

### Appendix III - analytical solution to the error propagation

The deduction of the analytical solution of the error propagation from  $Q_1$ ,  $Q_2$  and  $Q_3$  to the H:D has its milestone in the Taylor series expansion (Taylor 1955) of the H:D model. Basically this is stating the H:D estimated in point b as a function of the H:D estimated in point a and its increment from a to b due to each variable. This way the H:D residuals correspond to the sum of all the n order terms in the Taylor series:

$$\begin{aligned} \text{var}(H : D) &= \frac{1}{n-1} \sum [f(\bar{Q}_1 + h_1, \bar{Q}_2 + h_2, \bar{Q}_3 + h_3) - f(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3)]^2 \\ &= \frac{1}{n-1} \sum \left[ f(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3) + h_1 \frac{\partial f}{\partial Q_1} + h_2 \frac{\partial f}{\partial Q_2} + h_3 \frac{\partial f}{\partial Q_3} + \right. \\ &\quad \left. + h_1 h_2 \frac{\partial^2 f}{\partial Q_1 \partial Q_2} + h_1 h_3 \frac{\partial^2 f}{\partial Q_1 \partial Q_3} + h_2 h_3 \frac{\partial^2 f}{\partial Q_2 \partial Q_3} - f(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3) \right]^2 \end{aligned} \quad (\text{eqn.1})$$

where f is the H:D,  $h_i$  is each population's residual from its own  $Q_i$  to the average  $Q_i$ , both f and its partial derivatives are evaluated at the point  $(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3)$  and

$$\frac{\partial f}{\partial Q_1} = Q_2 \cdot Q_3 \quad (\text{eqn.2}) \quad \frac{\partial f}{\partial Q_2} = Q_1 \cdot Q_3 \quad (\text{eqn.3}) \quad \frac{\partial f}{\partial Q_3} = Q_1 \cdot Q_2 \quad (\text{eqn.4})$$

$$\frac{\partial^2 f}{\partial Q_1 \partial Q_2} = Q_3 \quad (\text{eqn.5}) \quad \frac{\partial^2 f}{\partial Q_1 \partial Q_3} = Q_2 \quad (\text{eqn.6}) \quad \frac{\partial^2 f}{\partial Q_2 \partial Q_3} = Q_1 \quad (\text{eqn.7})$$

The second order terms come from

$$\begin{aligned} &\frac{1}{2!} \left[ \left( h_1 \frac{\partial}{\partial Q_1} + h_2 \frac{\partial}{\partial Q_2} + h_3 \frac{\partial}{\partial Q_3} \right)^2 f \right] = \\ &= \frac{1}{2!} \left[ h_1^2 \frac{\partial^2 f}{\partial Q_1^2} + h_2^2 \frac{\partial^2 f}{\partial Q_2^2} + h_3^2 \frac{\partial^2 f}{\partial Q_3^2} + 2h_1 h_2 \frac{\partial^2 f}{\partial Q_1 \partial Q_2} + 2h_1 h_3 \frac{\partial^2 f}{\partial Q_1 \partial Q_3} + 2h_2 h_3 \frac{\partial^2 f}{\partial Q_2 \partial Q_3} \right] \end{aligned} \quad (\text{eqn.8})$$

but were left out of equation 1 all the second order terms of the form  $h_i^2 \frac{\partial^2 f}{\partial Q_i^2}$  because

$$\frac{\partial^2 f}{\partial Q_i^2} = 0.$$

All the third order partial derivatives equal zero and so, also do all the third order and higher terms of the series. The H:D can then be fully expressed by its second order Taylor series transformation with the remainder equalling zero. Equation 1 can be further developed into:

$$\begin{aligned} \text{var}(H : D) = & \frac{1}{n-1} \sum \left( h_1^2 \left( \frac{\partial f}{\partial Q_1} \right)^2 + h_2^2 \left( \frac{\partial f}{\partial Q_2} \right)^2 + h_3^2 \left( \frac{\partial f}{\partial Q_3} \right)^2 + \right. \\ & \left. + 2h_1h_2 \frac{\partial f}{\partial Q_1} \frac{\partial f}{\partial Q_2} + 2h_1h_3 \frac{\partial f}{\partial Q_1} \frac{\partial f}{\partial Q_3} + 2h_2h_3 \frac{\partial f}{\partial Q_2} \frac{\partial f}{\partial Q_3} \right) \end{aligned} \quad (\text{eqn.9})$$

where several terms were left out which can be divided into two groups:

those of the form  $h_i^2 h_j \frac{\partial f}{\partial Q_i} \frac{\partial f}{\partial Q_i \partial Q_j}$  or  $h_i^2 h_j h_k \frac{\partial f}{\partial Q_i \partial Q_j} \frac{\partial f}{\partial Q_i \partial Q_k}$  that were left out

because  $\lim_{n \rightarrow \infty} \sum h_i^2 h_j = 0$  and  $\lim_{n \rightarrow \infty} \sum h_i^2 h_j h_k = 0$  and those of the form

$h_i^2 h_j^2 \left( \frac{\partial f}{\partial Q_1 \partial Q_2} \right)^2$  or  $h_i h_j h_k \frac{\partial f}{\partial Q_i} \frac{\partial f}{\partial Q_j \partial Q_k}$  that were left out because for the present

case was observed that  $\lim_{n \rightarrow \infty} \sum h_i^2 h_j^2 \cong 0$  and  $\lim_{n \rightarrow \infty} \sum h_i h_j h_k \cong 0$ .

Knowing that  $\sum \frac{h_i^2}{n-1} = \text{var}(Q_i)$ , that  $\sum \frac{h_i h_j}{n-1} = \text{cov ar}(Q_i Q_j)$  and from equations 2 to

7, finally equation 9 can be developed to the form

$$\begin{aligned} \text{var}(H : D) = & \\ = & \text{var}(Q_1).(Q_2.Q_3)^2 + \text{var}(Q_2).(Q_1.Q_3)^2 + \text{var}(Q_3).(Q_1.Q_2)^2 + \quad (\text{eqn.10}) \\ & + 2.\text{cov}(Q_1, Q_2).Q_1.Q_2.Q_3^2 + 2.\text{cov}(Q_1, Q_3).Q_1.Q_2^2.Q_3 + 2.\text{cov}(Q_2, Q_3).Q_1^2.Q_2.Q_3 \end{aligned}$$