NON-LINEAR FINITE ELEMENT ANALYSIS OF THIN COMPOSITE STRUCTURES

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SUMMARY: A non-conforming three-node triangular finite element with 18 degree of freedom , is used in conjugation with the Kirchhoff theory for the non-linear analysis of thin composite plate-shell structures. The formulation of the geometrically non-linear analysis is based on an updated Lagrangian formulation associated with the Newton-Raphson iterative technique, which incorporates an automatic arc-length control procedure. Several illustrative test cases of thin and multilaminated plate-shell structures are analysed .The results of large displacements analyses are compared with alternative numerical solutions. From the applications a good accuracy has been found. The versatility of the element to model arbitrary thin structures and the simplicity of the formulation makes it a very computational efficient numerical tool.

KEYWORDS: non-linear analysis, finite elements, multilaminate shells

INTRODUCTION

In the recent years , the use of laminated composite materials has increased in a variety of engineering structures , due to their high strength to weight and high stiffness to weight ratios, which can be tailored through the variation of fibre orientation and lamination sequence.

The objective of this paper is to present the formulation, implementation and illustrative applications of a simple versatile finite element model, based on a flat multilaminated triangular element by employing the superposition of in-plane and bending effects to obtain the stiffness matrix. To carry out the large displacement analysis an updated Lagrangian formulation [1 ,2] has been implemented. This formulation is used in association with the Newton-Raphson incremental-iterative method having an automatic arc-length load control to contemplate general shell analysis [3].

The analytical formulation is based on Kirchhoff plate theory , applied to the geometrical non-linear analysis, considering anisotropy of the structure , with symmetric and asymmetric lay-ups, enabling coupling between membrane and bending and twisting.

The model is developed, through the generalisation of a pioneering non-conforming plate triangular element with 3 degrees of freedom per node, used on linear statics, free vibration and buckling analyses of isotropic and unilayered orthotropic plates [4,5,6]. The present multilaminated plate-shell element has 6 degrees of freedom per node, three displacements and three rotations, which includes an additional degree of freedom to enable the analysis of general structures, as an assembly of flat elements [7]. Related with the present study, but for isotropic shell structures, a comparative study has been carried out by Talbot and Dhatt [8] to assess flat triangular elements. Recently, Madenci and Barut [9], carried out a brief review on non-linear analysis of multilaminated shell structures. They developed a flat shell triangular element based on the concept of free formulation, using a co-rotational form of the updated Lagrangian formulation. Numerical validation problems of anisotropic plate/shell structures with symmetric and non-symmetric lay-ups are discussed to assess the effect of loading, geometry and material properties.

The performance of the present model is established through validation problems involving large deflections. The present results are compared with alternative numerical solutions [9,11,12] .

Displacements and Strains

The classical Kirchhoff Theory is considered. The displacement components (Figure 1) at a generic point in the laminate are assumed to be of the form :

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\,\theta_y \\ v(x,y,z) &= v_0(x,y) + z\,\theta_x \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

where (u_0, v_0, w_0) are the displacements of the point on the reference plane of the laminate, and $\theta_x = -\dfrac{\partial w}{\partial y}$ and $\theta_y = \dfrac{\partial w}{\partial x}$ are the rotations about the x and y axes respectively.

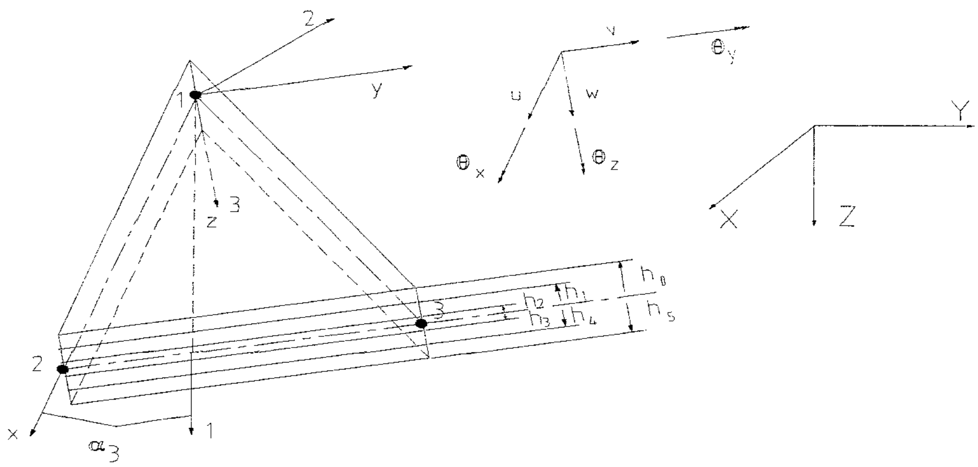

Fig. 1: Element with Local and Global co-ordinate systems

The present theory considers large displacements with small strains. The Green's strains components associated with the displacement in Eqn 1 are given by :

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\,\frac{\partial \theta_y}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial v_0}{\partial x}\right)^2 + \left(\frac{\partial w_0}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial \theta_x}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial w_0}{\partial y}\right)^2\right] \tag{2}$$

$$\varepsilon_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + z\left(\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x}\right) + \left(\frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right)$$

Laminate Analysis

For the case of a laminate of n layers, schematically shown in Figure1, the stress-strain relations of the laminate are based on the stress-strain relations in each layer, for which principal orthogonal material axes (1,2,3) are defined. A local orthogonal reference system (x, y, z), is applied to the laminate.The constitutive equations, in terms of material axes, for the kth layer can be written as [10] :

$$\sigma_k = Q_k\,\varepsilon_k \qquad \sigma_k = [\sigma_{11}\,\sigma_{22}\,\sigma_{12}]^T \qquad \varepsilon_k = [\varepsilon_{11}\,\varepsilon_{22}\,\gamma_{12}]^T \tag{3}$$

Where σ_{11} σ_{22} and σ_{12} are the normal and shearing stresses, ε_{11} ε_{22} and γ_{12} are the normal and distortion strains, and Q_k is the elasticity matrix, whose non - zero elements are given by:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \;,\quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \;,\quad Q_{12} = Q_{21} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \;,\quad Q_{33} = G_{12} \tag{4}$$

where E_1 and E_2 are the Young's modulis referred to 1 and 2 material axes respectively, G_{12} the shear modulus, ν_{12} the major Poisson's ratio of kth orthotropic layer, and $\nu_{21} = \nu_{12}\ (E_2 / E_1)$. Relating the principal material axis 1 to the x axis of the reference system of the laminate, through an angle α_k, the stress-strain relations for the kth orthotropic layer are given by:

$$\bar{\sigma}_k = \bar{Q}_k\bar{\varepsilon} \;,\quad \bar{\sigma}_k = [\sigma_{xx}\,\sigma_{yy}\,\sigma_{xy}]^T \;,\quad \bar{\varepsilon} = [\varepsilon_{xx}\,\varepsilon_{yy}\,\gamma_{xy}]^T \tag{5}$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\alpha + 2(Q_{12} + 2Q_{33})\cos^2\alpha\sin^2\alpha + Q_{22}\sin^4\alpha \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{33})\cos^2\alpha\sin^2\alpha + Q_{12}(\cos^4\alpha + \sin^4\alpha) \\
\bar{Q}_{22} &= Q_{11}\sin^4\alpha + 2(Q_{12} + 2Q_{33})\cos^2\alpha\sin^2\alpha + Q_{22}\cos^4\alpha \\
\bar{Q}_{33} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{33})\cos^2\alpha\sin^2\alpha + Q_{33}(\sin^4\alpha + \cos^4\alpha) \\
\bar{Q}_{13} &= (Q_{11} - Q_{12} - 2Q_{33})\cos^3\alpha\sin\alpha + (Q_{12} - Q_{22} + 2Q_{33})\sin^3\alpha\cos\alpha \\
\bar{Q}_{23} &= (Q_{11} - Q_{12} - 2Q_{33})\sin^3\alpha\cos\alpha + (Q_{12} - Q_{22} + 2Q_{33})\cos^3\alpha\sin\alpha
\end{aligned} \tag{6}$$

The strains for an arbitrary point of the kth layer given by Eqn. (2) can be written in the form:

$$\bar{\varepsilon} = \varepsilon_0{}^L + z\kappa + \varepsilon_0{}^{NL} \quad ; \quad \varepsilon_0^L = \left[\varepsilon_{0_{xx}}^L \ \varepsilon_{0_{yy}}^L \ \varepsilon_{0_{xy}}^L\right]^T$$

$$\kappa = \left[\kappa_{xx} \ \kappa_{yy} \ \kappa_{xy}\right]^t \quad ; \quad \varepsilon_0{}^{NL} = \left[\varepsilon_{0_{xx}}^{NL} \ \varepsilon_{0_{yy}}^{NL} \ \varepsilon_{0_{xy}}^{NL}\right]^T \tag{7}$$

where $\varepsilon_0{}^L$ is the linear strain vector of the middle surface, κ the vector of curvatures and $\varepsilon_0{}^{NL}$ the vector of non linear strains. From Eqn 5 and 7 yields:

$$\bar{\sigma}_k = \bar{Q}_k\left(\varepsilon_0{}^L + \varepsilon_0{}^{NL} + z\,\kappa\right) \tag{8}$$

The resultant forces and moments acting on the laminate are obtained by integrating the stresses in each layer through the laminate thickness :

$$N = \left[N_{xx}\, N_{yy} N_{xy}\right]^T = \int_{-h/2}^{h/2}\left[\sigma_{xx}\sigma_{yy}\sigma_{xy}\right]^T dz = \sum_{K=1}^{n}\int_{h_{k-1}}^{h_k}\left[\sigma_{xx}\sigma_{yy}\sigma_{xy}\right]^T dz$$

$$M = \left[M_{xx}\, M_{yy} M_{xy}\right]^T = \int_{-h/2}^{h/2}\left[\sigma_{xx}\sigma_{yy}\sigma_{xy}\right]^T zdz = \sum_{K=1}^{n}\int_{h_{k-1}}^{h_k}\left[\sigma_{xx}\sigma_{yy}\sigma_{xy}\right]^T zdz \tag{9}$$

where h is the overall thickness of the laminate. After integration one obtains in the condensed form the matrix relation:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \varepsilon_0{}^L + \varepsilon_0{}^{NL} \\ \kappa \end{Bmatrix} \quad ; \quad \hat{\sigma} = \hat{D}\,\bar{\varepsilon} \tag{10}$$

with

$$A_{ij} = \sum_{K=1}^{n}\bar{Q}_{ij}\left(h_k - h_{k-1}\right) \quad ; \quad B_{ij} = \sum_{K=1}^{n}\bar{Q}_{ij}\left(h_k^2 - h_{k-1}^2\right)/2 \quad ; \quad D_{ij} = \sum_{K=1}^{n}\bar{Q}_{ij}\left(h_k^3 - h_{k-1}^3\right)/3 \tag{11}$$

where A_{ij}, B_{ij}, D_{ij} are called extensional stiffness, extension-bending/twisting coupling stiffness, and bending stiffness respectively.

The present model has been developed for isotropic plate and shell structures by Bazeley et al[4], Zienkiewicz [7]and Mirza et al. [5]among others. More recently Malhotra et al [6]applied it to orthotropic plates. In the present work this element is extended to the geometrically non-linear analysis of general multilayered thin composite plate-shell type structures. As is shown in Figure 1, the element has three nodes and six degrees of freedom per node, the displacements u_i v_i w_i and rotations $\theta_{xi}, \theta_{yi}, \theta_{zi}$. It requires the introduction of fictitious stiffness coefficients $K_{\theta z}$, corresponding to rotations θ_z, which does not enter in the formulation in the local coordinate system [7].

The application of the virtual work principle to an element yields in the incremental large displacement analysis :

$$\left[K_L^e + K_\sigma^e\right]^{(i-1)} \{\Delta d_e\}^{(i)} = \{F_{ext}^e\} - \{F_{int}^e\}^{(i-1)} \tag{19}$$

where the linear stiffness matrix K_L^e, the geometrical stiffness matrix K_σ^e, the external load vector F_{ext}^e and internal load vector $F_{int}^{e\,(i-1)}$, are given by :

$$\left[K_L^e\right] = \int_A \left[B^L\right]^T \left[\hat{D}\right] \left[B^L\right] dA \quad ; \quad \left[K_\sigma^e\right] = \int_A [G]^T [\tau]\,[G]\,dA$$

$$\{F_{ext}^e\} = \int_A [N]^T \{p\}\, dA + \int_S N^T \{t\}\, dS + \{P\} \quad ; \quad \{F_{int}^e\}^{(i-1)} = \int_A \left[B^L\right] \hat{\sigma}^{(i-1)}\, dA \tag{20}$$

In these equations $\left[B^L\right]$ is the membrane-bending linear strain-displacement matrix , $[G]$ is the non-linear strain-displacement matrix, $[\tau]$ is the matrix of the total updated membrane force components, $\{p\}$, $\{t\}$ and $\{P\}$ are the surface ,side distributed in-plane and concentrated load vectors, respectively.The sub-matrices $\left[G_i\right]$ and matrix $[\tau]$, are given by :

$$[G_i] = \begin{bmatrix} \frac{\partial L_i}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial L_i}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial L_i}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial L_i}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial_1 N_i}{\partial x} & \frac{\partial_2 N_i}{\partial x} & \frac{\partial_3 N_i}{\partial x} & 0 \\ 0 & 0 & \frac{\partial_1 N_i}{\partial y} & \frac{\partial_2 N_i}{\partial y} & \frac{\partial_3 N_i}{\partial y} & 0 \end{bmatrix} \tag{21}$$

$$[\tau] = \begin{bmatrix} N_{xx} & N_{xy} & 0 & 0 & 0 & 0 \\ N_{xy} & N_{yy} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{xx} & N_{xy} & 0 & 0 \\ 0 & 0 & N_{xy} & N_{yy} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{xx} & N_{xy} \\ 0 & 0 & 0 & 0 & N_{xy} & N_{yy} \end{bmatrix} \tag{22}$$

To obtain the strain-displacements matrices, differentiation of the shape functions ${}_jN_i$ with respect to Cartesian co-ordinates , needs to be carried out. For example, with respect to x ,this is as follows:

$$\frac{\partial\, {}_jN_i}{\partial x} = \frac{\partial L_1}{\partial x}\frac{\partial\, {}_jN_i}{\partial L_1} + \frac{\partial L_2}{\partial x}\frac{\partial\, {}_jN_i}{\partial L_2} + \frac{\partial L_3}{\partial x}\frac{\partial\, {}_jN_i}{\partial L_3} \tag{23}$$

The system of equations of equilibrium are obtained in the usual way, after the local (x,y,z) - global (X,Y,Z) transformations [7] are carried out , yielding :

$$[K_L + K_\sigma]^{(i-1)} \{\Delta q\}^i = \{F_{ext}\} - \{F_{int}\}^{(i-1)} \tag{24}$$

where $\{\Delta q\}$ is the system displacement vector.

For shell analysis the arc-length method [3] is used. The Eqn 24 is then written in the form:

$$[K_L + K_\sigma]^{(i-1)} \{\Delta q\}^i = (\lambda^{(i-1)} + \Delta\lambda^{(i)}) \{F^0_{ext}\} - \{F_{int}\}^{(i-1)} \tag{25}$$

and an additional equation is employed to constrain the length of a load step :

$$\{\Delta q\}^{i^T} \{\Delta q\}^i = \Delta l^2 \tag{26}$$

where F^0_{ext} is a fixed load vector, λ is a load factor , $\Delta\lambda$ is the incremental load factor within the load step, and Δl is the arc-length .

A simply-supported square (axa) laminated plate with two layers , subjected to an uniform pressure load $p = \lambda\, p_0$ ($p_0 = 100\,N/m^2$) is considered , for two different cases of fibre orientation ,$[0°/90°]$ and $[45°/-45°]$.The geometry and material properties are : a = 1.0 m , h = 0.002 m , E_1= 250 GPa , E_2 =20 GPa , G_{12} =10 GPa , ν_{12} = 0.25.The boundary conditions are:

for cross-ply laminates :

along $x = 0$: $v_0 = 0$, $w = 0$, $\theta_x = 0$
along $x = a$: $v_0 = 0$, $w = 0$, $\theta_x = 0$
along $y = 0$: $u_0 = 0$, $w = 0$, $\theta_y = 0$
along $y = a$: $u_0 = 0$, $w = 0$, $\theta_y = 0$

for angle-ply laminates :

along $x = 0$: $u_0 = 0$, $w = 0$, $\theta_x = 0$
along $x = a$: $u_0 = 0$, $w = 0$, $\theta_x = 0$
along $y = 0$: $v_0 = 0$, $w = 0$, $\theta_y = 0$
along $y = a$: $v_0 = 0$, $w = 0$, $\theta_y = 0$

In Figure 2 ,are shown the results obtained in the present study ,using a (8x8) element full plate mesh, and those obtained by Barbero and Reddy [11] that used a higher order displacement model. It can be observed a very closed agreement for both lay-ups.

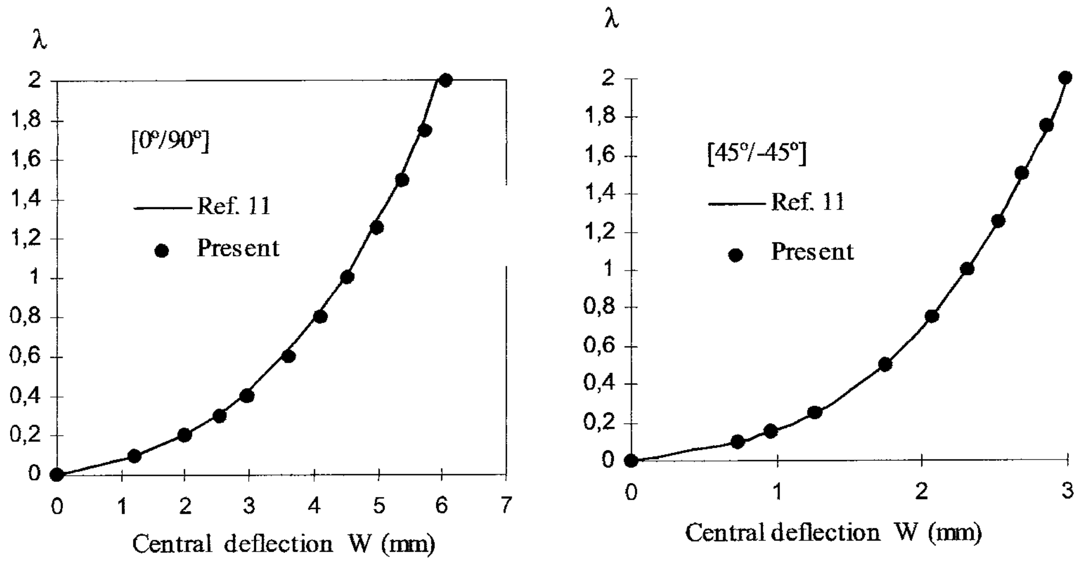

Fig. 2 : Load - Deflection curves for simply - supported plates under transverse load.

A clamped square (axa) laminated plate with a stacking sequence $[\pm 45°/0°_2/\pm 45°/90°_2]_s$, under uniform pressure load $p = p_0\, a^2 / E_2 h^2$ is studied. The geometric dimensions and the material properties are : $a = 254$ mm , $h = 2.114$ mm , $E_1 = 131.0$ GPa , $E_2 = 13.03$ GPa , $G_{12} = 6.41$ GPa , and $\nu_{12} = 0.38$.

The results of the present study obtained with (10x10) full plate mesh, and its comparison with the finite element solution (10x10 full plate mesh) obtained by Madenci and Barut [9] are presented in Figure 3. An excellent agreement is observed between both solutions.

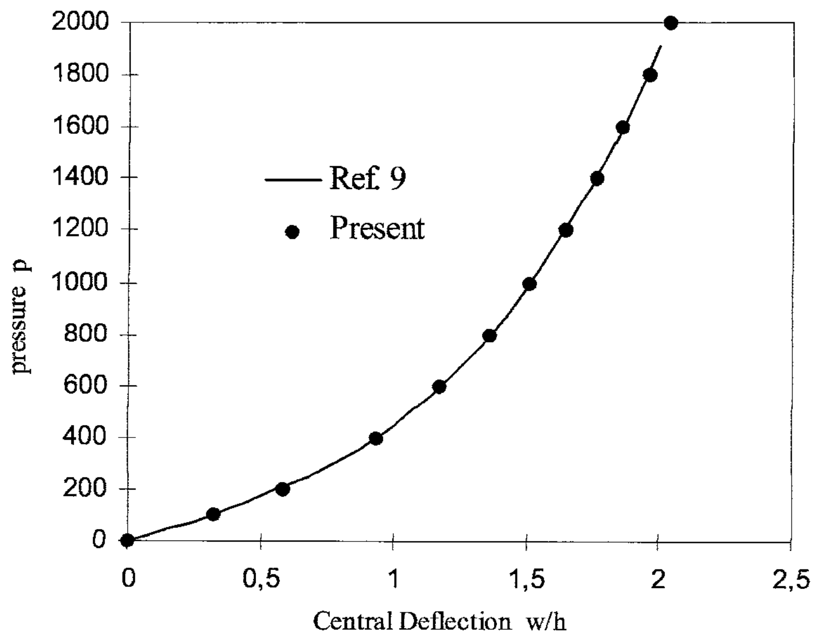

Fig. 3 : Load-Deflection curves for a clamped laminated square plate under uniform pressure

Hinged-free symmetric and antisymmetric cylindrical panels (Figure 4), with lay-ups $[90°/0°/90°]$ and $[-45°/45°]$, respectively, subjected to a point load, are analysed. The geometry and material properties are : $R = 2540$ mm, $L = 508$ mm, $h = 12.6$ mm, $\theta = 0.1$ rad, $E_1 = 3.3$ GPa, $E_2 = 1.1$ GPa, $G_{12} = 0.66$ GPa, and $\nu_{12} = 0.25$. The straight edges are hinged and the curved edges are free. A model discretization with (8x8) elements is used. The non-linear load deflection curves obtained with the present non-conforming CPT triangular element and those obtained by Madenci and Barut [9], are shown in Figure 5. A good agreement is found between both solutions.

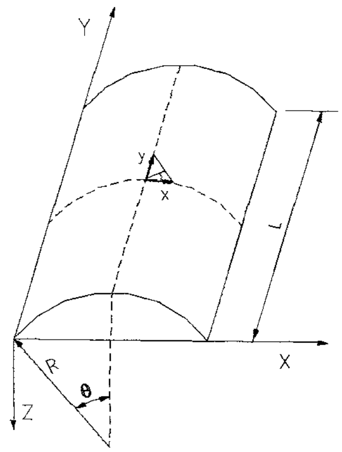

Fig. 4 : Cylindrical shell.

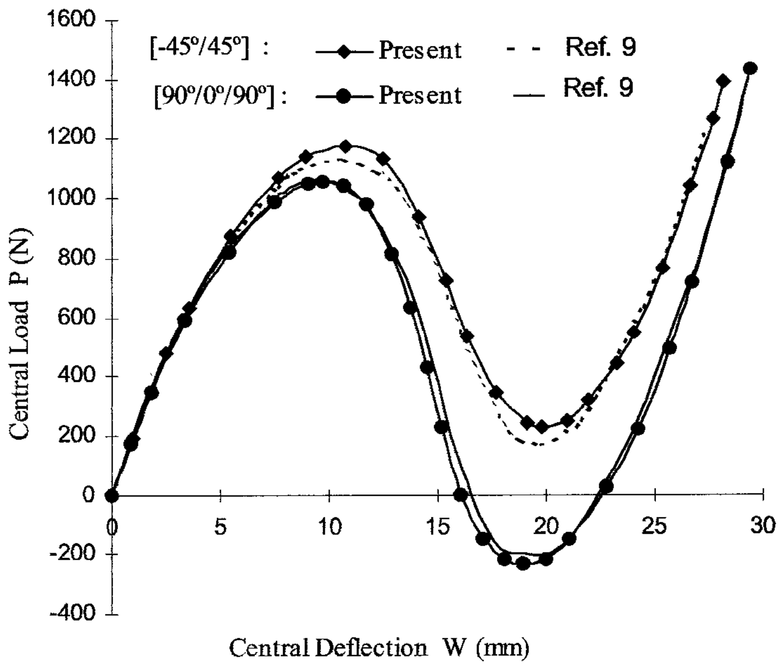

Fig. 5 : Load-Deflection curves of laminated shells under point load.

A nine-layer symmetric cross-ply $[0°/90°/.../0°]$ laminated spherical shallow shell (Figure 6), subjected to a normal pressure loading $p = p_0 R^2 / E_2 h^2$, and with the following geometric and material data: $R/a = 10$, $a/h = 1000$, $E_1/E_2 = 40$, $G_{12}/E_2 = 0.6$, $v_{12} = 0.25$, is considered. The shell has simply supported edges which were restricted from moving in the direction of the edge line but were free to move in a direction perpendicular to the edge line. A mesh of (8x8) elements has been used. The solution obtained by using the present CPT triangular flat element is compared , Figure 7 , with the solution obtained by Saigal et al [12] that used a 48 degree-of-freedom quadrilateral curved shell element. A very good agreement is found between both solutions.

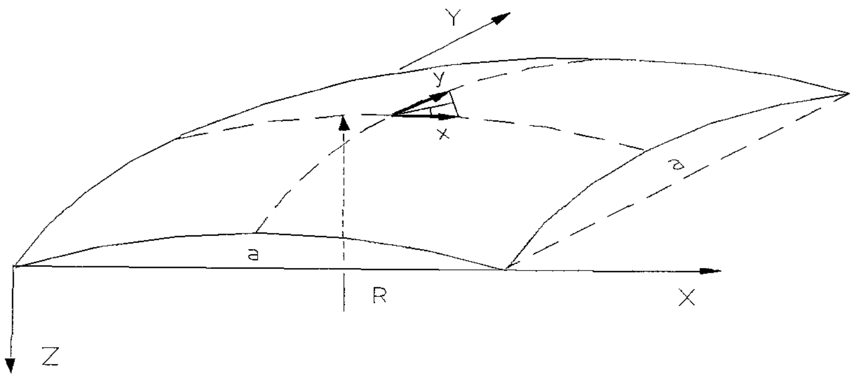

Fig. 6 : Spherical shell

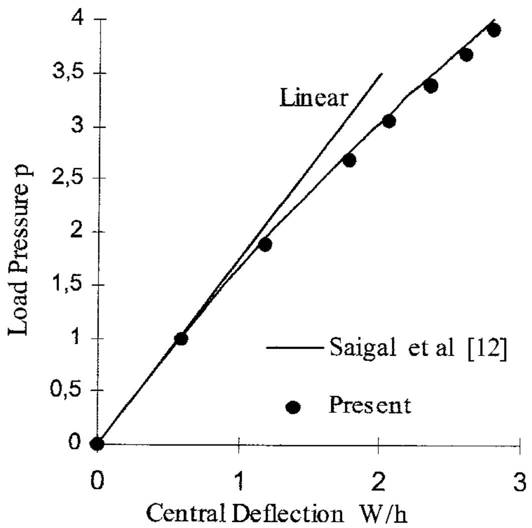

Fig. 7 :Load - Deflection curves of a shallow spherical shell under uniform pressure loading

A non-conforming flat triangular plate/shell element, based on Kirchhoff theory, is developed to carry out the geometrically non-linear analysis of general multilaminated anisotropic thin shell structures. The updated Lagrangian formulation has been implemented. This simple model has proved to be very efficient for predicting the non-linear response of thin multilaminated plate and shell structures. The comparison of the results with alternative numerical solutions shows that the model is accurate and versatile.

The authors thank the financial support received from H.C.M. Project CHRTX-CT93-0222 , and Fundação Calouste Gulbenkian .

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