PARABOLIC REACTION-DIFFUSION SYSTEMS WITH NONLOCAL COUPLED DIFFUSIVITY TERMS

J. FERREIRA[1] H.B. DE OLIVEIRA[2,3]

[1] *DCE - Universidade Federal Fluminense, Rio de Janeiro, Brazil*

[2] *FCT - Universidade do Algarve, Faro, Portugal*

[3] *CMAFCIO - Universidade de Lisboa, Lisboa, Portugal*

Abstract. In this work we study a system of parabolic reaction-diffusion equations which are coupled not only through the reaction terms but also by way of nonlocal diffusivity functions. For the associated initial problem, endowed with homogeneous Dirichlet or Neumann boundary conditions, we prove the existence of global solutions. We also prove the existence of local solutions but with less assumptions on the boundedness of the nonlocal terms. The uniqueness result is established next and then we find the conditions under which the existence of strong solutions is assured. We establish several bow-up results for the strong solutions to our problem and we give a criterium for the convergence of these solutions towards a homogeneous state.

Key words and phrases: Reaction-diffusion systems, nonlocal coupled diffusivity terms, local and global weak solutions, uniqueness, strong solutions, blow-up, asymptotic stability.

MSC 2010: 35K57, 35D30, 35D35, 35B44, 35B35.

1. Introduction

In this work, we are interested in studying systems of reaction-diffusion equations of the form

$$\frac{\partial u}{\partial t} - a_1(p(u), q(v))\Delta u = f(u, v) \quad \text{in } Q_T \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_2(r(u), s(v))\Delta v = g(u, v) \quad \text{in } Q_T \tag{1.2}$$

$$u = u_0, \quad v = v_0 \quad \text{in } \Omega\,, \quad \text{when } t = 0 \tag{1.3}$$

in a space and time cylinder

$$Q_T := \Omega \times [0, T)$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $T \in (0, \infty]$. Here, p, q, r and s are functions depending locally on the time variable t and non-locally on the density variables u and v. In (1.1)-(1.3), a_1 and a_2 are functions expressing possibly different diffusions in each of the nonlocal functions $p(u)$, $q(v)$ and $r(u)$, $s(v)$, respectively. The functions f and g express distinct interacting reactions between u and v. We supplement the system (1.1)-(1.3) with the following general boundary conditions

$$\tau\, \nabla u \cdot \mathbf{n} + (1 - \tau)u = 0 \quad \text{on } \Gamma_T\,, \tag{1.4}$$

$$\tau\, \nabla v \cdot \mathbf{n} + (1 - \tau)v = 0 \quad \text{on } \Gamma_T\,, \tag{1.5}$$

where $\tau = 0$ or $\tau = 1$, and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. Throughout this work, we will consider either the case of Dirichlet boundary conditions, *i.e.* $\tau = 0$ in (1.4)-(1.5), or of Neumann boundary conditions, *i.e.* $\tau = 1$ in (1.4)-(1.5). In the final part, we will distinguish situations where it is important to consider specific boundary conditions. Note that the existence of the unit normal to $\partial\Omega$ in almost all points of $\partial\Omega$ implies that $\partial\Omega$ is sufficiently regular, for instance Lipschitz-continuous.

E-mail address: ferreirajorge2012@gmail.com, CORRESPONDING AUTHOR: holivei@ualg.pt.

Systems of reaction-diffusion equations are very important in the applied sciences to model interesting and very distinct phenomena, where many chemical and biological processes are in the first line of its applications. On the other hand, the combination of the coupling diffusions together with the coupling reactions produces many mathematical features. In particular, systems of reaction-diffusion equations lead to the possibility of many threshold phenomena that we cannot expect they happen if we consider only one reaction-diffusion equation. An interesting feature of the reaction-diffusion equations (1.1)-(1.2) observed in many models, arise when the diffusion coefficient, say for simplicity $p(u)$, is given by a local quantity. However, in many applications this assumption is incompatible with the physical notion of measure, since we are not able to measure pointwisely the diffusivity of a pointwise density. One possibility to overcome this difficulty, consists in choosing a point x in the space and then constructing a ball $B := B(x, \epsilon)$ centered at x with radius ϵ and replacing $p(u)$ by

$$p\left(\fint_{B\cap\Omega} |u|^{\gamma}\, dy\right), \tag{1.6}$$

for some $\gamma \geq 1$, where $\fint_{B\cap\Omega} := \frac{1}{\mathcal{L}^N(B\cap\Omega)}\int_{B\cap\Omega}$ and $\mathcal{L}^N$ denotes the N-Lebesgue measure. This makes the mathematical analysis of the corresponding reaction-diffusion equation more feasible around the chosen point x (see *e.g.* [18]). On the other hand, systems of reaction-diffusion equations, but with nonlocal reaction terms instead of diffusion ones, were recently proposed to describe the motion of particle densities under the presence of some chemical reactions (see [16]) and to model the evolution of a population under chemotactic effects (see [19]). Though our motivation to study the system (1.1)-(1.3) is primarily mathematical, we can find some interesting aspects of its applications in population dynamics. See, for instance, the references [3, 8, 17, 18].

The exact formulation of the nonlocal functions a_i, $i = 1,\ 2$, we will consider here, relies on the assumption that

$$p,\ q,\ r \text{ and } s \ \text{ are continuous linear functionals over } \ L^{\gamma_p}(\Omega_p),\ L^{\gamma_q}(\Omega_q),\ L^{\gamma_r}(\Omega_r) \text{ and } L^{\gamma_s}(\Omega_s)\,, \tag{1.7}$$

respectively, for some bounded subdomains $\Omega_p,\ \Omega_q,\ \Omega_r,\ \Omega_s \subset \Omega$ and for some real numbers $\gamma_p,\ \gamma_q,\ \gamma_r,\ \gamma_s \geq 1$. Observe that, in view of (1.7), we can use Riesz representation theorem to infer the existence of unique $u_p^* \in L^{\gamma_p'}(\Omega_p)$, $v_q^* \in L^{\gamma_q'}(\Omega_q)$, $u_r^* \in L^{\gamma_r'}(\Omega_r)$ and $v_s^* \in L^{\gamma_s'}(\Omega_s)$, where $1/\gamma_i + 1/\gamma_i' = 1$ for $i \in \{p, q, r, s\}$, such that

$$p(u) = \int_{\Omega_p} u_p^* u\, dx \quad \forall\ u \in L^{\gamma_p}(\Omega_p), \qquad q(v) = \int_{\Omega_q} v_q^* v\, dx \quad \forall\ v \in L^{\gamma_q}(\Omega_q),$$
$$r(u) = \int_{\Omega_r} u_r^* u\, dx \quad \forall\ u \in L^{\gamma_r}(\Omega_r), \qquad s(v) = \int_{\Omega_s} v_s^* v\, dx \quad \forall\ v \in L^{\gamma_s}(\Omega_s).$$

Moreover, we have

$$\|p\|_{(L^{\gamma_p}(\Omega_p))'} = \|u_p^*\|_{L^{\gamma_p'}(\Omega_p)}, \quad \|q\|_{(L^{\gamma_q}(\Omega_q))'} = \|v_q^*\|_{L^{\gamma_q'}(\Omega_q)},$$
$$\|r\|_{(L^{\gamma_r}(\Omega_r))'} = \|u_r^*\|_{L^{\gamma_r'}(\Omega_r)}, \quad \|s\|_{(L^{\gamma_s}(\Omega_s))'} = \|v_s^*\|_{L^{\gamma_s'}(\Omega_s)}.$$

To the best of our knowledge, the first works on the mathematical analysis of partial differential equations, with nonlocal diffusivity terms as mentioned above, were studied in [8, 18]. However, we should note that it was proposed earlier, in [17], an abstract framework to handle hyperbolic problems with similar nonlocal diffusivity terms, previously and independently introduced by Dickey and Pohožaev (see the exact references in [17]).

During the last decades a lot of attention has been devoted to nonlocal diffusion and reaction-diffusion problems. In [4, 18], the existence and uniqueness of local and global solutions to the following parabolic diffusion problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f & \text{in } \Omega \times (0, T) \\ u = u_0 \quad \text{in } \Omega \quad \text{when } t = 0. \end{cases} \tag{1.8}$$

has been proved. Here, the diffusivity a is some function from $\mathbb{R}$ into $(0, +\infty)$ and l is a continuous mapping from $L^2(\Omega)$ into $\mathbb{R}$. The authors have worked on different problems for distinct diffusivity terms, but always depending on $\int_\Omega u\, dx$, and under different boundary conditions: Dirichlet, Neumann and mixed boundary conditions. In [5], besides proving the existence and uniqueness of solutions, the same authors have analyzed the asymptotic behavior of the solutions as well. These issues were extensively investigated in [7] where, in particular, the convergence of the solutions to a steady state was proved. Several extensions and modifications of the problem (1.8) were deeply studied in [2], where many interesting examples were given as well. Again the authors of [4, 5, 18], considered, in [6], a class of nonlocal elliptic and parabolic problems related to (1.8), now with homogeneous Dirichlet boundary conditions, for which they proved existence and uniqueness results. The analysis of the problem (1.8), considered with a nonlocal diffusivity depending on the Dirichlet integral $\int_\Omega |\nabla u|^p \, dx$, was carried out in [10, 23] for $p = 2$ and in [9] for a general p (and for the p-Laplacian). The asymptotic behavior of the solutions to the problem (1.8), considered with a nonlocal diffusivity written as a kernel, *i.e.* $l(u) = \int_\Omega g\, u\, dx$, where g is a given function in $L^2(\Omega)$, has been performed in [23] too. Reaction-diffusion analogues of the parabolic problem (1.8) were considered by the authors of [1, 13] in the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) \quad \text{in } \Omega \times (0,T) \\ u = u_0 \quad \text{in } \Omega \quad \text{when } t = 0. \end{cases} \tag{1.9}$$

In [1], the problem (1.9) was considered in a rather general Banach space and the authors worked on the case $a(u) = \left(\int_\Omega u\, dx\right)^{-1}$. This assumption led them to an equivalent reaction-diffusion problem with a nonlocal diffusivity, now multiplied by the reaction term $f(u)$. For these problems, the authors established local existence and uniqueness results and, in addition, they found conditions on the initial data in order to obtain time properties of finite extinction or persistency of the solutions. In [13], the authors extended the results of [4, 5, 6, 7, 10, 18] to the case of the reaction-diffusion problem (1.9). In particular, they considered both stationary and transient situations, where the nonlinearity appears, not only in the nonlocal diffusivity term $a(l(u))$, but also on the right-hand side in which one has the nonlinear function $f(u)$.

The outline of our work is the following. In Section 2, we define the notion of weak solutions to the problem (1.1)-(1.5) and we present Theorem 2.1 where is established the existence of weak solutions. The proof of Theorem 2.1 is carried out in Section 3 by using Galerkin approximations together with compactness arguments. In Section 4, we drop the boundedness condition on the nonlocal functions (see (2.2)) to prove a local existence result in Theorem 4.1. Section 5 is devoted to prove the uniqueness result and in Section 6 we find the conditions under which we prove the existence of strong solutions. In Section 7, we establish several bow-up results for the strong solutions to the problem (1.1)-(1.5). Finally, in Section 8, we give a criterium for the convergence of these strong solutions towards a homogeneous state by using the theory of invariant regions.

The notation used throughout this work is largely standard in the field of Partial Differential Equations and we address the reader to the monographs [3, 15, 20, 22] for any question related with this matter.

2. Weak formulation

To define the notion of weak solutions we are interested here, we shall assume that:

$$\text{the functions } a_i : \mathbb{R}^2 \to \mathbb{R}^+, \text{ wih } i = 1,\ 2 \text{ are continuous;} \tag{2.1}$$

$$\forall\, i \in \{1,2\}\ \exists\, m_i,\ M_i > 0 : 0 < m_i \le a_i(\xi,\eta) \le M_i < \infty\ \forall\, \xi,\ \eta \in \mathbb{R}. \tag{2.2}$$

Condition $a_i(\xi,\eta) > 0$ expresses the fact that we will consider uniformly parabolic equations (1.1)-(1.2). On the reaction functions, f and g, we assume that

$$|f(u_1,v_1) - f(u_2,v_2)| \le C_{L_1}|(u_1,v_1) - (u_2,v_2)| \quad \forall\, (u_1,v_1),\ (u_2,v_2) \in \mathbb{R}^2\,, \tag{2.3}$$

$$\text{with } f(0,0) = 0 \tag{2.4}$$

and

$$|g(u_1,v_1) - g(u_2,v_2)| \le C_{L_2}|(u_1,v_1) - (u_2,v_2)| \quad \forall\, (u_1,v_1),\ (u_2,v_2) \in \mathbb{R}^2\,, \tag{2.5}$$

$$\text{with } g(0,0) = 0\,, \tag{2.6}$$

where C_{L_1} and C_{L_2} are the correspondingly positive Lipschitz constants. For each $\tau \in \{0,\ 1\}$, we consider the following function space

$$V_\tau := \text{ closure of } \left\{\phi \in C^\infty(\overline{\Omega}) : \tau\, \nabla u \cdot \mathbf{n} + (1-\tau)u = 0 \quad \text{on } \partial\Omega\,, \quad \tau \int_\Omega \phi\, dx = 0\right\} \text{ in } H^1(\Omega)\,.$$

In any case, V_τ is a closed subspace of $H^1(\Omega)$, with its norm satisfying to

$$C_1\|\nabla\phi\|_{L^2(\Omega)} \leq \|\phi\|_{V_\tau} \leq C_2\|\nabla\phi\|_{L^2(\Omega)}$$

for some positive constants C_1 and C_2.

Definition 2.1. *Let $N \geq 2$ and assume that conditions (2.1)-(2.6) hold. We say (u,v) is a weak solution to the problem (1.1)-(1.5), for either $\tau = 0$ or $\tau = 1$, if:*

1. *$u,\ v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_\tau)$;*
2. *$u(0) = u_0$ and $v(0) = v_0$;*
3. *For every $\varphi,\ \eta \in V_\tau$*

$$\frac{d}{dt}\int_\Omega u(t)\varphi\,dx + a_1(p(u(t)),q(v(t)))\int_\Omega \nabla u(t)\cdot\nabla\varphi\,dx = \int_\Omega f(u(t),v(t))\varphi\,dx\,, \tag{2.7}$$

$$\frac{d}{dt}\int_\Omega v(t)\eta\,dx + a_2(r(u(t)),s(v(t)))\int_\Omega \nabla v(t)\cdot\nabla\eta\,dx = \int_\Omega g(u(t),v(t))\eta\,dx\,, \tag{2.8}$$

which hold in $\mathcal{D}'(0,T)$.

In order to prove the existence of weak solutions to the problem (1.1)-(1.5), we have to impose a suitable restriction related with the Poincaré inequality. We assume that the constants of uniform parabolicity m_i and of Lipschitz continuity C_{L_i} are related by

$$m_i\lambda_P > C_{L_1} + C_{L_2}\,, \quad i = 1,\ 2\,, \tag{2.9}$$

where λ_P is the principal (positive) eigenvalue for the Laplacian problem

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{in } \Omega \\ \tau\,\nabla\phi\cdot\mathbf{n} + (1-\tau)\phi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

for $\tau = 0$ or $\tau = 1$. Observe that, in the case of Neumann boundary conditions, *i.e.* $\tau = 1$ in (2.10), 0 is clearly an eigenvalue, with the associated eigenfunction given by any constant, which in turn can be fixed by a normalization such as $\overline{\phi} = 0$, where $\overline{\phi} = \fint_\Omega \phi\,dx$. In any case, the Rayleigh quotient allows one to characterize the principal (positive) eigenvalue of (2.10) with the following minimum principle,

$$\lambda_P = \min_{\phi\in V_\tau,\ \phi\neq 0} \frac{\|\nabla\phi\|^2_{L^2(\Omega)}}{\|\phi\|^2_{L^2(\Omega)}}. \tag{2.11}$$

It is well know that the minimum in (2.11) is attained for a function $\phi \in V_\tau$ such that $\phi > 0$ in Ω (see *e.g* [15]). Associated with the problem (2.10)-(2.11), we recall the following Poincaré inequalities (see *e.g.* [20, Theorem 11.11]) that will be used in the sequel:

$$\|\nabla\,\phi\|^2_{L^2(\Omega)} \geq \lambda_P\|\phi\|^2_{L^2(\Omega)} \quad \text{if } \phi \in H^1(\Omega) \text{ and } \phi = 0 \text{ on } \partial\Omega\,; \tag{2.12}$$

$$\|\nabla\,\phi\|^2_{L^2(\Omega)} \geq \lambda_P\|\phi - \overline{\phi}\|^2_{L^2(\Omega)} \quad \text{if } \phi \in H^1(\Omega) \text{ and } \nabla\,\phi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\,; \tag{2.13}$$

$$\|\Delta\,\phi\|^2_{L^2(\Omega)} \geq \lambda_P\|\nabla\,\phi\|^2_{L^2(\Omega)} \quad \text{if } \phi \in H^2(\Omega) \text{ and } \nabla\,\phi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\,. \tag{2.14}$$

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume that conditions (1.7), (2.1)-(2.6) and (2.9) hold. If*

$$u_0,\ v_0 \in L^2(\Omega)\,, \tag{2.15}$$

then, for any $T > 0$, there exists, at least, a weak solution (u,v) to the problem (1.1)-(1.5), for either $\tau = 0$ or $\tau = 1$, in the sense of Definition 2.1. In addition,

$$u,\ v \in \mathrm{C}([0,T];L^2(\Omega)), \tag{2.16}$$

$$\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t} \in L^2(0,T;V'_\tau). \tag{2.17}$$

The proof of Theorem 2.1 will be established in the next section.

3. Proof of Theorem 2.1

3.1. Existence of approximative solutions. In order to use the Galerkin method, let $\{\phi_i\}_{i=1}^{\infty}$ be a set of non-trivial solutions ϕ_i, associated to the eigenvalues $\lambda_i > 0$, $i = 1, 2, ...$, to the following spectral problem:

$$\begin{cases} \displaystyle\int_\Omega \nabla\phi_i \cdot \nabla\psi\, dx = \lambda_i \int_\Omega \phi_i \psi\, dx \qquad \text{in } \Omega\,, \quad \forall\, \psi \in V_\tau, \\ \tau\, \nabla\phi_i \cdot \mathbf{n} + (1-\tau)\phi_i = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The family $\{\phi_i\}_{i=1}^{\infty}$ is orthogonal in V_τ and can be chosen as being orthonormal in $L^2(\Omega)$ (see *e.g.* [15]). Given $m \in \mathbb{N}$, let us consider the correspondingly m-dimensional space V_τ^m spanned by $\phi_1, \phi_2, ..., \phi_m$. For each $m \in \mathbb{N}$, we search for an approximative solution $(u^m(t), v^m(t))$ of (2.7)-(2.8) in the form

$$u^m(t) = \sum_{k=1}^{m} c_k^m(t)\phi_k\,, \qquad v^m(t) = \sum_{k=1}^{m} d_k^m(t)\phi_k\,. \tag{3.1}$$

where $\phi_k \in V_\tau^m$ are given and $c_k^m(t)$ and $d_k^m(t)$ are the functions we look for. These functions are found by solving the following system of $2m$ nonlinear ordinary differential equations, with respect to the $2m$ unknowns $c_1^m, \dots, c_m^m$ and $d_1^m, \dots, d_m^m$, obtained from (2.7)-(2.8):

$$\frac{d}{dt}\int_\Omega u^m(t)\varphi\, dx + a_1(p(u^m(t)), q(v^m(t)))\int_\Omega \nabla u^m(t)\cdot\nabla\varphi\, dx = \int_\Omega f(u^m(t), v^m(t))\varphi\, dx\,, \tag{3.2}$$

$$\frac{d}{dt}\int_\Omega v^m(t)\eta\, dx + a_2(r(u^m(t)), s(v^m(t)))\int_\Omega \nabla v^m(t)\cdot\nabla\eta\, dx = \int_\Omega g(u^m(t), v^m(t))\eta\, dx\,, \tag{3.3}$$

for all $\varphi,\ \eta \in \{\phi_1, \dots, \phi_m\}$, and with

$$u^m(0) = u_0^m \quad \text{and} \quad v^m(0) = v_0^m\,, \tag{3.4}$$

where both u_0^m and v_0^m are chosen in such a way that

$$u_0^m \to u_0 \quad \text{and} \quad v_0^m \to v_0 \quad \text{strongly in } L^2(\Omega), \quad \text{as } m \to \infty. \tag{3.5}$$

Attending to the continuity of a_1, a_2 and f, g on u and v (see (2.1) and (2.3)-(2.4), (2.5)-(2.6)), we can use Peano's theorem to prove the existence of $t^m \in (0, T)$ and $(\mathbf{c}^m(t), \mathbf{d}^m(t))$, with $\mathbf{c}^m(t) := (c_1^m(t), \dots, c_N^m(t))$ and $\mathbf{d}^m(t) := (d_1^m(t), \dots, d_N^m(t))$, and such that $(\mathbf{c}^m(t), \mathbf{d}^m(t))$ is a solution to the system (3.2)-(3.4) in the interval $[0, t^m]$. To show that this solution holds for all the interval $[0, T]$, we shall establish an *a priori* estimate. To do it so, we multiply (3.2) by c_k^m and (3.3) by d_k^m, where in both it is taken $\varphi = \varphi_k$ and $\eta = \varphi_k$, we add up the resulting equations from $k = 1$ until $k = m$ and then we integrate them between 0 and t, with $t \in (0, t^m)$, to obtain

$$\begin{aligned} &\frac{1}{2}\|u^m(t)\|_{L^2(\Omega)}^2 + \int_0^t a_1(p(u^m(\varsigma)), q(v^m(\varsigma)))\int_\Omega |\nabla u^m(\varsigma)|^2\, dx d\varsigma \\ &= \int_0^t\int_\Omega f(u^m(\varsigma), v^m(\varsigma))u^m(\varsigma)\, dx d\varsigma + \frac{1}{2}\|u^m(0)\|_{L^2(\Omega)}^2\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\frac{1}{2}\|v^m(t)\|_{L^2(\Omega)}^2 + \int_0^t a_2(r(u^m(\varsigma)), s(v^m(\varsigma)))\int_\Omega |\nabla v^m(\varsigma)|^2\, dx d\varsigma \\ &= \int_0^t\int_\Omega g(u^m(\varsigma), v^m(\varsigma))u^m(\varsigma)\, dx d\varsigma + \frac{1}{2}\|v^m(0)\|_{L^2(\Omega)}^2\,. \end{aligned} \tag{3.7}$$

Adding the equations (3.6)-(3.7) and then taking the essential supreme in $[0, T]$ in the resulting equation and using the assumptions (2.2)-(2.4), one obtains

$$\begin{aligned} &\frac{1}{2}\left(\|u^m\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|v^m\|_{L^\infty(0,T;L^2(\Omega))}^2\right) + m_1\|\nabla u^m\|_{L^2(Q_T)}^2 + m_2\|\nabla v^m\|_{L^2(Q_T)}^2 \le \\ &C_{L_1}\int_{Q_T} |(u^m, v^m)|\, |u^m| dx dt + C_{L_2}\int_{Q_T} |(u^m, v^m)|\, |v^m| dx dt + \frac{1}{2}\left(\|u^m(0)\|_{L^2(\Omega)}^2 + \|v^m(0)\|_{L^2(\Omega)}^2\right)\,. \end{aligned} \tag{3.8}$$

Then we use Cauchy's inequality together with the Poincaré inequalities (2.12)-(2.13) on the fifth and sixth terms of (3.8), which yield

$$\int_{Q_T} |(u^m, v^m)|\, |u^m| dxdt \leq \frac{1}{2\lambda_P}\left(\|\nabla u^m\|^2_{L^2(Q_T)} + \|\nabla v^m\|^2_{L^2(Q_T)}\right) + \frac{1}{2\lambda_P}\|\nabla u^m\|^2_{L^2(Q_T)}\,, \tag{3.9}$$

$$\int_{Q_T} |(u^m, v^m)|\, |v^m| dxdt \leq \frac{1}{2\lambda_P}\left(\|\nabla u^m\|^2_{L^2(Q_T)} + \|\nabla v^m\|^2_{L^2(Q_T)}\right) + \frac{1}{2\lambda_P}\|\nabla v^m\|^2_{L^2(Q_T)}\,. \tag{3.10}$$

Observe that, by the definition of $u^m(t)$, $v^m(t)$ and of V^m_τ set forth in (3.1), we can use (2.13) with $⨍_\Omega u^m(t)dx = 0$ and $⨍_\Omega v^m(t)dx = 0$. Now we use the information of (3.9)-(3.10) in (3.8) which, in view of (3.5), yields

$$\begin{aligned}
&\frac{1}{2}\|u^m\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(m_1 - \frac{C_{L_1}}{\lambda_P} - \frac{C_{L_2}}{2\lambda_P}\right)\|\nabla u^m\|^2_{L^2(Q_T)} + \\
&\frac{1}{2}\|v^m\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(m_2 - \frac{C_{L_2}}{\lambda_P} - \frac{C_{L_1}}{2\lambda_P}\right)\|\nabla v^m\|^2_{L^2(Q_T)} \leq \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right)\,.
\end{aligned}$$

Finally, assumption (2.9) guaranties that

$$\|u^m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|v^m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^m\|^2_{L^2(Q_T)} + \|\nabla v^m\|^2_{L^2(Q_T)} \leq C_0\,, \tag{3.11}$$

where, by the assumption (2.15), $C_0 = C(\|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)}, m_1, m_2, C_{L_1}, C_{L_2}, \lambda_P)$ is a positive constant not depending on m. Thus, from the Theory of the ODEs, we can take $t^m = T$.

3.2. **Convergence of the approximative solutions.** Due to (3.11) and by means of separability and reflexivity, there exist subsequences (still denoted by) u^m and v^m, and there exist $u,\ v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_\tau)$ such that

$$u^m \longrightarrow u \quad \text{and} \quad v^m \longrightarrow v \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \quad \text{as } m\to\infty, \tag{3.12}$$

$$u^m \longrightarrow u \quad \text{and} \quad v^m \longrightarrow v \quad \text{weakly in } L^2(0,T;V_\tau), \quad \text{as } m\to\infty. \tag{3.13}$$

On the other hand, by using the equations (3.2)-(3.3) together with (3.11) and with the assumptions (2.2), (2.3)-(2.4) and (2.5)-(2.6) and still using the Poincaré inequalities (2.12)-(2.13), it can be proved the existence of positive constants $C_1 = C(M_1, C_0)$ and $C_2 = C(M_2, C_0)$, where C_0 is the constant from the inequality (3.11), such that

$$\left\|\frac{\partial u^m}{\partial t}\right\|_{L^2(0,T;V'_\tau)} \leq C_1 \quad \text{and} \quad \left\|\frac{\partial v^m}{\partial t}\right\|_{L^2(0,T;V'_\tau)} \leq C_2\,. \tag{3.14}$$

Hence, by means of reflexivity,

$$\frac{\partial u^m}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial v^m}{\partial t} \longrightarrow \frac{\partial v}{\partial t} \quad \text{weakly in } L^2(0,T;V'_\tau), \quad \text{as } m\to\infty. \tag{3.15}$$

Now, due to (3.13) and (3.15), and observing the compact and continuous imbeddings $V_\tau \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow V'_\tau$ hold, we can use Aubin-Lions compactness lemma to prove that

$$u^m \longrightarrow u \quad \text{and} \quad v^m \longrightarrow v \quad \text{strongly in } L^2(Q_T), \quad \text{as } m\to\infty. \tag{3.16}$$

Thus, from the assumptions (2.3)-(2.4) and (2.5)-(2.6), we have

$$f(u^m, v^m) \longrightarrow f(u,v) \quad \text{strongly in } L^2(Q_T), \quad \text{as } m\to\infty\,, \tag{3.17}$$

$$g(u^m, v^m) \longrightarrow g(u,v) \quad \text{strongly in } L^2(Q_T), \quad \text{as } m\to\infty\,. \tag{3.18}$$

On the other hand, from the continuity of p, q, r and s (see (1.7)) and from the continuity of a_1 and a_2 (2.1), we can use (3.16) to prove that

$$a_1(p(u^m), q(v^m)) \longrightarrow a_1(p(u), q(v)) \quad \text{strongly in } L^2(0,T), \quad \text{as } m\to\infty\,, \tag{3.19}$$

$$a_2(r(u^m), s(v^m)) \longrightarrow a_2(r(u), s(v)) \quad \text{strongly in } L^2(0,T), \quad \text{as } m\to\infty\,. \tag{3.20}$$

Then, from Riesz-Fischer theorem we have, up to some subsequences,

$$a_1(p(u^m), q(v^m)) \longrightarrow a_1(p(u), q(v)) \quad \text{a.e. in } [0,T], \quad \text{as } m\to\infty\,, \tag{3.21}$$

$$a_2(r(u^m), s(v^m)) \longrightarrow a_2(r(u), s(v)) \quad \text{a.e. in } [0,T], \quad \text{as } m\to\infty\,. \tag{3.22}$$

Finally, using the convergence results (3.13), (3.15), (3.17)-(3.18) and (3.21)-(3.22), we can pass (3.2) and (3.3) to the limit $m \to \infty$ to prove that (2.7) and (2.8) hold in $\mathcal{D}'(0,T)$, first for all $\varphi,\ \eta \in \{\phi_1, \dots, \phi_m\}$, then, by linearity, for all $\varphi,\ \eta \in V_\tau^m$ and next, by continuity, for all $\varphi,\ \eta \in V_\tau$. In particular, and once that by (3.13) $u(t),\ v(t) \in V_\tau$ for a.e. $t \in [0,T]$, we can take $\varphi = u(t)$ in (2.7) and $\eta = v(t)$ in (2.8) to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)|^2\,dx + a_1(p(u(t)),q(v(t)))\int_\Omega |\nabla u(t)|^2\,dx = \int_\Omega f(u(t),v(t))u(t)\,dx \quad \text{in} \quad \mathcal{D}'(0,T)\,, \tag{3.23}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v(t)|^2\,dx + a_2(r(u(t)),s(v(t)))\int_\Omega |\nabla v(t)|^2\,dx = \int_\Omega g(u(t),v(t))v(t)\,dx \quad \text{in} \quad \mathcal{D}'(0,T)\,. \tag{3.24}$$

Then, arguing as we did for (3.11), but taking the supreme, we obtain from (3.23)-(3.24) that

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q_T)} \le C\,, \tag{3.25}$$

$$\sup_{t\in[0,T]} \|v(t)\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(Q_T)} \le C\,. \tag{3.26}$$

As a consequence of (3.25)-(3.26), we have $u, v \in \mathrm{C}([0,T]; L^2(\Omega))$.

On the other hand, observe that we can write

$$\int_\Omega (u^m(t) - u^m(0))\,\varphi\,dx = \int_0^t\int_\Omega \frac{\partial u^m}{\partial s}\varphi\,dx d\varsigma \quad \text{for a.e. } t \in [0,T]$$

Using (3.4)-(3.5) and (3.15), we can pass the above equation to the limit $m \to \infty$ to obtain

$$\int_\Omega (u(t) - u_0)\,\varphi\,dx = \int_0^t\int_\Omega \frac{\partial u}{\partial s}\varphi\,dx d\varsigma = \int_\Omega (u(t) - u(0))\,\varphi\,dx \quad \text{for a.e. } t \in [0,T]. \tag{3.27}$$

Consequently $u(0) = u_0$. By a completely analogous reasoning, we also have $v(0) = v_0$. The proof of Theorem 2.1 is thus concluded.

4. Local existence

In this section, we establish a local version of Theorem 2.1. This result shall be proved under the assumptions that the nonlocal functions a_1 and a_2 are strictly positive in some neighborhoods. Before we establish the existence result of this section, let us fix some notation first. For each $i \in \{1,\ 2\}$, we consider the open ball $B_{\delta_i}(\xi_i,\eta_i)$ and the closed ball $\overline{B}_{\delta_i}(\xi_i,\eta_i)$ centered at $(\xi_i,\eta_i) \in \mathbb{R}^2$ and with radius δ_i. We stress here that the functions a_1 and a_2 have the arguments satisfying to (1.7).

Theorem 4.1. *Assume that all the conditions of Theorem 2.1 are satisfied, with the exception of (2.2). In addition, assume that*

$$a_1 : \overline{B}_{\delta_1}(\xi_1,\eta_1) \to (0,\infty), \tag{4.1}$$

$$a_2 : \overline{B}_{\delta_2}(\xi_2,\eta_2) \to (0,\infty) \tag{4.2}$$

for some $(\xi_1,\eta_1),\ (\xi_2,\eta_2) \in \mathbb{R}^2$ *and for some* $\delta_1,\ \delta_2 > 0$. *If*

$$(p(u_0),q(v_0)) \in B_{\delta_1}(\xi_1,\eta_1)\,, \tag{4.3}$$

$$(r(u_0),s(v_0)) \in B_{\delta_2}(\xi_2,\eta_2)\,, \tag{4.4}$$

then there exists $T_0 > 0$*, and a weak solution* (u,v) *to the problem (1.1)-(1.5), for either* $\tau = 0$ *or* $\tau = 1$*, such that* $u,\ v \in C([0,T_0];L^2(\Omega)) \cap L^2(0,T_0;V_\tau)$*,* $u_t,\ v_t \in L^2(0,T_0;V_\tau')$*,* $u(0) = u_0$ *and* $v(0) = v_0$*, and the integral identities (2.7) and (2.8) hold in* $\mathcal{D}'(0,T_0)$ *and for all* $\varphi,\ \eta \in V_\tau$.

In the proof of this result, we follow the approach of [5].

Proof. For each $i \in \{1,\ 2\}$, let us consider the following radial extension of a_i,

$$A_i(\xi,\eta) = \begin{cases} a_i(\xi,\eta) & \text{if } (\xi,\eta) \in \overline{B}_{\delta_i}(\xi_i,\eta_i) \\ a_i(\delta_i\cos(\theta_i),\delta_i\sin(\theta_i)) & \text{if } (\xi,\eta) = (\delta\cos(\theta_i),\delta\sin(\theta_i)) \text{ and } \delta > \delta_i\,. \end{cases} \tag{4.5}$$

From the assumptions (2.1) and by its definition set in (4.5), it can be proved that A_i is continuous and bounded in $\mathbb{R}^2$ for any $i \in \{1,\ 2\}$ (see [22, p. 153]). In particular, by Weierstrass theorem, we have

$$0 < \overline{m}_i := \min_{(\xi,\eta)\in\overline{B}_{\delta_i}(\xi_i,\eta_i)} a_i(\xi,\eta) \le A_i(\xi,\eta) \le \max_{(\xi,\eta)\in\overline{B}_{\delta_i}(\xi_i,\eta_i)} a_i(\xi,\eta) := \overline{M}_i < \infty \tag{4.6}$$

for all $\xi,\ \eta \in \mathbb{R}$ and for any $i \in \{1,\ 2\}$. Then , in view of Theorem 2.1 and for any $T > 0$, the problem (1.1)-(1.5), with A_1 and A_2 in the places of a_1 and a_2, has a weak solution (u, v), with $u,\ v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V_\tau)$, $u(0) = u_0$ and $v(0) = v_0$, and such that (2.7)-(2.8), with A_1 and A_2 in the places of a_1 and a_2, hold in $\mathcal{D}'(0, T)$. Moreover (2.16)-(2.17) also hold. In particular, from (2.16) and from (1.7), we also have

$$p(u),\ q(v),\ r(u),\ s(v) \in C([0, T])\,. \tag{4.7}$$

As a consequence of the assumptions (4.3)-(4.4) and of (4.7), $(p(u(t)), q(v(t))$ and $(r(u(t)), s(v(t))$ will remain in some neighborhoods of $(p(u_0), q(v_0))$ and $(r(u_0), s(v_0))$, respectively, for t sufficiently close to 0. Therefore, there exist positive times T_0^1 and T_0^2, sufficiently close to 0, such that

$$\begin{aligned}(p(u(t)), q(v(t)) &\in B_{\delta_1}(\xi_1, \eta_1) \quad \forall\ t \in [0, T_0^1]\,,\\ (r(u(t)), s(v(t)) &\in B_{\delta_2}(\xi_2, \eta_2) \quad \forall\ t \in [0, T_0^2]\,.\end{aligned}$$

Finally, we take $T_0 = \min\{T_0^1, T_0^2\}$ which concludes the proof of Theorem 4.1. □

5. Uniqueness

Here, we will adapt the results of [8, 13] to establish an uniqueness result. Lipschitz conditions on the nonlocal diffusivity terms and on the reaction functions (already assumed at (2.3) and (2.5)) play a fundamental role.

Theorem 5.1. *Let (u_1, v_1) and (u_2, v_2) be two weak solutions to the problem (1.1)-(1.5), for either $\tau = 0$ or $\tau = 1$, in the sense of Definition 2.1. Let the conditions (2.2), (2.3) and (2.5) be fulfilled, and assume that (1.7) is satisfied with*

$$1 \le \gamma_p,\ \gamma_q,\ \gamma_r,\ \gamma_s \le 2\,. \tag{5.1}$$

If for each $i \in \{1,\ 2\}$, there exists a positive constant C_{a_i} such that

$$|a_i(\xi_1, \eta_1) - a_i(\xi_2, \eta_2)| \le C_{a_i}|(\xi_1, \eta_1) - (\xi_2, \eta_2)| \quad \forall\ (\xi_1, \eta_1),\ (\xi_2, \eta_2) \in \mathbb{R}^2\,, \tag{5.2}$$

then $(u_1, v_1) = (u_2, v_2)$.

Proof. By the Definition 2.1, $u_1(t)$, $u_2(t)$, $v_1(t)$ and $v_2(t)$ are in V_τ for a.e. $t \in [0, T]$. Thus, we can take $\varphi = u(t) := u_2(t) - u_1(t)$ and $\eta = v(t) := v_2(t) - v_1(t)$ in (2.7) and (2.8), considered for (u_1, v_1) and (u_2, v_2) separately. After some algebraic manipulations, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(|u(t)|^2 + |v(t)|^2\right)\,dx +\\ &a_1(p(u_2(t)), q(v_2(t)))\int_\Omega |\nabla u(t)|^2\,dx + a_2(r(u_2(t)), s(v_2(t)))\int_\Omega |\nabla v(t)|^2\,dx\\ =&\int_\Omega \left(f(u_2(t), v_2(t)) - f(u_1(t), v_1(t))\right) u(t)\,dx + \int_\Omega \left(g(u_2(t), v_2(t)) - g(u_1(t), v_1(t))\right) v(t)\,dx +\\ &[a_1(p(u_1(t)), q(v_1(t))) - a_1(p(u_2(t)), q(v_2(t)))]\int_\Omega \nabla u_1(t)\cdot\nabla u(t)\,dx +\\ &[a_2(r(u_1(t)), s(v_1(t))) - a_2(r(u_2(t)), s(v_2(t)))]\int_\Omega \nabla v_1(t)\cdot\nabla v(t)\,dx\,.\end{aligned} \tag{5.3}$$

Let us denote by I_k the term that appears in the k-th position in this equation. By the assumption (2.2), we have

$$m\left(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)}\right) \le I_2 + I_3, \tag{5.4}$$

where $m := \min\{m_1, m_2\}$. Using the Schwarz and Cauchy inequalities together with the assumptions (2.3) and (2.5), we have

$$I_4 + I_5 \le C_L \int_\Omega \left(|u(t)|^2 + |v(t)|^2\right)\,dx, \tag{5.5}$$

where $C_L := \frac{3}{2}\max\{C_{L_1}, C_{L_2}\}$. For the two reminder terms, we first observe that we can use the assumptions (5.2) and (1.7) together with Hölder's inequality and assumption (5.1) in order to get

$$\begin{aligned} &|a_1(p(u_1(t)), q(v_1(t))) - a_1(p(u_2(t)), q(v_2(t)))| \le C_{11}\left(|p(u(t)| + |q(v(t)|\right) \\ &\le C_{12}\left(\|u(t)\|_{L^{\gamma_p}(\Omega_p)} + \|v(t)\|_{L^{\gamma_q}(\Omega_q)}\right) \le C_{13}\left(\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\right), \end{aligned} \tag{5.6}$$

where $C_{11} = C_{a_1}$, $C_{12} = C\left(C_{11}, \|p\|_{(L^{\gamma_p}(\Omega_p))'}, \|q\|_{(L^{\gamma_q}(\Omega_q))'}\right)$ and $C_{13} = C\left(C_{12}, \gamma_p, \gamma_q, \Omega_p, \Omega_q\right)$ are positive constants. Arguing in the same way, we obtain

$$\begin{aligned} &|a_2(r(u_1(t)), q(s_1(t))) - a_2(r(u_2(t)), s(v_2(t)))| \le C_{21}\left(|r(u(t)| + |s(v(t)|\right) \\ &\le C_{22}\left(\|u(t)\|_{L^{\gamma_r}(\Omega_r)} + \|v(t)\|_{L^{\gamma_s}(\Omega_s)}\right) \le C_{23}\left(\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\right). \end{aligned} \tag{5.7}$$

where here $C_{21} = C_{a_2}$, $C_{22} = C\left(C_{21}, \|r\|_{(L^{\gamma_r}(\Omega_r))'}, \|s\|_{(L^{\gamma_s}(\Omega_s))'}\right)$ and $C_{23} = C\left(C_{21}, \gamma_r, \gamma_s, \Omega_r, \Omega_s\right)$. Then plugging (5.6)-(5.7) into the sixth and seventh terms of (5.3), and using in addition Cauchy's inequality, we obtain

$$\begin{aligned} I_6 \le& C_{13}\left(\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\right)\|\nabla u(t)\|_{L^2(\Omega)}\|\nabla u_1(t)\|_{L^2(\Omega)} \\ \le& \frac{m}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + C_{13}(t)\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{5.8}$$

where C_{13} is the constant from the inequality (5.6) and

$$C_{13}(t) := \frac{C_{13}^2}{m}\|\nabla u_1(t)\|^2_{L^2(\Omega)},$$

and

$$\begin{aligned} I_7 \le& C_{23}\left(\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\right)\|\nabla v(t)\|_{L^2(\Omega)}\|\nabla v_1(t)\|_{L^2(\Omega)} \\ \le& \frac{m}{2}\|\nabla v(t)\|^2_{L^2(\Omega)} + C_{23}(t)\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{5.9}$$

where, in this case, C_{23} is the constant from the inequality (5.7) and

$$C_{23}(t) := \frac{C_{23}^2}{m}\|\nabla v_1(t)\|^2_{L^2(\Omega)}.$$

Now, gathering the information of (5.4)-(5.9) in (5.3), we get

$$\begin{aligned} &\frac{d}{dt}\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right) + m\left(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)}\right) \le \\ &\frac{m}{2}\left(\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)}\right) + C(t)\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right) \end{aligned} \tag{5.10}$$

where $C(t) := \max\{C_{13}(t), C_{23}(t)\}$. From (5.10), we readily obtain

$$\frac{d}{dt}\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right) \le C(t)\left(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right) \tag{5.11}$$

Observing that, by the Definition 2.1, $\|\nabla u_1\|^2_{L^2(\Omega)}$, $\|\nabla v_1\|^2_{L^2(\Omega)} \in L^1([0,T])$, we have for the coefficient function $C(t)$ defined at(5.11) that $C \in L^1[0,T]$. Hence, a simple integration, between 0 and an arbitrary $\varsigma \in (0,T]$, of (5.11) leads us to

$$\|u(\varsigma)\|^2_{L^2(\Omega)} + \|v(\varsigma)\|^2_{L^2(\Omega)} \le \left(\|u(0)\|^2_{L^2(\Omega)} + \|v(0)\|^2_{L^2(\Omega)}\right)e^{\int_0^s C(t)\,dt}.$$

Finally, since $u(0) = u_2(0) - u_1(0) = 0$ and $v(0) = v_2(0) - v_1(0) = 0$, we have $u_1 = u_2$ and $v_1 = v_2$. □

6. Strong solutions

In this section we will find conditions on the data of the problem (1.1)-(1.5) under which the solutions found in the previous sections are more regular. We prove that the time derivatives u_t and v_t are square sumable in Q_T and we establish a result that gives us more spatial regularity for the solution (u,v).

Theorem 6.1. *Let (u,v) be a weak solution to the problem (1.1)-(1.5) in the conditions of Theorem 4.1 such that*

$$u_0,\ v_0 \in H^1(\Omega). \tag{6.1}$$

If

$$a_i \in C^1(0,T) \quad \textit{and} \quad a_i \textit{ is non-increasing in } t, \quad \textit{for all } i\in\{1,\ 2\}\,, \tag{6.2}$$

then

$$u_t,\ v_t \in L^2(0,T;L^2(\Omega))\,, \tag{6.3}$$

$$u,\ v \in L^2(0,T;H^2(\Omega)). \tag{6.4}$$

Proof. To prove (6.3), we start by considering the Galerkin approximations $u^m(t)$ defined at (3.1)-(3.2). Here, and in addition to (3.4)-(3.5), we assume these approximations satisfy to

$$\nabla u^m(0) = D_0^m\,, \tag{6.5}$$

where D_0^m is chosen in such a way that

$$D_0^m \to \nabla u_0 \quad \text{strongly in } L^2(\Omega), \quad \text{as } m\to\infty. \tag{6.6}$$

Next, we consider a sequence of functions $\varrho_n \in C^1(0,T)$, with $n\in\mathbb{N}$, such that

$$\varrho_n(t) = \begin{cases} 1 & \text{if } 0\le t\le T_n := T-\frac{T}{n+2} \\ 0 & \text{if } T-\frac{T}{n+1} := T_{n+1}\le t\le T \end{cases} \qquad \text{and} \quad 0\le\varrho_n\le 1\,, \quad -1\le\varrho_n'\le 0,$$

for any $n\in\mathbb{N}$. We take

$$\varphi = \varphi_k \frac{d\,c_k^m}{d\,t}\varrho_n$$

in (3.2) and we add up the resulting equation from $k=1$ until $k=m$. Hence, we obtain

$$\begin{aligned} &\int_\Omega |u_t^m(t)|^2\varrho_n(t)\,dx + a_1(p(u^m(t)),q(v^m(t)))\int_\Omega \frac{d\,|\nabla u^m(t)|^2}{d\,t}\varrho_n(t)\,dx \\ =&\int_\Omega f(u^m(t),v^m(t))\,u_t^m(t)\varrho_n(t)\,dx \end{aligned} \tag{6.7}$$

for a.e. $t\in[0,T]$. Integrating in $[0,T_{n+1}]$, we obtain

$$\begin{aligned} &\frac12\int_0^{T_{n+1}}\int_\Omega |u_t^m|^2\varrho_n\,dxdt - \int_0^{T_{n+1}} a_1'(p(u^m(t)),q(v^m(t)))\int_\Omega |\nabla u^m|^2\varrho_n(t)\,dxdt \\ &-\int_0^{T_{n+1}} a_1(p(u^m(t)),q(v^m(t)))\int_\Omega |\nabla u^m|^2\varrho_n'(t)\,dxdt \\ \le& a_1(p(u_0^m),q(v_0^m))\int_\Omega |D_0^m|^2dx + \frac12\int_0^{T_{n+1}}\int_\Omega |f(u^m,v^m)|^2\varrho_n\,dxdt\,, \end{aligned} \tag{6.8}$$

where we have used integration by parts on the second term of (6.7) together with (6.5). The assumption that $a_1\in C^1(0,T)$ (see (6.2)) and the Cauchy-Schwarz inequality (on the last term) were also used in the derivation of (6.8). Then, using the assumptions (2.2) and (2.3) together with the properties of the sequence ϱ_n, and the fact that a_1 is non-increasing in t (see (6.2)), we obtain

$$\begin{aligned} &\frac12\int_0^{T_n}\int_\Omega |u_t^m|^2\,dxdt + m_1\int_0^{T_n}\int_\Omega |\nabla u^m|^2\,dxdt \\ \le&\frac12\int_0^{T_{n+1}}\int_\Omega |u_t^m|^2\rho_n\,dxdt + m_1\int_0^{T_{n+1}}\int_\Omega |\nabla u^m|^2\rho_n\,dxdt \\ \le& M_1\int_\Omega |D_0^m|^2dx + C_{L_1}\int_0^T\int_\Omega (|u^m|^2+|v^m|^2)\,dxdt\,. \end{aligned}$$

Letting $n\to\infty$ first and then making $m\to\infty$, we obtain

$$\begin{aligned} &\frac12\int_{Q_T} |u_t|^2dxdt + m_1\int_{Q_T} |\nabla u|^2\,dxdt \\ \le& M_1\int_\Omega |\nabla u_0|^2dx + C_{L_1}\int_{Q_T} (|u|^2+|v|^2)\,dxdt\,, \end{aligned} \tag{6.9}$$

where we have used (3.16) and (6.6). Finally, due to (3.13) and to (6.1), we conclude that $u_t\in L^2(0,T;L^2(\Omega))$. Analogously, it can be proved that $v_t\in L^2(0,T;L^2(\Omega))$.

The next step is to prove (6.4). To prove this, let us consider a fixed, but arbitrary, open bounded domain $U \subset\subset \Omega$ and let us choose another open bounded domain W such that $U \subset\subset W \subset\subset \Omega$. Then we consider a function $\zeta \in C^\infty(\mathbb{R}^N)$ such that

$$\zeta(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \in \mathbb{R}^N \setminus W \end{cases} \qquad \text{and} \quad 0 \le \zeta \le 1\,.$$

We consider the difference quotient $D_k^h u(t)$ of the (weak) partial derivative $u(t)$, defined by

$$D_k^h u(x,t) := \frac{u(x+he_k,t) - u(x,t)}{h}\,, \quad k = 1,\dots,N\,,$$

for $x \in U$ and $h \in \mathbb{R}\setminus\{0\}$ such that $|h| < \operatorname{dist}(U, \partial\Omega)$. Then we take for test function in (2.7)

$$\varphi := -D_k^{-h}(\zeta^2 D_k^h u(t)) \quad \text{a.e. in } t\,.$$

We observe that whenever the following relations are possible, we have

$$\begin{aligned} &D_k^h(\theta\vartheta) = \theta^h D_k^h\vartheta + \vartheta D_k^h\theta\,, \quad \text{where } \theta^h := \theta(x+he_k)\,, \\ &(D_k^h\theta)_{x_i} = D_k^h\theta_{x_i}\,, \quad i = 1,\dots,N\,, \\ &\int_\Omega \theta D_k^{-h}\vartheta\,dx = -\int_\Omega D_k^h\theta\,\vartheta\,dx \end{aligned}$$

for all admissible functions θ and ϑ (see *e.g.* [15]). Hence, choosing $k \in \{1,\dots,N\}$ we have for a.e. $t \in [0,T]$ that

$$\begin{aligned} &a_1(p(u(t)), q(v(t))) \sum_{i=1}^N \int_\Omega D_k^h u_{x_i}(t) D_k^h u_{x_i}(t)\zeta^2\,dx \\ =&\int_\Omega [u_t(t) - f(u(t),v(t))]\left[(\zeta^2)^{-h} D_k^h u(t) + \zeta^2 D_k^{-h}(D_k^h u(t))\right] dx \\ &- a_1(p(u(t)), q(v(t))) \sum_{i=1}^N \int_\Omega D_k^h u_{x_i}(t) D_k^h u(t) 2\zeta\zeta_{x_i}\,dx\,. \end{aligned}$$

Now, using the assumptions (2.2) and (2.3) together with the Cauchy-Schwarz inequality, and observing that $0 \le \zeta \le 1$ and $|\nabla\zeta| \le C$, where C is a positive constant, we have

$$\begin{aligned} &m_1 \int_\Omega |D_k^h(\nabla u(t))|^2\zeta^2\,dx \\ \le&\epsilon \int_\Omega |D_k^{-h}(D_k^h u(t))|^2\,dx + C(\epsilon)\int_\Omega |u_t(t)|^2 dx + C(\epsilon)C_{L_1}^2 \int_\Omega \left(|u(t)|^2 + |v(t)|^2\right) dx \\ &+ M_1\epsilon \int_\Omega |D_k^h(\nabla u(t))|^2\zeta^2\,dx + M_1 C(\epsilon) C^2 \int_\Omega |D_k^h u(t)|^2 dx\,. \end{aligned} \tag{6.10}$$

Then, observe that, by the properties of the difference quotients (see *e.g.* [15]), there exists a constant C_0 such that

$$\begin{aligned} \int_\Omega |D_k^{-h}(D_k^h u(t))|^2\,dx \le& C_0 \int_U |\nabla(D_k^h u(t))|^2\,dx \le C_0 \int_W |D_k^h(\nabla u(t))|^2\zeta^2\,dx \\ \le& C_0 \int_\Omega |\nabla(D_k^h u(t))|^2\zeta^2\,dx\,. \end{aligned} \tag{6.11}$$

Gathering the information of (6.10)-(6.11), choosing an ϵ such that $0 < \epsilon < \frac{m_1}{C_0 + M_1}$ and using the reasoning of (6.11) on the last term of (6.10), we get

$$\begin{aligned} \int_U |D_k^h(\nabla u(t))|^2\,dx \le& \int_W |D_k^h(\nabla u(t))|^2\zeta^2\,dx \le \int_\Omega |D_k^h(\nabla u(t))|^2\zeta^2\,dx \\ \le& C\left(\int_\Omega |u_t(t)|^2 dx + \int_\Omega \left(|u(t)|^2 + |v(t)|^2\right) dx + \int_\Omega |\nabla u(t)|^2 dx\right), \end{aligned} \tag{6.12}$$

where the positive constant C depends on m_1, C_{L_1} and C_0. Using a well-known result of the difference quotients (see *e.g.* [15, Theorem 5.8.3]), we obtain for a.a. $t \in [0,T]$

$$\int_\Omega |D^2 u(t)|^2 dx \leq C \left(\int_\Omega |u_t(t)|^2 dx + \int_\Omega \left(|u(t)|^2 + |v(t)|^2 \right) dx + \int_\Omega |\nabla u(t)|^2 dx \right).$$

Integrating the last relation in the interval $[0,T]$ and using (3.13) and (6.3), we prove finally that $u \in L^2(0,T;H^2(\Omega))$. Analogously, it can be proved that $v \in L^2(0,T;H^2(\Omega))$. □

7. Existence of Blow-up

In this section, we will establish several bow-up results for the strong solutions to the problem (1.1)-(1.5). By a strong solution, we mean here a solution (u,v) in the conditions of Theorem 6.1. For a given solution (u,v) to the reaction-diffusion system (1.1)-(1.5), for either $\tau = 0$ or $\tau = 1$, we define

$$t_* := \sup \{ t : (u,v) \text{ is bounded in } \Omega \times [0,t), \text{ and satisfies to (1.1)-(1.5) there} \}$$

If $t_* = \infty$, the solution (u,v) is global, since, as in the local problem, it can be shown (see e.g. [14]) that u and v can be continued for all times $t > 0$. On the other hand, if $t_* < \infty$, we have

$$\limsup_{t \to t_*^-} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \right) = \infty. \tag{7.1}$$

When this happens, we say the solution (u,v) under consideration blows up in the finite time t_*.

Blow-up criteria for systems of parabolic equations are normally more difficult to find than for the scalar case. The following version of Jensen's inequality will allow us to develop some blow-up criteria to our reaction-diffusion system (1.1)-(1.5).

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and assume that $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is convex. Then for every $u,\ v \in L^1(\Omega)$,*

$$F\left(⨍_\Omega u\,dx, ⨍_\Omega v\,dx \right) \leq ⨍_\Omega F(u,v)\,dx. \tag{7.2}$$

Proof. Due to the convexity of F, for each $(x_1,x_2) \in \mathbb{R}^2$ there exists $(z_1,z_2) \in \mathbb{R}^2$ such that

$$F(y_1,y_2) \geq F(x_1,x_2) + z_1(y_1 - x_1) + z_2(y_2 - x_2)$$

holds for all $(y_1,y_2) \in \mathbb{R}^2$, *i.e.* the graph of F lies above its supporting hyperplane at (x_1,x_2). In this inequality, let us take $x_1 = ⨍_\Omega u\,dx$, $x_2 = ⨍_\Omega v\,dx$, $y_1 = u$ and $y_2 = v$. This yields

$$F(u,v) \geq F\left(⨍_\Omega u\,dx, ⨍_\Omega v\,dx \right) + z_1 \left(u - ⨍_\Omega u\,dx \right) + z_2 \left(v - ⨍_\Omega v\,dx \right).$$

Then, integrating over Ω, with respect to x, and observing that the terms which are multiplied by z_1 and z_2 vanish, we immediately arrive at (7.2). □

As a first example of the utility of the Lemma 7.1, we have the following blow-up result under Neumann boundary conditions.

Theorem 7.1. *Let (u,v) be a couple of strong solutions to the reaction-diffusion system (1.1)-(1.5) endowed with the Neumann boundary conditions,* i.e. *with $\tau = 1$ in (1.4)-(1.5). Assume that*

1. *f and g are convex functions,*
2. *$f(u,v) + g(u,v) \geq h(u+v)$ for all $(u,v) \in \mathbb{R}^2$ and for some function $h : \mathbb{R} \longmapsto \mathbb{R}$ such that $h(w) > 0$ for all $w \geq \overline{u_0 + v_0}$.*

If

$$t_* := \int_{\overline{u_0+v_0}}^\infty \frac{1}{h(\varsigma)} d\varsigma < \infty, \quad \text{where} \quad \overline{u_0 + v_0} = ⨍_\Omega u_0 + v_0\,dx\,, \tag{7.3}$$

then the solution (u,v) to the reaction-diffusion system (1.1)-(1.5) with $\tau = 1$ blows-up in the finite time t_*.

Remark 7.1. *Some functions satisfying to condition (2) are, in the case of* $u_0+v_0>0$, $f(s,r)=|r|^p$ *and* $g(s,r)=|s|^q$ *for suitable* $p\geq q>1$, *or still more general* $f(s,r)=a|r|^p+b|s|^q$ *and* $g(s,r)=c|r|^{\overline{p}}+d|s|^{\overline{q}}$, *where* a, b, c, d *are positive real constants and* p, q, $\overline{p}$, $\overline{q}>1$. *In the case of* $u_0+v_0\leq 0$, *one should consider, for instance, examples of the form* $f(u,v)=g(u,v)=h(u+v)=(1+|u+v|)^p$, *for some* $p>1$, *or* $f(u,v)=g(u,v)=h(u+v)=e^{u+v}$.

Proof. Adding up the equations (1.1) and (1.2), we obtain

$$(u+v)_t-a_1(p(u(t)),q(v(t)))\Delta u-a_2(q(u),r(v))\Delta v=f(u,v)+g(u,v).$$

Let use the notations $\overline{u(t)}=\fint_\Omega u(t)\,dx$ and $\overline{v(t)}=\fint_\Omega v(t)\,dx$. Integrating the above equation over Ω, using Gauss-Green's theorem together with (1.4)-(1.5) with $\tau=1$, and invoking the nonlocal character of a_1 and a_2, we obtain

$$\frac{d\,\overline{u(t)+v(t)}}{dt}=\fint_\Omega f(u,v)\,dx+\fint_\Omega g(u,v)\,dx\,.$$

Then, Lemma 7.1 and assumption (2) yield

$$\frac{d\,\overline{u(t)+v(t)}}{dt}\geq f\left(\overline{u(t)},\overline{v(t)}\right)+g\left(\overline{u(t)},\overline{v(t)}\right)\geq h(\overline{u(t)+v(t)})\,.$$

Finally, integrating between 0 and $t>0$ and using (1.3) together with (7.3), we obtain

$$\begin{aligned} t &\leq \int_0^t \frac{1}{h(\overline{u(t)+v(t)})}\frac{d\,\overline{u(t)+v(t)}}{d\tau}\,d\tau \\ &= \int_{\overline{u_0+v_0}}^{\overline{u(t)+v(t)}}\frac{d\varsigma}{h(\varsigma)}\leq\int_{\overline{u_0+v_0}}^{\infty}\frac{1}{h(\varsigma)}d\varsigma<\infty\,. \end{aligned}$$

Then, from a well-known result (see *e.g.* [3, Theorem 13.11]), we conclude that $\overline{u+v}$ will blow up in the finite time t_* provided that $h(w)>0$ for all $w\geq\overline{u_0+v_0}$. That (u,v) blows up in the sense of (7.1), is an immediate consequence. □

In the next result, we establish a blow-up criterium under Dirichlet boundary conditions.

Theorem 7.2. *Let* (u,v) *be a couple of strong solutions to the reaction-diffusion system (1.1)-(1.5) endowed with the Dirichlet boundary conditions,* i.e. *with* $\tau=0$ *in (1.4)-(1.5). Assume that*

(1) f *is convex,*

(2) $f(u,v)\geq f(u,0)$ *for all* $(u,v)\in\mathbb{R}^2$,

(3) $f(w,0)+\lambda_P a_1 w>0$ *for all* $w\geq\mu(0)$, *where*

$$\mu(t):=\fint_\Omega u(t)\phi\,dt\,,\quad a_1(t):=a_1(p(u(t)),q(v(t)))\,,$$

λ_P and ϕ are the principal eigenvalue and the associated eigenfunction of the Laplacian problem (2.10), restricted to the case of

$$\int_\Omega\phi\,dx=1\,. \tag{7.4}$$

If

$$t_*:=\int_{\mu(0)}^{\infty}\frac{d\mu}{f(\mu,0)+\lambda_P a_1\mu}<\infty\,, \tag{7.5}$$

then the first component of the solution (u,v) *to the reaction-diffusion system (1.1)-(1.5), with* $\tau=0$, *blows-up in the finite time* t_*.

Proof. We start by multiplying the equation (1.1) by φ, we integrate over Ω and we use (1.1), with $\tau=0$, and (2.10) together with the nonlocal character of a_1. After all, we obtain

$$\frac{d}{dt}\int_\Omega u(t)\varphi\,dx-\lambda_P a_1(t)\int_\Omega u(t)\varphi\,dx=\int_\Omega f(u(t),v(t))\varphi\,dx\,.$$

Observing (7.4), we can use Jensen's inequality (7.2), to prove that

$$\fint_\Omega f(u,v)\varphi\,dx \geq f\left(\fint_\Omega u\,\varphi\,dx, \fint_\Omega v\,\varphi\,dx\right).$$

Replacing this into the previous equation and, in addition, using the hypothesis that $f(u,v) \geq f(u,0)$ for all $(u,v) \in \mathbb{R}^2$, we get

$$\mu'(t) - \lambda_P a_1(t)\mu(t) \geq f(\mu(t), 0).$$

It should be noted that $\mu(t)$ is well defined on the existence interval of the solution u. Then, integrating the last inequality between 0 and $t > 0$, and using the fact that $\mu(0) \geq 0$ and hypothesis (7.5), we obtain

$$t \leq \int_0^t \frac{\mu'(\tau)}{f(\mu(\tau),0) + \lambda_P a_1(\tau)\mu(\tau)}\,d\tau = \int_{\mu(0)}^{\mu(t)} \frac{d\mu}{f(\mu,0)+\lambda_P a_1 \mu}$$
$$\leq \int_{\mu(0)}^{\infty} \frac{d\mu}{f(\mu,0)+\lambda_1 a_1\mu} < \infty\,.$$

Then, from [3, Theorem 13.11], we conclude that $\mu(t)$, and consequently u, will blow up in a finite time provided that $f(w,0) + \lambda_P a_1 w > 0$ for all $w \geq \mu(0)$. □

Remark 7.2. *We observe that according to the proof of the last result, we had no need to use the boundary condition $v = 0$ on $\partial\Omega$. Therefore, we still have blow up of the first component even if v is not prescribed at the boundary.*

We end this section by giving a criterium of blow up of both components of the solution (u,v) to the reaction-diffusion system (1.1)-(1.5) endowed with Neumann boundary conditions. It should be remarked that, in the case of condition (7.1) is satisfied, there is, *a priori*, no reason for both components of the system (1.1)-(1.5) to blow up. Indeed it may happen that one of the components of (u,v) blows up as $t \to t_*^-$, while the other remains bounded on $[0, t_*^-)$. Thus condition (7.1) only implies that

$$\limsup_{t\to t_*^-} \|u(t)\|_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \limsup_{t\to t_*^-} \|v(t)\|_{L^\infty(\Omega)} = \infty. \tag{7.6}$$

If

$$\limsup_{t\to t_{1*}^-} \|u(t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \limsup_{t\to t_{2*}^-} \|v(t)\|_{L^\infty(\Omega)} = \infty,$$

for possibly distinct times t_{1*} and t_{2*}, we shall say that both u and v blow up in finite times. When this happens at the same time t_*, *i.e.* when $t_* = t_{1*} = t_{2*}$, we say that u and v blow simultaneously (in the finite time t_*).

Theorem 7.3. *Let (u,v) be a couple of strong solutions to the reaction-diffusion system (1.1)-(1.5) endowed with the Neumann boundary conditions,* i.e. *with $\tau = 1$ in (1.4)-(1.5). Assume that*

1. *$f(u,v) = f(v)$ and $g(u,v) = f(u)$, or $f(u,v) = f(u)$ and $g(u,v) = f(v)$,*
2. *f is a convex function,*
3. *$f(w) > 0$ for all $w \geq \min\{\overline{u_0}, \overline{v_0}\}$.*

If

$$t_* := \int_{\frac{\overline{u_0+v_0}}{2}}^{\infty} \frac{d\varsigma}{f(\varsigma)} < \infty\,, \quad \textit{where} \quad \overline{u_0+v_0} = \fint_\Omega u_0 + v_0\,dx\,, \tag{7.7}$$

then both u and v blow up, one in the finite time t_* and the other in another instant that can be posterior.

Proof. Arguing as we did in the first part of the proof of Theorem 7.1, we obtain

$$\frac{d}{dt}\left(\frac{\overline{u(t)+v(t)}}{2}\right) = \frac{1}{2}\fint_\Omega \left(f(u(t)) + f(v(t))\right)\,dx$$
$$\geq \frac{1}{2} f(\overline{u(t)}) + \frac{1}{2} f(\overline{v(t)}) \geq f\left(\frac{\overline{u(t)+v(t)}}{2}\right).$$

Recall that $\overline{u(t)} = \fint_\Omega u(t)\,dx$ and $\overline{v(t)} = \fint_\Omega v(t)\,dx$, and observe that in the last inequality we have again made use of the convexity of f. Then, integrating between 0 and $t > 0$ and using (7.7), we obtain

$$t \leq \int_0^t \frac{1}{f\left(\overline{\frac{u(t)+v(t)}{2}}\right)} \frac{d\,\overline{\frac{u(t)+v(t)}{2}}}{d\tau}\,d\tau$$
$$= \int_{\overline{\frac{u_0+v_0}{2}}}^{\overline{\frac{u(t)+v(t)}{2}}} \frac{d\varsigma}{f(\varsigma)} \leq \int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{1}{f(\varsigma)} d\varsigma < \infty\,.$$

Again, as in the proof of Theorem 7.1, we conclude that $\overline{\frac{u(t)+v(t)}{2}}$ will blow up in the finite time t_* provided that $f(w) > 0$ for all $w \geq \overline{\frac{u_0+v_0}{2}}$, condition that is assured by assumption (3) and once that

$$\min\{\overline{u_0}, \overline{v_0}\} \leq \overline{\frac{u_0+v_0}{2}} \leq \max\{\overline{u_0}, \overline{v_0}\}\,. \tag{7.8}$$

Consequently (u, v) blows up in the finite time t_* and therefore (7.1) holds, which in turn only implies (7.6).

Now, in order to show that both u and v blow up in finite times, we will argue by contradiction. If we assume, for instance, that v blows-up at the time t_* and u does not blow up in any finite time, then we would have

$$\limsup_{t\to t_*^-} \|v(t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \int_{\overline{u_0}}^{\infty} \frac{d\varsigma}{f(\varsigma)} = \infty\,.$$

In the case of $\overline{u_0} \geq \overline{v_0}$, then we would get, in view of (7.8), that

$$\int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{d\varsigma}{f(\varsigma)} \geq \int_{\overline{u_0}}^{\infty} \frac{d\varsigma}{f(\varsigma)} = \infty\,,$$

which contradicts (7.7). For the case of $\overline{u_0} \leq \overline{v_0}$, then, and again in view of (7.8),

$$2\int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{d\varsigma}{f(\varsigma)} \geq \int_{\overline{\frac{u_0+v_0}{2}}}^{\infty} \frac{d\varsigma}{f(\varsigma)} + \int_{\overline{\frac{u_0+v_0}{2}}}^{\overline{u_0}} \frac{d\varsigma}{f(\varsigma)} = \int_{\overline{u_0}}^{\infty} \frac{d\varsigma}{f(\varsigma)} = \infty\,,$$

which cannot happen due to (7.7). □

Remark 7.3. *Under the assumptions of Theorem 7.3 and for suitable reaction terms, it is possible to prove the blow-up of u and v is simultaneous. In fact, modifying the arguing of* [11, 21]*, we can prove the simultaneous blow-up of u and v in the case of reaction terms with the same shape and such that*

$$f(w) = 0 \Leftrightarrow w = 0 \quad \text{and} \quad f\left(\frac{w}{\lambda^{2\alpha}}\right) = \frac{1}{\lambda^{2(\alpha+1)}} f(w)$$

for some positive constants α and λ. An example of such a situation is the reaction function $f(w) = |w|^p$, which satisfies to the above conditions for $\alpha = \frac{1}{p-1}$, $p > 1$, and for any positive constant λ. The same reasoning can be applied to non-local problems with reaction terms similar to the ones considered in the works [11, 21].

Remark 7.4. *In the particular case of Theorem 7.3 with assumption (1) restricted to the case of $f(u,v) = f(v)$ and $g(u,v) = f(u)$, we can easily prove the simultaneous blow-up of u and v. To see this let us assume thatu and v blow-up at distinct times t_{1*} and t_{2*}, respectively, with $t_{1*} < t_{2*}$.*

8. Asymptotic stability

In this section, we shall consider strong solutions (u, v) to the Neumann problem (1.1)-(1.5), in the conditions of Theorem 6.1, in a cylinder Q_{T_0} for some $T_0 > 0$. The aim of the present section, is to give a criterium for the convergence of these solutions towards a homogeneous state. In particular, we will show the non-existence of nonconstant steady state solutions if certain conditions are satisfied. Our approach will be based on a criterium for the existence of invariant regions and then to exploit this idea to study the asymptotic behavior of the solutions (see *e.g.* [20, Chapter 14]). We recall that a bounded subset Σ in the $uv-$plane is called an invariant region for a strong solution (u, v) to the problem (1.1)-(1.5) in Q_{T_0}, if the following property is verified:

$$(u_0, v_0) \text{ and the boundary values of } (u,v) \text{ on } \partial\Omega \text{ lie in } \Sigma \quad \Rightarrow \quad (u,v) \text{ lies in } \Sigma \text{ for all } (x,t) \in Q_{T_0}.$$

Let us assume that Σ is prescribed through m functions $h_i:\mathbb{R}^2\to\mathbb{R}$, $i=1,\ \dots,\ m$ and $m\in\mathbb{N}$, as

$$\Sigma=\bigcap_{i=1}^{m}\{(u,v)\in\mathbb{R}^2: h_i(a_1(p(u),q(v))u,a_2(r(u),s(v))v)\le 0\}\,, \tag{8.1}$$

with $\partial\Sigma=\bigcup_{i=1}^{m}\{(u,v)\in\mathbb{R}^2: h_i(a_1(p(u),q(v))u,a_2(r(u),s(v))v)=0\}$, and where, for each $i\in\{1,\dots,\ m\}$, h_i is supposed to be a smooth real-valued function on an open subset $U_i\subset\mathbb{R}^2$ and such that

$$\nabla_{(u,v)}h_i\neq(0,0)\quad\forall(u,v)\in U_i\,.$$

Now, if we assume the existence of a solution (u,v) to the problem (1.1)-(1.5) in Q_{T_0}, with boundary data in Σ and also with initial data $u_0(x)$ and $v_0(x)$ in Σ for all $x\in\Omega$, which is not in Σ for all $t>0$, then, in view of the definition of Σ set forth in (8.1), there is a function h_i, for some $i\in\{1,\dots,\ m\}$, a time $t_0>0$ and a point $x_0\in\Omega$ such that

$$h_i(a_1(p(u(t)),q(v(t)))\,u(x,t),a_2(r(u(t)),s(v(t)))\,v(x,t))\le 0\quad\text{for } x\in\Omega \text{ and } t\le t_0,$$

and

$$\forall\,\epsilon>0\ \ \exists\, t'\in(t_0,t_0+\epsilon): h_i(a_1(p(u(t')),q(v(t')))\,u(x_0,t'),a_2(r(u(t')),s(v(t')))\,v(x_0,t'))>0\,.$$

Thus we may characterize the invariant regions for (u,v) as follows. If, for an arbitrary $(x_0,t_0)\in Q_{T_0}$, the assumptions

$$h_i(a_1(p(u(t)),q(v(t)))\,u(x_0,t),a_2(r(u(t)),s(v(t)))\,v(x_0,t))<0\quad\text{for } 0\le t<t_0 \tag{8.2}$$

and

$$h_i(a_1(p(u(t_0)),q(v(t_0)))\,u(x_0,t_0),a_2(r(u(t_0)),s(v(t_0)))\,v(x_0,t_0))=0 \tag{8.3}$$

together imply that

$$\frac{\partial\,h_i(u,v)}{\partial\,t}<0\quad\text{at } (x_0,t_0) \tag{8.4}$$

for all $i=1,\dots,m$, then Σ must be an invariant region for (u,v).

Theorem 8.1. *Assume that (u,v) is a strong solution to the problem (1.1)-(1.5) in Q_{T_0}, for some $T_0>0$, and let us consider the bounded domain Σ defined at (8.1). If*

(1) (u_0,v_0), $(0,0)\in\Sigma$,
(2) *h_i is quasi-convex for all $i\in\{1,\dots,m\}$,*
(3) *$(f(u,v),g(u,v))\cdot\mathbf{n}<0$ on $\partial\Sigma$, where $\mathbf{n}$ is the outward unit normal to $\partial\Sigma$,*

then Σ is an invariant region for (u,v).

Observe that the assumption of h_i to be quasi-convex for all $i\in\{1,\dots,m\}$, implies that Σ is a convex domain. In particular, a rectangular domain of the form $[a,b]\times[c,d]$ satisfies this condition. On the other hand, the condition $(f(u,v),g(u,v))\cdot\mathbf{n}<0$ on $\partial\Sigma$ means that $(f(u,v),g(u,v))$ points to the interior of Σ on $\partial\Sigma$.

Proof. Using the reasoning aforementioned at (8.2)-(8.4), we assume that, for an arbitrary $i\in\{1,\dots,m\}$, the assumptions (8.2)-(8.3) hold for some $x_0\in\Omega$ and for some $t_0>0$. Let us prove now that (8.4) is verified at (x_0,t_0). In the general case, we have by the assumption (3), and due to the fact that (u,v) is a strong solution, that

$$\begin{aligned}\frac{\partial\,h_i(u,v)}{\partial\,t}&=\nabla_{(u,v)}h_i\cdot(u_t,v_t)\\&<\nabla_{(u,v)}h_i\cdot(a_1(p(u),q(v))\Delta u,a_2(r(u),s(v))\Delta v)\,.\end{aligned} \tag{8.5}$$

Then, by using (8.2)-(8.3), together with the assumption (2) and with the spatial nonlocal character of a_1 and a_2, we can show that the right-hand side of (8.5) cannot be positive at (x_0,t_0). To prove this, we will adapt some of the arguments of the proof of [20, Theorem 14.7]. In order to simplify the exposition, we consider the case of only one space dimension, *i.e.* we assume that $\Omega\subset\mathbb{R}$. Defining

$$\mathcal{H}_i(x):=h_i(a_1(p(u(t_0)),q(v(t_0)))\,u(x,t_0),a_2(r(u(t_0)),s(v(t_0)))\,v(x,t_0))\,,$$

we readily see that

$$\mathcal{H}_i'(x) = \nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t_0)), q(v(t_0)))\, u_x(x,t_0), a_2(r(u(t_0)), s(v(t_0)))\, v_x(x,t_0)\right), \tag{8.6}$$

$$\mathcal{H}_i(x_0) = 0\,, \tag{8.7}$$

the last due to (8.3). First we will show that

$$\mathcal{H}_i'(x_0) = 0\,, \tag{8.8}$$

arguing by contradiction. If we had $\mathcal{H}_i'(x_0) > 0$, then, in view of (8.7), we would have $\mathcal{H}_i(x) > 0$ for $x > x_0$ sufficiently close to x_0. Due to the definition of $\mathcal{H}_i$, we also would have

$$h_i(a_1(p(u(t_0)), q(v(t_0)))\, u(x,t_0), a_2(r(u(t_0)), s(v(t_0)))\, v(x,t_0)) > 0$$

for $x > x_0$ sufficiently close to x_0, and consequently

$$h_i(a_1(p(u(t)), q(v(t)))\, u(x,t), a_2(r(u(t)), s(v(t)))\, v(x,t)) > 0$$

for some x and for t sufficiently close to t_0. In particular, we would have

$$h_i(a_1(p(u(t)), q(v(t)))u(x_0,t), a_2(r(u(t)), s(v(t)))v(x_0,t)) > 0 \quad \text{for some } t < t_0, \tag{8.9}$$

which violates (8.2). In the case of $\mathcal{H}_i'(x_0) < 0$, we would have, in view of (8.7), that $\mathcal{H}_i(x) > 0$ but now for $x < x_0$ sufficiently close to x_0. By the same reasoning used in the previous case, we would also end up in (8.9). As a consequence, (8.8) holds.

Next, we will show that

$$\mathcal{H}_i''(x_0) \leq 0. \tag{8.10}$$

We start by observing that, in view of the assumption (2) and due to (8.6) and (8.8), we can use the theory of quasi-convex functions to prove that

$$\begin{aligned}\mathcal{H}_i''(x_0) = & \begin{bmatrix} a_1(p(u(t_0)), q(v(t_0)))u_x(x_0,t_0) \\ a_2(r(u(t_0)), s(v(t_0)))v_x(x_0,t_0) \end{bmatrix}^T \mathrm{H}_{(u,v)}(h_i)(x_0,t_0) \begin{bmatrix} a_1(p(u(t_0)), q(v(t_0)))u_x(x_0,t_0) \\ a_2(r(u(t_0)), s(v(t_0)))v_x(x_0,t_0) \end{bmatrix} \\ & + \nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t_0)), q(v(t_0)))\, u_{xx}(x_0,t_0), a_2(r(u(t_0)), s(v(t_0)))\, v_{xx}(x_0,t_0)\right) \\ & \geq \nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t_0)), q(v(t_0)))\, u_{xx}(x_0,t_0), a_2(r(u(t_0)), s(v(t_0)))\, v_{xx}(x_0,t_0)\right),\end{aligned} \tag{8.11}$$

where $\mathrm{H}_{(u,v)}(h_i)(x_0,t_0)$ denotes the Hessian matrix of h_i, with respect to (u,v), evaluated at the point (x_0,t_0). Then, gathering the information of (8.10) and (8.11), we prove that the left-hand side of (8.5) is negative at (x_0,t_0). Consequently the conclusion of the theorem follows from the characterization (8.2)-(8.4) of an invariant region of the type (8.1).

Finally, it last to show that (8.10) holds. Arguing again by contradiction, we assume that we had $\mathcal{H}_i''(x_0) > 0$. Then, in view of (8.8), we would have $\mathcal{H}_i'(x) > 0$ for $x > x_0$ sufficiently close to x_0. Due to the expression of $\mathcal{H}_i'$ (see (8.6)), we also would have

$$\nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t_0)), q(v(t_0)))\, u_x(x,t_0), a_2(r(u(t_0)), s(v(t_0)))\, v_x(x,t_0)\right) > 0$$

for $x > x_0$ sufficiently close to x_0, and consequently

$$\nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t)), q(v(t)))\, u_x(x,t), a_2(r(u(t)), s(v(t)))\, v_x(x,t)\right) > 0$$

for some x and for t sufficiently close to t_0. In particular, we would have

$$\nabla_{(u,v)} h_i \cdot \left(a_1(p(u(t)), q(v(t)))u_x(x_0,t), a_2(r(u(t)), s(v(t)))v_x(x_0,t)\right) > 0 \quad \text{for some } t < t_0$$

which violates (8.8).

This proof can be carried over to any space dimension, though the exposition becomes too heavy. □

Next we will use the notion of invariant regions to study the asymptotic behavior of the solutions to the system (1.1)-(1.3) in the case of Neumann boundary conditions, *i.e.* when $\tau = 1$ in (1.4)-(1.5). To proceed with this study, we assume that (1.1)-(1.2) admits a bounded invariant region Σ and we use the notation

$$M := \max_{(u,v)\in\Sigma} \left(|\nabla_{(u,v)} f| + |\nabla_{(u,v)} g|\right)\,, \tag{8.12}$$

where the reaction functions f and g are assumed to be sufficiently regular, and the subscript (u,v) means that the gradient of f and g is taken with respect to these variables. Observe that, by the characterization of the invariant regions set forth in (8.1), Σ is compact and therefore $M<\infty$. We also fix the notation

$$\sigma := m\lambda_P - 2M\,, \tag{8.13}$$

where λ_P is the principal eigenvalue to the Laplacian problem (2.10) with Neumann boundary conditions and $m:=\min\{m_1,m_2\}$, being m_1 and m_2 defined at (2.2).

Theorem 8.2. *Let (u,v) be a strong solution to the problem (1.1)-(1.5) in Q_{T_0}, for some $T_0>0$, endowed with Neumann boundary conditions. Assume that (1.1)-(1.2) admits a bounded invariant region $\Sigma\subset\mathbb{R}^N$ such that*

$$(u_0(x),v_0(x))\in\Sigma\quad\forall\ x\in\Omega\,.$$

If the constant σ defined in (8.13) is positive, then there exist positive constants C_1 and C_2 such that

$$\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla v(t)\|^2_{L^2(\Omega)}\leq C_1e^{-2\sigma t}\quad\forall\ t>0\,, \tag{8.14}$$

$$\|u(t)-\overline{u(t)}\|^2_{L^2(\Omega)}+\|v(t)-\overline{v(t)}\|^2_{L^2(\Omega)}\leq C_2e^{-2\sigma t}\quad\forall\ t>0\,. \tag{8.15}$$

Proof. We start by multiplying the equations (1.1) and (1.2) by $\Delta\,u$ and $\Delta\,v$, respectively, and we integrate over Ω. Next we add up the resulting equations and we use the Cauchy-Schwarz inequality together with (2.2) and (8.12), and we obtain

$$\frac{1}{2}\frac{d}{d\,t}\left(\int_\Omega|\nabla u(t)|^2dx+\int_\Omega|\nabla v(t)|^2dx\right)\leq$$
$$-\,m_1\int_\Omega|\Delta u(t)|^2dx-m_2\int_\Omega|\Delta v(t)|^2dx+2M\left(\int_\Omega|\nabla u(t)|^2dx+\int_\Omega|\nabla v(t)|^2dx\right)\,.$$

Then, we use Poincaré's inequality (2.14) together with the notation of (8.13) which yield

$$\frac{d}{d\,t}\left(\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla v(t)\|^2_{L^2(\Omega)}\right)\leq-2\sigma\left(\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla v(t)\|^2_{L^2(\Omega)}\right)\,.$$

Integrating the last relation between 0 and $t>0$, and using the fact that $u_0\,,\ v_0\in H^1(\Omega)$ (see (6.1)), we obtain

$$\|\nabla u(t)\|^2_{L^2(\Omega)}+\|\nabla v(t)\|^2_{L^2(\Omega)}\leq\left(\|\nabla u_0\|^2_{L^2(\Omega)}+\|\nabla v_0\|^2_{L^2(\Omega)}\right)e^{-2\sigma t}\,,$$

which proves (8.14). Now, using the Poincaré inequality (2.13), we obtain, from the last relation, that

$$\|u(t)-\overline{u(t)}\|^2_{L^2(\Omega)}+\|v(t)-\overline{v(t)}\|^2_{L^2(\Omega)}\leq\frac{\|\nabla u_0\|^2_{L^2(\Omega)}+\|\nabla v_0\|^2_{L^2(\Omega)}}{\lambda_P}e^{-2\sigma t}\,,$$

and (8.15) follows. □

Arguing as in [12], the exponential decay (8.15) can be strengthened to

$$\|u(t)-\overline{u(t)}\|^2_{L^\infty(\Omega)}+\|v(t)-\overline{v(t)}\|^2_{L^\infty(\Omega)}\leq C_3e^{-2\sigma t}\quad\forall\ t>0\,,$$

for some positive constant C_3. The main consequence of the previous theorem, is that the elliptic problem

$$\begin{cases}-a_1(p(u),q(v))\Delta u=f(u,v) & \text{in }\Omega\\ -a_2(r(u),s(v))\Delta v=g(u,v) & \text{in }\Omega\\ \nabla u\cdot\mathbf{n}=0\,,\quad\nabla v\cdot\mathbf{n}=0 & \text{on }\partial\Omega\end{cases}$$

has no nonconstant solutions, because these solutions depend only on x and, by (8.15), they must tend to solutions independent of x.

Acknowledgements

We would like to thank to the anonymous referee for the detailed and concise review that his/her has made and which enabled us to improve this work. We would also like to thank to Professor Eduardo Espejo for valuable discussions and comments on some parts of this manuscript and for bringing our attention to the references [16, 19] where similar problems, modelling real world phenomena, are considered. The first author was partially supported by the Research Project CAPES - Grant BEX 2478-12-8 and MEC/MCTI/CAPES/CNPq/FAPs no. 71/2013, Grant 88881.030388/2013-01, Brazil. The second author was partially supported by Fundação para a Ciência e a Tecnologia, UID/MAT/04561/2013, Portugal.

References

[1] A.S. Ackleh and L. Ke. Existence-uniqueness and long time behavior for a class of nonlocal nonlinear parabolic evolution equations. *Proc. Am. Math. Soc.* **128** (2000), 3483–3492.

[2] N.-H. Chang and M. Chipot. Nonlinear nonlocal evolution problems. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **97** (2003), no. 3, 423–445.

[3] M. Chipot. *Elements of nonlinear analysis.* Birkhäuser Verlag, Basel, 2000.

[4] M. Chipot and B. Lovat. Some remarks on nonlocal elliptic and parabolic problems. *Nonlinear Anal. Theory Methods Appl.* **30** (1997), no. 7, 4619–4627.

[5] M. Chipot, and B. Lovat. On the asymptotic behaviour of some nonlocal problems. *Positivity* **3** (1999), 65–81.

[6] M. Chipot and B. Lovat. Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **8** (2001), no. 1, 35–51.

[7] M. Chipot and L. Molinet. Asymptotic behaviour of some nonlocal diffusion problems. *Appl. Anal.* **80** (2001), no. 3-4, 279–315.

[8] M. Chipot and J.F. Rodrigues. On a class of nonlocal nonlinear elliptic problems. *RAIRO Modél. Math. Anal. Numér.* **26** (1992), no. 3, 447–467.

[9] M. Chipot and T. Savitska. Nonlocal p-Laplace equations depending on the L^p norm of the gradient. *Adv. Differential Equations* **19** (2014), no. 11-12, 997–1020.

[10] M. Chipot, V. Valente and G. Vergara Caffarelli. Remarks on a nonlocal problem involving the Dirichtlet energy. *Rend. Sem. Mat. Univ. Padova.* **110** (2003), 199–220.

[11] M. Chlebík and M. Fila. From critical exponents to blow-up rates for parabolic problems. *Rend. Mat. Appl. Ser VII* **19** (1999), 449–470.

[12] E. Conway, D. Hoff and J. Smoller. Large time behavior of solutions of systems of nonlinear reaction-diffusion equations. *SIAM J. Appl. Math.* **35** (1978), 1–16.

[13] F.J.S.A. Corrêa, S.D.B. Menezes and J. Ferreira. On a class of problems involving a nonlocal operator. *Appl. Math. Comput.* **147** (2004), 475-489.

[14] M. Escobedo and M. Herrero. Boundedness and blow up for a semilinear reaction diffusion system. *J. Differential Equations* **89** (1991), 176–202.

[15] L.C. Evans. *Partial differential equations.* Graduate Studies in Math. **19**, American Mathematical Society, Providence, RI, 1998.

[16] K. Ichikawa, M. Rouzimaimaiti and T. Suzuki. Reaction diffusion equation with non-local term arises as a mean field limit of the master equation. *Discrete Contin. Dyn. Syst. S*, 5-1, Special Issue (2012), 115–126.

[17] J.-L. Lions. On some questions in boundary value problems of mathematical physics. In *Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro*, North-Holland Math. Stud. **30**, 284–346, North-Holland, Amsterdam-New York, 1978.

[18] B. Lovat. *Etudes de quelques problemes paraboliques non locaux.* Thèse présentée pour l'obtention du doctorat en Mathématiques, Université de Metz (1995).

[19] M. Negreanu and J.I. Tello. On a competitive system under chemotactic effects with non-local terms. Nonlinearity **26** (2013), 1083–1103.

[20] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York (1983).

[21] Ph. Souplet and S. Tayachi. Optimal condition for non-simultaneous blow-up in a reaction-diffusion system. *J. Math. Soc. Japan* **56** (2004), no. 2, 571–584.

[22] I. Vrabie. *Compactness methods for non-linear evolutions.* Pitman Longman, London, 1987.

[23] S. Zheng and M. Chipot. Asymptotic behavior to nonlinear parabolic equations with nonlocal terms. *Asymptot. Anal.* **45** (2005), no. 3-4, 301-312.