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RESEARCH PAPER

A NOTE ON RIESZ FRACTIONAL INTEGRALS IN THE LIMITING CASE $\alpha(x)p(x) \equiv n$

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Dedicated to Professor Francesco Mainardi
on the occasion of his 70th anniversary

Abstract

We show that the Riesz fractional integration operator $I^{\alpha(\cdot)}$ of variable order on a bounded open set in $\Omega \subset \mathbb{R}^n$ in the limiting Sobolev case is bounded from $L^{p(\cdot)}(\Omega)$ into $BMO(\Omega)$, if $p(x)$ satisfies the standard log-condition and $\alpha(x)$ is Hölder continuous of an arbitrarily small order.

MSC 2010: Primary 46E30; Secondary 26A33, 43A85

Key Words and Phrases: fractional integral, Riesz potential, variable exponent Lebesgue space, variable order, BMO

1. Introduction

We consider the Riesz fractional integral

$$I^{\alpha(\cdot)} f(y) = \int\limits_{\Omega} \frac{f(z)\, dz}{|y-z|^{n-\alpha(y)}}, \qquad y \in \Omega,$$

of variable order, on an open set $\Omega \subset \mathbb{R}^n$, in variable exponent spaces $L^{p(\cdot)}(\Omega)$. For the theory of these spaces we refer to the papers [4], [5], [11] and the book [3].

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In [7] it was proved that the Sobolev type theorem on the boundedness of the operator $I^{\alpha(\cdot)}$ on bounded sets Ω, from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \quad \text{and} \quad \sup_{x\in\Omega} \alpha(x)p(x) < n$$

holds for exponents $p(x)$, $1 < p_- \leq p(x) \leq p_+ < \infty$, with the log-condition under the assumption that the maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{B(x,r)} \int\limits_{\widetilde{B}(x,r)} |f(y)|\, dy, \qquad \widetilde{B}(x,r) := B(x,r) \cap \Omega,$$

is bounded in $L^{p(\cdot)}(\Omega)$. The boundedness of the latter for log-continuous exponents was later proved in [2].

For unbounded domains the variable exponent Sobolev theorem is known for constant α, see Theorem 6.1.9 in [3].

Meanwhile a question of interest in the variable exponent setting is to cover the case where

$$\alpha(x)p(x) \equiv n.$$

In the case of constant exponents, it is known that the Riesz fractional integration operator acts in this case from L^p to the space

$$BMO = \{f : \mathcal{M}^\sharp f \in L^\infty\}, \quad \|f\|_{BMO} := \|\mathcal{M}^\sharp f\|_\infty,$$

where

$$\mathcal{M}^\sharp f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{\widetilde{B}(x,r)} \left| f(y) - f_{\widetilde{B}(x,r)} \right| dy,$$

is the sharp maximal function and $f_{\widetilde{B}(x,r)} = \frac{1}{|\widetilde{B}(x,r)|} \int_{\widetilde{B}(x,r)} f(z)\, dz$. This goes back to a result in [13], where it was given in convolution terms; for weighted versions of such a result for Riesz potentials we refer to [6].

In this note we show that the $L^{p(\cdot)} \to BMO$-boundedness holds also in the variable exponent setting when $\alpha(x)p(x) \equiv n$, at the least in the case of bounded sets Ω.

Observe that the case of unbounded sets needs special treatment not only because of the known problems related to infinity well known in the variable exponent analysis, but also by a reason which goes back to the case of constant exponents. Even when α and p are constant, the operators $\int_\Omega f(y)|y-x|^{\alpha-n}$ on an unbounded set Ω is not well defined by this direct definition (as an absolutely convergent integral) for all the functions on the whole space $L^p(\Omega)$ when $\alpha p = n$ since $(n-\alpha)p' = n$ in this case, although it may be treated as a continuous continuation from a dense set in L^p, as an operator acting from L^p to BMO. The operator may be also treated in this case via distributional interpretation. Since Schwartz test function space is not invariant with respect to the Riesz fractional integration, other space (known as the Lizorkin test function space, see Chapter 2 of [10]) is used. In [9] there was shown that the Riesz fractional operator I^α of functions $f \in L^p(\mathbb{R}^n)$ with $\alpha \geq \frac{n}{p}$, interpreted in the distributional sense is a regular distribution. More precisely, any distribution $I^\alpha f, 0 < \alpha < \infty$, generated by a function $f \in L_p,\ 1 \leq p < \infty$, is a regular distribution and even belongs to $L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Besides this, finite differences of the distribution $I^\alpha f$ are well known to be better globally defined on $L^p(\mathbb{R}^n)$, which in a sense is reflected in the BMO-language for the range $I^\alpha(L^p(\mathbb{R}^n))$ when $\alpha p = n$.

2. Preliminaries

Recall that the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the space of measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int\limits_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\} < \infty. \tag{2.1}$$

The local log-condition well known in the variable exponent analysis has the form

$$|p(x) - p(y)| \leq \frac{A}{ln\frac{1}{|x-y|}} \quad \text{for all} \quad x, y \in \Omega \quad \text{with} \quad |x - y| \leq \frac{1}{2}, \tag{2.2}$$

where $A > 0$ does not depend on x and y. In case of a bounded set Ω the condition (2.2) may be also equivalently written in the form

$$|p(x) - p(y)| \leq \frac{NA}{ln\frac{N}{|x-y|}} \qquad x, y \in \Omega, \qquad N = 2\, diam\, \Omega. \tag{2.3}$$

The condition

$$|p(x) - p(\infty)| \leq \frac{A}{ln(e + |x|)}, \quad x \in \Omega \tag{2.4}$$

in case of unbounded sets, is known as the decay condition.

We always assume that

$$1 < p_- \leq p(x) \leq p_+ < \infty. \tag{2.5}$$

3. Theorem on $L^{p(\cdot)}(\Omega) \to BMO$-boundedness

Let

$$\mathcal{M}^{\beta(\cdot)} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta(x)}} \int\limits_{B(x,r)} |f(y)|\, dy$$

be the variable order fractional maximal function. In the following lemma a set Ω is admitted to be unbounded, but the main statement in Theorem **3.1** relates only to bounded sets.

LEMMA **3.1**. *Let the exponent p satisfy the conditions (2.2) and (2.5). In case Ω is unbounded we also suppose that (2.4) holds and $p_- = p(\infty)$. Then*

$$\|\mathcal{M}^{\beta(\cdot)} f\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}$$

for any measurable function $\beta(x)$ such that

$$\frac{n}{p(x)} \leq \beta(x)$$

when Ω is bounded, and

$$\frac{n}{p(x)} \leq \beta(x) \leq \frac{n}{p(\infty)},$$

when Ω is unbounded.

P r o o f. By the Hölder inequality for variable exponents, we have

$$\mathcal{M}^{\beta(\cdot)} f(x) \leq \sup_{r>0} r^{\beta(x)-n} \|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(\Omega)} \|f\|_{L^{p(\cdot)}(\Omega)}.$$

It is known ([8], Theorem 2.21; see also [12]) that

$$\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(\Omega)} \leq C r^{\frac{n}{p'(x)}}, \tag{3.1}$$

when Ω is bounded, and

$$\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(\Omega)} \leq C r^{\frac{n}{p'_r(x)}},$$

when Ω is unbounded, where

$$p_r(x) := \begin{cases} p(x), & \text{if } \ 0 < r < 1 \\ p(\infty), & \text{if } \ r \geq 1, \end{cases}$$

see Corollary 4.5.9 in [3]. Therefore,

$$\mathcal{M}^{\beta(\cdot)} f(x) \leq C \sup_{0<r<\operatorname{diam}\Omega} r^{\beta(x)-n+\frac{n}{p'(x)}} \|f\|_{L^{p(\cdot)}(\Omega)}$$

when Ω is bounded, and

$$\mathcal{M}^{\beta(\cdot)} f(x) \leq C \sup_{r>0} r^{\beta(x)-n+\frac{n}{p'_r(x)}} \|f\|_{L^{p(\cdot)}(\Omega)}$$

when Ω is unbounded. Hence the statement of the lemma follows. □

By $H^\lambda(\Omega)$ we denote the space of functions f on Ω satisfying the Hölder condition: $|f(x)-f(y)| \leq C|x-y|^\lambda$, $0 < \lambda \leq 1$. Let $H(\Omega) = \cup_{0<\lambda\leq 1} H^\lambda(\Omega)$.

THEOREM **3.1**. *Let Ω be a bounded open set and p satisfy the conditions in (2.2) and* (2.5). *Let also $\alpha \in H(\Omega)$. If $\alpha(x)p(x) \equiv n$, then the Riesz potential operator is bounded from $L^{p(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

P r o o f. Suppose that $f(z) \geq 0$. We continue the function f as zero outside Ω whenever necessary. For $r > 0$ we split the function f as $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = f(z)\chi_{B(x,2r)}(z), \qquad f_2(z) = f(z)\chi_{\Omega\setminus B(x,2r)}(z)$$

and then

$$I^{\alpha(\cdot)}f(y) = I^{\alpha(\cdot)}f_1(y) + I^{\alpha(\cdot)}f_2(y) =: F_1(y) + F_2(y).$$

Estimation of $F_1(y)$:
When $y \in B(x,r)$, we have $|x-y| < 3r$ for $z \in B(x,2r)$ so that

$$F_1(y) \leq \int\limits_{|y-z|<3r} \frac{f(z)\,dz}{|z-y|^{n-\alpha(y)}} \leq \frac{2^n(3r)^{\alpha(y)}}{2^{\alpha(y)}-1}\,\mathcal{M}f(y) \leq Cr^{\alpha(y)}\mathcal{M}f(y)$$

for $y \in B(x,r)$, by the well known inequality. Then

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leq Cr^{\alpha(x)-n}\int\limits_{B(x,r)}\mathcal{M}f(y)\,dy.$$

We apply Hölder inequality and obtain

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leq Cr^{\alpha(x)-n}\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}}\|\mathcal{M}f\|_{L^{p(\cdot)}},$$

whence by (3.1) and the boundedness of the maximal operator in $L^{p(\cdot)}(\Omega)$, see for instance [1], we get

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leq Cr^{\alpha(x)-n+\frac{n}{p'(x)}}\|f\|_{L^{p(\cdot)}} = C\|f\|_{L^{p(\cdot)}}. \tag{3.2}$$

Estimation of $F_2(y)$:
We denote

$$c_f = F_2(x) = \int\limits_{\Omega\setminus B(x,2r)} \frac{f(z)\,dz}{|x-z|^{n-\alpha(x)}}$$

and have

$$|F_2(y) - c_f| = \left|\int\limits_{\Omega\setminus B(x,2r)} f(z)\left[\frac{1}{|y-z|^{n-\alpha(y)}} - \frac{1}{|x-z|^{n-\alpha(x)}}\right]dz\right|,$$

whence

$$|F_2(y) - c_f| \leq \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left| \frac{1}{|y-z|^{n-\alpha(x)}} - \frac{1}{|x-z|^{n-\alpha(x)}} \right| dz$$

$$+ \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left| \frac{1}{|y-z|^{n-\alpha(y)}} - \frac{1}{|y-z|^{n-\alpha(x)}} \right| dz =: G_1 + G_2.$$

To estimate G_1, we use the inequality

$$|a^{-\gamma} - b^{-\gamma}| \leq |\gamma| \cdot |a-b| (\min\{a,b\})^{-\gamma-1}, \quad a>0,\ b>0, \quad \gamma \in \Omega, \tag{3.3}$$

and observe that $|y-x| < r$ and $|z-y| > 2r$ imply $|x-z| < \frac{3}{2}|y-z|$, so that

$$G_1 \leq C|x-y| \int\limits_{\Omega\setminus B(x,2r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)+1}}$$

$$= c|x-y| \sum_{k=1}^{\infty} \int\limits_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)+1}}.$$

By the inequality

$$\int\limits_{B(x,2r)\setminus B(x,r)} \frac{f(z)\,dz}{|x-z|^{n-\alpha(x)+1}} \leq \frac{2^{n-\alpha(x)}}{r} \mathcal{M}^{\alpha(\cdot)} f(x)$$

valid for $0 < \alpha(x) < n$, we get

$$G_1 \leq C \frac{|x-y|}{r} \mathcal{M}^{\alpha(\cdot)} f(x) \leq C \mathcal{M}^{\alpha(\cdot)} f(x) \|f\|_{L^{p(\cdot)}(\Omega)}$$

Therefore,

$$\|G_1\|_{L^\infty} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}, \tag{3.4}$$

by Lemma **3.1**.

For G_2 we use the inequality

$$\left| t^a - t^b \right| \leq |a-b| \begin{cases} t^{\min\{a,b\}}, & \text{if } \ 0 < t \leq 1 \\ t^{\max\{a,b\}}, & \text{if } \ \ t \geq 1. \end{cases}$$

and obtain

$$G_2 \leq |\alpha(x) - \alpha(y)| \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left(\frac{1}{|y-z|^{n-\alpha(y)}} + \frac{1}{|y-z|^{n-\alpha(x)}} \right) dz.$$

Since $|y-z| \geq \frac{2}{3}|x-z|$ and $\Omega\setminus B(x,2r) \subseteq \Omega\setminus B(y,r)$, we obtain

$$G_2 \leq C|\alpha(x) - \alpha(y)| \left(\int\limits_{\Omega\setminus B(y,r)} \frac{|f(z)|\,dz}{|y-z|^{n-\alpha(y)}} + \int\limits_{\Omega\setminus B(x,2r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)}} \right)$$

$$=: C|\alpha(x)-\alpha(y)|[H(y)+H(x)].$$

Since y runs the ball $B(x,r)$ centered at x, it suffices to deal only with the term $H(y)$.

Let $\delta \in (0, p_- - 1)$ be a small number. We apply the Hölder inequality with the variable exponent $p_\delta(x) = \frac{p(x)}{1+\delta}$ and have

$$|\alpha(x)-\alpha(y)|H(y) \le C|\alpha(x)-\alpha(y)|\|f\|_{L^{p_\delta(\cdot)}} \left\| \frac{\chi_{\Omega\setminus B(y,2r)}}{|z-y|^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}}.$$

The estimate

$$\left\| \frac{\chi_{\Omega\setminus B(y,2r)}}{|z-y|^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}} \le Cr^{-\frac{n\delta}{p(y)}} \le Cr^{-\frac{n\delta}{p_-}}$$

is valid ([7], Theorem 1.8). Therefore,

$$|\alpha(x)-\alpha(y)|[H(y)+H(x)] \le C \sup_{|x-y|<r} |\alpha(x)-\alpha(y)| r^{-\frac{n\delta}{p_-}}.$$

Which provides the boundedness of $|\alpha(x)-\alpha(y)|[H(y)+H(x)]$ provided $\alpha(x)$ has the corresponding Hölder property. Since δ may be chosen arbitrarily small, it is sufficient to suppose that α is Hölderian of an arbitrarily small order.

Taking also the embedding $\|f\|_{L^{p_\delta(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}$ into account, we obtain

$$\|G_2\|_{L^\infty} \le C\|f\|_{L^{p(\cdot)}}. \tag{3.5}$$

Consequently,

$$\|F_2 - c_f\|_{L^\infty} \le C\|f\|_{L^{p(\cdot)}}. \tag{3.6}$$

by (3.4) and (3.5).

It remains to gather the estimates (3.2) and (3.6). □

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