E_7 and E_8 , see (b), for each two-sided cell J there exists a diagonal H -cell \mathcal{H} such that $\mathcal{A}_H \simeq \mathcal{V}ect((\mathbb{Z}/2\mathbb{Z})^k)$ for some $k \in \mathbb{N}$, or $\mathcal{A}_H \simeq \mathcal{R}ep(G)$ for G being S_3 , S_4 or S_5 . This follows from (the arguments in the proof of) [4, Theorem 4].

For all of these, the classification of the associated simple transitive 2-representation works as follows. Let G be a finite group and let $\Omega(G)$ denote the set of subgroups of G up to conjugacy, K a choice of representative of $[K] \in \Omega(G)$, and $H^2(K, \mathbb{C}^\times)$ the second group cohomology of K with values in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, whose non-trivial generators are called *Schur multipliers*. By,

for example, [15, Example 7.4.10 and Corollary 7.12.20], we have

$$\left\{ \begin{array}{l} \text{equivalence classes of simple transitive} \\ 2\text{-representations of } \mathcal{V}\text{ect}(G) \text{ or } \mathcal{R}\text{ep}(G) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} ([K], \varpi) \mid [K] \in \Omega(G), \\ \varpi \in H^2(K, \mathbb{C}^\times) \end{array} \right\}.$$

The simple transitive 2-representations of $\mathcal{V}\text{ect}(G)$ have rank $\#G/\#K$ and the ones for $\mathcal{R}\text{ep}(G)$ are the ϖ -twisted representation categories $\mathcal{R}\text{ep}^\varpi(K)$ (in particular, their rank is equal to the rank of the character ring of K for trivial ϖ). Hence, we need to analyze the simple transitive 2-representations of $\mathcal{V}\text{ect}(G)$ or $\mathcal{R}\text{ep}(G)$, which are given by (conjugacy classes of) subgroups of $K \subset G$, their number $\#$, and the Schur multipliers in $H^2(K, \mathbb{C}^\times)$ of these subgroups. We additionally list their rank rk . Listing the data that we need is easy (calculating the subgroups and their numbers for $(\mathbb{Z}/2\mathbb{Z})^k$ is a pleasant exercise, while the Schur multipliers of these subgroups were already determined by Schur, see, for example, [3, Theorem 4] for a more modern reference; the data for the other three cases, S_3, S_4 and S_5 , can be calculated by computer). In the row rk in (8.1), two entries correspond to two different simple transitive 2-representations, one for each Schur multiplier with the one for the trivial Schur multiplier listed first.

$\mathcal{V}\text{ect}((\mathbb{Z}/2\mathbb{Z})^k)$					$\mathcal{R}\text{ep}(S_3)$				
K	$(\mathbb{Z}/2\mathbb{Z})^l$				K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
$\#$	$\binom{k}{l}$				$\#$	1	1	1	1
H^2	$(\mathbb{Z}/2\mathbb{Z})^{l(l-1)/2}$				H^2	1	1	1	1
rk	k/l				rk	1	2	3	3

$\mathcal{R}\text{ep}(S_4)$									
K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	S_3	D_4	A_4	S_4
$\#$	1	2	1	1	2	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	3	5, 2	4, 3	5, 3

$\mathcal{R}\text{ep}(S_5)$																
K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
$\#$	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

(Here $GA(1, 5)$ is the general affine group of rank one over \mathbb{F}_5 .)

- (b) *Finite Weyl type: exceptional case.* Type E_7 contains one and type E_8 two so-called *exceptional cells*. For these, by [51, Theorem 1.1], we have $\mathcal{A}_H \simeq \mathcal{V}\text{ect}^\zeta(\mathbb{Z}/2\mathbb{Z})$, having its 2-structure twisted by the non-trivial element ζ in the third group cohomology group $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$. In this case, \mathcal{A}_H has only one associated simple transitive 2-representation, which is the regular 2-representation of rank 2, see, for example, [49, Theorem 3.1].
- (c) *Finite dihedral type (including G_2).* We have $\mathcal{A}_H \simeq \mathcal{V}\text{ect}$ for the cells containing the identity element or the longest element, or $\mathcal{A}_H \simeq \mathcal{S}\mathcal{O}(3)_k$ for the middle cell, by [10, Theorem 2.15].

By, for example, [25, Theorem 6.1; 50, Theorem 6.1], we have

$$\left\{ \begin{array}{l} \text{equivalence classes of simple transitive} \\ 2 - \text{representations of } \mathcal{S}\mathcal{O}(3)_k \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{bicolored ADE diagrams} \\ \text{with Coxeter number } k + 2 \end{array} \right\}. \quad (8.2)$$

The corresponding simple transitive 2-representations have rank equal to the number of vertices of the associated ADE diagram.

(d) *Types H_3 and H_4 .* We do not know what \mathcal{A}_H is in general. For details see below.

What the main theorem covers

Recall that any (graded) simple transitive 2-representation of \mathcal{S} has an apex \mathcal{J} in W . Then, Theorem 7.1 together with Theorem 2.20, implies that

$$\mathcal{S}\text{-gstmod}_{\mathcal{J}} \simeq \mathcal{A}_H\text{-stmod}'$$

holds for any diagonal \mathcal{H} -cell $\mathcal{H} \subset \mathcal{J}$. Thus, we get

$$\left\{ \begin{array}{l} \text{equivalence classes of (graded)} \\ \text{simple transitive 2 - representations} \\ \text{of } \mathcal{S} \text{ with apex } \mathcal{J} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{simple transitive 2 - representations} \\ \text{of } \mathcal{A}_H \text{ (including grading shifts)} \end{array} \right\}.$$

(Note that Corollary 3.12 also gives us the rank of the simple transitive 2-representations of \mathcal{S}_H associated to the ones from \mathcal{A}_H . However, the corresponding simple transitive 2-representations for \mathcal{S} will have greater rank in general.) Thus, the above shows that only certain cells in Coxeter types H_3 and H_4 — most prominently, the cell (8.8) in type H_4 given below — would remain open with respect to a complete classification of (graded) simple transitive 2-representations of \mathcal{S} . For all other cases, Theorem 7.1 gives a complete classification and parameterization of the (graded) simple transitive 2-representations of \mathcal{S} , as we will summarize now.

What we cover

Let us give some details of what is covered by the results in this paper. We say that a type is ‘done’ if we can identify \mathcal{A}_H and parameterize its simple transitive 2-representations for at least one diagonal \mathcal{H} -cell $\mathcal{H} \subset \mathcal{J}$.

For this purpose, we use what we call a *full cell matrix*:

4 _{5,5}	1 _{5,5}	1 _{5,20}	2 _{5,25}	2 _{5,25}
1 _{5,5}	4 _{5,5}	1 _{5,20}	2 _{5,25}	2 _{5,25}
1 _{20,5}	1 _{20,5}	4 _{20,20}	2 _{20,25}	2 _{20,25}
2 _{25,5}	2 _{25,5}	2 _{25,20}	4 _{25,25}	1 _{25,25}
2 _{25,5}	2 _{25,5}	2 _{25,20}	1 _{25,25}	4 _{25,25}

(8.3)

Here we indicate the number of elements in left or right cells, where, for example, 2_{20,25} is to be understood as a 20-by-25 matrix containing only the entry 2 (thus, having 1000 elements), that is, it is $2Id_{20,25}$, where $Id_{20,25}$ is the 20-by-25 identity matrix. The shaded boxes are (matrices of) diagonal \mathcal{H} -cells.

The full cell matrices are block matrices, but we also view them as matrices containing only the scalars n in $nId_{20,25}$ and call them *cell matrices*. The difference between the two is that the cell matrix actually encodes equivalence classes of cell 2-representation: each column of the full cell matrix corresponds to a (left) cell 2-representation, while the columns of the cell matrix itself correspond to (left) cell 2-representation up to equivalence.

Here are a few more examples, which have appeared under various names in the literature; the first and second cases are $k = 0$, respectively, $k = 1$ below.

$$\begin{array}{l}
 \text{strongly regular: } \begin{array}{|c|} \hline 1_{a,a} \\ \hline \end{array}, \quad \mathcal{A}_{\mathcal{H}} \simeq \mathcal{Vect}, \\
 \text{---} \\
 \text{nice: } \begin{array}{|c|c|} \hline 2_{b,b} & 1_{c,b} \\ \hline 1_{b,c} & 2_{c,c} \\ \hline \end{array}, \quad \mathcal{A}_{\mathcal{H}} \simeq \mathcal{Vect}(\mathbb{Z}/2\mathbb{Z}), \\
 \text{---} \\
 \text{exceptional: } \begin{array}{|c|} \hline 2_{d,d} \\ \hline \end{array}, \quad \mathcal{A}_{\mathcal{H}} \simeq \mathcal{Vect}^{\mathcal{S}}(\mathbb{Z}/2\mathbb{Z}), \\
 \text{---} \\
 \text{dihedral: } \begin{array}{|c|c|} \hline \frac{m-1}{2} & \frac{m-1}{2} \\ \hline m-1 & m-1 \\ \hline 2 & 2 \\ \hline \end{array}, \quad m \text{ odd,} \\
 \begin{array}{|c|c|} \hline \frac{m}{2} & \frac{m-2}{2} \\ \hline m-2 & \frac{m}{2} \\ \hline 2 & 2 \\ \hline \end{array}, \quad m \text{ even,} \\
 \end{array} \tag{8.4}$$

where $a^2, 2(b^2 + c^2 + bc), 2d^2$ or $2(m - 1)$, respectively, is the size of the cell in question. (Knowing the size of the cells, one can recover a, b, c, d since there is always a unique solution in positive integers.) Note for example that in the nice case there are only two columns in the cell matrix, thus, only two cell 2-representations up to equivalence. However, there are $b + c$ cell 2-representations as encoded in the columns of the full cell matrix.

Type A_n . This type is **done** for all n , since all cells are strongly regular. In this case, there is one equivalence class of simple transitive 2-representation per two-sided cell, all of which are cell 2-representations.

Type B_n . This type is **done** for all n .

- (i) For all \mathcal{H} , we have $G = (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$ with $k(k + 1) \leq n$. The classification in this case is given by (8.1).
- (ii) The diagonal of the cell matrix is 2^k , all other entries are 2^l for $l < k$.
- (iii) Type B_5 is the first case not covered by previous classification results.
- (iv) B_6 is the smallest example (that is, the example of smallest rank) in classical type where we have a non-cell simple transitive 2-representation; see below.

Type B_5 is:

cell	0	1	2	3	4	5	6	7	7'	6'	5'	4'	3'	2'	1'	0'
size	1	42	150	100	225	152	600	650	650	600	152	225	100	150	42	1
a	0	1	2	3	3	4	4	5	6	7	9	10	10	11	16	25
$\mathcal{A}_{\mathcal{H}}$	0	1	1	0	0	1	1	1	1	1	1	0	0	1	1	0

Here and throughout: from left to right, we have listed the numbered cells, paired $\mathcal{J} \leftrightarrow \mathcal{J}' = \mathcal{J}w_0$ (with 0 being the minimal containing 1 and 0' the maximal cell containing w_0). From top to

bottom, we have listed their sizes, the \mathbf{a} -values, and \mathcal{A}_H , where the number k means that $\mathcal{A}_H \simeq \mathcal{V}\text{ect}(G)$ for $G = (\mathbb{Z}/2\mathbb{Z})^k$.

Type B_6 is:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12=12'	13=13'	11'	10'	9'	8'	7'	6'	5'	4'	3'	2'	1'	0'
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	3150	650	576	342	62	1
\mathbf{a}	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	11	11	16	12	15	17	18	25	36
\mathcal{A}_H	0	1	1	0	1	1	1	0	1	1	0	0	2	1	0	0	1	1	0	1	1	1	0	1	1	0

The cell 12 is displayed in (8.3). In this case, we have $G = (\mathbb{Z}/2\mathbb{Z})^2$, which has the (non-conjugate) subgroups $1, K_1, K_2, K_3, G$. The subgroups 1 and $K_1 \cong K_2 \cong K_3 \cong \mathbb{Z}/2\mathbb{Z}$ all have trivial second group cohomology, but $H^2(G, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, we have six equivalence classes of (graded) simple transitive 2-representations of ranks 1, 1, 2, 2, 2 and 4, respectively, see (8.1). It follows from Theorem 7.1 that this case gives a non-cell simple transitive 2-representation. The same happens repeatedly for higher ranks.

Type D_n . This type is **done** for all n .

- (i),(ii) As for type B_n , but with $(k + 1)^2 \leq n$.
- (iii) Type D_4 is the first case not covered by previous results.
- (iv) In type D_9 the first non-cell simple transitive 2-representations appear.

Type $I_2(m)$. This type is **done** for all $m > 2$.

- (i) Every \mathcal{J} contains a w_0^I ; there are only three two-sided cells.
- (ii) The bottom and top cell are strongly regular.
- (iii) The middle cell has a dihedral cell matrix, see (8.4), and the classification is given in (8.2). This is the smallest example with non-cell simple transitive 2-representations, starting from type $I_2(6) = G_2$.

Type E_6 . This type is **done**:

cell	0	1	2	3	4	5	6	7	8=8'	7'	6'	5'	4'	3'	2'	1'	0'
size	1	36	400	1350	4096	3600	6561	576	18600	576	6561	3600	4096	1350	400	36	1
\mathbf{a}	0	1	2	3	4	5	6	6	7	12	10	11	13	15	20	25	36
\mathcal{A}_H	0	0	0	1	0	0	0	0	(8.5)	0	0	0	0	1	0	0	0

Here we write k for $\mathcal{A}_H \simeq \mathcal{V}\text{ect}(G)$ with $G = (\mathbb{Z}/2\mathbb{Z})^k$. The corresponding cells are strongly regular or nice, except

$$\begin{array}{|c|c|c|} \hline 3_{10,10} & 2_{50,10} & 1_{20,10} \\ \hline 2_{10,50} & 3_{50,50} & 3_{20,50} \\ \hline 1_{10,20} & 3_{50,20} & 6_{20,20} \\ \hline \end{array}, \quad 3_{10,10} : \mathcal{A}_H \simeq \mathcal{R}\text{ep}(S_3). \tag{8.5}$$

Again, we get non-cell simple transitive 2-representations, see (8.1).

Type E_7 . This type is **done**, and it is quite similar to type E_6 :

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
size	1	49	729	4802	441	25650	35721	44100	11025	28224	35721	262150	246402	142884	44100	296352	11025	524288
a	0	1	2	3	3	4	5	6	6	6	7	7	8	9	10	10	12	11
\mathcal{A}_H	0	0	0	1	0	1	0	0	0	0	0	(8.5)	1	0	0	1	0	ζ

cell	16'	15'	14'	13'	12'	11'	10'	9'	8'	7'	6'	5'	4'	3'	2'	1'	0'
size	11025	296352	44100	142884	246402	262150	35721	28224	11025	44100	35721	25650	441	4802	729	49	1
a	15	13	13	14	15	16	20	21	21	21	22	25	36	30	37	46	63
\mathcal{A}_H	0	1	0	0	1	(8.5)	0	0	0	0	0	1	0	1	0	0	0

Cell 17 is exceptional with $\mathcal{A}_H \simeq \mathcal{V} \text{ect}^{\zeta}(\mathbb{Z}/2\mathbb{Z})$, see [51, Theorem 1.1]. The cells 11 and 11' are as in (8.5) (with diagonals $3_{70,70}$, $3_{210,210}$ and $6_{35,35}$), giving non-cell simple transitive 2-representations. All other cells are strongly regular or nice.

Type E_8 . This type is **done**, and it is similar to type E_7 :

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22=22'	
size	1	64	1225	20384	72200	313600	321489	740000	4986240	5696250	10497600	7768224	7683200	33554432	275625	29635200	12740000	20575296	8037225	36905625	17640000	47360000	4410000	
a	0	1	2	3	4	5	6	6	7	8	9	10	10	11	12	12	13	13	14	14	15	15	20	
\mathcal{A}_H	0	0	0	1	1	0	0	1	(8.5)	(8.5)	0	1	1	ζ	0	1	1	0	0	0	0	0	1	0

cell	23=23'	21'	20'	19'	18'	17'	16'	15'	14'	13'	12'	11'	10'	9'	8'	7'	6'	5'	4'	3'	2'	1'	0'
size	202671840	47360000	17640000	36905625	8037225	20575296	12740000	29635200	275625	33554432	7683200	7768224	10497600	5696250	4986240	740000	321489	313600	72200	20384	1225	64	1
a	16	21	21	22	22	23	25	24	36	26	28	30	31	32	37	42	46	47	52	63	74	91	120
\mathcal{A}_H	(8.6)	1	0	0	0	0	1	1	0	ζ	1	1	0	(8.5)	(8.5)	1	0	0	1	1	0	0	0

As before, the majority of cells are strongly regular or nice. We also have exceptional cells with $\mathcal{A}_H \simeq \mathcal{V} \text{ect}^{\zeta}(\mathbb{Z}/2\mathbb{Z})$, see [51, Theorem 1.1], and also cells as in (8.5) (with diagonals $3_{448,448}$, $3_{896,896}$ and $6_{56,56}$, or $3_{175,175}$, $3_{875,875}$ and $6_{350,350}$), giving non-cell, simple transitive 2-representations. There is one remaining cell with $\mathcal{A}_H \simeq \mathcal{R}ep(S_5)$ for appropriate \mathcal{H} , giving again non-cell simple transitive 2-representations. Its cell matrix is

7 _{420,420}	5 _{756,420}	6 _{1596,420}	5 _{168,420}	3 _{378,420}	4 _{1092,420}	2 _{70,420}
5 _{420,756}	8 _{756,756}	7 _{1596,756}	7 _{168,756}	8 _{378,756}	8 _{1092,756}	7 _{70,756}
6 _{420,1596}	7 _{756,1596}	12 _{1596,1596}	8 _{168,1596}	9 _{378,1596}	13 _{1092,1596}	11 _{70,1596}
5 _{420,168}	7 _{756,168}	8 _{1596,168}	12 _{168,168}	7 _{378,168}	12 _{1092,168}	12 _{70,168}
3 _{420,378}	8 _{756,378}	9 _{1596,378}	7 _{168,378}	15 _{378,378}	14 _{1092,378}	19 _{70,378}
4 _{420,1092}	8 _{756,1092}	13 _{1596,1092}	12 _{168,1092}	14 _{378,1092}	21 _{1092,1092}	24 _{70,1092}
2 _{420,70}	7 _{756,70}	11 _{1596,70}	12 _{168,70}	19 _{378,70}	24 _{1092,70}	39 _{70,70}

7_{420,420} : $\mathcal{A}_H \cong \mathcal{R}ep(S_5)$.

(8.6)

The simple transitive 2-representations of $\mathcal{R}ep(S_5)$ can be obtained from (8.1).

Type F_4 . This type is **done**:

cell	0	1	2	3	4	5=5'	4'	3'	2'	1'	0'
size	1	24	81	64	64	684	64	64	81	24	1
a	0	1	2	3	3	4	9	9	10	13	24
\mathcal{A}_H	0	1	0	0	0	(8.7)	0	0	0	1	0

where we write k for $\mathcal{A}_H \simeq \mathcal{V}ect(G)$ with $G = (\mathbb{Z}/2\mathbb{Z})^k$, as before, and the cell matrices are strongly regular or nice, see (8.4). In the remaining case we have (for appropriate H):

5 _{3,3}	3 _{3,3}	4 _{3,4}	5 _{3,1}	2 _{3,1}
3 _{3,3}	5 _{3,3}	4 _{3,4}	2 _{3,1}	5 _{3,1}
4 _{4,3}	4 _{4,3}	9 _{4,4}	6 _{4,1}	6 _{4,1}
5 _{1,3}	2 _{1,3}	6 _{1,4}	9 _{1,1}	3 _{1,1}
2 _{1,3}	5 _{1,3}	6 _{1,4}	3 _{1,1}	9 _{1,1}

$$5_{3,3} : \mathcal{A}_H \simeq \mathcal{R}ep(S_4). \tag{8.7}$$

For the list of equivalence classes of simple transitive 2-representations of $\mathcal{R}ep(S_4)$, see (8.1). This is the second smallest example in Weyl type with a non-cell simple transitive 2-representation.

Type H_3 . This type needs **more** work:

cell	0	1	2	3=3'	2'	1'	0'
size	1	18	25	32	25	18	1
a	0	1	2	3	5	6	15
\mathcal{A}_H	(a)	(b)	(a)	(c)	(a)	(b)	(a)

- (a) These cases are strongly regular two-sided cells.
- (b) In these cases, the cell is $2_{3,3}$, and the Grothendieck rings of \mathcal{A}_H and $\mathcal{S}\mathcal{O}(3)_3$ coincide.
- (c) Cell 3 is $2_{4,4}$, and the Grothendieck rings of \mathcal{A}_H and $\mathcal{V}ect(\mathbb{Z}/2\mathbb{Z})$ coincide.

By [48, Subsection 2.5], the only two possibilities for (b) are $\mathcal{A}_H \simeq \mathcal{S}\mathcal{O}(3)_3$ or $\mathcal{A}_H \simeq \mathcal{M}(2, 5)$ (in the notation of Ostrik). Similarly, by [48, Subsection 2.4], the only two possibilities for (c) are $\mathcal{A}_H \simeq \mathcal{V}ect(\mathbb{Z}/2\mathbb{Z})$ or $\mathcal{A}_H \simeq \mathcal{V}ect^t(\mathbb{Z}/2\mathbb{Z})$. However, only in the case of the cell 1 do we know which option it is, namely $\mathcal{A}_H \simeq \mathcal{S}\mathcal{O}(3)_3$, since this case is covered by [23, Theorem 28]. In this case, the classification is as in (8.2).

Type H_4 . This type needs **much more** work:

cell	0	1	2	3	4	5	6=6'	5'	4'	3'	2'	1'	0'
size	1	32	162	512	625	1296	9144	1296	625	512	162	32	1
a	0	1	2	3	4	5	6	15	16	18	22	31	60
\mathcal{A}_H	(a)	(b)	(b)	(c)	(a)	(a)	(8.8)	(a)	(a)	(c)	(b)	(b)	(a)

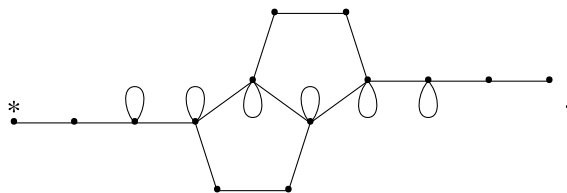
(a),(b),(c) are similar to (a),(b),(c) in type H_3 , and the same remark as in (d) holds. In the remaining case we have:

14 _{8,8}	13 _{10,8}	14 _{6,8}
13 _{8,10}	18 _{10,10}	18 _{6,10}
14 _{8,6}	18 _{10,6}	24 _{6,6}

$$\tag{8.8}$$

We were not able to find \mathcal{A}_H in the literature. In fact, for none of the diagonal \mathcal{H} -cells do we know what \mathcal{A}_H is; we only know the multiplication tables of their Grothendieck rings with respect to the asymptotic Kazhdan–Lusztig bases $\{a_w \mid w \in \mathcal{H}\}$, see also [1]. For example, if \mathcal{H} is in the 14-

by-14 block, then the Grothendieck ring of \mathcal{A}_H is not commutative, \mathcal{A}_H has Perron–Frobenius dimension $120(9 + 4\sqrt{5})$ and a simple generating 1-morphism of Perron–Frobenius dimension $1 + \sqrt{5}$ and fusion graph



Remark 8.1. Note that the two-sided cell 6 in type H_4 , whose structure is detailed in (8.8), contains a diagonal \mathcal{H} -cell for which the fusion algebra \mathcal{A}_H is non-commutative. This implies, of course, that the pivotal fusion category \mathcal{A}_H , whose precise structure is unknown, is not braided. For completeness, we list all non-commutative \mathcal{A}_H in all finite Coxeter types below.

- In types E_6 , the \mathcal{H} of order 6 in the two-sided cell 8.
- In type E_7 , the \mathcal{H} of order 6 in the two-sided cells 11 and 11'.
- In type E_8 , the \mathcal{H} of order 6 in the two-sided cells 8, 8', 9, 9', and the \mathcal{H} of order > 8 in the two-sided cell 23.
- In type F_4 , the \mathcal{H} of order 9 in the two-sided cell 5.
- In type H_4 , all \mathcal{H} in the two-sided cell 6.

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