

Research Article

Stefan Samko*

Variable exponent fractional integrals in the limiting case $\alpha(x)p(x) \equiv n$ on quasimetric measure spaces

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Abstract: We show that the fractional operator $I^{\alpha(\cdot)}$, of variable order on a bounded open set in Ω , in a quasimetric measure space (X, d, μ) in the case $\alpha(x)p(x) \equiv n$ (where n comes from the growth condition on the measure μ), is bounded from the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ into $BMO(\Omega)$ under certain assumptions on $p(x)$ and $\alpha(x)$.

Keywords: Riesz potential, variable exponent spaces, Sobolev type theorem, BMO results, quasimetric measure space

MSC 2010: 46E30

1 Introduction

The results of this paper lie in an area which has been extensively developed in the last two decades and continues to attract the attention of researchers from various fields of mathematics. It suffices to refer to the books [1, 2, 7, 8, 11].

Many problems of variable exponent analysis have been solved both in the classical setting, i.e., in the Euclidean case, and in the general setting of functions defined on quasimetric measure spaces of general nature, including fractional upper and lower dimensions; the references may be found in the above-cited books.

In this paper we consider fractional integrals in such a general setting. As is well known, fractional operators I^α , defined on functions on domains in \mathbb{R}^n , are bounded from L^p to L^q , where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ in the so-called prelimiting case $1 < p < \frac{n}{\alpha}$, bounded to spaces of Hölder continuous functions in the overlimiting case $p > \frac{n}{\alpha}$ and to BMO in the case $\alpha p = n$ (see, for instance, [10, 14] for the latter case).

The extension of such results to the case of variable exponents $p(x)$ and orders $\alpha(x)$, in the case where $\sup \alpha(x)p(x) < n$, is known both in the Euclidean setting for open sets $\Omega \subset \mathbb{R}^n$ (see [12] or [7, 8, Chapter 2]) and in a more general setting of quasimetric measure spaces, see [4, 5]. The case $\alpha(x)p(x) \equiv n$ was considered in [13] in the Euclidean setting.

In this paper we consider the fractional integral

$$I^{\alpha(\cdot)}f(y) = \int_{\Omega} \frac{f(z) d\mu(z)}{d(y, z)^{n-\alpha(y)}}, \quad y \in \Omega,$$

of variable order $\alpha(x)$ over a bounded open set Ω in a quasimetric measure space (X, d, μ) , with quasidistance d and measure μ , where the “dimension” n comes from the growth condition on the measure μ (the necessary preliminaries on the notions related to quasimetric measure spaces are given in Section 2.1). We show that in

*Corresponding author: **Stefan Samko**, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Faro, Portugal, e-mail: ssamko@ualg.pt

this setting the result of [13] on the action of the operator $I^{\alpha(\cdot)}$ from $L^{p(\cdot)}(\Omega)$ to BMO, in the case $\alpha(x)p(x) \equiv n$, extends to the general case.

2 Preliminaries

2.1 Preliminaries on quasimetric measure spaces

The basics on quasimetric measure spaces may be found for instance in the books [3, 6].

In the sequel, (X, d, μ) denotes a quasimetric space, with the quasidistance d satisfying the triangle inequality

$$d(x, y) \leq k[d(x, z) + d(z, y)], \quad k \geq 1,$$

and μ being the Borel regular measure. For simplicity, we deal with symmetrical quasidistances, that is, $d(x, y) = d(y, x)$.

We denote by $\Omega \subset X$ an open set. It will always be assumed to be bounded, i.e., $0 < \ell < \infty$, $\ell = \text{diam } \Omega$.

We use the notation

$$B(x, r) = \{y \in X : d(x, y) < r\}, \quad \bar{B}(x, r) = B(x, r) \cap \Omega.$$

The following standard conditions will be assumed to be satisfied:

- (1) All the balls $B(x, r)$ are measurable and $\mu S(x, r) = 0$ for all the spheres $S(x, r) = \{y \in X : d(x, y) = r\}$, $x \in X$, $r \geq 0$.
- (2) The space $C(X)$ of uniformly continuous functions on X is dense in $L^1(X, \mu)$.

The measure μ is called doubling if

$$\mu B(x, 2r) \leq C\mu B(x, r),$$

where $C > 0$ does not depend on $r > 0$ and $x \in X$.

The conditions

$$\mu(B(x, r)) \leq c_1 r^n \tag{2.1}$$

and

$$\mu B(x, r) \geq c_0 r^N, \tag{2.2}$$

imposed on the measure μ , are known as the *upper and lower Ahlfors conditions*; the first one of which is also referred to as the *growth condition*. Note that the exponents n and N are not necessarily integers.

The Hardy–Littlewood maximal function of a locally μ -integrable function $f: \Omega \rightarrow \mathbb{R}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{\bar{B}(x, r)} |f(y)| d\mu(y).$$

As was shown in [9], every quasimetric space (X, d) admits an equivalent quasidistance d_1 with the Lipschitz property, i.e., there exists an exponent $\theta \in (0, 1]$ such that

$$|d_1(x, z) - d_1(y, z)| \leq M d_1^\theta(x, y) \{d_1(x, z) + d_1(y, z)\}^{1-\theta}, \tag{2.3}$$

and the quasi-distance d_1 is the power of a distance, namely,

$$d_1(x, y) = \mathbf{d}(x, y)^{\frac{1}{\theta}}, \tag{2.4}$$

i.e., the triangle inequality for $\mathbf{d}(x, y)$ holds with $k = 1$:

$$\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z).$$

Note that relation (2.4) being given, property (2.3) is an immediate consequence of (2.4) and holds with $M = \frac{1}{\theta}$.

Definition 2.1. We say that the quasimetric d is regular of order $\theta \in (0, 1]$ if it itself satisfies property (2.3).

2.2 Preliminaries on variable Lebesgue spaces $L^{p(\cdot)}$

Let $\Omega \subseteq X$. The space $L^{p(\cdot)}(\Omega)$, with μ -measurable exponent $p: \Omega \rightarrow [1, \infty)$, is defined by the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} d\mu(x) \leq 1 \right\}.$$

The important role of the log-continuity of $p(x)$ is well known in variable analysis. In the quasimetric measure spaces setting, we may use two forms of the log-condition. We denote by $\mathcal{P}^{\log}(\Omega)$ and $\mathcal{P}_{\mu}^{\log}(\Omega)$ the sets of μ -measurable exponents $p: \Omega \rightarrow [1, \infty)$ which satisfy the following log-conditions on Ω :

$$|p(x) - p(y)| \leq \frac{C_p}{-\ln d(x, y)}, \quad d(x, y) \leq \frac{1}{2}, \quad x, y \in \Omega, \quad (2.5)$$

$$|p(x) - p(y)| \leq \frac{A}{-\ln \mu B(x, d(x, y))} \quad (2.6)$$

for all $x, y \in \Omega$ such that $\mu B(x, d(x, y)) < \frac{1}{2}$, respectively.

Let $\mathbb{P}^{\log}(\Omega)$ denote the set of exponents $p \in \mathcal{P}^{\log}(\Omega)$ such that

$$1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty.$$

The log-condition in form (2.6), coinciding with (2.5) in the Euclidean case, is more suitable in the context of general quasimetric measure spaces because in some results it allows us to put less restrictions on (X, d, μ) .

Lemma 2.2. *Let (X, d, μ) be a quasimetric measure space and $\Omega \subseteq X$. If the lower Ahlfors condition (2.2) holds, then*

$$\mathcal{P}^{\log}(\Omega) \subseteq \mathcal{P}_{\mu}^{\log}(\Omega). \quad (2.7)$$

If the upper Ahlfors condition (2.1) holds, then

$$\mathcal{P}_{\mu}^{\log}(\Omega) \subseteq \mathcal{P}^{\log}(\Omega).$$

Proof. The proof is a matter of direct verification. □

We will need the estimation of the following norm of truncated potential kernels:

$$A_{\beta, p}(x, r) = \|d(x, \cdot)^{-\beta(x)} \chi_r(x, \cdot)\|_{L^{p(\cdot)}(\Omega)},$$

where $\chi_r(x, y) = \chi_{\Omega \setminus B(x, r)}(y)$. The estimate given in the theorem below is known. For a more general kernel than the constant α , such an estimate was proved in [4, Lemma 4.2]. The proof with variable $\alpha(x)$ is given in [12] for the Euclidean case. The proof for quasimetric measure spaces follows the same lines.

Theorem 2.3. *Let Ω be a bounded open set in X , let the growth condition (2.1) hold and let $p \in \mathbb{P}^{\log}(\Omega)$ and $\beta \in L^{\infty}(\Omega)$. If*

$$\gamma := \inf_{x \in \Omega} [\beta(x)p(x) - n(x)] > 0,$$

then

$$A_{\beta, p}(x, r) \leq Cr^{\frac{n(x)}{p(x)} - \beta(x)}$$

for all $x \in \Omega$, $0 < r < \ell = \text{diam } \Omega$, where $C > 0$ does not depend on x and r .

3 Auxiliary statements

The estimate of the following lemma is known under various assumptions on (X, d, μ) and $p(x)$. We give its proof under our assumptions for completeness of the presentation.

Lemma 3.1. *Let Ω be an open bounded set in a quasimetric measure space (X, d, μ) . If $p \in \mathcal{P}_\mu^{\log}$, then*

$$\|\chi_{B(x,r)}\|_{p(\cdot)} \leq C[\mu B(x,r)]^{\frac{1}{p(x)}} \quad (3.1)$$

for all $r \in [0, \text{diam } \Omega]$, where $C > 0$ does not depend on x and r . Estimate (3.1) is valid also for $p \in \mathcal{P}^{\log}$ if the lower Ahlfors condition holds.

Proof. Let $x \in \Omega$ and $0 < r < \text{diam } \Omega$. For $p \in \mathcal{P}_\mu^{\log}$, it is easy to check that

$$\frac{1}{C} \mu B(x,r) \leq [\mu B(x,r)]^{\frac{p(y)}{p(x)}} \leq C \mu B(x,r)$$

for all $y \in \bar{B}(x,r)$. Hence, for $C_1 = C^{\frac{1}{p^-}}$, we have

$$\int_{\bar{B}(x,r)} \frac{d\mu(y)}{C_1^{p(y)} [\mu B(x,r)]^{\frac{p(y)}{p(x)}}} \leq \int_{\bar{B}(x,r)} \frac{d\mu(y)}{\mu B(x,r)} \leq 1.$$

Then

$$\|\chi_{\bar{B}(x,r)}\|_{p(\cdot)} = \inf \left\{ \eta > 0 : \int_{\bar{B}(x,r)} \eta^{-p(y)} d\mu(y) \leq 1 \right\} \leq C_1 [\mu B(x,r)]^{\frac{1}{p(x)}}.$$

When $p \in \mathcal{P}^{\log}$, it suffices to refer to (2.7). □

Recall that we are interested in the case

$$\alpha(x)p(x) \geq n.$$

where n is from the growth condition (2.1).

The space BMO is defined as

$$\text{BMO} = \{f : \mathcal{M}^\# f \in L^\infty\}, \quad \|f\|_{\text{BMO}} := \|\mathcal{M}^\# f\|_\infty,$$

where

$$\mathcal{M}^\# f(x) := \sup_{r>0} \frac{1}{\mu B(x,r)} \int_{\bar{B}(x,r)} |f(y) - f_{\bar{B}(x,r)}| d\mu(y)$$

is the sharp maximal function and

$$f_{\bar{B}(x,r)} = \frac{1}{|\bar{B}(x,r)|} \int_{\bar{B}(x,r)} f(z) d\mu(z).$$

We consider two forms of the fractional maximal function:

$$\mathcal{M}^{\beta(\cdot)} f(x) = \sup_{r>0} \frac{r^{\beta(x)}}{\mu(B(x,r))} \int_{\bar{B}(x,r)} |f(y)| d\mu(y)$$

and

$$\mathcal{M}_{\beta(\cdot)} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta(x)}} \int_{\bar{B}(x,r)} |f(y)| d\mu(y),$$

where n is from the growth condition. The first one serves better in the case of the lower Ahlfors condition, the second one in the case of the upper Ahlfors condition.

Lemma 3.2. *Let Ω be bounded and $p \in \mathcal{P}_\mu^{\log}(\Omega)$. Then*

$$\|\mathcal{M}^{\beta(\cdot)} f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^{p(\cdot)}(\Omega)} \quad \text{if } \beta(x)p(x) \geq N \quad (3.2)$$

and

$$\|\mathcal{M}_{\beta(\cdot)} f\|_{L^\infty(\Omega)} \leq C \|f\|_{L^{p(\cdot)}(\Omega)} \quad \text{if } \beta(x)p(x) \geq n, \quad (3.3)$$

under the assumption that (X, d, μ) satisfies the lower Ahlfors condition in case (3.2) and the upper Ahlfors condition in case (3.3).

Proof. By the Hölder inequality for variable exponents and estimate (3.1), we have

$$\mathcal{M}^{\beta(\cdot)} f(x) \leq \sup_{r>0} \frac{r^{\beta(x)}}{\mu(B(x, r))} \|\chi_{B(x, r)}\|_{L^{p'(\cdot)}(\Omega)} \|f\|_{L^{p(\cdot)}(\Omega)} \leq \sup_{r>0} \frac{r^{\beta(x)}}{[\mu(B(x, r))]^{\frac{1}{p(x)}}} \|f\|_{L^{p(\cdot)}(\Omega)},$$

whence (3.2) follows.

Estimate (3.3) is obtained similarly. \square

Lemma 3.3. *Let (X, d) be a quasimetric space, regular of order $\theta \in (0, 1]$. Then*

$$|d(x, z)^{-\gamma} - d(y, z)^{-\gamma}| \leq \frac{\gamma}{\theta} d(x, y)^\theta \min\{d(x, z), d(y, z)\}^{-\gamma-\theta}, \quad \gamma > 0. \quad (3.4)$$

Proof. By (2.4), we have

$$|d(x, z)^{-\gamma} - d(y, z)^{-\gamma}| = |\mathbf{d}(x, z)^{-\frac{\gamma}{\theta}} - \mathbf{d}(y, z)^{-\frac{\gamma}{\theta}}|,$$

where \mathbf{d} is a distance, so that $|\mathbf{d}(x, z) - \mathbf{d}(y, z)| \leq \mathbf{d}(x, y)$. By the inequality

$$|a^{-\gamma} - b^{-\gamma}| \leq \gamma \cdot |a - b| (\min\{a, b\})^{-\gamma-1}, \quad a > 0, b > 0, \gamma > 0,$$

we then arrive at (3.4). \square

4 Theorem on the $L^{p(\cdot)}(\Omega) \rightarrow \text{BMO}$ -boundedness

We denote by $H^\lambda(\Omega)$ the space of functions f on Ω satisfying the Hölder condition $|f(x) - f(y)| \leq Cd(x, y)^\lambda$, $0 < \lambda \leq 1$. Let $H(\Omega) = \bigcup_{0 < \lambda \leq 1} H^\lambda(\Omega)$.

Remark 4.1. Note that assumption (4.1), used in the theorem below, may seem to be cumbersome as containing the lower bound depending on α and p . However, a “more natural” assumption like (2.2) is more restrictive, since we must then assume that $\alpha(x)p(x) \geq N$ instead of $\alpha(x)p(x) \geq n$ and $N \geq n$. Consequently, we prefer to base ourselves on the condition $\alpha(x)p(x) \geq n$ and use the lower bound in form (4.1). In the case $N = n$, assumption (4.1) can be replaced by (2.2).

Remark 4.2. In the following theorem, we assume that $\alpha(x)p(x) \geq n$ instead of $\alpha(x)p(x) \equiv n$, due to the fact that Ω is bounded. This assumption has some disadvantage because at the points $x \in \Omega$, where it may happen that $\alpha(x)p(x) > n$, we should expect that the fractional integral is better than just a BMO function. However, an advantage of the assumption $\alpha(x)p(x) \geq n$ is that we should not require that $p(x)$ is Hölder continuous, which would immediately follow from the assumption that $\alpha(x)p(x) \equiv n$.

Theorem 4.3. *Let (X, d, μ) be regular of order $\theta \in (0, 1]$, let μ be doubling, satisfy the upper Ahlfors condition (2.1) and a variable lower Ahlfors condition of the form*

$$\mu(B(x, r)) \geq cr^{\alpha(x)p(x)}, \quad (4.1)$$

let Ω be a bounded open set in X and $p \in \mathbb{P}^{\log}(\Omega)$, and let $\alpha \in H(\Omega)$. If $\alpha(x)p(x) \geq n$, then the fractional operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $\text{BMO}(\Omega)$.

Proof. We may assume that $f(z) \geq 0$. We continue the function f as zero outside Ω whenever necessary. For $r > 0$, we split the function f as $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = f(z)\chi_{B(x, 2kr)}(z), \quad f_2(z) = f(z)\chi_{\Omega \setminus B(x, 2kr)}(z),$$

where k comes from the triangle inequality, and then

$$I^{\alpha(\cdot)} f(y) = I^{\alpha(\cdot)} f_1(y) + I^{\alpha(\cdot)} f_2(y) =: F_1(y) + F_2(y).$$

Estimation of $F_1(y)$. When $y \in B(x, r)$, we have $d(z, y) < 3kr$ for $z \in B(x, 2kr)$, so that

$$F_1(y) \leq \int_{B(y, 3kr)} \frac{f(z) d\mu(z)}{d(z, y)^{n-\alpha(y)}} \leq Cr^{\alpha(y)} \mathcal{M}f(y)$$

for $y \in B(x, r)$, where the last inequality is obtained in the standard way via the dyadic decomposition and use of the growth condition. Then

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} F_1(y) d\mu(y) \leq \frac{Cr^{\alpha(x)}}{\mu(B(x, r))} \int_{B(x, r)} \mathcal{M}f(y) dy.$$

We apply the Hölder inequality and estimate (3.1) to obtain

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} F_1(y) d\mu(y) \leq \frac{Cr^{\alpha(x)}}{\mu(B(x, r))^{\frac{1}{p(x)}}} \|\mathcal{M}f\|_{L^{p(\cdot)}}.$$

Since the maximal operator is bounded in $L^{p(\cdot)}$ on bounded doubling spaces (see [5]), in view of (4.1), we get

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} F_1(y) d\mu(y) \leq C\|f\|_{L^{p(\cdot)}}. \quad (4.2)$$

Estimation of $F_2(y)$. We set

$$c_f = F_2(x) = \int_{\Omega \setminus B(x, 2kr)} \frac{f(z) d\mu(z)}{d(x, z)^{n-\alpha(x)}}$$

and have

$$|F_2(y) - c_f| = \left| \int_{\Omega \setminus B(x, 2kr)} f(z) \left[\frac{1}{d(y, z)^{n-\alpha(y)}} - \frac{1}{d(x, z)^{n-\alpha(x)}} \right] d\mu(z) \right|,$$

whence

$$\begin{aligned} |F_2(y) - c_f| &\leq \int_{\Omega \setminus B(x, 2kr)} f(z) \left| \frac{1}{d(y, z)^{n-\alpha(x)}} - \frac{1}{d(x, z)^{n-\alpha(x)}} \right| d\mu(z) \\ &+ \int_{\Omega \setminus B(x, 2kr)} f(z) \left| \frac{1}{d(y, z)^{n-\alpha(y)}} - \frac{1}{d(y, z)^{n-\alpha(x)}} \right| d\mu(z) =: G_1 + G_2. \end{aligned}$$

For G_1 , we use Lemma 3.3 and observe that $d(y, x) < r$ and $d(y, z) > 2kr$ imply $d(x, z) < (k + \frac{1}{2})d(y, z)$, and get

$$G_1 \leq Cd(x, y)^\theta \int_{\Omega \setminus B(x, 2kr)} \frac{f(z) d\mu(z)}{d(x, z)^{n-\alpha(x)+\theta}} = cd(x, y)^\theta \sum_{j=1}^{\infty} \int_{B(x, 2^{j+1}r) \setminus B(x, 2^j r)} \frac{f(z) d\mu(z)}{d(x, z)^{n-\alpha(x)+\theta}}.$$

It is easy to see that

$$\int_{B(x, 2kr) \setminus B(x, r)} \frac{f(z) d\mu(z)}{d(x, z)^{n-\alpha(x)+\theta}} \leq \frac{C}{r^\theta} \mathcal{M}_{\alpha(\cdot)} f(x)$$

and then

$$G_1 \leq C \frac{d(x, y)^\theta}{r^\theta} \mathcal{M}_{\alpha(\cdot)} f(x) \leq C \mathcal{M}_{\alpha(\cdot)} f(x).$$

Therefore,

$$\|G_1\|_{L^\infty} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}, \quad (4.3)$$

by Lemma 3.2.

To estimate G_2 , we use the inequality

$$|t^a - t^b| \leq |a - b| \begin{cases} t^{\min\{a, b\}} & \text{if } 0 < t \leq 1, \\ t^{\max\{a, b\}} & \text{if } t \geq 1, \end{cases}$$

and obtain

$$G_2 \leq |\alpha(x) - \alpha(y)| \int_{\Omega \setminus B(x, 2kr)} f(z) \left(\frac{1}{d(y, z)^{n-\alpha(y)}} + \frac{1}{d(y, z)^{n-\alpha(x)}} \right) d\mu(z).$$

Since $d(y, z) \geq \frac{2}{2k+1}d(x, z)$ and $\Omega \setminus B(x, 2kr) \subseteq \Omega \setminus B(y, r)$, we obtain

$$\begin{aligned} G_2 &\leq C|\alpha(x) - \alpha(y)| \left(\int_{\Omega \setminus B(y, r)} \frac{f(z) d\mu(z)}{d(y, x)^{n-\alpha(y)}} + \int_{\Omega \setminus B(x, 2kr)} \frac{f(z) d\mu(z)}{d(x, x)^{n-\alpha(x)}} \right) \\ &=: C|\alpha(x) - \alpha(y)|[\mathcal{H}(y) + \mathcal{H}(x)]. \end{aligned}$$

Since y runs the ball $B(x, r)$ centered at x , it suffices to deal only with the term $\mathcal{H}(y)$.

Let $\delta \in (0, p_- - 1)$ be a small number. We apply the Hölder inequality with the variable exponent $p_\delta(x) = \frac{p(x)}{1+\delta}$ and have

$$|\alpha(x) - \alpha(y)|\mathcal{H}(y) \leq C|\alpha(x) - \alpha(y)| \|f\|_{L^{p_\delta(\cdot)}} \left\| \frac{\chi_{\Omega \setminus B(y, 2kr)}}{d(y, z)^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}}.$$

The estimate

$$\left\| \frac{\chi_{\Omega \setminus B(y, 2kr)}}{d(z, y)^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}} \leq Cr^{-\frac{n\delta}{p(y)}} \leq Cr^{-\frac{n\delta}{p_-}}$$

follows from Theorem 2.3. Therefore,

$$|\alpha(x) - \alpha(y)|[\mathcal{H}(y) + \mathcal{H}(x)] \leq C \sup_{d(x, y) < r} |\alpha(x) - \alpha(y)| r^{-\frac{n\delta}{p_-}},$$

which implies the boundedness of $|\alpha(x) - \alpha(y)|[\mathcal{H}(y) + \mathcal{H}(x)]$, provided $\alpha(x)$ has the corresponding Hölder property. Since δ may be chosen arbitrarily small, it is sufficient to suppose that α is Hölderian of an arbitrarily small order.

Taking also the embedding $\|f\|_{L^{p_\delta(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}$ into account, we obtain

$$\|G_2\|_{L^\infty} \leq C\|f\|_{L^{p(\cdot)}}. \tag{4.4}$$

Consequently,

$$\|F_2 - c_f\|_{L^\infty} \leq C\|f\|_{L^{p(\cdot)}}, \tag{4.5}$$

by (4.3) and (4.4).

By estimates (4.2) and (4.5), the proof is completed. □

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