Bounding anisotropic Lorentz invariance violation from measurements of the effective energy scale of quantum gravity

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(Received 11 July 2025; accepted 31 July 2025; published 4 November 2025)

Observations of energy-dependent photon time delays from distant flaring sources provide significant constraints on Lorentz invariance violation (LIV). Such effects originate from modified vacuum dispersion relations causing differences in propagation times for photons emitted simultaneously from gamma-ray bursts, active galactic nuclei, or pulsars. These modifications are often parametrized within a general framework by an effective quantum gravity energy scale $E_{QG,n}$. While such general constraints are well established in the LIV literature, their translation into specific coefficients of alternative theoretical frameworks, such as the Standard-Model extension (SME), is rarely carried out. In particular, existing bounds on the quadratic case ($n = 2$) of $E_{QG,n}$ can be systematically converted into constraints on the nonbirefringent, CPT-conserving SME coefficients $c^{(6)}_{(I)jm}$. This work provides a concise overview of the relevant SME formalism and introduces a transparent conversion method from $E_{QG,2}$ to SME parameters. We review the most stringent time-of-flight-based bounds on $E_{QG,n}$ and standardize them by accounting for systematics, applying missing prefactors, and transforming results into two-sided Gaussian uncertainties where needed. We then use these standardized constraints, along with additional bounds from the literature, to improve bounds on the individual SME coefficients of the photon sector by about an order of magnitude. A consistent methodology is developed to perform this conversion from the general LIV framework to the SME formalism.

DOI: 10.1103/k3xg-wkrc

I. INTRODUCTION

The most stringent bounds on a modified dispersion relation of the photon in vacuum have been derived from astrophysical tests of energy-dependent time delays of photons from very distant but fast-varying (flaring) sources. In this scenario, simultaneously emitted photons of different energies would be detected on Earth at different times, due to possible Lorentz-invariance-violating (LIV) properties of vacuum [1–5]. In the most generic approach, the modified vacuum dispersion relation for the photon is expanded into a power-law series of the following form [3]:

$$E^2 \simeq p^2c^2 \times \left[1 - \sum_{n=1}^{\infty} s_\pm \left(\frac{E}{E_{QG,n}}\right)^n\right]. \qquad (1)$$

Here, c is the speed of light, $s_\pm$ defines the sign of the (LIV) effect ($s_\pm = -1$ for superluminal and $s_\pm = +1$ for subluminal behavior), and $E_{QG,n}$ refers to an effective energy scale of quantum gravity [3]. The parameter n denotes the expansion order; i.e., $n = 1$ is used for a deviation from the trivial case that scales linearly with energy, while $n = 2$ is for the quadratic case. From Eq. (1), the photon's group velocity can be obtained assuming the dominant lowest-order term is n:

$$u_{ph}(E) = \frac{\partial E}{\partial p} \simeq c \times \left[1 - s_\pm \frac{n+1}{2}\left(\frac{E}{E_{QG,n}}\right)^n\right]. \qquad (2)$$

Hence, two photons with different observed energies E_h and E_l ($E_h > E_l$), emitted simultaneously from the same

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source, will reach Earth with a time delay, which for large distances depend on the cosmological model. In its first version presented [6], the time delay is expressed as

$$\Delta t \simeq (E_h^n - E_l^n) s_\pm \frac{(n+1)}{2H_0} \frac{1}{E_{QG,n}^n} \times \int_0^z \frac{(1+z')^n}{\sqrt{\Omega_\Lambda + \Omega_\mathrm{M}(1+z')^3}} dz', \tag{3}$$

but other, more complicated formulations of the redshift dependence have been proposed [7,8].

Bounds on $E_{QG,1}$ up to and well beyond the Planck energy have been obtained with this method from gamma-ray bursts (GRBs) [9–14] and active galactic nuclei (AGNs) [15–18]. Furthermore, the same sources and pulsars [19] have contributed to strong bounds on $E_{QG,2}$, although those are still orders of magnitude below the Planck energy scale.

Moving to a more specific case, various scenarios have been formulated that may result in the generation of a LIV-induced vacuum dispersion relation for photons. It is important to note that all these scenarios assume quantum properties of gravity [20–25].

In a complementary approach proposed by Kostelecký and co-workers, violations of Lorentz symmetry at attainable energies are described by the Standard-Model extension (SME) [26–28]. This is an effective field theory allowing for the possibility of Lorentz and CPT violation in the Standard Model of particles coupled to general relativity. The SME is generally chosen to preserve energy-momentum conservation, observer Lorentz invariance, hermiticity, microcausality, gauge invariance, and reparametrization invariance. Each Lorentz-violating term in the Lagrangian density of the SME is an observer scalar density formed by contracting a Lorentz-violating operator of a certain mass dimension with a coefficient governing the size of the effect. These can, in principle, be measured by appropriate experiments. All possible SME terms affecting photon propagation have been constructed explicitly [29].

A large, ever-expanding set of experimental bounds on the SME coefficients has been published. An overview can be found in a set of data tables elaborated by Kostelecký and Russell [30].[1] However, it is useful to note that publications leading to entries in these tables are not independently checked for quality of the methodology and analysis, as is clearly stated in the introduction of the tables. At the same time, the astrophysical LIV community rarely converts their bounds on $E_{QG,n}$ to those that match the SME framework (with the commendable exceptions of [14,17,31]). In particular, the bounds on $E_{QG,2}$ could and should also be converted to the nonbirefringent and CPT-conserving coupling SME coefficients.

The purpose of this paper is to provide an overview of the SME formalism leading to an easy and straightforward conversion recipe from $E_{QG,2}$ to SME coefficients. In this process, we critically review publications in which such a conversion has been made, including the data tables [30]. We will correct inconsistencies introduced during early papers on $E_{QG,n}$ and present a consistent collection of bounds on the SME coefficients $c_{(I)jm}^{(6)}$, which differ from the current version of the data tables and extend those. Finally, we provide a simple new recipe to deconvolute a combination of bounds on $E_{QG,2}$ obtained from sources at different directions into a set of bounds on the individual SME coefficients.

The article is organized as follows: Section II provides a review of the SME formalism in the photon sector, while Sec. III outlines the procedure to convert constraints on the quantum gravity scale $E_{QG,2}$ into bounds on SME coefficients. In Sec. IV, a compilation and critical comparison of the most stringent existing bounds, including necessary corrections, is presented; also, individual constraints on the SME coefficients $k_{(I)jm}^{(6)}$ are derived through a statistical combination from multiple astrophysical sources. Finally, Sec. V draws the conclusions.

II. THE PHOTON SECTOR OF THE STANDARD-MODEL EXTENSION

Numerous studies have proposed that one of the physical effects of quantum gravity may be the violation of Lorentz invariance [2,5,32–37]. One of the most useful frameworks for studying these effects is the SME [27,28,38]. In this work, we are interested in the photon sector developed in detail in a series of works [27,29,39,40]. In this section, we will take the opportunity to present a short review of the most relevant aspects and formulas. For a more complete treatment, we refer to the relevant literature, in particular to [29].

The pure photon sector of the SME is entirely described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\kappa\lambda} (\hat{k}_F)^{\kappa\lambda\mu\nu} F_{\mu\nu} + \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} A_\lambda (\hat{k}_{AF})_\kappa F_{\mu\nu}, \tag{4}$$

where A_μ is the usual magnetic vector potential while $F_{\mu\nu}$ is the field-strength tensor. The operator-valued coefficients in the second and third terms on the right-hand side are defined by [29]

$$(\hat{k}_F)^{\kappa\lambda\mu\nu} = \sum_{d=\mathrm{even}} (k_F^{(d)})^{\kappa\lambda\mu\nu\alpha_1\ldots\alpha_{d-4}} \partial_{\alpha_1} \ldots \partial_{\alpha_{d-4}}, \tag{5}$$

[1] They are updated on a yearly basis on a preprint server (see https://arxiv.org/abs/0801.0287).

$$(\hat{k}_{AF})_\kappa = \sum_{d=\mathrm{odd}} (k_{AF}^{(d)})_\kappa{}^{\alpha_1\ldots\alpha_{d-3}}\partial_{\alpha_1}\ldots\partial_{\alpha_{d-3}}, \tag{6}$$

with the mass dimension $d \geq 3$ defined by partial spacetime derivatives contracted with the (constant) coefficients $(k_F^{(d)})^{\kappa\lambda\mu\nu\alpha_1\ldots\alpha_{d-4}}$ and $(k_{AF}^{(d)})_\kappa{}^{\alpha_1\ldots\alpha_{d-3}}$. The latter parametrize, respectively, the most general CPT-even and CPT-odd operators in the photon sector of the SME with mass dimension d.[2]

Of prime interest is the photon dispersion relation, which follows from the equations of motion corresponding to the Lagrangian density Eq. (4). Adopting the ansatz

$$A_\mu(x) = A_\mu(p)e^{-ix\cdot p}, \tag{7}$$

the equation of motion can be expressed as

$$M^{\mu\nu}A_\nu = 0, \tag{8}$$

with

$$M^{\mu\nu} = (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + 2(\hat{k}_F)^{\mu\alpha\nu\beta})p_\alpha p_\beta - 2i\epsilon^{\mu\nu\alpha\beta}(\hat{k}_{AF})_\alpha p_\beta, \tag{9}$$

where it is understood that each occurrence of ∂_ρ in the operators $\hat{k}_{AF}$ and $\hat{k}_F$ is replaced by $-ip_\rho$. The CPT-violating coefficients $(\hat{k}_{AF})_\alpha$ lead to photon birefringence [29]. The bounds obtained from the absence of vacuum birefringence via analogous sources of x-ray and gamma-ray polarimetry [31,42] are by several orders of magnitude more stringent than those from a modified dispersion relation. For this reason, the present study will focus exclusively on nonbirefringent photon propagation, thereby excluding the other terms from the subsequent analysis.

The coefficient $(\hat{k}_F)^{\mu\alpha\nu\beta}$ has the symmetries of the Riemann tensor. It can therefore be expanded in a Weyl decomposition as

$$(\hat{k}_F)^{\mu\alpha\nu\beta} = \frac{1}{2}(\eta^{\mu\nu}\hat{c}_F^{\alpha\beta} + \eta^{\alpha\beta}\hat{c}_F^{\mu\nu} - \eta^{\nu\alpha}\hat{c}_F^{\mu\beta} - \eta^{\mu\beta}\hat{c}_F^{\nu\alpha}) + \hat{C}^{\mu\alpha\nu\beta} \tag{10}$$

where the tensor $\hat{C}^{\mu\alpha\nu\beta}$ corresponds to the Weyl component satisfying $\hat{C}^{\mu\alpha\nu\beta}\eta_{\alpha\beta} = 0$, while the terms corresponding to the Ricci component involve the tensor

$$\hat{c}_F^{\alpha\beta} = (\hat{k}_F)^{\alpha\lambda\beta}{}_\lambda - \frac{1}{6}\eta^{\alpha\beta}(\hat{k}_F)^{\mu\nu}{}_{\mu\nu}. \tag{11}$$

The Weyl component leads to birefringence [29], while (at leading order) the Ricci component does not. Therefore, we will consider, in the following, only the case $\hat{C}^{\mu\alpha\nu\beta} = 0$.

To first order in the coefficients $\hat{c}_F^{\alpha\beta}$, the equation of motion (8) can be written as

$$((\hat{g}^{-1})^{\mu\nu}(\hat{g}^{-1})^{\alpha\beta} - (\hat{g}^{-1})^{\mu\alpha}(\hat{g}^{-1})^{\nu\beta})p_\alpha p_\beta A_\nu = 0 \tag{12}$$

where we introduced the "inverse effective metric"

$$(\hat{g}^{-1})^{\mu\nu} = \eta^{\mu\nu} + \hat{c}_F^{\mu\nu}. \tag{13}$$

The equation of motion (12) is invariant under gauge transformations $A_\mu \to A_\mu + \epsilon p_\mu$, allowing us to impose the "deformed Lorenz gauge" $p_\mu(\hat{g}^{-1})^{\mu\nu}A_\nu = 0$. The equation of motion (12) then reduces to

$$p_\alpha(\hat{g}^{-1})^{\alpha\beta}p_\beta(\hat{g}^{-1})^{\mu\nu}A_\nu = 0 \tag{14}$$

from which we immediately deduce the dispersion relation

$$p_\alpha(\hat{g}^{-1})^{\alpha\beta}p_\beta = p^\mu p_\mu + \hat{c}_F^{\mu\nu}p_\mu p_\nu = 0. \tag{15}$$

When working to leading order in the Lorentz-violating coefficients, we are free to replace p^0 in $\hat{c}_F^{\mu\nu}p_\mu p_\nu$ by its vacuum plane-wave value $\omega = p \equiv |\vec{p}|$ [29,40] (note that the coefficients $\hat{c}_F^{\mu\nu}$ themselves are momentum dependent). The dispersion relation (15) can then be expressed as

$$p^0 = (1 - \varsigma^0)p \tag{16}$$

with

$$\varsigma^0 = \frac{1}{2}(\hat{c}_F)^{\mu\nu}p_\mu p_\nu/\omega^2. \tag{17}$$

Note that we adopted the metric signature $(+---)$. An analysis that also takes into account the Weyl part of the coefficients $\hat{k}_F$ and the coefficients $\hat{k}_{AF}$ yields a generalization of relations (15) and (16); see [43].

It is useful to note that the coefficients $(\hat{c}_F)^{\mu\nu}$ are momentum dependent and can be expanded in accordance with Eqs. (11) and (5):

$$\hat{c}_F^{\mu\nu}p_\mu p_\nu = \sum_{d=\mathrm{even},d\geq 4} (\tilde{c}_F^{(d)})^{\mu\nu\alpha_1\ldots\alpha_{d-4}}p_\mu p_\nu p_{\alpha_1}\ldots p_{\alpha_{d-4}}. \tag{18}$$

[2]Silently, it is assumed here that we are working in Minkowski space. In fact, we will be considering Friedmann-Lemaître-Robertson-Walker-type cosmological spacetimes. Strictly, in such a curved-space context, one should consider a more general expansion involving covariant derivatives and contractions, as well as the Riemann tensor. Such a more general framework for the SME has been developed recently [41]. However, the relevant curved-space corrections, which will involve powers of the Hubble constant, are extremely tiny compared to the photon momenta we will be considering in this work. Therefore, the Lagrangian density [Eq. (4)] is perfectly adequate for our purposes.

The structure of Eq. (18) shows that the coefficients $(\tilde{c}_F^{(d)})^{\mu\nu\alpha_1\ldots\alpha_{d-4}}$ can be taken totally symmetric in all its indices. This can be enforced by imposing the constraint

$$\partial_p^{[\mu}\hat{c}_F^{\rho]\nu} = 0, \tag{19}$$

where $\partial_p^\mu \equiv \partial/\partial p_\mu$ and the square brackets denote antisymmetrization. A natural way to satisfy condition (19) is by introducing a scalar potential $\hat{\Phi}_F$ and taking

$$\hat{c}_F^{\mu\nu} = \partial_p^\mu \partial_p^\nu \hat{\Phi}_F. \tag{20}$$

It is then convenient to expand $\hat{\Phi}_F$ in spherical harmonics,

$$\hat{\Phi}_F = \sum_{d,n,j,m} \omega^{d-2-n} p^n {}_0Y_{jm}(\hat{p})(c_F^{(d)})_{njm}^{(0E)}. \tag{21}$$

Here, the index ranges satisfy $d = 4, 6, 8, \ldots$; $0 \leq n \leq d-2$; $j = n, n-2, n-4, \ldots$ $(j \geq 0)$; $-j \leq m \leq j$. Note that the expansion (21) does not assume (yet) the leading-order approximation $\omega = p$. It then follows from Eqs. (17) and (20), after some algebra, that

$$\varsigma^0 = \sum_{d,j,m} \omega^{d-4}(-1)^j {}_0Y_{jm}(\hat{p}) c_{(I)jm}^{(d)} \tag{22}$$

where

$$c_{(I)jm}^{(d)} = \frac{1}{2}(d-2)(d-3)(-1)^j \sum_n (c_F^{(d)})_{njm}^{(0E)} \tag{23}$$

are a set of $(d-1)^2$ coefficients parametrizing the nonbirefringent part at dimension d of the $\hat{k}_F^{\mu\alpha\nu\beta}$ coefficients. Note that in Eq. (23) we imposed the leading-order "vacuum-model" approximation $\omega = p$ for the terms on the right-hand side [29].

Sometimes, the vacuum model is restricted to the isotropic subset of LIV operators that are invariant under spatial rotations. Note that the notion of spatial rotations (as opposed to boosts) is frame dependent; in other words, it assumes the presence of a preferred observer Lorentz frame. This model is obtained by setting all coefficients to 0, except for those with $j = 0$, which means that the total angular momentum is 0. Therefore, the only nonzero coefficient that we are interested in for our case is

$$(\mathring{c}_F^{(d)})_n = (c_F^{(d)})_{n00}^{(0E)}, \tag{24}$$

since it controls the nonbirefringent operators in the preferred frame. The index n determines the wavelength or frequency dependence, while the other two indices correspond to $j = 0$ and $m = 0$, the total angular momentum and its z-component, respectively. The superscript 0 refers to the spin weight of the operator, and with E parity, that is, it transforms as $(-1)^{j+1}$ under parity.

III. DISPERSION TESTS

In this section, we explore how time delays between high-energy photons from distant astrophysical sources can be used to test for possible violations of Lorentz invariance. If photons of different energies travel at slightly different speeds, even small differences can lead to observable delays over cosmological distances. These delays provide a way to probe modified dispersion relations and place constraints on SME coefficients. We show how previous bounds on $E_{QG,n}$ can be translated into bounds on SME coefficients.

Assuming that the temporal emission of the source is well understood, we can use the time delays to constrain the velocities depending on the photon energies. Setting to zero all the birefringent coefficients, the group velocity of the photon becomes

$$\begin{aligned} v_{gr} &\simeq 1 - (d-3)\varsigma^0 \\ &= 1 - (d-3)\sum_{djm} E^{d-4} {}_0Y_{jm}(\hat{n}) c_{(I)jm}^{(d)}. \end{aligned} \tag{25}$$

As demonstrated by the above equation, the minimum SME, which only includes $d = 4$, is not enough to test a modified dispersion relation, since there is no difference in velocity for different energies. Consequently, considering higher-dimensional operators becomes imperative.

Assuming that the photons are emitted simultaneously by the same source, the comoving distance between the source and Earth is the same for both photons. However, due to the difference in group velocity, two photons observed with energies E_l and E_h will have a time delay given by [43]

$$\Delta t \approx (d-3)(E_h^{d-4} - E_l^{d-4}) \int_0^z \frac{(1+z)^{d-4}}{H_z} dz \sum_{jm} {}_0Y_{jm} c_{(I)jm}^{(d)} \tag{26}$$

where H_z is the Hubble expansion rate at redshift z defined as[3]

$$\begin{aligned} H_z = H_0[&\Omega_r(1+z)^4 + \Omega_M(1+z)^3 \\ &+ \Omega_k(1+z)^2 + \Omega_\Lambda]^{1/2}. \end{aligned} \tag{27}$$

Hence, by comparing Eq. (3) with Eq. (26), one can achieve the objective of this work: to establish a straightforward conversion between the general LIV parameters

[3]Note that Ω_r and Ω_k have been frequently neglected [6–8] due to their relative smallness [44].

and the SME coefficients, leading to the following formal substitution:

$$\frac{s_{\pm}}{2E_{QG,d-4}^{d-4}} \rightarrow \sum_{jm} Y_{jm} c_{(I)jm}^{(d)} \tag{28}$$

whereby convention $s_{\pm} = +1$ would apply for a positive sum on the right side and $s_{\pm} = -1$ for a negative sum. Recalling that the corresponding SME coefficient of $E_{QG,1}$ is birefringent, we will only focus on the cases of $(d-4)$ even and greater than or equal to 2. The bounds treated in this article apply to the particular case of $E_{QG,2}$.

IV. RESULTS

This section presents the translation of existing bounds on the quantum gravity energy scale $E_{QG,2}$ to constraints on the SME coefficients. An extensive literature review has been conducted to collect data on $E_{QG,2}$ based on the time-of-flight method. To ensure consistency across the different studies, we first homogenized the reported bounds on $E_{QG,2}$ in terms of confidence level, systematic uncertainties, and possible discrepancies in the series expansion, Eq. (2), of the group velocity of the photon.

Constraints on the sum of the nonbirefringent coefficients of the photon sector, $c_{(I)jm}^{(6)}$, are obtained following the formalism introduced in the previous section. Then, we compare the bounds with those reported in the data tables [30]. Afterward, we present a method to derive bounds on the individual coefficients $c_{(I)jm}^{(6)}$ and list the obtained bounds.

A. Bounds on the nonbirefringent coefficients of the photon sector

A summary of the most stringent lower bounds on $E_{QG,1}$ and $E_{QG,2}$ based on time-of-flight studies is given in Table I, which includes both superluminal ($s_{\pm} = -1$) and subluminal ($s_{\pm} = +1$) scenarios. All bounds were derived following the cosmological model proposed by Jacob and Piran [6] and expressed at a 95% confidence level (CL). Only results obtained using the maximum-likelihood method [45] have been considered.

Several bounds presented in Table I required corrections to be comparable. Specifically, those bounds marked with an ([a]) had been initially published omitting the prefactor $(n+1)/2$ in the quadratic term of the photon group velocity expansion [Eq. (2)]. We incorporated that factor *post hoc* to their original publication. In addition, the entries marked with a ([b]) account for corrections due to systematic uncertainties in the energy scale that were omitted in the original publication. These systematic uncertainties depend on the instrument: A 10% uncertainty is considered for Fermi-LAT [14] and an uncertainty of 12% is assumed for the Water Cherenkov Detector Array as suggested by the LHAASO experiment [51], while a 20% uncertainty is applied for Imaging Atmospheric Cherenkov Telescopes (IACTs), like MAGIC and H.E.S.S. [49,50]. The corresponding bounds on the sum of the SME coefficients $c_{(I)jm}^{(d)}$ were then derived using Eq. (28) and are also listed in Table I.

Comparing our results with those presented in the data tables [30], we observe the following differences: First, this work includes several new astrophysical sources not formerly covered in the data tables, specifically the Crab Pulsar [19], the AGNs PG $1553+113$ [46] and Mrk 421 [15], and the gamma-ray bursts GRB 190114C [10] and GRB 221009A [12]. Among these, the results on Mrk 421 and GRB 221009A have been published only recently. We carefully revised the corresponding bounds on the quantum gravity energy scale $E_{QG,n}$ from these sources and converted both superluminal and subluminal on $E_{QG,2}$ bounds to nonbirefringence SME parameters using Eq. (28). Notably, the original studies of GRB 190114C [10] and GRB 221009A [12] did not account for systematic uncertainties in the energy scale; we have incorporated these corrections by applying 20% and 12% uncertainty, respectively.

It is also important to highlight the inclusion of a pulsar—namely, the Crab Pulsar—for the first time in such an analysis. Due to their high stability and predictable emission profiles, pulsars serve as particularly valuable benchmarks for time-of-flight LIV studies. Their inclusion broadens the scope of this work, increasing the diversity of source types and enhancing the overall robustness of the analysis. Moreover, we have updated the bounds from Mrk 501, originally presented in the data tables based on MAGIC observations [52], to reflect the more recent and stringent results from the H.E.S.S. Collaboration [48], who reported a stronger flare in June 2014 with an extended energy range up to 20 TeV. A correction for the prefactor $(n+1)/2$ in the quadratic term of the photon group velocity was required and implemented accordingly. Similarly, for PKS 2155-304 [16], which is found in the data tables, we applied the same correction for the prefactor $(n+1)/2$, yielding a tighter bound on $E_{QG,2}$ and consequently on the nonbirefringent coefficients of the photon sector.

For the bounds obtained from GRB 090510 [14], GRB 080916C [14], GRB 090902B [14], and GRB 090926A [14], we applied a correction corresponding to 10% systematic uncertainty on the energy scale of Fermi-LAT. We also note that only results obtained via the maximum-likelihood method from Vasileiou *et al.* [14] were selected and translated into bounds on the SME coefficients. When attempting to reproduce the bounds on SME coefficients reported by Vasileiou *et al.* [14]—which, to our knowledge, is the only publication in the literature directly converting time-of-flight results to the SME framework—found several discrepancies. A possible missing factor of $(d-3)$ may explain the mismatch only

TABLE I. List of the most stringent lower bounds on $E_{QG,1}$ and $E_{QG,2}$ based on time-of-flight studies. The table is divided by source type: pulsars (first row), AGNs (second to fifth rows), and GRBs (sixth to last rows). $s_\pm = -1$ and $s_\pm = +1$ represent superluminal and subluminal behaviors, respectively. All lower bounds are expressed at 95% CL. The final column indicates whether the detailed likelihood profile for each source is publicly available: ✓ denotes complete availability, ✓ (?) indicates partial or incomplete availability, and ✗ indicates it is unavailable. The term "R.A." stands for "Right Ascension" and "Dec." stands for "Declination".

Source	R.A. (deg)	Dec. (deg)	Redshift	Type	$\mathbf{E_{QG,1}}$ ($\times 10^{19}$ GeV)	$\mathbf{E_{QG,2}}$ ($\times 10^{10}$ GeV)	$\sum_{jm} Y_{jm} \mathbf{c}^{(6)}_{(I)jm}$ ($\times 10^{-22}$ GeV^{-2})	$\sum_{jm} Y_{jm} \mathbf{c}^{(6)}_{(I)jm}$ ($\times 10^{-22}$ GeV^{-2}) two-sided	Likelihood availability
Crab Pulsar[c] [19]	83.63	22.01	2.0 kpc	$s_\pm = +1$ $s_\pm = -1$	>0.055 >0.045	>5.9 >5.3	<1.4 >−1.8	<1.7 >−2.1	✓
PKS 2155-304 [16]	329.72	−30.22	0.12	$s_\pm = +1$ $s_\pm = -1$	>0.21 >1.5	>6.4 >7.0	>1.2 >−1.0	>1.2 >−1.0	✓
PG $1553 + 113$[c] [46,47]	238.94	11.19	0.43	$s_\pm = +1$ $s_\pm = -1$	>0.041 >0.028	>2.1 >1.7	<11 >−18	<14 >−21	✗
Mrk 501[a,c] [48]	253.47	39.76	0.034	$s_\pm = +1$ $s_\pm = -1$	>0.036 >0.026	>10.4 >8.9	<0.46 >−0.63	<0.57 >−0.73	✓
Mrk 421 [15]	166.08	38.19	0.031	$s_\pm = +1$ $s_\pm = -1$	>0.036 >0.027	>2.5 >2.6	<8.0 >−7.4	<8.0 >−7.4	✓
GRB 090510[b,c] [14]	333.55	−26.58	0.90	$s_\pm = +1$ $s_\pm = -1$	>4.7 >10	>7.7 >8.5	<0.84 >−0.70	<0.98 >−0.85	✓ (?)
GRB 080916C[b,c] [14]	119.85	−56.64	4.4	$s_\pm = +1$ $s_\pm = -1$	>0.20 >0.18	>0.32 >0.31	$<5.0 \times 10^2$ $>-5.3 \times 10^2$	$<6.0 \times 10^2$ $>-6.3 \times 10^2$	✓ (?)
GRB 090902B[b,c] [14]	264.94	27.32	1.8	$s_\pm = +1$ $s_\pm = -1$	>0.11 >0.33	>0.58 > 0.58	$<1.5 \times 10^2$ $>-1.5 \times 10^2$	$<1.8 \times 10^2$ $>-1.8 \times 10^2$	✓ (?)
GRB 090926A[b,c] [14]	353.40	−66.32	2.1	$s_\pm = +1$ $s_\pm = -1$	>1.1 >0.15	>0.43 >0.28	$<2.7 \times 10^2$ $>-6.4 \times 10^2$	$<3.6 \times 10^2$ $>-7.3 \times 10^2$	✓ (?)
GRB 190114C[b] [10]	54.51	−26.95	0.42	$s_\pm = +1$ $s_\pm = -1$	>0.46 >0.44	>5.0 >4.5	<2.0 >−2.5	<2.0 >−2.5	✓
GRB 221009A[b] [12]	288.26	19.77	0.15	$s_\pm = +1$ $s_\pm = -1$	>8.0 >8.8	>55 >56	<0.014 >−0.013	<0.014 >−0.013	✓ (?)

[a]Refers to the correction due to the missing term $(n+1)/2$ for the photon group velocity.
[b]Indicates the correction to the energy scale due to systematics. A 10% systematic uncertainty due to instrumental effects is used for Fermi-LAT [14]) and a 12% is considered for LHAASO [12], while 20% is applied to IACTs ([49,50]).
[c]Refers those analyses originally carried out using a one-sided Gaussian distribution, which were converted into equivalent two-sided Gaussian bounds. Those two-sided, nonbirefringent SME coefficients are then shown in the penultimate column.

partially. In summary, the corrected bounds from GRB 090510, GRB 080916C, GRB 090902B, and GRB 090926A lead to slightly tighter constraints on the SME coefficients compared to Vasileiou *et al.* [14].

Several time-of-flight studies used one-sided Gaussian confidence intervals (CIs). We have converted these into their equivalent two-sided Gaussian CIs to ensure consistency across all bounds. Analyses that originally employed a one-sided Gaussian distribution are denoted with a ([c]) in the table. These include the Crab Pulsar [19], PG $1553 + 113$ [46], Mrk 501 [48], GRB 090510 [14], GRB 080916C [14], GRB 090902B [14], and GRB 090926A [14]. To perform the conversion, we reviewed the profile likelihoods associated with each source and marked their availability in the final column of Table I. Whereas the majority of the studies report the profile likelihood—with the exception of PG $1553 + 113$ [46]—publications do not provide the best-fit values that minimize the profile likelihood, making it difficult to perform an accurate two-sided conversion, especially when these profile likelihoods are not well behaved. In particular, Vasileiou *et al.* [14] and Cao *et al.* [12] did not publish the calibrated likelihoods, i.e., the reference likelihood functions from which the subsequent bound on $E_{QG,2}$ has been derived. Therefore, to ensure coherence across all datasets, we assumed symmetric and well-behaved profile likelihoods, using the central values of the reported bounds as proxies for the best-fit values. Under this assumption, we have converted the one-sided 95% CL bounds into their equivalent for a two-sided 95% CL.

In summary, new and updated bounds on nonbirefringent SME coefficients of the photon sector derived from time-of-flight studies have been presented in this work. The corrected bounds yield slightly more stringent constraints on LIV effects compared to those in the data tables compilation [30]. While GRB 221009A [12] stands out for providing the most stringent bound to date, it remains an isolated event.

B. Bounds on individual nonbirefringent SME coefficients

Any time delay measurement from photons arriving from a certain source in the sky observed from direction $\hat{n}_i$ leads, in some form, to a posterior probability distribution for the weighted sum of SME coefficients $\theta_i \coloneqq \sum_{j,m} {}_0Y_{jm}(\hat{n}_i) c^{(d)}_{(I)jm}$,

$$P_i(\theta_i; \overrightarrow{\nu_i} | D_i), \qquad i = 1, \ldots, s, \tag{29}$$

in which D_i denotes the ith of a total s time delay measurements and $\vec{\nu}_i$ a possible set of nuisance parameters for D_i, for instance, unknowns in the instrumental acceptance or intrinsic source-dependent effects. If the nuisance parameters have been marginalized or a profile likelihood [53] calculated, a probability distribution depending only on data can be calculated. In the case of large samples, the likelihood $\mathcal{L}_i(D_i|\theta_i)$ will approach a normal distribution [45] for 1 degree of freedom, with a given central mode $\theta_{m,i}$ (which maximizes $\mathcal{L}_i$) and width σ_{θ_i}. In such a case, $\theta_{m,i}$ will be found close to the expectation value $E[\theta_i]$ and $\sigma^2_{\theta_i}$ close to its variance $s^2[\theta_i]$. Unfortunately, very often [14,19,31], statistics of those events with the highest energies—which receive a strong weight in the likelihood in the case of $d > 5$ [see Eq. (26)]—is low, also due to the typical power-law spectra of the involved sources. Together with asymmetric systematics $\vec{\nu}_i$, this leads to not "well-behaved" profile likelihoods, which show serious skewness and tails exceeding the Gaussian expectation. In the best cases, a calibration of $\mathcal{L}_i$ is carried out [14,16,19], e.g., through simulations or bootstrapping methods, to match the coverage properties of the actual likelihood with those expected from a normal distribution.[4] Authors then publish (asymmetric) confidence intervals around $\theta_{m,i}$, or only just the 95% CL bounds, assuming that a null measurement was the most plausible outcome of the measurement. The posterior probability distribution P_i and the likelihood $\mathcal{L}_i$ are related through Bayes theorem and may incorporate prior knowledge on the coefficients. In the absence of prior knowledge about the coefficient, a flat prior (as used, for instance, in Wei *et al.* [31]) is not recommended [54,55] because of its lack of scale and parameter transformation invariance. Better priors have been proposed [56,57], but are not always used.

Although it is standard to combine likelihoods [sum $\ln(\mathcal{L}_i)$] from different measurements of one parameter to obtain a combined likelihood from all measurements, the inverse is not possible to our knowledge, that is, to combine likelihoods of one or various parameters θ_i that are by themselves linear combinations of unknowns. The latter is the case for the coefficients $c^{(d)}_{(I)jm}$, on which likelihoods have been derived only for their linear combination $\sum_{j,m} {}_0Y_{jm}(\hat{n}_i) c^{(d)}_{(I)jm}$. The spherical harmonics ${}_0Y_{jm}(\hat{n}_i)$ and the coefficients $c^{(d)}_{(I)jm}$ are moreover complex and satisfy the relations

$${}_0Y_{jm}(\hat{n}_i) = (-1)^m {}_0Y_{j(-m)}(\hat{n}_i)^*, \tag{30}$$

$$c^{(d)}_{(I)jm} = (-1)^m c^{(d)*}_{(I)j(-m)}. \tag{31}$$

Therefore, $c^{(d)}_{(I)j0}$ is real, while for $m \neq 0$ the pair of complex coefficients $c^{(d)}_{(I)jm}$ and $c^{(d)}_{(I)j(-m)}$ are codependent but represent 2 real degrees of freedom. For fixed (even) d, the number of real degrees of freedom for the set of coefficients $c^{(d)}_{(I)jm}$ is therefore $N = (d-1)^2$. We can show them explicitly by expressing the left-hand side of Eq. (29) as

$$\sum_{j=0}^{d-2} \Bigg({}_0Y_{j0}(\hat{n}_i) c^{(d)}_{(I)j0} + 2 \sum_{m=1}^{j} \big((\mathrm{Re}\,{}_0Y_{jm}(\hat{n}_i))(\mathrm{Re}\,c^{(d)}_{(I)jm}) - (\mathrm{Im}\,{}_0Y_{jm}(\hat{n}_i))(\mathrm{Im}\,c^{(d)}_{(I)jm}) \big) \Bigg). \tag{32}$$

Identifying the coefficients $c^{(d)}_{(I)j0}$, $\mathrm{Re}\,c^{(d)}_{(I)jm}$, and $\mathrm{Im}\,c^{(d)}_{(I)jm}$ $(m > 0)$ with the (real) N-dimensional vector x_j, we calculate a global probability distribution function for the set of independent coefficients

$$P(x_1, \ldots, x_N) = \prod_{i=1}^{s} P_i(x_1, \ldots, x_N | D_i). \tag{33}$$

We assume a normal distribution as a best guess for $P_i(x_1, \ldots, x_N | D_i)$, leading to

$$P(x_1, \ldots, x_N) \propto \exp\left[-\sum_{i=1}^{s} \frac{(\sum_{j=1}^{N} a_{ij} x_j - \mu_i)^2}{2\sigma_i^2} \right] \tag{34}$$

with a_{ij} being the values of the spherical harmonics in the direction of the source. Now, as outlined in D'Agostini [58,59], $\theta_{m,i}$ would be a wrong approximation

[4]Although unfortunately, the likelihood calibration recipes are often only applied to derive the corresponding bounds, without publishing the recipe itself (see Sec. IV and Table I).

for μ_i in the case of skewed likelihoods. We adopt, therefore, the prescriptions of D'Agostini [59] to approximate instead $E[\theta_i]$ and $s^2(\theta_i)$ from the asymmetric bounds provided and (if available) the central value $\theta_{m,i}$. Whenever publications become available that include the full information on the calibrated likelihoods, including expectation values and variances, this part of our analysis can be improved in that respect.

For convenience, redefining $\tilde{a}_{ij} = a_{ij}/\sigma_i$ and $\tilde{g}_i = \mu_i/\sigma_i$, we have

$$\begin{aligned} P(x_1, \ldots, x_N) &\propto \exp\left[-\frac{1}{2}\sum_{i=1}^{s}\left(\sum_{j=1}^{N}\tilde{a}_{ij}x_j - \tilde{g}_i\right)^2\right] \\ &= \exp\left[-\frac{1}{2}x^T(\tilde{a}^T\tilde{a})x + (\tilde{g}^T\tilde{a})x - \frac{1}{2}\tilde{g}^T\tilde{g}\right] \\ &= \exp\left[-\frac{1}{2}((x-\langle x\rangle)^T A(x-\langle x\rangle))\right] \end{aligned} \quad (35)$$

where the last two lines are expressed using matrix notation. We see that $P(x_1, \ldots, x_N)$ amounts to a multidimensional Gaussian distribution that is controlled by the $N \times N$ symmetric matrix $\tilde{a}^T\tilde{a}$ and the (co)vector $\tilde{g}^T\tilde{a}$. In the last line of Eq. (35), we reformulate $P(x_1, \ldots, x_N)$ in a more convenient form with $A = \tilde{a}^T\tilde{a}$ and $\langle x\rangle = A^{-1}\tilde{a}^T\tilde{g}$.

The easiest way to understand the form of the probability distribution (35) is by diagonalizing the matrix $\tilde{a}^T\tilde{a}$. Note that this can always be done for a symmetric matrix by applying an orthogonal transformation Q: $x \to y = Qx$, $\tilde{a}^T\tilde{a} \to M = Q\tilde{a}^T\tilde{a}Q^{-1} = Q\tilde{a}^T\tilde{a}Q^T$ ($Q^{-1} = Q^T$ for orthogonal Q), so that[5]

$$M_{ij} = \begin{pmatrix} M_{11} & 0 & 0 & \cdots & 0 \\ 0 & M_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & M_{nn} \end{pmatrix}. \quad (36)$$

The eigenvalues M_{ii} are guaranteed to be non-negative.[6] The rank of the matrix M, $\mathrm{rank}(M) = r$, is equal to the number of positive eigenvalues, which can at most be equal to the minimum of the values of N and s. Since the number of measurements $s \geq N$, we can make $r = N$.

It is convenient to arrange the eigenvalues so that $M_{ii} > 0$ for $i = 1, \ldots, r$, while $M_{jj} = 0$ for $j = r+1, \ldots, N$. We have

$$P(y) \propto \exp\left[-\frac{1}{2}(y - \langle y\rangle)^T M(y - \langle y\rangle)\right]. \quad (37)$$

This distribution is a direct product of Gaussians for the components y_i, $i = 1, \ldots, r$, such that the variance for the coordinate y_i is equal to M_{ii}^{-1}. Moreover, it is easy to see that the distribution yields an average value for y_i equal to

$$\langle y_i\rangle = \frac{(Q\tilde{a}^T\tilde{g})_i}{M_{ii}}, \qquad i = 1, \ldots, r. \quad (38)$$

We can obtain the distribution in terms of the components of the vector x by applying the orthogonal transformation (i.e., multidimensional rotation) Q^T. It can be visualized as a multidimensional ellipsoid, centered around the expectation value of the vector x,

$$\langle x\rangle = Q^T\langle y\rangle \quad (39)$$

where the components of the vector $\langle y\rangle$ are given by Eq. (38). Equation (43) provides the matrix Q which leads to rotated variables with a precision of better than 0.5%.

A reduced distribution function for any of the components x_i of the vector x can be obtained by integrating over (marginalizing) all the remaining components, for instance,

$$\tilde{P}(x_1) = \int dx_2 \ldots dx_n P(x_1, \ldots, x_n). \quad (40)$$

This integral is well defined if $r = N$, in which case it leads to

$$\tilde{P}(x_1) \propto \exp\left[-\frac{1}{2}(x_1 - \langle x_1\rangle)^2/((\tilde{a}^T\tilde{a})^{-1})_{11}\right] \quad (41)$$

which is a Gaussian distribution with variance $s^2(x_1) = ((\tilde{a}^T\tilde{a})^{-1})_{11}$.[7]

A similar reasoning using the properties of the expectation value $E[\sum_i a_i x_i] = \sum_i a_i E[x_i]$, $\sigma^2[\sum_i a_i x_i] = \sum_i a_i^2\sigma^2[x_i]$ leads to the same solutions for $\langle x_i\rangle$ and $s^2(x_i)$. Note that Kislat and Krawczynski [17] used only the second equation, on 95% asymmetric bounds $\sigma^2[\sum_i a_i x_i]$, without specifying how the asymmetry was treated, and without taking into account the expectation values $E[\sum_i a_i x_i]$.

If $r < N$, it is not possible to obtain reduced distribution functions for the individual components x_i. In that case, the best one can do is to determine the eigenvectors in x space corresponding to the nonzero eigenvalues $M_{11}, \ldots, M_{rr}$. They are given by the first r columns of the matrix Q^T (or, equivalently, the first r rows of the matrix Q). We can then express the probability distribution in the form

[5] Note that MPMATH's EIGHE function returns Q^T, aside M, instead of Q.

[6] If $(\tilde{a}^T\tilde{a})x = \lambda x$ for nonzero eigenvector x, it follows that $\lambda(x^Tx) = x^T(\tilde{a}^T\tilde{a})x = (\tilde{a}x)^T(\tilde{a}x)$. As $x^Tx > 0$ and $(\tilde{a}x)^T(\tilde{a}x) \geq 0$ (both expressions are equal to a sum of squares), it follows that $\lambda \geq 0$.

[7] Note that $((\tilde{a}^T\tilde{a})^{-1})_{11} \neq ((\tilde{a}^T\tilde{a})_{11})^{-1}$.

$$P(x_1,\ldots,x_n) = \exp\left[-\frac{1}{2}\sum_{i=1}^{r} M_{ii}\left(\sum_{j=1}^{N} Q_{ij}(x_j - \langle x_j\rangle)\right)^2\right] \tag{42}$$

which shows explicitly that the linear combination $\sum_{j=1}^{N} Q_{ij}x_j$, for $i = 1,\ldots,r$, is bounded with variance $(M_{ii})^{-1}$.

We have solved both Eqs. (35) and (37) for their expectation values and variances shown in Tables II and III. For the inversions, we adopted the estimated expectation values for $\sum_{j,m\,0} Y_{jm}(\hat{n}_i) c^{(d)}_{(I)jm}$ of Table I for μ_i $(i = 1,\ldots,11)$, following $E[x_i] \approx 1/2\cdot(UL_i + LL_i)$, where UL and LL correspond to $s_\pm = +1$ and $s_\pm = -1$ of the column two-sided of Table I, and $\sigma_i \approx 1/4\cdot(UL_i + LL_i)$. For the remaining bounds needed to fill r up to the minimum of $N \geq 25$ measurements, we have adopted the 24 measurements of Table III of Kislat and Krawczynski [17], the bounds of Table 1 of Wei *et al.* [60] (corrected for their fit χ^2/ndf), 29 bounds on GRB021206B from Boggs *et al.* [11], Kostelecký and Mewes [43], and the bounds from GRBs of Table 2 of Wei *et al.* [31], following the prescription of D'Agostini [59]: $E[x_i] \approx \theta_{m,i} + 1/2\cdot(\Delta_{+,i} + \Delta_{-,i})$ and $\sigma_i \approx 1/4\cdot(\Delta_{+,i} + \Delta_{-,i})$, where $\Delta_{+,i}$ and $\Delta_{-,i}$ represent the asymmetric 95% CL uncertainties with respect to the mode $\theta_{m,i}$. With these additions, a total of 65 datasets are available. The approximated expectation values and standard deviations of the 54 added datasets for this part of the analysis are shown in Tables IV and V.

TABLE II. Expectation values (second column), standard deviations (third column), 95% CL lower (fourth column), and upper bounds (last column) for the 25 nonbirefringent coefficients of the SME photon sector for $d = 6$.

Coefficient	$\langle \mathbf{x_i} \rangle$ ($\times 10^{-15}$ GeV^{-2})	$\sqrt{\mathbf{A_{ii}^{-1}}}$ ($\times 10^{-15}$ GeV^{-2})	LL (95% CL) ($\times 10^{-15}$ GeV^{-2})	UL (95% CL) ($\times 10^{-15}$ GeV^{-2})
$c^{(6)}_{(I)00}$	−0.5	1.5	−3.4	2.4
$c^{(6)}_{(I)10}$	0.0	0.8	−1.5	1.6
$\mathrm{Re}\,c^{(6)}_{(I)11}$	0.1	0.7	−1.3	1.4
$\mathrm{Im}\,c^{(6)}_{(I)11}$	−0.3	1.1	−2.5	1.9
$c^{(6)}_{(I)20}$	0.2	1.1	−2.0	2.4
$\mathrm{Re}\,c^{(6)}_{(I)21}$	0.0	0.5	−1.0	1.0
$\mathrm{Im}\,c^{(6)}_{(I)21}$	−0.2	0.4	−1.0	0.7
$\mathrm{Re}\,c^{(6)}_{(I)22}$	0.1	0.5	−1.0	1.2
$\mathrm{Im}\,c^{(6)}_{(I)22}$	0.2	1.0	−1.7	2.1
$c^{(6)}_{(I)30}$	−0.2	0.8	−1.8	1.4
$\mathrm{Re}\,c^{(6)}_{(I)31}$	−0.4	0.9	−2.2	1.3
$\mathrm{Im}\,c^{(6)}_{(I)31}$	0.4	0.7	−1.1	1.8
$\mathrm{Re}\,c^{(6)}_{(I)32}$	−0.3	0.3	−0.9	0.3
$\mathrm{Im}\,c^{(6)}_{(I)32}$	0.5	0.5	−0.5	1.5
$\mathrm{Re}\,c^{(6)}_{(I)33}$	−0.1	0.9	−1.9	1.8
$\mathrm{Im}\,c^{(6)}_{(I)33}$	−0.1	0.7	−1.6	1.3
$c^{(6)}_{(I)40}$	−0.7	0.7	−2.0	0.6
$\mathrm{Re}\,c^{(6)}_{(I)41}$	−0.2	0.2	−0.7	0.2
$\mathrm{Im}\,c^{(6)}_{(I)41}$	−0.3	0.4	−1.0	0.4
$\mathrm{Re}\,c^{(6)}_{(I)42}$	0.1	0.4	−0.7	1.0
$\mathrm{Im}\,c^{(6)}_{(I)42}$	−0.0	0.5	−1.0	1.0
$\mathrm{Re}\,c^{(6)}_{(I)43}$	−0.2	0.2	−0.6	0.2
$\mathrm{Im}\,c^{(6)}_{(I)43}$	−0.2	0.2	−0.6	0.1
$\mathrm{Re}\,c^{(6)}_{(I)44}$	0.1	0.3	−0.5	0.7
$\mathrm{Im}\,c^{(6)}_{(I)44}$	0.0	0.4	−0.9	0.9

TABLE III. Expectation values (second column), standard deviations (third column), 95% CL lower (fourth column), and upper bounds (last column) for the 25 nonbirefringent rotated coefficients of the SME photon sector for $d = 6$.

Coefficient	$\langle \mathbf{y_i} \rangle$ ($\times 10^{-22}$ GeV^{-2})	$\sqrt{\mathbf{M}^{-1}_{\mathbf{ii}}}$ ($\times 10^{-22}$ GeV^{-2})	LL (95% CL) ($\times 10^{-22}$ GeV^{-2})	UL (95% CL) ($\times 10^{-22}$ GeV^{-2})
y_1	-9.6×10^{-5}	3.5×10^{-3}	-7.0×10^{-3}	6.8×10^{-3}
y_2	1.8×10^{-2}	1.7×10^{-1}	-3.2×10^{-1}	3.5×10^{-1}
y_3	6.7×10^{-2}	1.9×10^{-1}	-3.2×10^{-1}	4.5×10^{-1}
y_4	4.4×10^{-2}	4.9×10^{-1}	-9.3×10^{-1}	1.0
y_5	-1.6×10^{-1}	6.2×10^{-1}	-1.4	1.1
y_6	3.9×10^{-2}	1.8	-3.5	3.6
y_7	1.7×10^{-1}	2.2	-4.2	4.5
y_8	1.8×10^{0}	5.0	-8.1	1.2×10^{1}
y_9	-1.7×10^{1}	1.3×10^{2}	-2.8×10^{2}	2.5×10^{2}
y_{10}	6.6×10^{1}	1.5×10^{2}	-2.3×10^{2}	3.7×10^{2}
y_{11}	-1.0×10^{2}	2.0×10^{2}	-5.0×10^{2}	3.0×10^{2}
y_{12}	-1.3×10^{3}	2.5×10^{3}	-6.4×10^{3}	3.7×10^{3}
y_{13}	-1.9×10^{4}	3.7×10^{4}	-9.3×10^{4}	5.6×10^{4}
y_{14}	7.0×10^{3}	6.5×10^{4}	-1.2×10^{5}	1.4×10^{5}
y_{15}	-1.5×10^{3}	1.5×10^{5}	-3.0×10^{5}	2.9×10^{5}
y_{16}	3.5×10^{4}	1.6×10^{5}	-2.9×10^{5}	3.6×10^{5}
y_{17}	6.5×10^{4}	2.1×10^{5}	-3.5×10^{5}	4.8×10^{5}
y_{18}	9.9×10^{4}	4.6×10^{5}	-8.2×10^{5}	1.0×10^{6}
y_{19}	-1.6×10^{4}	1.1×10^{6}	-2.3×10^{6}	2.2×10^{6}
y_{20}	-2.8×10^{6}	3.9×10^{6}	-1.1×10^{7}	4.9×10^{6}
y_{21}	7.3×10^{6}	5.2×10^{6}	-3.1×10^{6}	1.8×10^{7}
y_{22}	-1.6×10^{6}	5.9×10^{6}	-1.3×10^{7}	1.0×10^{7}
y_{23}	1.0×10^{7}	8.9×10^{6}	-7.6×10^{6}	2.8×10^{7}
y_{24}	-4.0×10^{6}	1.6×10^{7}	-3.5×10^{7}	2.7×10^{7}
y_{25}	-4.1×10^{6}	2.9×10^{7}	-6.1×10^{7}	5.3×10^{7}

TABLE IV. Approximated expectation values and standard deviations of the 24 AGNs of Kislat and Krawczynski [17] used to increase the data sample for the inversion of the individual coefficients $c^{(6)}_{(I)jm}$.

Source name	R.A. (deg)	Dec. (deg)	$\mathrm{E}[\sum_{jm} {}_0Y_{jm}(\theta,\phi) c^{(6)}_{(I)jm}]$ (GeV^{-2})	$s(\sum_{jm} {}_0Y_{jm}(\theta,\phi) c^{(6)}_{(I)jm})$ (GeV^{-2})
3C454.3	343.5	16.1	2.3×10^{-19}	4.3×10^{-19}
PKS1222 + 216	186.2	21.4	2.4×10^{-18}	7.0×10^{-18}
3C279	194.0	−5.8	2.3×10^{-18}	9.4×10^{-18}
PKS1502 + 106	226.1	10.5	0.6×10^{-17}	1.3×10^{-17}
PKS1510-089	228.2	−9.1	0.3×10^{-17}	1.3×10^{-17}
PKS1424-41	217.0	−42.1	-0.2×10^{-17}	2.3×10^{-17}
PKS1830-211	278.4	−21.1	-0.3×10^{-17}	2.8×10^{-17}
S41849 + 67	282.3	67.1	-0.5×10^{-16}	1.3×10^{-16}
3C273	187.3	2.1	0.2×10^{-16}	1.7×10^{-16}
B21520 + 31	230.5	31.7	-1.8×10^{-16}	3.3×10^{-16}
PKS0426-380	67.2	−37.9	0.5×10^{-16}	6.7×10^{-16}
PKS0716 + 714	110.5	71.3	-6.6×10^{-16}	7.8×10^{-16}
GB1310 + 487	198.2	48.5	7.1×10^{-16}	9.9×10^{-16}
PKS2326-502	352.2	−49.9	1.3×10^{-15}	1.7×10^{-15}
PKS0454-234	74.3	−23.4	0.8×10^{-15}	1.7×10^{-15}
PKS1633 + 382	248.8	38.1	-0.09×10^{-15}	1.7×10^{-15}

(Table continued)

TABLE IV. *(Continued)*

Source name	R.A. (deg)	Dec. (deg)	$\mathrm{E}[\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm}]$ (GeV^{-2})	$s(\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm})$ (GeV^{-2})
B31343 + 451	206.4	44.9	-0.06×10^{-15}	2.4×10^{-15}
S30218 + 35	35.3	35.9	-0.1×10^{-15}	3.7×10^{-15}
PKS2233-148	339.1	−14.6	-0.9×10^{-15}	4.3×10^{-15}
4C14.23	111.3	14.4	-0.02×10^{-15}	4.5×10^{-15}
PMNJ2345-1555	356.3	−15.9	0.6×10^{-15}	4.7×10^{-15}
PKS0235 + 164	39.7	16.6	3.0×10^{-15}	6.5×10^{-15}
4C28.07	39.5	28.8	-0.08×10^{-14}	1.3×10^{-14}
3C66A	35.7	43.0	0.5×10^{-14}	2.1×10^{-14}

TABLE V. Extracted values of $\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm}$ for those 29 GRBs of Wei *et al.* [31] used to increase the data sample for the inversion of the individual coefficients $c^{(6)}_{(I)jm}$. In the last-but-one and last columns, an approximation of the expectation value and standard deviation has been made, following the prescription of D'Agostini [59].

Source name	R.A. (deg)	Dec. (deg)	$\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm}$ (GeV^{-2})	$\mathrm{E}[\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm}]$ (GeV^{-2})	$s(\sum_{jm} {}_0Y_{jm}(\theta,\phi)c^{(6)}_{(I)jm})$ (GeV^{-2})
GRB 210619B	319.7	+33.9	$1.20^{+0.45}_{-0.46}\times10^{-15}$	12.0×10^{-16}	2.3×10^{-16}
GRB 160625B[a]	308.6	+6.9		1.6×10^{-15}	1.4×10^{-15}[b]
GRB 180720B	0.59	−3.0	$-0.03^{+0.70}_{-0.65}\times10^{-14}$	-0.1×10^{-15}	3.4×10^{-15}
GRB 200829A	251.1	+72.4	$3.3^{+0.9}_{-1.1}\times10^{-14}$	31.8×10^{-15}	4.9×10^{-15}
GRB 130427A	173.1	+27.7	$4.7^{+5.7}_{-6.1}\times10^{-14}$	4.5×10^{-14}	2.9×10^{-14}
GRB 130518A	355.7	+47.5	$7.0^{+6.3}_{-5.8}\times10^{-14}$	7.2×10^{-14}	3.0×10^{-14}
GRB 150314A	126.7	+63.8	$1.3^{+2.4}_{-2.7}\times10^{-13}$	1.1×10^{-13}	1.3×10^{-13}
GRB 091003A	251.5	−36.6	$2.7^{+3.7}_{-2.3}\times10^{-13}$	3.4×10^{-13}	1.5×10^{-13}
GRB 131108A	156.5	+9.7	$0.1^{+4.5}_{-2.9}\times10^{-13}$	0.9×10^{-13}	1.8×10^{-13}
GRB 150403A	311.5	−62.7	$7.1^{+8.9}_{-8.0}\times10^{-13}$	7.6×10^{-13}	4.2×10^{-13}
GRB 140206A	145.3	+66.8	$0.9^{+1.0}_{-0.8}\times10^{-12}$	9.8×10^{-13}	4.5×10^{-13}
GRB 160509A	310.1	+76.0	$5.6^{+11.9}_{-10.9}\times10^{-13}$	6.1×10^{-13}	5.7×10^{-13}
GRB 140508A	255.5	+46.8	$1.7^{+1.6}_{-1.5}\times10^{-12}$	17.4×10^{-13}	7.8×10^{-13}
GRB 201216C	16.4	+16.5	$0.9^{+2.5}_{-2.5}\times10^{-12}$	0.9×10^{-12}	1.3×10^{-12}
GRB 141028A	322.6	−0.2	$2.1^{+4.4}_{-2.8}\times10^{-12}$	3.0×10^{-12}	1.8×10^{-12}
GRB 120119A	120.0	−9.8	$3.1^{+3.9}_{-3.3}\times10^{-12}$	3.4×10^{-12}	1.8×10^{-12}
GRB 131231A	10.6	−1.6	$1.04^{+0.42}_{-0.44}\times10^{-11}$	10.3×10^{-12}	2.2×10^{-12}
GRB 180703A	6.5	−67.1	$2.4^{+7.4}_{-3.3}\times10^{-12}$	4.5×10^{-12}	2.7×10^{-12}
GRB 081221	15.8	−24.5	$6.5^{+6.9}_{-5.6}\times10^{-12}$	7.2×10^{-12}	3.1×10^{-12}
GRB 090328	155.7	+33.4	$7.5^{+6.8}_{-6.7}\times10^{-12}$	7.5×10^{-12}	3.4×10^{-12}
GRB 200613A	153.0	+45.8	$0.59^{+0.84}_{-0.72}\times10^{-11}$	6.5×10^{-12}	3.9×10^{-12}
GRB 100728A	88.8	−15.3	$-0.7^{+10.3}_{-5.8}\times10^{-12}$	1.5×10^{-12}	4.0×10^{-12}
GRB 210610B	243.9	+14.4	$-0.8^{+10.8}_{-8.4}\times10^{-12}$	0.4×10^{-12}	4.8×10^{-12}
GRB 210204A	109.1	+9.7	$0.2^{+11.9}_{-8.9}\times10^{-12}$	1.7×10^{-12}	5.2×10^{-12}
GRB 090618	294.0	+78.4	$1.5^{+1.4}_{-1.3}\times10^{-11}$	15.8×10^{-12}	6.7×10^{-12}
GRB 150514A	74.8	−60.9	$1.6^{+2.4}_{-1.6}\times10^{-11}$	2.0×10^{-11}	1.0×10^{-11}
GRB 171010A	66.6	−10.5	$3.5^{+4.1}_{-12.5}\times10^{-11}$	-0.7×10^{-11}	4.1×10^{-11}
GRB 150821A	341.9	−57.9	$0.7^{+1.2}_{-0.6}\times10^{-10}$	9.5×10^{-11}	4.4×10^{-11}
GRB 130925A	41.2	−26.1	$2.9^{+12.4}_{-7.0}\times10^{-11}$	5.6×10^{-11}	4.8×10^{-11}

[a]Due to the apparently problematic fit of the spectral lags of this GRB to the data in Fig. 1 of Wei *et al.* [31] (without the corresponding fit χ^2/ndf provided), we preferred to derive expectation value and standard deviation of GRB 160625B from the upper bounds provided in Wei *et al.* [60].

[b]The standard deviation has been corrected with the square root of the fit χ^2/ndf.

On the technical side of the implementation, we observed that an internal precision of at least 23 decimal places[8] is required for stable solutions.

We observed that the direct bounds on $c^{(d)}_{(I)jm}$ do slightly improve as more and more of the less constraining bounds $N > 25$ are added, although they also become more asymmetric, due to the inherent asymmetry of many of the GRB measurements of Wei *et al.* [31]. Table II shows the solution for all 65 measurements used. If only 25 are taken, the bounds worsen by about 10%. On the contrary, the bounds on the rotated parameters y_i of Table III remain almost unaffected by the addition of further measurements beyond $N = 25$, except for the very last parameter y_{25}, as expected.

$$Q_{ij} = \left(\begin{array}{ccccccccccccccccccccccccc}
0.5 & 0.06 & 0.1 & 0.4 & -0.4 & -0.1 & -0.08 & 0.02 & 0.3 & 0.2 & -0.2 & -0.1 & 0.05 & -0.1 & -0.3 & 0.2 & -0.07 & 0.03 & 0.08 & 0.1 & -0.1 & 0.05 & 0.004 & -0.06 & -0.1 \\
-0.2 & -0.4 & 0.3 & -0.01 & -0.2 & 0.2 & -0.2 & -0.3 & 0.2 & 0.1 & 0.2 & -0.3 & 0.1 & 0.08 & -0.07 & -0.3 & 0.3 & 0.06 & 0.1 & -0.06 & 0.1 & 0.01 & 0.007 & 0.09 & 0.04 \\
-0.2 & 0.1 & 0.2 & -0.05 & 0.2 & 0.09 & -0.1 & -0.2 & 0.4 & 0.1 & -0.4 & 0.3 & 0.04 & 0.3 & -0.1 & -0.3 & -0.3 & -0.09 & -0.03 & 0.04 & -0.2 & -0.1 & -0.03 & 0.1 & 0.01 \\
-0.07 & 0.1 & -0.1 & -0.01 & 0.04 & 0.005 & -0.3 & -0.03 & -0.3 & 0.5 & 0.2 & 0.009 & 0.08 & -0.2 & -0.04 & -0.2 & -0.2 & 0.2 & 0.07 & 0.4 & 0.06 & 0.1 & -0.02 & 0.3 & -0.1 \\
-0.3 & -0.2 & -0.06 & -0.08 & -0.4 & -0.03 & -0.3 & 0.2 & -0.01 & -0.3 & -0.07 & -0.1 & -0.2 & 0.2 & -0.1 & 0.1 & -0.3 & -0.1 & -0.3 & 0.3 & 0.1 & -0.1 & -0.1 & 0.04 & -0.1 \\
-0.03 & 0.2 & -0.2 & 0.1 & -0.1 & -0.05 & -0.1 & -0.2 & -0.09 & -0.3 & -0.08 & 0.06 & 0.3 & -0.08 & 0.03 & -0.2 & 0.4 & -0.07 & -0.3 & 0.2 & -0.3 & -0.2 & 0.4 & 0.02 & -0.1 \\
-0.2 & -0.3 & 0.08 & 0.1 & -0.4 & 0.1 & 0.1 & -0.06 & -0.08 & -0.1 & 0.06 & 0.4 & 0.08 & -0.3 & 0.09 & 0.008 & -0.3 & 0.3 & 0.1 & -0.2 & -0.3 & 0.08 & 0.09 & -0.09 & 0.07 \\
0.1 & -0.01 & -0.2 & 0.06 & 0.02 & 0.4 & 0.2 & -0.05 & -0.1 & -0.1 & 0.2 & 0.1 & 0.01 & 0.1 & -0.4 & -0.06 & -0.1 & -0.1 & 0.1 & -0.2 & 0.1 & -0.1 & -0.03 & 0.1 & -0.5 \\
-0.2 & 0.3 & -0.3 & 0.09 & -0.2 & 0.1 & -0.2 & -0.1 & -0.09 & 0.02 & -0.3 & -0.2 & 0.2 & 0.1 & -0.1 & -0.002 & -0.05 & 0.09 & -0.05 & -0.4 & 0.3 & 0.4 & 0.08 & -0.1 & 0.1 \\
-0.005 & -0.2 & 0.02 & 0.2 & 0.2 & -0.4 & -0.002 & 0.04 & -0.1 & -0.4 & 0.007 & -0.3 & 0.02 & -0.01 & -0.09 & -0.4 & -0.3 & -0.2 & 0.4 & 0.08 & -0.01 & 0.2 & 0.2 & -0.2 & -0.02 \\
0.3 & -0.4 & -0.1 & -0.4 & 0.02 & -0.1 & 0.2 & -0.2 & -0.2 & 0.1 & -0.1 & -0.1 & 0.4 & 0.1 & -0.08 & 0.1 & -0.2 & -0.08 & -0.2 & -0.02 & -0.2 & 0.2 & -0.01 & 0.1 & 0.07 \\
0.2 & -0.03 & 0.2 & 0.03 & -0.09 & -0.2 & 0.1 & 0.08 & -0.2 & 0.2 & 0.07 & 0.09 & -0.2 & 0.5 & -0.03 & -0.2 & -0.004 & 0.3 & -0.2 & -0.1 & 0.1 & -0.07 & 0.4 & -0.1 & -0.04 \\
-0.2 & 0.03 & -0.2 & 0.1 & 0.2 & -0.08 & 0.1 & -0.4 & 0.2 & 0.05 & 0.3 & -0.04 & -0.08 & 0.1 & -0.3 & 0.3 & -0.1 & 0.2 & -0.1 & 0.3 & -0.06 & 0.02 & 0.06 & -0.4 & 0.2 \\
-0.2 & -0.1 & 0.3 & 0.2 & 0.1 & 0.2 & 0.04 & -0.1 & -0.3 & 0.2 & -0.2 & -0.04 & -0.3 & -0.08 & 0.05 & 0.2 & 0.1 & -0.3 & -0.1 & 0.1 & -0.2 & 0.3 & 0.1 & -0.2 & -0.2 \\
0.3 & -0.4 & -0.1 & 0.2 & 0.4 & 0.1 & -0.4 & 0.05 & 0.01 & -0.2 & -0.1 & 0.3 & -0.01 & -0.09 & -0.08 & 0.06 & 0.08 & 0.2 & -0.2 & -0.07 & 0.2 & 0.1 & -0.004 & 0.02 & 0.04 \\
-0.05 & -0.05 & 0.1 & 0.2 & -0.07 & -0.4 & -0.1 & -0.2 & 0.08 & 0.1 & 0.3 & 0.3 & 0.1 & -0.2 & 0.07 & 0.1 & -0.2 & -0.5 & -0.2 & -0.3 & 0.4 & -0.08 & 0.1 & 0.1 & -0.07 \\
-0.2 & -0.2 & -0.1 & 0.2 & 0.03 & -0.05 & 0.2 & -0.005 & -0.06 & 0.03 & -0.3 & -0.01 & -0.1 & 0.09 & -0.1 & 0.3 & 0.07 & 0.05 & 0.3 & 0.09 & 0.1 & -0.2 & 0.3 & 0.6 & 0.2 \\
-0.2 & -0.07 & -0.2 & 0.2 & 0.2 & -0.02 & -0.0008 & 0.2 & 0.3 & 0.2 & 0.1 & -0.4 & 0.05 & 0.06 & 0.3 & 0.06 & -0.2 & 0.1 & -0.2 & -0.3 & -0.3 & 0.004 & 0.1 & 0.2 & -0.3 \\
0.2 & 0.2 & 0.4 & 0.1 & 0.06 & 0.4 & 0.06 & 0.06 & 0.008 & -0.2 & 0.2 & -0.09 & 0.3 & 0.1 & 0.2 & 0.2 & -0.3 & 0.01 & -0.09 & 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.2 \\
0.0005 & -0.07 & 0.05 & -0.06 & 0.1 & 0.01 & 0.2 & -0.3 & 0.03 & -0.03 & -0.3 & -0.2 & 0.04 & -0.3 & 0.3 & 0.04 & -0.2 & 0.3 & -0.02 & 0.1 & 0.4 & -0.4 & 0.01 & -0.2 & -0.3 \\
0.3 & -0.1 & -0.3 & 0.3 & -0.1 & 0.2 & -0.1 & -0.3 & -0.2 & 0.1 & -0.008 & 0.02 & -0.1 & 0.3 & 0.4 & -0.1 & -0.1 & -0.2 & 0.1 & 0.03 & -0.07 & -0.2 & -0.2 & -0.1 & 0.2 \\
-0.1 & -0.2 & -0.03 & -0.2 & 0.07 & 0.1 & -0.3 & 0.3 & -0.02 & 0.2 & -0.02 & 0.09 & 0.4 & 0.2 & -0.02 & 0.3 & 0.004 & -0.1 & 0.3 & 0.01 & -0.002 & -0.2 & 0.3 & -0.4 & -0.04 \\
0.08 & -0.2 & -0.3 & -0.06 & -0.09 & 0.2 & 0.3 & 0.2 & 0.4 & 0.2 & -0.06 & 0.2 & -0.06 & -0.1 & 0.1 & -0.3 & 0.01 & -0.2 & -0.08 & 0.3 & 0.2 & 0.3 & 0.3 & -0.1 & 0.07 \\
-0.2 & -0.08 & 0.05 & 0.3 & -0.02 & -0.2 & 0.2 & 0.07 & 0.009 & 0.01 & -0.05 & 0.2 & 0.4 & 0.3 & 0.2 & 0.04 & 0.2 & 0.1 & 0.06 & 0.2 & 0.1 & 0.3 & -0.4 & -0.0002 & -0.3 \\
-0.1 & -0.09 & 0.1 & 0.3 & 0.1 & 0.08 & 0.2 & 0.3 & -0.2 & 0.2 & -0.04 & -0.1 & 0.3 & -0.2 & -0.3 & -0.2 & -0.001 & -0.1 & -0.3 & -0.05 & 0.04 & -0.3 & -0.2 & -0.1 & 0.3
\end{array}\right) \tag{43}$$

V. CONCLUSIONS

In this work, we tackled the lack of a clear connection between the commonly used constraints on LIV, typically expressed via the effective quantum gravity scale $E_{QG,n}$, and the coefficients of the SME in the pure photon sector. Although these bounds are widely used in the literature, they are rarely translated into the SME framework. Here, we developed a consistent and straightforward formalism to perform this conversion and showed how dispersion relations from the general LIV approach can be systematically mapped onto the SME description.

To make meaningful comparisons and enable a coherent global analysis, we carried out a critical review and standardization of some of the most stringent $E_{QG,1}$ and $E_{QG,2}$ bounds available. This included adding missing prefactors accounting for systematic uncertainties in the energy scale, and converting one-sided bounds into two-sided Gaussian uncertainties where appropriate. The resulting methodology lays a solid foundation for expressing time-of-flight constraints in terms of SME coefficients and sets the stage for combining results from a broad range of astrophysical sources.

The best bound converted from LHAASO's analysis of GRB 221009A data [12] leads to an improvement of bounds on $\sum_{jm} Y_{jm} c^{(6)}_{(I)jm}$ by one and a half order of magnitude with respect to previous ones, albeit possible

[8]Available, e.g., in Matlab or the MPMATH library in Python.

intrinsic energy-dependent photon time delays were not considered in their analysis. We may, hence, not exclude the possibility that the photon detection times were potentially affected by intrinsic delays in the source, and nature has conspired to compensate for a LIV-induced effect to provide a final null measurement. It is therefore essential to expand the sample through analyses of similar or better sensitivity from a variety of astrophysical sources at different distances and of different origins to eliminate such a scenario.

A decent statistical treatment, beyond the most straightforward approach used in Kislat and Krawczynski [17], of a large variety of measurements, many of which use a different treatment of uncertainties and (choice of) underlying systematics, has revealed challenges. Apart from the fact that none of the most sensitive analyses have incorporated a statistical treatment of intrinsic time delays or a correct marginalization of the sometimes quite considerable detector-related systematics, the standards themselves differ. We have tried, as best as we could with the available information, to standardize at least confidence intervals, detector systematics, inversion formulas, and conversion of the measurement of best and asymmetric confidence intervals to expectation values and standard deviations. Our wish is to have provided a first step toward a community-wide standardization in all of these aspects. We urge the community to publish their full likelihoods, plus expectation values and variances.

Our new bounds on the individual nonbirefringent coefficients of the photon sector for $d = 6$ improve over previous ones [17] by about an order of magnitude. This is a direct consequence of the improvement in the least sensitive measurement in the sample of the best 25, which determines the order of magnitude of the bounds on all coefficients $c^{(d)}_{(I)jm}$. Note that the largest standard deviation on the rotated coefficients, $s(y_{25})$, provides an upper bound to the standard deviations of the original coefficients, which themselves vary by only within a factor of ~ 8.

Reversing this argument, a set of 14 additional competitive bounds from very-high-energy or ultrahigh-energy gamma-ray observatories could improve sensitivity to all $c^{(d)}_{(I)jm}$ by another 5 orders of magnitude.

ACKNOWLEDGMENTS

It is a pleasure to thank Alan Kostelecký and Matt Mewes for discussions and helpful suggestions. We also thank Michele Doro for helpful discussions and suggestions on the statistics part. M. G. and R. P. acknowledge support by the Center for Research and Development in Mathematics and Applications (CIDMA) the Portuguese Foundation for Science and Technology (FCT–Fundação para a Ciência e a Tecnologia) the Multi-Annual Financing Program for R&D Units (UID/04106). M. G. acknowledges the support in part by a fellowship by the Hanse-Wissenschaftskolleg—Institute for Advanced Study and Horizon Europe staff exchange program HORIZON-MSCA2021-SE-01 Grant No. NewFunFiCO-101086251. M. G. and A. C.-O. acknowledge funding from the Spanish Grant No. PID2022-139117NB-C43 funded by MCIN/705AEI/10.13039/501100011033/FEDER, UE.

DATA AVAILABILITY

The data that support the findings of this article are openly available [61].

[1] A. Addazi *et al.*, Quantum gravity phenomenology at the dawn of the multi-messenger era-a review, Prog. Part. Nucl. Phys. **125**, 103948 (2022).

[2] G. Amelino-Camelia, Quantum-spacetime phenomenology, Living Rev. Relativity **16**, 5 (2013).

[3] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Tests of quantum gravity from observations of γ-ray bursts, Nature (London) **393**, 763 (1998).

[4] J. Bolmont *et al.*, First combined study on Lorentz Invariance violation from observations of energy-dependent time delays from multiple-type gamma-ray sources. I. Motivation, method description, and validation through simulations of H.E.S.S., MAGIC, and VERITAS data sets, Astrophys. J. **930**, 75 (2022).

[5] D. Mattingly, Modern tests of Lorentz invariance, Living Rev. Relativity **8**, 5 (2005).

[6] U. Jacob and T. Piran, Lorentz-violation-induced arrival delays of cosmological particles, J. Cosmol. Astropart. Phys. 01 (2008) 031.

[7] S. Caroff *et al.*, Discriminating between different modified dispersion relations from gamma-ray observations, Phys. Rev. D **111**, 083021 (2025).

[8] C. Pfeifer, Redshift and lateshift from homogeneous and isotropic modified dispersion relations, Phys. Lett. B **780**, 246 (2018).

[9] A. A. Abdo *et al.*, A limit on the variation of the speed of light arising from quantum gravity effects, Nature (London) **462**, 331 (2009).

[10] V. A. Acciari *et al.*, Bounds on Lorentz invariance violation from MAGIC observation of GRB 190114C, Phys. Rev. Lett. **125**, 021301 (2020).
[11] S. E. Boggs, C. B. Wunderer, K. Hurley, and W. Coburn, Testing Lorentz invariance with GRB 021206, Astrophys. J. **611**, L77 (2004).
[12] Z. Cao *et al.*, Stringent tests of Lorentz invariance violation from LHAASO observations of GRB 221009A, Phys. Rev. Lett. **133**, 071501 (2024).
[13] J. Ellis, R. Konoplich, N. E. Mavromatos, L. Nguyen, A. S. Sakharov, and E. K. Sarkisyan-Grinbaum, Robust constraint on Lorentz violation using Fermi-LAT gamma-ray burst data, Phys. Rev. D **99**, 083009 (2019).
[14] V. Vasileiou, A. Jacholkowska, F. Piron, J. Bolmont, C. Couturier, J. Granot, F. W. Stecker, J. Cohen-Tanugi, and F. Longo, Constraints on Lorentz invariance violation from Fermi-Large Area Telescope observations of gamma-ray bursts, Phys. Rev. D **87**, 122001 (2013).
[15] S. Abe *et al.*, Constraints on Lorentz invariance violation from the extraordinary Mrk 421 flare of 2014 using a novel analysis method, J. Cosmol. Astropart. Phys. 07 (2024) 044.
[16] H. E. S. S. Collaboration, Search for Lorentz invariance breaking with a likelihood fit of the PKS 2155-304 flare data taken on MJD 53944, Astropart. Phys. **34**, 738 (2011).
[17] F. Kislat and H. Krawczynski, Search for anisotropic Lorentz invariance violation with γ -rays, Phys. Rev. D **92**, 045016 (2015).
[18] MAGIC Collaboration, Probing quantum gravity using photons from a flare of the active galactic nucleus Markarian 501 observed by the MAGIC telescope, Phys. Lett. B **668**, 253 (2008).
[19] MAGIC Collaboration, Constraining Lorentz invariance violation using the Crab Pulsar emission observed up to TeV energies by MAGIC, Astrophys. J. Suppl. Ser. **232**, 9 (2017).
[20] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Loop quantum gravity and light propagation, Phys. Rev. D **65**, 103509 (2002).
[21] G. Amelino-Camelia, N. Loret, and G. Rosati, Speed of particles and a relativity of locality in κ-Minkowski quantum spacetime, Phys. Lett. B **700**, 150 (2011).
[22] S. Das, E. C. Vagenas, and A. F. Ali, Discreteness of space from GUP II: Relativistic wave equations, Phys. Lett. B **690**, 407 (2010).
[23] J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Derivation of a vacuum refractive index in a stringy space time foam model, Phys. Lett. B **665**, 412 (2008).
[24] R. Gambini and J. Pullin, Nonstandard optics from quantum space-time, Phys. Rev. D **59**, 124021 (1999).
[25] A. Kempf, G. Mangano, and R. B. Mann, Hilbert space representation of the minimal length uncertainty relation, Phys. Rev. D **52**, 1108 (1995).
[26] D. Colladay and V. A. Kostelecký, *CPT* violation and the standard model, Phys. Rev. D **55**, 6760 (1997).
[27] D. Colladay and V. A. Kostelecký, Lorentz-violating extension of the standard model, Phys. Rev. Lett. **58**, 116002 (1998).
[28] V. A. Kostelecky, Gravity, Lorentz violation, and the standard model, Phys. Rev. D **69**, 105009 (2004).
[29] V. A. Kostelecký and M. Mewes, Electrodynamics with Lorentz-violating operators of arbitrary dimension, Phys. Rev. D **80**, 015020 (2009).
[30] V. A. Kostelecký and N. Russell, Data tables for Lorentz and *CPT* violation, Rev. Mod. Phys. **83**, 11 (2011).
[31] J. N. Wei, Z.-K. Liu, J.-J. Wei, B.-B. Zhang, and X.-F. Wu, Exploring anisotropic Lorentz invariance violation from the spectral-lag transitions of gamma-ray bursts, Universe **8**, 519 (2022).
[32] J. Ellis and N. E. Mavromatos, Probes of Lorentz violation, Astropart. Phys. **43**, 50 (2013).
[33] V. A. Kostelecký and R. Potting, *CPT* and strings, Nucl. Phys. **B359**, 545 (1991).
[34] V. A. Kostelecký and R. Potting, *CPT*, strings, and meson factories, Phys. Rev. D **51**, 3923 (1995).
[35] V. A. Kostelecký and S. Samuel, Gravitational phenomenology in higher-dimensional theories and strings, Phys. Rev. D **40**, 1886 (1989).
[36] N. E. Mavromatos, Quantum-gravity induced Lorentz violation and dynamical mass generation, Phys. Rev. D **83**, 025018 (2011).
[37] J. D. Tasson, What do we know about Lorentz invariance?, Rep. Prog. Phys. **77**, 062901 (2014).
[38] R. Bluhm, Overview of the SME: Implications and phenomenology of Lorentz violation, Lect. Notes Phys. **702**, 191 (2006).
[39] V. A. Kostelecky and M. Mewes, Cosmological constraints on Lorentz violation in electrodynamics, Phys. Rev. Lett. **87**, 251304 (2001).
[40] V. A. Kostelecký and M. Mewes, Signals for Lorentz violation in electrodynamics, Phys. Rev. D **66**, 056005 (2002).
[41] A. Kostelecky and Z. Li, Backgrounds in gravitational effective field theory, Phys. Rev. D **103**, 024059 (2021).
[42] F. Kislat and H. Krawczynski, Planck-scale constraints on anisotropic Lorentz and *CPT* invariance violations from optical polarization measurements, Phys. Rev. D **95**, 083013 (2017).
[43] V. A. Kostelecký and M. Mewes, Astrophysical tests of Lorentz and *CPT* violation with photons, Astrophys. J. Lett. **689**, L1 (2008).
[44] S. Navas *et al.*, Review of particle physics, 25. Cosmological parameters, Phys. Rev. D **110**, 030001 (2024).
[45] S. S. Wilks, The large-sample distribution of the likelihood ratio for testing composite hypotheses, Ann. Math. Stat. **9**, 60 (1938).
[46] A. Abramowski *et al.*, The 2012 flare of PG 1553 + 113 seen with H.E.S.S. and FERMI-LAT, Astrophys. J. **802**, 65 (2015).
[47] J. Dorigo Jones *et al.*, Improving blazar redshift constraints with the edge of the Ly α forest: 1ES 1553 + 113 and implications for observations of the WHIM, Mon. Not. R. Astron. Soc. **509**, 4330 (2022).
[48] H. Abdalla *et al.*, The 2014 TeV γ-ray flare of Mrk 501 seen with H.E.S.S.: Temporal and spectral constraints on Lorentz invariance violation, Astrophys. J. **870**, 93 (2019).
[49] J. Aleksić *et al.*, The major upgrade of the MAGIC telescopes, Part II: A performance study using observations of the Crab Nebula, Astropart. Phys. **72**, 76 (2016).
[50] T. Ashton *et al.*, A NECTAr-based upgrade for the Cherenkov cameras of the H.E.S.S. 12-meter telescopes, Astropart. Phys. **118**, 102425 (2020).

[51] F. Aharonian *et al.*, Calibration of the air shower energy scale of the water and air Cherenkov techniques in the LHAASO experiment, Phys. Rev. D **104**, 062007 (2021).

[52] MAGIC Collaboration, Probing quantum gravity using photons from a flare of the active galactic nucleus Markarian 501 observed by the MAGIC telescope, Phys. Lett. B **668**, 253 (2008).

[53] S. A. Murphy and A. W. van der Vaart, On profile likelihood, J. Am. Stat. Assoc. **95**, 449 (2000).

[54] R. J. Barlow, Practical statistics for particle physics, CERN Yellow Rep. School Proc. **5**, 149 (2020).

[55] L. Lista, Practical statistics for particle physicists, in *Proceedings of the 2016 European School of High-Energy Physics* (2017), arXiv:1609.04150.

[56] L. Demortier, S. Jain, and H. B. Prosper, Reference priors for high energy physics, Phys. Rev. D **82**, 034002 (2010).

[57] H. Jeffreys, An invariant form for the prior probability in estimation problems, Proc. R. Soc. A **186**, 453 (1946).

[58] G. D'Agostini, *Bayesian Reasoning in Data Analysis* (World Scientific, Singapore, 2003).

[59] G. D'Agostini, Asymmetric uncertainties: Sources, treatment and potential dangers, arXiv:physics/0403086.

[60] J. J. Wei, X.-F. Wu, B.-B. Zhang, L. Shao, P. Mészáros, and V. A. Kostelecký, Constraining anisotropic Lorentz violation via the spectral-lag transition of GRB 160625B, Astrophys. J. **842**, 115 (2017).

[61] https://github.com/mgaug/bounding_coeffs.