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Physics with non-unital algebras? *An invitation* to the Okubo algebra

Alessio Marrani[1], **Daniele Corradetti**[2,3,*] **and Francesco Zucconi**[4]

[1] Dep.to de Física, Instituto de Física Teorica, Universidad de Murcia, Campus de Espinardo, E-30100 Murcia, Spain
[2] Departamento de Matemática, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal
[3] Grupo de Física Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
[4] Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Universitá di Udine, Udine 33100, Italy

E-mail: a55499@ualg.pt, alessio.marrani@um.es and francesco.zucconi@uniud.it

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Abstract

This paper presents some preliminary discussion on the possible relevance of the Okubonions, i.e. the real Okubo algebra $\mathcal{O}$, in quantum chromodynamics (QCD). The Okubo algebra lacks a unit element and sits in the adjoint representation of its automorphism group $\mathrm{SU}_{\mathcal{O}}$, thus being fundamentally different from the better-known octonions $\mathbb{O}$. While these latter may represent quarks (and color singlets), the Okubonions are conjectured to represent the gluons, i.e. the gauge bosons of the QCD $\mathrm{SU}(3)$ color symmetry. However, it is shown that the $\mathrm{SU}(3)$ groups pertaining to Okubonions and octonions are distinct and inequivalent subgroups of $Spin(8)$ that share no common $\mathrm{SU}(2)$ subgroup. The unusual properties of Okubonions may be related to peculiar QCD phenomena like asymptotic freedom and color confinement, though the actual mechanisms remain to be investigated.

Keywords: non-associative algebras, Okubo algebra, quantum chromodynamics, representation theory, octonions, Gauge theory

* Author to whom any correspondence should be addressed.

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Introduction

Quantum chromodynamics (QCD) is the theory of the strong interaction between *quarks* mediated by *gluons*, and it is an important part of the *Standard Model of particle physics* (henceforth abbreviated as SM); for a nice recent introduction and a list of references, see e.g. [1, 2]. QCD is a non-Abelian (Yang-Mills) gauge theory, with exact symmetry group $\mathrm{SU}(3)$, whose charge is named *color*, such that the QCD gauge group is notated as $\mathrm{SU}(3)_{\mathrm{color}}$. *Quarks* are fundamental, massive fermions, with spin $1/2$, carrying a color charge in the fundamental representation $\mathbf{3}$ of $\mathrm{SU}(3)_{\mathrm{color}}$, along with a fractional electric charge (either $-1/3$ or $+2/3$). They participate in the weak interactions as part of the weak isospin doublets, and come in six flavours, denoted as u (up), d (down) , c (charm), s (strange), t (top), b (bottom); each type of quark has a corresponding antiquark, whose (color and electric) charges are exactly opposite : e.g. antiquarks transform in the conjugate $\mathrm{SU}(3)_{\mathrm{color}}$ -representation to quarks, denoted $\overline{\mathbf{3}}$. On the other hand, gluons are fundamental, massless bosons, with spin 1, also carrying color charges, but lying in the adjoint representation $\mathbf{8}$ of $\mathrm{SU}(3)_{\mathrm{color}}$; they have no electric charge, and do not participate in the weak interactions.

QCD exhibits a number of counterintuitive, 'weird' features, such as *color confinement* [3] and *asymptotic freedom* [4, 5]. Thus, the following question arises quite naturally : *can some of the features exhibited by QCD be explained by modeling the corresponding fundamental particles, namely quarks and gluons, in terms of some algebra?*

In this work, we will not attempt at answering this intriguing question in a complete and satisfactory way, but we will rather aim at suggesting a further ingredient—which exhibits some quite unique features—within the quest for an algebraic modeling of QCD: the 8-dimensional division algebra of *Okubonions* (*aka* the Okubo algebra) $\mathcal{O}$. This algebra is non-alternative and lacks the unit element; moreover, it has the quite unique feature to sit in the adjoint representation of its automorphism group. It was formally discovered by Petersson in the late'60 s, and then independently rediscovered by Okubo almost ten years later [6, 7] (see also [8] for a review and a list of references). However, Okubonions did not receive much attention by mathematicians (and physicists as well) until quite recently, when they were investigated by Elduque and Myung in a series of works [9–15].

In the present paper, we will proceed by contrasting the properties of the (real) Okubo algebra $\mathcal{O}$ (which is non-unital and non-alternative) to the properties of real (division) octonions $\mathbb{O}$ (*aka* Cayley numbers). This latter is a 8-dimensional non-associative (but alternative and unital) Hurwitz algebra, which has fascinated mathematicians and physicists for decades, and which has recently witnessed a renewed interest and application within the algebraic modeling of SM interactions (see e.g. [16] for a nice review and a list of references).

The plan of the paper is as follows.

In section 1 we review the main properties of all the three 8-dimensional division and composition algebras (over $\mathbb{R}$): the octonions $\mathbb{O}$, the para-octonions $p\mathbb{O}$ (not to be confused with the split-octonions $\mathbb{O}_s$), and the Okubonions $\mathcal{O}$. In section 2 we recall the interpretation of $\mathbb{O}$ in QCD. Then, in section 3 we put forward an interpretation of $\mathcal{O}$ within the same framework, whereas in section 4 we prove a crucial difference between the $\mathrm{SU}(3)$ symmetries pertaining to $\mathbb{O}$ and $\mathcal{O}$. Finally, some conclusions are drawn in section 5.

Before starting, a few general remarks. Since we work over $\mathbb{R}$, we will call the Cayley algebra over the field of real as the 'octonions' $\mathbb{O}$ (rigorously identified by the triplet $(\mathbb{O},\cdot,n)$), and the Okubo algebra $\mathcal{O}$ (rigorously identified by the triplet $(\mathbb{O},*,n)$) over the reals as 'Okubonions' (or simply 'Okubo algebra'). Moreover, we will be using the usual physicists' notation of irreducible representations (irreprs.) of Lie groups and algebras, which identifies an irrepr. with its dimension in bold, along with some further marks if needed.

1. The three 8-dimensional, real, division, composition algebras: $\mathbb{O}$, $p\mathbb{O}$, and $\mathcal{O}$

An algebra A is a vector space endowed with a bilinear product. The specific properties of such bilinear product (which we will denote by $\cdot$) lead to various classifications: an algebra A is said to be*commutative* if $x \cdot y = y \cdot x$ for every $x, y \in A$; is *associative* if satisfies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; is *alternative* if $x \cdot (y \cdot y) = (x \cdot y) \cdot y$; and finally, *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. If the algebra is also equipped with a norm n, then it is a normed algebra. Moreover, if the norm respects the multiplicative structure, i.e. $n(x \cdot y) = n(x)\, n(y)$, the algebra is called a *composition algebra*. Thus, a composition algebra is fully identified by the triple $(A, \cdot, n)$.

Composition algebras split into *unital*, i.e. where exists a unit element 1 such that $x \cdot 1 = 1 \cdot x = x$, *para-unital*, i.e. where exists an involution $x \longrightarrow \overline{x}$ and an element called para-unit $\mathbf{1}$ such that $x \cdot \mathbf{1} = \mathbf{1} \cdot x = \overline{x}$, and *non-unital*, namely algebras that do not possess neither a unit element nor a para-unit element. A complete classification is provided by the Generalized Hurwitz Theorem [15], which states that only sixteen composition algebras exist (e.g.on $\mathbb{R}$) : seven are unital and called *Hurwitz algebras*, including the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ along with their split companions $\mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$; another seven are para-unital, closely related to the Hurwitz algebras and termed *para-Hurwitz algebras*; finally, there are two composition algebras, one division and one split, that are both non-unital and 8-dimensional, known as the *Okubo algebras* $\mathcal{O}$ and $\mathcal{O}_s$ [10].

Out of such sixteen composition algebras, in the present treatment we will be interested only in the 8-dimensional division ones, namely in the following three (mutually non-isomorphic) algebras : the octonions $\mathbb{O}$ (which are alternative and unital), the para-octonions $p\mathbb{O}$ (non-alternative and para-unital) and the Okubo algebra $\mathcal{O}$ (non-alternative and non-unital). Their properties are summarized in table 1.

1.1. The algebra of octonions $\mathbb{O} \equiv (\mathbb{O}, \cdot, n)$

The *octonions* $(\mathbb{O}, \cdot, n)$ are a Hurwitz algebra, and they can be defined as the 8-dimensional real vector space with basis $\{e_0 = 1, e_1, \ldots, e_7\}$, endowed with a bilinear product $\cdot$ encoded through the *Fano plane* and explained in figure 1. The resulting algebra is non-associative and non-commutative, but alternative (and thus flexible).

Given an element $x \in \mathbb{O}$ with decomposition $x = x_0 e_0 + x_1 e_1 + \ldots + x_7 e_7$, one can define its conjugate element as $\overline{x} := x_0 e_0 - x_1 e_1 - \ldots - x_7 e_7$; as in any unital composition algebra, $\overline{x}$ is the image of x under an order-two homomorphism given by a canonical involution, named *conjugation*. The norm n is the obvious Euclidean one, defined by

$$n(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2, \tag{1.1}$$

and, therefore,

$$n(x) = x \cdot \overline{x} = \overline{x} \cdot x, \tag{1.2}$$

as it holds for every Hurwitz algebra. (1.1) yields that $n(x) = 0 \Leftrightarrow x = 0$ and thus the inverse of a non-zero element of the octonions is easily found as $x^{-1} = \overline{x}/n(x)$. Also, the polarization of (1.2) yields the definition of the octonionic inner product as $\langle x, y\rangle = x \cdot \overline{y} + y \cdot \overline{x}$, so that $\langle x, x\rangle = 2n(x)$. Note that $\mathbb{O}$ is a division algebra, since if $\langle x, y\rangle = 0$, then either x or y are zero.

Table 1. Synoptic table of the algebraic properties of octonions $\mathbb{O}$, para-octonions $p\mathbb{O}$ and the real Okubo algebra $\mathcal{O}$.

Property	$\mathbb{O}$	$p\mathbb{O}$	$\mathcal{O}$
Unital	Yes	No	No
Para-unital	Yes	Yes	No
Alternative	Yes	No	No
Flexible	Yes	Yes	Yes
Composition	Yes	Yes	Yes
Automorphism	G_2	G_2	SU(3)

$$\begin{pmatrix} e_0\mapsto 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_4 & e_7 & -e_2 & e_6 & -e_5 & -e_3 \\ e_2 & -e_4 & -1 & e_5 & e_1 & -e_3 & e_7 & -e_6 \\ e_3 & -e_7 & -e_5 & -1 & e_6 & e_2 & -e_4 & e_1 \\ e_4 & e_2 & -e_1 & -e_6 & -1 & e_7 & e_3 & -e_5 \\ e_5 & -e_6 & e_3 & -e_2 & -e_7 & -1 & e_1 & e_4 \\ e_6 & e_5 & -e_7 & e_4 & -e_3 & -e_1 & -1 & e_2 \\ e_7 & e_3 & e_6 & -e_1 & e_5 & -e_4 & -e_2 & -1 \end{pmatrix}$$

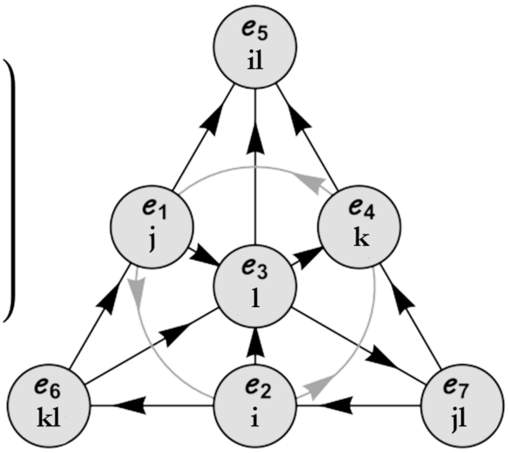

Figure 1. *On the left*: octonionic multiplication tables for the basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. *On the right*: a mnemonic representation on the Fano plane of the same octonionic multiplication rule with the equivalence with the Dickson notation $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{l}, \mathrm{il}, \mathrm{jl}, \mathrm{kl}\}$ according to [19].

By exploiting the unitality of $\mathbb{O}$, the octonionic conjugation can equivalently be defined using the orthogonal projection on the unit element 1 as

$$x \mapsto \overline{x} := \langle x, 1\rangle\, 1 - x. \tag{1.3}$$

This canonical involution has the distinctive property of being an antihomomorphism with respect to the product, i.e. $\overline{x\cdot y} = \overline{y}\cdot\overline{x}$.

1.2. The algebra of para-octonions $p\mathbb{O} \equiv (\mathbb{O}, \bullet, n)$

Starting from the Hurwitz algebra of octonions $(\mathbb{O}, \cdot, n)$, a para-Hurwitz algebra can be introduced by defining a new product

$$x \bullet y = \overline{x}\cdot\overline{y}, \tag{1.4}$$

for every $x, y \in \mathbb{O}$. The new algebra $(\mathbb{O}, \bullet, n)$ is again a composition algebra, in fact a para-Hurwitz algebra, called *para-octonions* and denoted with $p\mathbb{O}$ (not to be confused with the algebra of split-octonions $\mathbb{O}_s$, which is a Hurwitz algebra with zero divisors) [15, 17]. Para-octonions do not have a unit, but only a para-unit, i.e. $\mathbf{1} \in p\mathbb{O}$ such that $\mathbf{1}\bullet x = x\bullet\mathbf{1} = \overline{x}$. Note that $p\mathbb{O}$ is still a division algebra, since if $x \bullet y = 0$, then either $\bar{x}$ or $\bar{y}$ are zero, thus implying that either x or y are zero.

1.3. The Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$: Petersson's and Okubo's realizations

$\mathbb{O}$ also admits an order-three involutive automorphism τ; considering again the basis $\{e_0 = 1, e_1, \ldots, e_7\}$ of $\mathbb{O}$ as a vector space, τ can be defined as follows [17]:

$$\begin{aligned}
&\tau(e_k) = e_k, \qquad\qquad k = 0,1,3,7 \\
&\tau(e_2) = -\frac{1}{2}\left(e_2 - \sqrt{3}e_5\right), \tau(e_4) = -\frac{1}{2}\left(e_4 - \sqrt{3}e_6\right), \\
&\tau(e_5) = -\frac{1}{2}\left(e_5 + \sqrt{3}e_2\right), \tau(e_6) = -\frac{1}{2}\left(e_6 + \sqrt{3}e_4\right).
\end{aligned} \tag{1.5}$$

By starting from the Hurwitz algebra of octonions $(\mathbb{O}, \cdot, n)$, such a map can be exploited in order to obtain a new (non-isomorphic) algebra (belonging to the class of Petersson algebras [18]), whose product (denoted by $*$) is defined as

$$x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}), \tag{1.6}$$

for every $x, y \in \mathbb{O}$. The new algebra $(\mathbb{O}, *, n)$ is again a composition algebra, called the *Okubo algebra* $\mathcal{O}$. This algebra does not have a unit nor a para-unit, but it has idempotent elements. However, $\mathcal{O}$ is still a division algebra, because if $x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}) = 0$, then either $\tau(\bar{x})$ or $\tau^2(\bar{y})$ are zero, and recalling that τ is an automorphism, this implies that either x or y are zero.

An independent realisation of $\mathcal{O}$, equivalent to the above realization *á la Petersson* based on τ, was found by Okubo in the late'70 s, and it is based on a peculiar deformation of the Jordan product. Following [6, 10], $\mathcal{O}$ can indeed also be defined as the set of 3×3 Hermitian traceless matrices over the complex numbers $\mathbb{C}$, endowed with the product

$$x * y = \mu xy + \overline{\mu} yx - \frac{1}{3}\mathrm{Tr}(xy), \tag{1.7}$$

where $\mu = {}^1\!/_3\left(3 + \mathrm{i}\sqrt{3}\right)$ and the juxtaposition denotes the ordinary (non-commutative but associative) matrix product. It should be remarked that $*$ is bilinear, but non-symmetric, with its antisymmetric part proportional to $\mathrm{Im}\mu = \frac{\sqrt{3}}{6}$; as anticipated, $*$ can be regarded as a deformation of the Jordan product $x \circ y = \frac{1}{2}xy + \frac{1}{2}yx$, to which it reduces by setting $\mathrm{Im}\mu = 0 \Leftrightarrow \mu = {}^1\!/_2$ and neglecting the last term in the r.h.s. of (1.7). Nevertheless, the tracelessness property is not preserved under the Jordan product $\circ$, and thus the additional term $-{}^1\!/_3\mathrm{Tr}(xy)$ is actually needed for the closure of the algebra. It is amusing to notice that by simply setting $\mathrm{Im}\mu = 0$ (and retaining the trace term in the r.h.s. of (1.7)), one retrieves the traceless part $\mathfrak{J}_3(\mathbb{C})_0$ of the simple, cubic Jordan algebra $\mathfrak{J}_3(\mathbb{C})$, whose derivation Lie algebra is $\mathfrak{su}(3)$.

Analyzing (1.7), one can realize that the resulting algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$, of real dimension 8, is neither unital, nor associative, nor alternative. However, $\mathcal{O}$ admits idempotent elements, an example of which is

$$e = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1.8}$$

which indeed satisfies $e*e=e$. By setting $e=e_0$ and defining

$$e_1=\sqrt{3}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad e_2=\sqrt{3}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad e_3=\sqrt{3}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$
$$e_4=\sqrt{3}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad e_5=\sqrt{3}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\quad e_6=\sqrt{3}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$
$$e_7=\sqrt{3}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \tag{1.9}$$

one obtains an explicit realization of a basis for $\mathcal{O}$. In this realization, the norm n is defined through the matrix trace (denoted as Tr), namely[5]

$$n(x)=\frac{1}{6}\mathrm{Tr}\left(x^2\right), \tag{1.10}$$

where $x^2\equiv xx$ (i.e. the square of x under the usual matrix product), for every $x\in\mathcal{O}$. It is then easy to see that the (quadratic) norm n defined by (1.10) has Euclidean signature, and furthermore that it is associative and composition over $\mathcal{O}$ itself.

1.4. Transitions among $\mathbb{O}$, $p\mathbb{O}$ and $\mathcal{O}$

As evident by glancing at their rigorous triplet notations, $\mathbb{O}\equiv(\mathbb{O},\cdot,n)$, $p\mathbb{O}\equiv(\mathbb{O},\bullet,n)$ and $\mathcal{O}\equiv(\mathbb{O},*,n)$ are tightly related, as one can easily switch from one to the other by simply changing the definition of the bilinear product over the vector space of the algebra.

The transition $(\mathbb{O},*,n)\longrightarrow(\mathbb{O},\cdot,n)$ can be realized by starting from $\mathcal{O}\equiv(\mathbb{O},*,n)$ and defining the octonionic product $\cdot$ as follows:

$$x\cdot y=(e*x)*(y*e), \tag{1.11}$$

where $x,y\in\mathcal{O}$ and e is any idempotent element of $\mathcal{O}$ (e.g. given by (1.8)). Since $e*e=e$ and $n(e)=1$, the element e acts as a left and right identity, i.e.

$$x\cdot e=e*x*e=n(e)x=x, \tag{1.12}$$
$$e\cdot x=e*x*e=n(e)x=x. \tag{1.13}$$

Moreover, since $\mathcal{O}$ is composition, the same norm n enjoys the relation

$$n(x\cdot y)=n((e*x)*(y*e))=n(x)n(y), \tag{1.14}$$

which means that $(\mathcal{O},\cdot,n)$ is a unital composition algebra of real dimension 8. Since it is also a division algebra, then it must be isomorphic to that of octonions $\mathbb{O}$, as noted by Okubo himself [6, 7] : $(\mathcal{O},\cdot,n)\simeq\mathbb{O}$. On the other hand, the treatment of Okubonions given above explains how $(\mathbb{O},\cdot,n)\longrightarrow(\mathbb{O},*,n)$ can be realized (actually, in a twofold way).

The scenario with para-octonions $p\mathbb{O}$ is also straightforward. The transition $(\mathbb{O},\cdot,n)\longrightarrow(\mathbb{O},\bullet,n)$ has been discussed above (within the treatment of $p\mathbb{O}$ itself), whereas the transition $(\mathbb{O},\bullet,n)\longrightarrow(\mathbb{O},\cdot,n)$ can be realized through the aid of the para-unit $\mathbf{1}\in p\mathbb{O}$, such that

$$x\cdot y=(\mathbf{1}\bullet x)\bullet(y\bullet\mathbf{1})=\overline{x}\bullet\overline{y}. \tag{1.15}$$

[5] It can be proved that all the treatment is independent on the actual explicit expression of the idempotent e.

Table 2. In this table we see how to obtain the Okubonic product $*$, the para-octonionic product $\bullet$ and the octonionic product $\cdot$ from Okubo algebra $(\mathbb{O},*,n)$, para-octonions $(p\mathbb{O},\bullet,n)$ and octonions $(\mathbb{O},\cdot,n)$, respectively.

Algebra	$\mathcal{O} \equiv (\mathbb{O},*,n)$	$p\mathbb{O} \equiv (\mathbb{O},\bullet,n)$	$\mathbb{O} \equiv (\mathbb{O},\cdot,n)$
$x*y$	$x*y$	$\tau(x)\bullet\tau^2(y)$	$\tau(\bar{x})\cdot\tau^2(\bar{y})$
$x\bullet y$	$\tau^2(x)*\tau(y)$	$x\bullet y$	$\bar{x}\cdot\bar{y}$
$x\cdot y$	$(e*x)*(y*e)$	$(\mathbf{1}\bullet x)\bullet(y\bullet\mathbf{1})$	$x\cdot y$

Again, the new algebra $(p\mathbb{O},\cdot,n)$ is a 8 -dimensional composition algebra which is also unital and division; thus, for the Hurwitz theorem, it must necessarily be isomorphic to the octonions $\mathbb{O}$: $(p\mathbb{O},\cdot,n) \simeq \mathbb{O}$.

Finally, since $\tau(\bar{x}) = \overline{\tau(x)}$, the transition $(\mathbb{O},\bullet,n) \longrightarrow (\mathbb{O},*,n)$ is realized by defining the Okubonic product $*$ in terms of the para-octonionic product, by exploiting the order-3 involution τ:

$$x*y = \tau(x)\bullet\tau^2(y), \tag{1.16}$$

and reversely the transition $(\mathbb{O},*,n) \longrightarrow (\mathbb{O},\bullet,n)$ can be realized by defining the para-octonionic product $\bullet$ in terms of the Okubonic one, once again by using τ:

$$x\bullet y = \tau^2(x)*\tau(y). \tag{1.17}$$

Thus, we have shown how all the three 8-dimensional, real, division composition algebras are obtainable one from the other (see table 2 for a summary).

However, it is worth remarking once again that these algebras are not isomorphic.

1.5. Symmetries

Given an algebra $(A,\cdot,n)$, one can consider :

- The special orthogonal group $\mathrm{SO}(A)$, defined as the identity connected component of the group of endomorphisms of A that preserve the norm n (or equivalently the bilinear and symmetric inner product defined through its polarization), but not necessarily the algebraic structure defined by the product[6] $\cdot$:
$$\mathrm{SO}(A) := \left\{\varphi \in \mathrm{End}^0(A) : n(\varphi(x)) = n(x)\right\}. \tag{1.18}$$
As usual, in the subsequent treatment $\mathrm{Spin}(A)$ will denote the spin covering group of $\mathrm{SO}(A)$.
- The automorphism group $\mathrm{Aut}(A)$ is the group of (linear, identity connected) isomorphisms that preserves the algebraic structure :
$$\mathrm{Aut}(A) := \left\{\varphi \in \mathrm{End}^0(A) : \varphi \,\text{bijective, linear, } \varphi(x\cdot y) = \varphi(x)\cdot\varphi(y)\right\}. \tag{1.19}$$
While an element of Aut(A) is also an element of SO(A), the converse is not necessarily true. In fact, in general Aut(A) is a proper subgroup of SO(A):
$$\mathrm{Aut}(A) \subsetneq \mathrm{SO}(A). \tag{1.20}$$
- The triality group is the group of triples of orthogonal transformations of A which respect its algebraic structure:

[6] The upperscript 0 denotes the identity connected component of the Lie group.

Table 3. Symmetries of the octonions $\mathbb{O}$, para-octonions $p\mathbb{O}$ and Okubo algebra $\mathcal{O}$.

	Octonions $\mathbb{O}$	Para-Octonions $p\mathbb{O}$	Okubo $\mathcal{O}$
$\mathrm{Aut}\,(A)$	$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	SU(3)
$\mathrm{SO}\,(A)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$
$\mathrm{Tri}\,(A)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$	$\mathrm{Spin}\,(8)$

$$\mathrm{Tri}\,(A) := \left\{(\alpha,\beta,\gamma) \in \mathrm{SO}\,(A)^{\otimes 3} : \alpha\,(x\cdot y) = \beta\,(x)\cdot\gamma\,(y)\right\}. \tag{1.21}$$

In general, $\mathrm{Tri}(A)$ is a (proper) subgroup of the product group $\mathrm{SO}(A)^{\otimes 3}$, and it contains (or coincides with) $\mathrm{SO}(A)$ itself:

$$\mathrm{SO}\,(A) \subseteq \mathrm{Tri}\,(A) \subsetneq \mathrm{SO}\,(A)^{\otimes 3}. \tag{1.22}$$

The symmetries of $\mathbb{O}$, $p\mathbb{O}$ and $\mathcal{O}$ are well-known (e.g. see [14, 15, 17, 20]) and they are summarized in table 3. Both the appearance of the spin covering group Spin(8) (instead of SO(8)) and the fact that $\mathrm{Tri}(A) \simeq$Spin(8) are due to the triality of $\mathfrak{d}_4$ [20–22].

2. $(\mathbb{O},\cdot,n)$ and QCD

2.1. Symmetries

Tautologically, any algebra A sits in a linear (not necessarily irreducible) representation of its (linearly realized) automorphism Lie group, whose Lie algebra is the algebra of derivations, i.e. $\mathfrak{Der}\,(A) = \mathfrak{Lie}\,(\mathrm{Aut}\,(A))$. Considering $(\mathbb{O},\cdot,n)$, table 3 reports that

$$\mathrm{Aut}\,(\mathbb{O}) = \mathrm{G}_{2(-14)}, \tag{2.1}$$
$$\mathrm{Tri}\,(\mathbb{O}) = \mathrm{Spin}\,(8), \tag{2.2}$$
$$\mathrm{Spin}\,(\mathbb{O}) = \mathrm{Spin}\,(8), \tag{2.3}$$

and it holds that

$$\mathbb{O} \simeq \mathbf{1}\oplus\mathbf{7} \text{ of } \mathrm{G}_{2(-14)}. \tag{2.4}$$

Both Spin(8) and $\mathrm{G}_{2(-14)}$ have real representations [23]. The relation among these Lie groups is given by the following chain of maximal group embeddings:

$$\begin{array}{ccccc} \mathrm{Spin}\,(8) & \supset_{\mathrm{s}} & \mathrm{Spin}\,(7) & \supset_{\mathrm{ns}} & \mathrm{G}_{2(-14)} \\ \mathbf{8}_v & = & \mathbf{7}\oplus\mathbf{1} & = & \mathbf{7}\oplus\mathbf{1} \\ \mathbf{8}_s & = & \mathbf{8} & = & \mathbf{7}\oplus\mathbf{1} \\ \mathbf{8}_c & = & \mathbf{8} & = & \mathbf{7}\oplus\mathbf{1} \\ \mathbf{28} & = & \mathbf{21}\oplus\mathbf{7} & = & \mathbf{14}\oplus\mathbf{7}\oplus\mathbf{7}, \end{array} \tag{2.5}$$

where the subscripts 's' and 'ns' of $\subset$ respectively stand for 'symmetric' and 'non-symmetric' (embedding), and, as mentioned at the end of the Introduction, we recall that the physicists' notation of representations of Lie algebras and groups is used throughout this paper. In particular, $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ are the three eight-dimensional irreducible representations of *Spin*(8), namely the vector, (semi)spinor and conjugate (semi)spinor, respectively, whereas **28** denotes the adjoint irrepr. of *Spin*(8) itself.

We remark that the unital nature of $\mathbb{O}$ is related to the fact that Aut($\mathbb{O}$) is a (maximal) subgroup of Spin(7), and that one (say[7], $\mathbf{8}_v$) of the three 8-dimensional representations of Spin(8) contains a singlet (namely, the unit 1 of $\mathbb{O}$) when branched with respect to Spin(7); in fact, (2.5) defines a choice of 'polarization' within the triality symmetry of Spin(8) (see e.g. [25]). The existence of a unity implies the following decomposition of $\mathbb{O}$:

$$\mathbb{O} = \mathbb{R} \oplus \mathbb{O}', \tag{2.6}$$

where $\mathbb{O}' := \text{span}_{\mathbb{R}}(e_1, \ldots, e_7)$ denotes the *imaginary* octonions, whose basis is provided by the seven octonionic units $e_1, \ldots, e_7$, with multiplication rules given by a suitable realization of the discrete geometric of the Fano plane, as shown in figure 1.

2.2. $SU(3)_{\mathbb{O}} \subsetneq Aut(\mathbb{O})$

The automorphism group of the octonions Aut$(\mathbb{O})$ is the real compact form of the exceptional Lie group G_2, i.e. Aut($\mathbb{O}$) $= G_{2(-14)}$. The Lie group $G_{2(-14)}$ admits three maximal subgroups, two of Borel-de Siebenthal type, i.e. of maximal rank [26], namely SU(3) and SO(4) $\simeq$ SU(2) $\times$ SU(2), and one of non-maximal rank, SU(2). In the present treatment, we are interested in the[8] SU(3) maximal subgroup of $G_{2(-14)}$, which we will be denoting as $SU(3)_{\mathbb{O}}$, and in its representations stemming from the decomposition of the defining and adjoint irreprs. of $G_{2(-14)}$ itself:

$$\begin{array}{ccc} G_{2(-14)} & \supset_{\text{ns}} & SU(3)_{\mathbb{O}} \\ \mathbf{7} & = & \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}, \\ \mathbf{14} & = & \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}. \end{array} \tag{2.7}$$

Despite the fact that all representations of $G_{2(-14)}$ are real, SU(3) (and thus $SU(3)_{\mathbb{O}}$ defined in (2.7)) admits complex (i.e. reflexive) representations [23]. In the case under consideration, this implies the existence of a complex structure J in the exceptional (and non-symmetric) presentation S^6_J of the 6-sphere with isotropy group $SU(3)_{\mathbb{O}}$ (see e.g. [27] and references therein),

$$S^6_J \simeq \frac{G_{2(-14)}}{SU(3)_{\mathbb{O}}}. \tag{2.8}$$

Thus, Aut($\mathbb{O}$) can also be presented as

$$G_{2(-14)} \simeq SU(3)_{\mathbb{O}} \ltimes S^6_J. \tag{2.9}$$

There are seven maximal non-symmetric subgroups $SU(3)_{\mathbb{O}}$ of $G_{2(-14)}$, defined as invariance symmetries of one of the seven octonionic imaginary units; they are all equivalent under inner automorphisms of $G_{2(-14)}$ itself, and they all correspond to the following $SU(3)_{\mathbb{O}}$-covariant breaking of $\mathbb{O}$:

$$\begin{array}{ccc} G_{2(-14)} & \supset & SU(3)_{\mathbb{O}} \\ \mathbf{1} \oplus \mathbf{7} & = & \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}, \\ \mathbb{O} = \mathbb{R} \oplus \mathbb{O}' & = & \mathbb{C} \oplus \mathbb{C}^3, \end{array} \tag{2.10}$$

[7] This depends on the choice of 'spinor polarization' of the Spin(7) subalgebra [24] , but it is immaterial, due to $\mathfrak{d}_4$ triality symmetry.

[8] By SU(3) we denote the compact real form of the Lie group whose Lie algebra is $\mathfrak{a}_2$, which is also the maximal compact subgroup of the Lie group $SL(3, \mathbb{C})_{\mathbb{R}}$.

where one can identify

$$\mathbb{C} \simeq \mathbb{R} \oplus J\mathbb{R}, \tag{2.11}$$

with J denoting an arbitrary but fixed octonionic imaginary unit, i.e. $J \in \{e_1, \ldots, e_7\}$, which is $\mathrm{SU}(3)_{\mathbb{O}}$-invariant.

2.3. Application to QCD

The physical interpretation of the maximal subgroup $SU(3)_{\mathbb{O}}$ as the *color* gauge group $SU(3)_{\text{color}}$ of the strong interaction, i.e.

$$SU(3)_{\text{color}} \equiv SU(3)_{\mathbb{O}}, \tag{2.12}$$

has a long history, dating back to Gürsey *et al* [28–31] and Morita [32–34] between mid'70 s and early'80 s, and more recently revived by Bisht, Chanyal *et al* [35–37] and Wolk [38]. In a broader framework, the algebraic formulation of the particle content and the gauge symmetries of the SM and beyond, crucially involving the octonions and related (exceptional) algebraic and geometric structures, has been the object of a number of works along the years; without any claim of completeness of our list, here we confine ourselves to cite Dixon [39], Furey [40], Hughes [41, 42], Dray, Manogue and Wilson [43–45], Boyle [46], Krasnov [47], Todorov and Dubois-Violette [48], Singh [49, 50], Penrose [51], Castro [52], Rowlands [53], Masi [54], and two of the present authors with Chester, Aschheim and Irwin [55].

By virtue of (2.10), the identification (2.12) implies $\mathbb{O}$ to correspond to a color triplet (quark q) and its conjugate anti-color anti-triplet (anti-quark $\overline{q}$), plus two color singlets (whose possible identification will not be discussed here). Thus, the decomposition (2.10) can be interpreted in QCD as follows :

$$\begin{array}{ccc} \mathrm{G}_{2(-14)} & \supset & \mathrm{SU}(3)_{\mathbb{O}} \\ \mathbb{O} = \mathbf{1} \oplus \mathbf{7} & = & \underbrace{\mathbf{1} \oplus \mathbf{1}}_{\text{pair of QCD singlets}} \oplus \underset{q}{\mathbf{3}} \oplus \underset{\overline{q}}{\overline{\mathbf{3}}}, \end{array} \tag{2.13}$$

with q being any of the six quark flavours. Therefore, within this interpretation, a pair of $\mathrm{Tri}(\mathbb{O}) \simeq \mathrm{Spin}(8)$ -covariant (and $\mathfrak{d}_4$-triality-invariant) triplets of reciprocally independent octonions contains the color states of *all* quarks and antiquarks (namely, a $\mathbf{3} \oplus \overline{\mathbf{3}}$ for every quark flavour $q = u, d, c, s, t, b$) of the SM, plus a total of 12 color singlets[9]:

$$\{\mathbb{O}_{Av}, \mathbb{O}_{As}, \mathbb{O}_{Ac}\}_{A=1,2} = \underset{6\cdot(\mathbf{3}\oplus\overline{\mathbf{3}}) = \text{quark\&antiquark color states}}{\{u, \overline{u}, d, \overline{d}, c, \overline{c}, s, \overline{s}, t, \overline{t}, b, \overline{b},\}} \oplus \underset{12 \text{ color singlets}}{12 \cdot \mathbf{1}}. \tag{2.14}$$

On the other hand, the decomposition of the adjoint representation under the same embedding as (2.13), given by the second line of (2.7), suggests that the gauge group $\mathrm{SU}(3)_{\text{color}}$, within the interpretation (2.12), could be enhanced to the smallest exceptional Lie group, $\mathrm{G}_{2(-14)}$. This is something that physicists have tried for a long time, in their quest for a comprehensive Grand Unified Theory (GUT), finding an insurmountable obstruction in the fact that, as mentioned above, $\mathrm{G}_{2(-14)}$ has no complex representations, and thus cannot properly account for the SM fermions. Thus, despite recent attempts (see e.g. [54], and references therein), no consistent GUT with gauge group $\mathrm{G}_{2(-14)}$ can be formulated.

[9] We cannot help but observe that this formula seems to provide further evidence for the relevance of $\mathfrak{d}_4$-triality for the existence of three generations of matter fields in the SM, an intriguing research venue which has a quite long story [56–58].

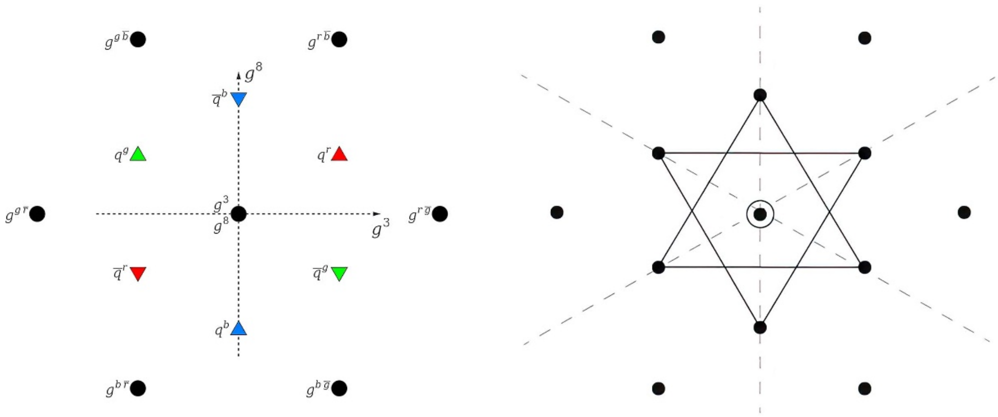

Figure 2. *On the left* : strong force symmetry pattern, pertaining to the **3** color states of a quark q, and the corresponding $\overline{\mathbf{3}}$ states of its anti-quark $\overline{q}$, along with the **8** gluon color states (with g^3 and g^8 coinciding in the center of the diagram, taken from [59]). *On the right* : 'Magic Star' projection of the root lattice of $\mathfrak{g}_{2(-14)}$ [60, 61], with the plane of the sheet being defined by the two Cartan generators of $\mathfrak{g}_{2(-14)}$ itself. Despite the fact that the Cartans of $\mathfrak{g}_{2(-14)}$ (on the right) and of $\mathfrak{su}(3)$ (on the left) coincide, the strong force pattern in the l.h.s. cannot be interpreted as the 'Magic Star' projection of $\mathfrak{g}_{2(-14)}$, and vice versa. This is ultimately due to the fact that the Lie groups $\mathrm{SU}(3)_{\mathcal{O}}$ (pertaining to the l.h.s) and $\mathrm{SU}(3)_{\mathbb{O}}$ (pertaining to the r.h.s.) are totally different subgroups of $\mathrm{SO}(\mathbb{O}) \simeq \mathrm{Spin}(8) \simeq \mathrm{SO}(\mathcal{O})$; see (4.3), and the discussion at the end of section 5.

Nevertheless, it is instructive, also in view of the treatment below, to report the pattern of strong charges for

- the three colors of a quark q in the irrepr. **3** of $\mathrm{SU}(3)_{\mathrm{color}}$;
- three anti-colors of the corresponding anti-quark $\overline{q}$ in the corresponding, conjugate irrepr. $\overline{\mathbf{3}}$ of $\mathrm{SU}(3)_{\mathrm{color}}$;
- eight gluons in the adjoint irrepr. $\mathbf{Adj} \equiv \mathbf{8}$ of $\mathrm{SU}(3)_{\mathrm{color}}$.

This pattern is reported in the left side of figure 2 with the two gluons g^3 and g^8 of zero color charge overlapping in the center of the pattern, and corresponding to the Cartan generators of $\mathfrak{g}_{2(-14)}$ or, equivalently, of $\mathfrak{su}(3)$. In this pattern, of course q can take any value $q \in \{u, d, c, s, t, b\}$.

3. $\mathcal{O}$ and QCD

3.1. Symmetries

As reported in table 3, the Okubo algebra $\mathcal{O} \equiv (\mathbb{O}, *, n)$ is characterized by the following symmetries:

$$\mathrm{Aut}(\mathcal{O}) = \mathrm{SU}(3) \equiv \mathrm{SU}(3)_{\mathcal{O}}, \tag{3.1}$$

$$\mathrm{Tri}(\mathcal{O}) = \mathrm{Spin}(8), \tag{3.2}$$

$$\mathrm{Spin}(\mathcal{O}) = \mathrm{Spin}(8), \tag{3.3}$$

where we introduced $\mathrm{SU}(3)_{\mathcal{O}}$ as the automorphism group $\mathrm{Aut}(\mathcal{O})$, in order to distinguish it from the $\mathrm{SU}(3)_{\mathbb{O}}$ introduced in (2.7). It also holds that

$$\mathcal{O} \simeq \mathbf{8} \equiv \mathbf{Adj} \text{ of } \mathrm{SU}(3)_{\mathcal{O}}. \tag{3.4}$$

3.2. $SU(3)_{\mathcal{O}} = Aut(\mathcal{O})$

Spin(8) and[10] $\mathrm{SU}(3)_{\mathcal{O}}$ are related by a maximal (and non-symmetric) embedding, under which the defining and the adjoint irrepr. of Spin(8) decompose as follows:

$$\begin{array}{ccc} \mathrm{Spin}(8) & \supset_{\mathrm{ns}} & \mathrm{SU}(3)_{\mathcal{O}} \\ \mathbf{8}_v & = & \mathbf{8} \\ \mathbf{8}_s & = & \mathbf{8} \\ \mathbf{8}_c & = & \mathbf{8} \\ \mathbf{28} & = & \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}, \end{array} \tag{3.5}$$

where $\mathbf{10} \equiv S^3\mathbf{3}$ is the rank-3 symmetric irrepr. of $\mathrm{SU}(3)_{\mathcal{O}}$. The maximal embedding (3.5) exists by virtue of a Theorem of Dynkin (see e.g. theorem 1.5 of [62]), and it is a consequence of the existence of the Cartan-Killing bilinear invariant form in the adjoint irrepr. **8** of the Lie group $\mathrm{SU}(3)_{\mathcal{O}}$.

We stress that the non-unital nature of $\mathcal{O}$ is deeply linked to the fact that $\mathrm{Aut}(\mathcal{O})$ is a *maximal* subgroup of Spin(8), and *not* a subgroup of Spin(7). The lack of a unity implies that $\mathcal{O}$ is *irreducible* under the action of its automorphism group $\mathrm{SU}(3)_{\mathcal{O}}$, as expressed by (3.4). Thus, differently from the (unique) unit element 1 in any unital algebra (which is invariant under the automorphism group of the unital algebra itself), in the basis of the Okubonic 8-dimensional vector space (whose a possible explicit realization is given by(1.9)), the (not unique, but rather threefold[11]) idempotent e is *not* invariant under $\mathrm{Aut}(\mathcal{O})$, but instead it corresponds to one generator inside the adjoint representation of $\mathrm{Aut}(\mathcal{O})$ itself.

Before discussing the possible relevance of Okubonions in QCD, we would like to comment that, intuitively, a unital algebra (such as $\mathbb{O} \equiv (\mathbb{O}, \cdot, n)$) might turn out to be more amenable at describing theories and phenomena which admit a perturbative description, whereas a non-unital algebra (such as $\mathcal{O} \equiv (\mathbb{O}, *, n)$) might result in a more consistent algebraic modeling of non-perturbative regimes and physical phenomena. For this reason, we will not be dealing with the seamless unital extension of $\mathcal{O}$, defined by $\mathcal{O}^+ := \mathcal{O} \oplus 1$, since this is spoiled of the peculiar property of non-unitality, and, for instance, due to the traceful nature of the unit element, it does *not* admit a realization as $\mathfrak{J}_3(\mathbb{C})_0$ (with suitable deformation of the Jordan product *á la Michel-Radicati*) anymore.

3.3. Application to QCD

Hinted by (3.1), we cannot help but put forward a conjectural physical interpretation of the Okubonic automorphism group $\mathrm{SU}(3)_{\mathcal{O}}$ (whose relation with (2.12) will be investigated below) as the *color* gauge group $\mathrm{SU}(3)_{\mathrm{color}}$ of the strong interaction within the SM, i.e.

$$\mathrm{SU}(3)_{\mathrm{color}} \equiv \mathrm{SU}(3)_{\mathcal{O}}. \tag{3.6}$$

[10] $SU(3)_{\mathcal{O}}$ admits two maximal subgroups, one of Borel—de Siebenthal (i.e. maximal rank) type, i.e. U(2), and one of non-maximal rank, i.e. SU(2). Their treatment will be considered elsewhere.

[11] From the very definition of $\mathcal{O}$ as the algebra of 3×3 traceless Hermitian matrices over $\mathbb{C}$ (namely, as $\mathfrak{J}_3(\mathbb{C})_0$) with a suitable deformation $*$ of the Michel-Radicati product [7], it follows that there are three independent idempotents in $\mathcal{O}$, which are however all equivalent under the order -three involutive automorphism τ of $\mathcal{O}$ (cfr (1.5)) [14, 63].

By virtue of (3.4) and (3.6) implies $\mathcal{O}$ to represent the octet of the *gluons*$\{g^i\}_{i=1,\ldots,8}$, namely of the gauge vector bosons mediating the strong interaction[12] in QCD :

$$\mathcal{O} \equiv (\mathbb{O}, *, n) \simeq \underset{\{g^i\}_{i=1,\ldots,8}}{\mathbf{8}} \equiv \mathbf{Adj} \text{ of } \mathrm{SU}(3)_{\mathcal{O}}. \tag{3.7}$$

Thus, there is a 1 : 1 correspondence between the elements of the basis of the Okubonic 8-dimensional vector space (whose a possible explicit realization is given by(1.9)) and the gluons $\{g_i\}_{i=1,\ldots,8}$ of QCD; this is consistent with the physical picture in which, since the color symmetry (3.6) is exact and unbroken, *all gluons are massless and stand on the same footing, color-wise*. The gluons are indeed in 1 : 1 correspondence with the generators of $\mathrm{SU}(3)$ or, realization-wise, with the eight Gell–Mann matrices λ^i's (see e.g. [64], pp 283–288 and 366–369 therein).

However, it is at this stage worth stressing the conjectural nature of the identification (3.6), which deserves further work and investigation. Nonetheless, it is suggestive to wonder whether any feature of QCD, such as asymptotic freedom and color confinement, might be traced back to the unusual properties of $\mathcal{O}$ itself, such as the non-alternativity and the absence of a unit element (i.e. non-unitality), despite $\mathcal{O}$ is still a division algebra.

4. $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ as different subgroups of Spin(8)

We have put forward a suggestive physical interpretation of the (real) Okubo algebra $\mathcal{O}$ in (an algebraic model of) QCD, given by (3.7) (within the identification (3.6)). By recalling its octonionic counterparts, i.e. (2.13) (within the identification (2.12)), it then seems that $\mathbb{O}$ and $\mathcal{O}$ have complementary[13] physical interpretations in QCD (namely, quarks/fermions from $\mathbb{O}$ *versus* gluons/bosons from $\mathcal{O}$).

In order to elucidate the relation between the physical identifications (2.12) and (3.6), in this section we will show that the two corresponding SU(3), namely $\mathrm{SU}(3)_{\mathbb{O}}$ and $\mathrm{SU}(3)_{\mathcal{O}}$, do *not* coincide, but they are rather totally different. This is a consequence of the different embeddings of $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ in $\mathrm{SO}(\mathbb{O}) \simeq \mathrm{Spin}(8) \simeq \mathrm{SO}(\mathcal{O})$. Indeed, from (2.5), (3.5) and (2.7), one can draw the following chains of maximal group embeddings from Spin(8) to $\mathrm{SU}(3)_{\mathcal{O}}$,

$$\begin{array}{ccc} \mathrm{Spin}(8) & \supset & \mathrm{SU}(3)_{\mathcal{O}} \\ \mathbf{8}_v & = & \mathbf{8} \\ \mathbf{8}_s & = & \mathbf{8} \\ \mathbf{8}_c & = & \mathbf{8} \\ \mathbf{28} & = & \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}, \end{array} \tag{4.1}$$

[12] It is here worth remarking that, within the physical interpretation (3.6), any $\mathrm{SU}(2)$ or $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(3)_{\mathcal{O}}$ *cannot* be regarded as the weak or electromagnetic gauge group, of course.

[13] Such a 'physical complementarity', corresponding to switching between the matter and the gauge sectors of QCD, is actually realized by the definition of alternative product within the vector space of the algebra (while keeping the norm n invariant); in fact, from its very definition, $\mathcal{O} \equiv (\mathbb{O}, *, n)$ (see section 1), and $\mathbb{O} \equiv (\mathcal{O}, \cdot, n)$ (cf the discussion below (1.14)).

and to $\mathrm{SU}(3)_{\mathbb{O}}$,

$$\begin{array}{ccccccc} \mathrm{Spin}(8) & \supset & \mathrm{Spin}(7) & \supset & G_{2(-14)} & \supset & \mathrm{SU}(3)_{\mathbb{O}} \\ \mathbf{8}_v & = & \mathbf{7}\oplus\mathbf{1} & = & \mathbf{7}\oplus\mathbf{1} & = & \mathbf{3}\oplus\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1} \\ \mathbf{8}_s & = & \mathbf{8} & = & \mathbf{7}\oplus\mathbf{1} & = & \mathbf{3}\oplus\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1} \\ \mathbf{8}_c & = & \mathbf{8} & = & \mathbf{7}\oplus\mathbf{1} & = & \mathbf{3}\oplus\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1} \\ \mathbf{28} & = & \mathbf{21}\oplus\mathbf{7} & = & \mathbf{14}\oplus 2\cdot\mathbf{7} & = & \mathbf{8}\oplus 3\cdot\mathbf{3}\oplus 3\cdot\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1}. \end{array} \tag{4.2}$$

Comparing the branchings of the $\mathbf{Adj}\equiv\mathbf{28}$ of Spin(8) along the chains (4.1) and (4.2), one immediately realizes that

$$\mathrm{SU}(3)_{\mathcal{O}}\neq\mathrm{SU}(3)_{\mathbb{O}}. \tag{4.3}$$

Even more interestingly, $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ do *not* even share a common $\mathrm{SU}(2)$ subgroup. This can be proved as follows. By continuing the chain (4.1) to the maximal subgroups $\mathrm{SU}(2)\times\mathrm{U}(1)$ resp. $\mathrm{SU}(2)$ of $\mathrm{SU}(3)_{\mathcal{O}}$, one respectively obtains

$$\begin{array}{l} \mathrm{Spin}(8)\supset\mathrm{SU}(3)_{\mathcal{O}}\supset\mathrm{SU}(2)\times\mathrm{U}(1), \\ \mathbf{28}=\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}=3\cdot\mathbf{3}_0\oplus 2\cdot\mathbf{2}_3\oplus 2\cdot\mathbf{2}_{-3}\oplus\mathbf{4}_3\oplus\mathbf{4}_{-3}\oplus\mathbf{1}_0\oplus\mathbf{1}_{-6}\oplus\mathbf{1}_6; \end{array} \tag{4.4}$$

$$\begin{array}{l} \mathrm{Spin}(8)\supset\mathrm{SU}(3)_{\mathcal{O}}\supset\mathrm{SU}(2), \\ \mathbf{28}=\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}=3\cdot\mathbf{3}\oplus\mathbf{5}\oplus 2\cdot\mathbf{7}. \end{array} \tag{4.5}$$

On the other hand, by continuing the chain (4.2) to the maximal subgroups $\mathrm{SU}(2)\times\mathrm{U}(1)$ resp. $\mathrm{SU}(2)$ of $\mathrm{SU}(3)_{\mathbb{O}}$, one respectively obtains

$$\begin{array}{l} \mathrm{Spin}(8)\supset\ldots\supset\mathrm{SU}(3)_{\mathbb{O}}\supset\mathrm{SU}(2)\times\mathrm{U}(1), \\ \mathbf{28}=\ldots=\mathbf{8}\oplus 3\cdot\mathbf{3}\oplus 3\cdot\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1}=\mathbf{3}_0\oplus\mathbf{2}_3\oplus\mathbf{2}_{-3}\oplus 3\cdot(\mathbf{2}_1\oplus\mathbf{2}_{-1})\oplus 3\cdot(\mathbf{1}_0\oplus\mathbf{1}_{-2}\oplus\mathbf{1}_2); \end{array} \tag{4.6}$$

$$\begin{array}{l} \mathrm{Spin}(8)\supset\ldots\supset\mathrm{SU}(3)_{\mathbb{O}}\supset\mathrm{SU}(2), \\ \mathbf{28}=\ldots=\mathbf{8}\oplus 3\cdot\mathbf{3}\oplus 3\cdot\overline{\mathbf{3}}\oplus 2\cdot\mathbf{1}=7\cdot\mathbf{3}\oplus\mathbf{5}\oplus 2\cdot\mathbf{1}. \end{array} \tag{4.7}$$

The comparison of the final breakings (4.4)–(4.7) implies that $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$, as subgroups of the same Spin(8) group, do *not* even share a common $\mathrm{SU}(2)$ subgroup, *q.e.d.*□.

Thus, (4.3) implies that *no Okubonic/gluonic interpretation of* $\mathfrak{su}(3)_{\mathbb{O}}$ (as found in (4.2)) is possible, because $\mathrm{SU}(3)_{\mathcal{O}}$ and $\mathrm{SU}(3)_{\mathbb{O}}$ are different, inequivalent subgroups of Spin(8). This implies that the QCD pattern reported in the l.h.s. of figure 2, which is intrinsically gluonic/Okubonic (according to the interpretation (3.6), *cannot* enjoy an algebraic interpretation as the 'Magic Star' decomposition of $\mathfrak{g}_{2(-14)}$, represented in the r.h.s. of figure 2, which is instead intrinsically pertaining to quarks/octonions (according to the interpretation (2.12)). This is ultimately due to (4.3). For the same reason, *no octonionic/quark interpretation of* $\mathfrak{su}(3)_{\mathcal{O}}$ (as found in (4.1)) is possible, of course.

5. Conclusions

In this work, for the first time to the best of our knowledge, a physical interpretation of the real[14], non-unital, non-alternative, composition and division algebra of the Okubonions

[14] The present paper only assumes $\mathbb{R}$ as ground field. We leave the treatment on $\mathbb{C}$ and over finite fields $\mathbb{F}_p$ for future investigation.

$\mathcal{O} \equiv (\mathbb{O}, *, n)$ has been put forward, within (the attempts at elaborating an algebraic model of) QCD. As investigated in [63, 65, 66], many algebraic and geometric structures, such as the the octonionic entries of the Tits-Freudenthal Magic Square and the Cayley plane, can be constructed by resorting to $\mathcal{O}$ only, without ever employing the algebra of the octonions.

Both $\mathbb{O}$ and $\mathcal{O}$ are non-associative, flexible, composition and division, real 8-dimensional algebras. Such two algebras are however very different: for instance, $\mathbb{O}$ is alternative and unital (Hurwitz), whereas $\mathcal{O}$ is non-alternative and non-unital (only admitting idempotent elements, though). Moreover, the Lie group $\text{Aut}(\mathcal{O}) = \text{SU}(3)_{\mathcal{O}}$ is much smaller than (and not a subgroup, albeit with the same rank, of) the exceptional Lie group $\text{G}_{2(-14)} = \text{Aut}(\mathbb{O})$. More precisely, $\text{Aut}(\mathcal{O}) = \text{SU}(3)_{\mathcal{O}}$ and $\text{Aut}(\mathbb{O}) = \text{G}_{2(-14)}$ are totally different, rank-2 subgroups of $\text{SO}(\mathbb{O}) \simeq \text{Spin}(8) \simeq \text{SO}(\mathcal{O})$: $\text{SU}(3)_{\mathcal{O}}$ is *maximal* and non-symmetric in $\text{Spin}(8)$ (as given by the embedding (4.1)), whereas $\text{SU}(3)_{\mathbb{O}}$ is *next-to-maximal* (and non-symmetric) in $\text{Spin}(8)$ (as given by (4.2)). We find this fact quite tantalizing, and possibly paving the way to surprising developments. Indeed, as shown in [67] and very recently in [63], by using the Okubonions one can obtain a number of exceptional structures, such as the cubic exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ (*aka* Albert algebra) or various real forms of the exceptional Lie groups of type F_4, E_6 and even E_7 and E_8, without employing the octonions $\mathbb{O}$, nor G_2, at all.

Having proved that the $\text{SU}(3)$ Lie groups pertaining to $\mathbb{O}$ and $\mathcal{O}$ are different, inequivalent subgroups of $\text{Spin}(8)$, we hope to have been able in this paper to convey the message that the uses of such composition algebras within the algebraic modeling of the strong interaction in the SM are deeply different one from the other, and, in a sense, complementary. In this regard, we would like to remark that, in the perspective of an algebraic modeling of the strong interaction, $\mathcal{O}$ may provide a more suitable algebraic structure, with a remarkably smaller number of generators than $\mathbb{O}$ itself. Indeed, $\text{Aut}(\mathcal{O}) = \text{SU}(3)$, thus possibly matching the exact unbroken gauge symmetry of QCD (within the conjectural interpretation (3.6)), without the need to introduce further (namely, six) massive gauge bosons corresponding to the generators of the (non-symmetric) coset $\text{G}_{2(-14)}/\text{SU}(3)$, as implied in G_2-based GUT extensions of the SM.

Moreover, it may well be that some 'weird', unusual properties (such as non-alternativity and non-unitality) of real Okubonions $\mathcal{O}$ might be related to peculiar QCD phenomena like asymptotic freedom and color confinement (analogously to what has been discussed for octonions in [37]), though the actual mechanisms remain to be investigated.

Before concluding this work, we would like to stress once again the conjectural nature of the identification (3.6), which surely deserves further work, e.g. in relation to the consistency of 'Okubonic fields', whose investigation seemingly calls for the Okubonic analogue of the treatment given for the octonions in [32–34]. Also, at present it is not known whether a representation of the Poincaré group can be found within the Okubo algebra, and whether this is consistent with the spin-1 (gluonic) interpretation put forward by (3.7). We leave these intriguing issues for further future work.

We would like to conclude by citing Susumu Okubo in the introduction of his book [8], dating back to 1995: '*Octonion algebra may surely be called a beautiful mathematical entity. Nevertheless, it has never been systematically utilized in physics in any fundamental fashion, although some attempts have been made toward this goal. However, it is still possible that non-associative algebras (other than Lie algebras) may play some essential future role in the ultimate theory, yet to be discovered.*' While there have been many advances on the physical applications of octonions since the publication of Okubo's book, non-associative algebras may still be a key player in the formulation of a ultimate theory of Physics. While octonions have been (with very good reason) dubbed *'the strangest numbers in string theory'* [68], in the present paper we have suggested a fierce (possibly even stranger!) opponent : the Okubonions.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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ORCID iD

Daniele Corradetti https://orcid.org/0000-0001-8086-0593

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