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Bethe states and Knizhnik-Zamolodchikov equations of the trigonometric Gaudin model with triangular boundary

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Abstract

We present a comprehensive treatment of the non-periodic trigonometric $s\ell(2)$ Gaudin model with triangular boundary, with an emphasis on specific freedom found in the local realization of the generators, as well as in the creation operators used in the algebraic Bethe ansatz. First, we give Bethe vectors of the non-periodic trigonometric $s\ell(2)$ Gaudin model both through a recurrence relation and in a closed form. Next, the off-shell action of the generating function of the trigonometric Gaudin Hamiltonians with general boundary terms on an arbitrary Bethe vector is shown, together with the corresponding proof based on mathematical induction. The action of the Gaudin Hamiltonians is given explicitly. Furthermore, by careful choice of the arbitrary functions appearing in our more general formulation, we additionally obtain: i) the solutions to the Knizhnik-Zamolodchikov equations (each corresponding to one of the Bethe states); ii) compact formulas for the on-shell norms of Bethe states; and iii) closed-form expressions for the off-shell scalar products of Bethe states.

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1. Introduction

Gaudin systems have been a subject of study for almost half a century. Gaudin originally introduced them as a quasi-classical limit of the Heisenberg spin chains [1–3]. Sklyanin used a suitable unitary classical r-matrix [4] in the study of the rational $s\ell(2)$ model [5]. A generalisation of these results to the skew-symmetric classical r-matrices of simple Lie algebras [6–8] and Lie superalgebras [9–12] was relatively straightforward as well as their Jordanian deformation [13–15].

Non-periodic Gaudin systems have also attracted considerable attention [16–30]. Particularly compelling is the approach based on the non-unitary r-matrices [31–33]. Recently we have studied the rational $s\ell(2)$ Gaudin model with general boundary K-matrices, both as a limit of the spin-chain model [34] and independently [35], and provided a number of additional interesting results, from the solutions to Knizhnik-Zamolodchikov (KZ) equations to compact formulas for on-shell norms and for off-shell scalar products of the Bethe states [36].

A number of results are already available also for the trigonometric $s\ell(2)$ Gaudin model with nontrivial boundary. The generating function of the trigonometric $s\ell(2)$ Gaudin Hamiltonians with triangular boundary terms was obtained in [37]. Moreover, we have shown that in this case a suitable non-unitary trigonometric classical r-matrix is an essential tool in the implementation of the algebraic Bethe ansatz [38] and we have conjectured the spectrum of the generating function and the corresponding Bethe equations [38]. Restricting boundary conditions to triangular K-matrices was here essential for the existence of unique lowest weight state, which then allowed straightforward application of the algebraic Bethe ansatz and finding the off-shell action of the generating function. The so-called modified Bethe ansatz for this model was studied in [39]. In a special case (when the number of excitations matched a specific value) this approach allowed a more general form of boundary conditions, while in the rest of the cases it was again necessary to restrict boundary conditions to triangular K-matrices (after which our earlier results [37,38] would be recovered). In spite of all this progress, a number of other important problems has remained hitherto open and we intend to address them in the present paper.

Although we have conjectured the form of full off-shell action of the generating function on Bethe vectors in [38], the strict mathematical proof of the obtained formulas was lacking. More importantly, one of the main goals here was to provide solutions to the corresponding KZ equations, as well as to find the formulas for on-shell norms of Bethe vectors and for their off-shell scalar products – the tasks we have already successfully accomplished in the rational case. However, we faced serious obstacles in pursuing these goals: the straightforward approach adapted from the rational case was failing to produce either KZ solutions or norm/scalar product formulas. It was not before we noticed and employed a combination of freedom in defining the local realization of Gaudin algebra generators and a freedom in defining the effective creation operators for the Bethe vectors, that we could achieve our established objectives. This realisation and the corresponding technique is another novel contribution of this paper.

The approach taken here differs in many aspects from our previous work on the subject [37, 38]. As a crucial step towards the complete proof (by mathematical induction) of the off-shell action of the generating function, we here define Bethe states through a particular recurrence relation. Next, our study is here based on the unitary, trigonometric $s\ell(2)$ classical r-matrix and the corresponding reflection K-matrix, both given in the so-called homogeneous gradation [21], as opposed to our previous papers [37,38] where we have used the trigonometric r-matrix and the classical reflection K-matrix in the principal gradation. Although the two formulations are equivalent, the motivation for making the present choice is related to the forementioned freedom in the local realization of the new set of generators of the generalized trigonometric $s\ell(2)$ Gaudin algebra. This approach yields a neat form of the off-shell action of the generating function on the Bethe vectors while at the same time enables the quest for the solutions to the corresponding Knizhnik-Zamolodchikov equations. Once this freedom – represented by two initially arbitrary functions – is fixed by solving appropriate differential equations, we not only find solutions to KZ equations, but also recover the compact determinant representation for the norms and the scalar products of the Bethe vectors, analogous to the expressions we have obtained in the rational case [36].

The composition of the paper is the following. In Section 2 the local realisation of the new generators of the relevant Gaudin algebra is given. The novel feature of this realisation is an arbitrary function of the local inhomogeneous parameter whose introduction does not affect the algebraic Bethe ansatz. The suitable creation operators and the complete details of the algebraic Bethe ansatz for the first two Bethe states are given in Section 3. It should be emphasised that the creation operator contains the second arbitrary function, this time of the rapidity parameter. In Section 4 we present the recurrent relation which defines the Bethe states of this system as well as the complete proof of the off-shell action of the generating function. The action of Gaudin Hamiltonians on the Bethe vectors is shown in Section 5. The solutions o the Knizhnik-Zamolodchikov equations are obtained in Section 6 by solving the appropriate differential equations for the two previously introduced arbitrary functions. The same choice for these functions allows us to derive, in Section 7, very compact relations for the on-shell norm and the scalar product of the Bethe vectors. Finally, our conclusions are presented in Section 8. For the sake of completeness, the fundamental structure on which this work is based (e.g. the classical r-matrix and the corresponding classical K-matrix) is presented in Appendix A. The standard definition of the Hilbert space of the system is given in Appendix B. The generalized Gaudin Lax operator as well as the Gaudin Hamiltonians with the boundary terms, both in the parametrization we have considered here, are given in Appendix C. In Appendix D we present the key formula in the proof of the off-shell action of the generating function of the trigonometric Gaudin Hamiltonians with boundary terms.

2. Generalized trigonometric Gaudin algebra

Our approach adopted here is established upon the non-unitary r-matrix (A.5) and the Lax operator (C.3), which in turn follow from the unitary, trigonometric $s\ell(2)$ classical r-matrix (A.1) and the corresponding reflection K-matrix (A.3), both given in the so-called homogeneous gradation [21]. We will focus on the triangular case when the boundary parameter ϕ is set to zero. As it follows from the linear bracket (C.4), the entries of the Lax matrix (C.3) generate the generalized $s\ell(2)$ trigonometric Gaudin algebra [38]. Furthermore, it is convenient to apply a linear triangular transformation that yields a particularly convenient set of generators which will be used below. These generators admit the following local realization

$$e(\lambda) = \sum_{m=1}^{N} \frac{(\xi e^{-2\alpha_m} + \nu)\, Z(\alpha_m)\, S_m^+}{\sinh(\lambda - \alpha_m)\sinh(\lambda + \alpha_m)}, \tag{2.1}$$

$$h(\lambda) = \sum_{m=1}^{N} \frac{S_m^3 + \dfrac{\psi}{2\xi\nu}\left(\xi e^{-2\alpha_m} - \nu\right) Z(\alpha_m)\, S_m^+}{\sinh(\lambda - \alpha_m)\sinh(\lambda + \alpha_m)}, \tag{2.2}$$

$$f(\lambda)=\sum_{m=1}^{N}\frac{\left(\xi e^{2\alpha_m}+\nu\right)\frac{S_m^-}{Z(\alpha_m)}-\frac{\psi}{\xi\nu}\left(\xi^2-\nu^2-2\xi\nu\sinh(2\alpha_m)\right)S_m^3+\frac{\psi^2}{\xi\nu}e^{-2\alpha_m}\left(\xi e^{2\alpha_m}+\nu\right)Z(\alpha_m)\,S_m^+}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)},\tag{2.3}$$

where $Z(\alpha_m)$ is an arbitrary function of the inhomogeneous parameter α_m. As it will be shown in the next two sections, the function $Z(\alpha_m)$ will remain arbitrary throughout the implementation of the algebraic Bethe ansatz. However, it will be of utmost importance both for finding the solutions to the Knizhnik-Zamolodchikov equations (6.7) and for finding formulas for norms and scalar products of Bethe vectors 7.

An important feature of the these generators is that their commutation relations are quite simple. Namely, the trivial commutation relations are

$$[e(\lambda),e(\mu)]=[h(\lambda),h(\mu)]=[f(\lambda),f(\mu)]=0,\tag{2.4}$$

and the only nontrivial ones are given by

$$[h(\lambda),e(\mu)]=\frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\left(e(\mu)-e(\lambda)\right),\tag{2.5}$$

$$\begin{aligned}[h(\lambda),f(\mu)]=&\frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\left(f(\lambda)-f(\mu)+\frac{\psi^2}{2\xi^2\nu^2}\times\right.\\&\left.\times\left(\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu)\right)e(\mu)-\left(\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)\right)e(\lambda)\right)\right),\end{aligned}\tag{2.6}$$

$$\begin{aligned}[e(\lambda),f(\mu)]=&\frac{2}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\times\\&\times\left(\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu)\right)h(\mu)-\left(\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)\right)h(\lambda)\right).\end{aligned}\tag{2.7}$$

Note that function the function $Z(\alpha_m)$ does not appear in the relations above.

Next, for the study of the algebraic Bethe ansatz it is essential to have the expression of the generating function $\tau(\lambda)$ (C.5) in terms of the generators (2.1)–(2.3).

$$\begin{aligned}\tau(\lambda)=&2\sinh^2(2\lambda)\left(h^2(\lambda)-\frac{4\xi\nu\,h(\lambda)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}-\frac{h'(\lambda)}{\sinh(2\lambda)}\right)\\&+2\sinh^2(2\lambda)\left(\frac{f(\lambda)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}-\frac{\psi^2}{4\xi^2\nu^2}e(\lambda)\right)e(\lambda).\end{aligned}\tag{2.8}$$

As we have shown in [38] the trigonometric $s\ell(2)$ Gaudin Hamiltonians with the boundary terms are obtained as the residues of the generating function $\tau(\lambda)$ at poles $\lambda=\pm\alpha_m$. For completeness, the explicit expressions of the Gaudin Hamiltonians H_m in the parametrisation used here are given in the Appendix C, more precisely, in the equation (C.7).

3. The algebraic Bethe ansatz

As it is well known [40–42], the existence of the pseudo-vacuum is essential for the algebraic Bethe ansatz. In the present case, the vector Ω_+ (see the definition in the Appendix B, in particular relations (B.3) and (B.4)), is annihilated by the generator $e(\lambda)$, i.e.

$$e(\lambda)\Omega_+ = 0\,, \tag{3.1}$$

and is an eigenvector of the generator $h(\lambda)$,

$$h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+\,, \quad \text{with} \quad \rho(\lambda) = \sum_{m=1}^{N} \frac{s_m}{\sinh(\lambda+\alpha_m)\sinh(\lambda-\alpha_m)}\,. \tag{3.2}$$

From the expression for the generating function $\tau(\lambda)$ that we have obtained in the previous section (2.8) and from the relations above (3.1) and (3.2) it is evident that the pseudo-vacuum is an eigenvector

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+, \tag{3.3}$$

with the eigenvalue $\chi_0(\lambda)$ given by

$$\chi_0(\lambda) = 2\sinh^2(2\lambda)\left(\rho^2(\lambda) - \frac{4\xi\nu\,\rho(\lambda)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \frac{\rho'(\lambda)}{\sinh(2\lambda)}\right). \tag{3.4}$$

One way to define the Bethe states of the system is to introduce the appropriate creation operators [11,42]. In this case, it is of interest to consider the following creation operators

$$\mathcal{C}_M(\lambda) = \mathcal{A}(\lambda)\left(f(\lambda) + 2\psi\left((2M-1) - \frac{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}{2\xi\nu}\left(h(\lambda) + \frac{\psi}{2\xi\nu}e(\lambda)\right)\right)\right), \tag{3.5}$$

where $\mathcal{A}(\lambda)$ is an arbitrary function of the parameter λ. Being an overall multiplier of the creation operator, it is not surprising that the function $\mathcal{A}(\lambda)$ will not be relevant for the algebraic Bethe ansatz. However, it will play an important role in relation to the solutions to the Knizhnik-Zamolodchikov equations, together with the function $Z(\alpha_m)$ that we have introduced in the local realization of the generators of the generalized $s\ell(2)$ trigonometric Gaudin algebra (2.1)–(2.3). The constant values for functions $\mathcal{A}(\lambda)$ and $Z(\alpha_m)$ would effectively (i.e. up to technical differences in used gradation) correspond to our earlier results [37,38]. (Similarly, with a suitable choice of these functions the creation operators (3.5) exactly correspond to those in the triangular boundary case of [39].)

In particular, it holds:

$$[\mathcal{C}_1(\lambda), \mathcal{C}_1(\mu)] = 4\psi\,(\mathcal{A}(\lambda)\mathcal{C}_1(\mu) - \mathcal{A}(\mu)\mathcal{C}_1(\lambda))\,, \tag{3.6}$$

as well as

$$[\mathcal{C}_1(\lambda), e(\mu)] = \frac{\mathcal{A}(\lambda)}{\xi\nu\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\left(\left(2\nu\xi\cosh(2\lambda)+\nu^2+\xi^2\right)\times\right.$$
$$\left.(\psi e(\lambda) - \psi e(\mu) + 2\nu\xi h(\lambda)) - 2\nu\xi h(\mu)\left(2\nu\xi\cosh(2\mu)+\nu^2+\xi^2\right)\right), \tag{3.7}$$

$$[\mathcal{C}_1(\lambda), h(\mu)] = \mathcal{A}(\lambda)\left(\frac{2\psi^2 e(\mu)}{\nu\xi} + \frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}(f(\mu)-f(\lambda))\right), \tag{3.8}$$

$$[\mathcal{C}_1(\lambda), f(\mu)] = \frac{\psi\mathcal{A}(\lambda)}{2\nu^3\xi^3\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\left(2\nu\xi\cosh(2\lambda)+\nu^2+\xi^2\right)\times$$
$$\left(\psi^2 e(\lambda)\left(2\nu\xi\cosh(2\lambda)+\nu^2+\xi^2\right) - \psi^2 e(\mu)\left(2\nu\xi\cosh(2\mu)+\nu^2+\xi^2\right) + \right.$$

$$
2\nu\xi\Big(-\nu\xi f(\lambda)+\nu\xi f(\mu)+\psi h(\lambda)\left(2\nu\xi\cosh(2\lambda)+\nu^2+\xi^2\right)-
\psi h(\mu)\left(2\nu\xi\cosh(2\mu)+\nu^2+\xi^2\right)\Big)\Big). \tag{3.9}
$$

Another useful relation is:

$$
\begin{aligned}
\tau(\lambda)&=2\sinh^2(2\lambda)\left(h^2(\lambda)-\frac{4\xi\nu\,h(\lambda)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}-\frac{h'(\lambda)}{\sinh(2\lambda)}\right)\\
&+2\sinh^2(2\lambda)\left(\frac{\mathcal{C}_1(\lambda)-2\psi\mathcal{A}(\lambda)}{\mathcal{A}(\lambda)(\xi^2+\nu^2+2\xi\nu\cosh(2\lambda))}+\frac{\psi}{\xi\nu}\left(h(\lambda)+\frac{\psi}{4\xi\nu}e(\lambda)\right)\right)e(\lambda).
\end{aligned} \tag{3.10}
$$

Now it is not difficult to show that the Bethe vector $\varphi_1(\mu)$ has the form

$$
\varphi_1(\mu)=\mathcal{C}_1(\mu)\Omega_+, \tag{3.11}
$$

where the operator $\mathcal{C}_1(\mu)$ is given in (3.5), with $M=1$. Evidently, the action of the generating function of the Gaudin Hamiltonians (2.8) reads

$$
\tau(\lambda)\varphi_1(\mu)=[\tau(\lambda),\mathcal{C}_1(\mu)]\,\Omega_++\chi_0(\lambda)\varphi_1(\mu). \tag{3.12}
$$

In this case, a direct calculation shows that the commutator in the first term on the right hand side of (3.12) is given by

$$
\begin{aligned}
&[\tau(\lambda),\mathcal{C}_1(\mu)]\,\Omega_+\\
&\quad=-\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\left(2\rho(\lambda)-\frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\right)\varphi_1(\mu)\\
&\quad+\frac{\mathcal{A}(\mu)}{\mathcal{A}(\lambda)}\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\times\\
&\quad\times\left(2\rho(\mu)-\frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}\right)\varphi_1(\lambda).
\end{aligned} \tag{3.13}
$$

Consequently, the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is given by

$$
\begin{aligned}
\tau(\lambda)\varphi_1(\mu)&=\chi_1(\lambda,\mu)\varphi_1(\mu)+\frac{\mathcal{A}(\mu)}{\mathcal{A}(\lambda)}\frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\times\\
&\times\left(\rho(\mu)-\frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}\right)\varphi_1(\lambda)\,,
\end{aligned} \tag{3.14}
$$

where

$$
\chi_1(\lambda,\mu)=\chi_0(\lambda)-\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu)\sinh(\lambda-\mu)}\left(2\rho(\lambda)-\frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\right). \tag{3.15}
$$

The unwanted term in (3.14) vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)} = 0. \tag{3.16}$$

Thus we have shown that $\varphi_1(\mu)$ (3.11) is a desired Bethe vector of the generating function $\tau(\lambda)$ with the eigenvalue $\chi_1(\lambda,\mu)$ (3.15), for arbitrary choices of the functions $\mathcal{A}(\mu)$ and $Z(\alpha_m)$.

As the next step, we will show that the Bethe vector $\varphi_2(\mu_1,\mu_2)$ can be given in a similar form:

$$\varphi_2(\mu_1,\mu_2) = \mathcal{C}_1(\mu_1)\varphi_1(\mu_2) + 4\psi\,\mathcal{A}(\mu_2)\varphi_1(\mu_1). \tag{3.17}$$

With the aim of obtaining the off-shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ we observe that

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) = [\tau(\lambda),\mathcal{C}_1(\mu_1)]\,\varphi_1(\mu_2) + \mathcal{C}_1(\mu_1)\,\tau(\lambda)\varphi_1(\mu_2) + 4\psi\,\mathcal{A}(\mu_2)\,\tau(\lambda)\varphi_1(\mu_1). \tag{3.18}$$

As the action of $\tau(\lambda)$ on $\varphi_1(\mu)$ known (3.14), now it is only necessary to calculate the first term on the right hand side of the equation above. Its straightforward to obtain

$$\begin{aligned}
&[\tau(\lambda), C_1(\mu_1)]\,\varphi_1(\mu_2) = -\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\times\\
&\times\left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \frac{2}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\right)\varphi_2(\mu_1,\mu_2)\\
&+\frac{\mathcal{A}(\mu_1)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\times\\
&\times\left(2\rho(\mu_1) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)} - \frac{2}{\sinh(\mu_1+\mu_2)\sinh(\mu_1-\mu_2)}\right)\varphi_2(\lambda,\mu_2)\\
&-\frac{\mathcal{A}(\mu_2)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_2)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\\
&\times\frac{2}{\sinh(\mu_2+\mu_1)\sinh(\mu_2-\mu_1)}\varphi_2(\mu_1,\lambda)\\
&+\frac{8\psi\,\mathcal{A}(\mu_2)\,\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\sinh(\mu_1+\mu_2)\sinh(\mu_1-\mu_2)}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\\
&\times\left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\right)\varphi_1(\mu_1)\\
&-\frac{8\psi\,\mathcal{A}(\mu_1)\,\mathcal{A}(\mu_2)\,\sinh^2(2\lambda)}{\mathcal{A}(\lambda)\,\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\\
&\times\left(2\rho(\mu_1) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)}\right)\varphi_1(\lambda)\\
&+\frac{8\psi\,\mathcal{A}(\mu_2)\,\sinh^2(2\lambda)}{\sinh(\lambda+\mu_2)\sinh(\lambda-\mu_2)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_2)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)}\\
&\times\left(2\rho(\mu_2) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_2)}\right)\varphi_1(\mu_1).
\end{aligned} \tag{3.19}$$

Then, after substituting this formula into the equation (3.18), twice using the action of $\tau(\lambda)$ on $\varphi_1(\mu)$ (3.14) and then rearranging some terms, the desired off-shell action of $\tau(\lambda)$ on the vector $\varphi_2(\mu_1, \mu_2)$ is obtained

$$
\begin{aligned}
&\tau(\lambda)\varphi_2(\mu_1,\mu_2) = \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) \\
&+ \sum_{j=1}^{2} \frac{\mathcal{A}(\mu_j)}{\mathcal{A}(\lambda)} \frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)} \frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \left(\rho(\mu_j) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \frac{1}{\sinh(\mu_j+\mu_{3-j})\sinh(\mu_j-\mu_{3-j})}\right) \\
&\times \varphi_2(\lambda,\mu_{3-j}),
\end{aligned}
\tag{3.20}
$$

with the eigenvalue

$$
\begin{aligned}
&\chi_2(\lambda,\mu_1,\mu_2) = \chi_0(\lambda) - \sum_{i=1}^{2} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \times \\
&\times \left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \frac{1}{\sinh(\lambda+\mu_{3-i})\sinh(\lambda-\mu_{3-i})}\right).
\end{aligned}
$$

The two unwanted terms in the action above (3.20) vanish when the Bethe equations are imposed on the parameters μ_1 and μ_2,

$$
\rho(\mu_j) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \frac{1}{\sinh(\mu_j+\mu_{3-j})\sinh(\mu_j-\mu_{3-j})} = 0, \tag{3.21}
$$

with $j = 1, 2$. In this way, we have demonstrated that, for an arbitrary choice of the functions $\mathcal{A}(\mu_j)$ and $Z(\alpha_m)$, the vector $\varphi_2(\mu_1, \mu_2)$, as defined in (3.17), is the Bethe state of the generating function of the Gaudin Hamiltonians with the eigenvalue $\chi_2(\lambda, \mu_1, \mu_2)$.

4. The general form of Bethe vectors

In this section we will obtain the general form of Bethe states as well as the off-shell action of the generating function on these states. For arbitrary natural number M, the Bethe states of the system can be defined by the following recurrence relation

$$
\begin{aligned}
\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) &= \mathcal{C}_1(\mu_1)\varphi_{M-1}(\mu_2,\mu_3,\ldots,\mu_M) \\
&+ 4\psi \sum_{j=2}^{M} \mathcal{A}(\mu_j)\, \varphi_{M-1}(\mu_1,\mu_2,\ldots,\widehat{\mu_j},\ldots,\mu_{M-1},\mu_M),
\end{aligned}
\tag{4.1}
$$

where the notation $\widehat{\mu}_j$ means that the argument μ_j is not present. Thus the action of the generating function of the Gaudin Hamiltonians on $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ can be obtained as follows

$$
\begin{aligned}
&\tau(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) \\
&\quad = [\tau(\lambda), \mathcal{C}_1(\mu_1)]\,\varphi_{M-1}(\mu_2,\ldots,\mu_M) + \mathcal{C}_1(\mu_1)\,\tau(\lambda)\varphi_{M-1}(\mu_2,\ldots,\mu_M) \\
&\quad + 4\psi \sum_{j=2}^{M} \mathcal{A}(\mu_j)\, \tau(\lambda)\varphi_{M-1}(\mu_1,\mu_2,\ldots,\widehat{\mu}_j,\ldots,\mu_{M-1},\mu_M).
\end{aligned}
\tag{4.2}
$$

According to the principle of mathematical induction, on the right hand side of the equation above, it is assumed that the action of $\tau(\lambda)$ on φ_{M-1} is known. Therefore the essential step in the proof consists in determining the off-shell action of the commutator between the generating function and the creation operator, i.e. $[\tau(\lambda), \mathcal{C}_1(\mu_1)]$, on the Bethe vector $\varphi_{M-1}(\mu_2, \ldots, \mu_M)$. This can be done by a straightforward calculation and the result is a somewhat cumbersome one-page formula given in Appendix D, equation (D.1). Then, after substituting this formula into the equation (4.2) and using the action of $\tau(\lambda)$ on φ_{M-1} in the remaining M terms on the right hand side of (4.2), it is also necessary to make some obvious steps which consist in rearranging some terms and simplifying others. Finally, as required by the mathematical induction, we obtain the action of the generating function on $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ in the desired form

$$\begin{aligned}
&\tau(\lambda)\varphi_M(\mu_1, \ldots, \mu_M) = \chi_M(\mu_1, \ldots, \mu_M)\varphi_M(\mu_1, \ldots, \mu_M) \\
&+ \sum_{j=1}^{M} \frac{\mathcal{A}(\mu_j)}{\mathcal{A}(\lambda)} \frac{4\sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)} \frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \left(\rho(\mu_j) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \sum_{k\neq j}^{M} \frac{1}{\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)}\right) \\
&\times \varphi_M(\lambda, \mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M),
\end{aligned} \tag{4.3}$$

where the notation $\widehat{\mu}_j$ means that the argument μ_j is not present and the eigenvalue $\chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)$ is given by

$$\begin{aligned}
&\chi_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M) = \chi_0(\lambda) - \sum_{j=1}^{M} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)} \times \\
&\times \left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \sum_{k\neq j}^{M} \frac{1}{\sinh(\lambda+\mu_k)\sinh(\lambda-\mu_k)}\right).
\end{aligned} \tag{4.4}$$

In order to guarantee that the M unwanted terms in (4.3) will vanish, the Bethe equations have to be imposed on the rapidity parameters μ_j:

$$\rho(\mu_j) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \sum_{k\neq j}^{M} \frac{1}{\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)} = 0, \tag{4.5}$$

with $j = 1, 2, \ldots, M$. In this way, we have concluded the proof.

While the form (4.1) for Bethe vectors is more convenient for the proof of the off-shell action relation (4.3), it is relatively straightforward to show that Bethe vectors can be also written in a much more compact form that was conjectured in [38]:

$$\varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\cdots\mathcal{C}_M(\mu_M)\Omega_+ \tag{4.6}$$

As it is evident from the formulas above, the choice of functions $\mathcal{A}(\mu_j)$ and $Z(\alpha_m)$ does not affect the algebraic Bethe ansatz and therefore they may still remain arbitrary. This will remain so until the Section 6 where will have to fix these functions in order to derive solutions to the related Knizhnik-Zamolodchikov equations.

5. The action of Gaudin Hamiltonians

To deal with KZ equations we must first find the off-shell action of the Gaudin Hamiltonians (C.7) on an arbitrary Bethe vector $\varphi_M(\mu_1,\ldots,\mu_M)$ (4.1). It is obtained by taking the residum of (4.3) at $\lambda=\alpha_m$ and dividing both sides of the equation by four

$$\begin{aligned}
&H_m\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)=\mathcal{E}_{m,M}\;\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)\\
&+\sum_{j=1}^{M}\frac{\mathcal{A}(\mu_j)}{\mathcal{A}(\alpha_m)}\frac{\sinh^2(2\lambda)}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}\times\\
&\times\left(\rho(\mu_j)-\frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}-\sum_{k\neq j}^{M}\frac{1}{\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)}\right)\times\\
&\times\operatorname*{Res}_{\lambda=\alpha_m}\varphi_M(\lambda,\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M),
\end{aligned}\tag{5.1}$$

where the eigenvalues of the Gaudin Hamiltonians (C.7) are given by

$$\begin{aligned}
&\mathcal{E}_{m,M}=\frac{1}{4}\operatorname*{Res}_{\lambda=\alpha_m}\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)=s_m(s_m+1)\coth(2\alpha_m)+s_m\sinh(2\alpha_m)\times\\
&\times\left(-\frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}+\sum_{n\neq m}^{N}\frac{s_n}{\sinh(\alpha_m+\alpha_n)\sinh(\alpha_m-\alpha_n)}\right)\\
&-\sum_{i=1}^{M}\frac{s_m\sinh(2\alpha_m)}{\sinh(\alpha_m+\mu_i)\sinh(\alpha_m-\mu_i)},
\end{aligned}\tag{5.2}$$

and, for example,

$$\begin{aligned}
&\operatorname*{Res}_{\lambda=\alpha_m}\varphi_M(\lambda,\mu_2,\ldots,\mu_M)=\operatorname*{Res}_{\lambda=\alpha_m}\mathcal{C}_1(\lambda)\,(\varphi_{M-1}(\mu_2,\ldots,\mu_M)\\
&+4\psi\sum_{j=2}^{M}\mathcal{A}(\mu_j)\,\varphi_{M-2}(\mu_2,\ldots,\widehat{\mu}_j,\ldots\mu_M)\\
&+(4\psi)^2\cdot 2\sum_{\substack{j,k=2\\k\neq j}}^{M}\mathcal{A}(\mu_j)\mathcal{A}(\mu_k)\,\varphi_{M-3}(\mu_2,\ldots,\widehat{\mu}_j,\ldots,\widehat{\mu}_k,\ldots,\mu_M)+\cdots\\
&+(4\psi)^{M-1}\cdot(M-1)!\,\mathcal{A}(\mu_2)\mathcal{A}(\mu_3)\cdots\mathcal{A}(\mu_M)\,\Omega_+\Big).
\end{aligned}\tag{5.3}$$

Here the residuum of the creation operator (3.5) at the pole $\lambda=\alpha_m$ is given by

$$\operatorname*{Res}_{\lambda=\alpha_m}\mathcal{C}_1(\lambda)=\frac{\mathcal{A}(\alpha_m)\left(\xi e^{2\alpha_m}+\nu\right)}{\sinh(2\alpha_m)}\left(\frac{S_m^-}{Z(\alpha_m)}-2\frac{\psi}{\nu}e^{-2\alpha_m}S_m^3-\frac{\psi^2}{\nu^2}e^{-4\alpha_m}Z(\alpha_m)\,S_m^+\right).\tag{5.4}$$

This action will be essential in obtaining solutions to the corresponding Knizhnik-Zamolodchikov equations in the following section.

6. Knizhnik-Zamolodchikov equations

Even before writing down the form of Knizhnik-Zamolodchikov equations in this context, we note that the results in this section will be obtained under the condition that the boundary parameter ψ is set to zero. Having in mind the local realization (2.1)–(2.3), it should be emphasised that then the creation operator (3.5) simplifies to become

$$\widetilde{\mathcal{C}}(\lambda) = \mathcal{C}_M(\lambda)\Big|_{\psi=0} = \mathcal{A}(\lambda) \sum_{m=1}^{N} \frac{\xi\, e^{2\alpha_m} + \nu}{Z(\alpha_m)\,\sinh(\lambda - \alpha_m)\,\sinh(\lambda + \alpha_m)}\, S_m^-, \tag{6.1}$$

and therefore the Bethe vectors (4.1) simplify as well

$$\widetilde{\varphi}_M(\mu_1, \mu_2, \ldots, \mu_M) = \varphi_M(\mu_1, \mu_2, \ldots, \mu_M)\Big|_{\psi=0} = \widetilde{\mathcal{C}}(\mu_1)\,\widetilde{\mathcal{C}}(\mu_2)\cdots\widetilde{\mathcal{C}}(\mu_M)\Omega_+. \tag{6.2}$$

Moreover, when $\psi = 0$, the Gaudin Hamiltonians (C.7) reduce to

$$\begin{aligned}
\widetilde{H}_m = H_m\Big|_{\psi=0} &= \sum_{n\neq m}^{N} \left(\coth(\alpha_m - \alpha_n)\, S_m^3 \cdot S_n^3 + \frac{e^{\alpha_m-\alpha_n}\, S_m^- \cdot S_n^+ + e^{-(\alpha_m-\alpha_n)}\, S_m^+ \cdot S_n^-}{2\sinh(\alpha_m - \alpha_n)} \right) \\
&+ \sum_{n=1}^{N} \coth(\alpha_m + \alpha_n) \left(\frac{S_m^3 \cdot S_n^3 + S_n^3 \cdot S_m^3}{2} \right) \\
&+ \sum_{n=1}^{N} \frac{e^{-(\alpha_m+\alpha_n)}}{\sinh(\alpha_m + \alpha_n)} \left(\frac{\xi\, e^{2\alpha_m} + \nu}{\xi\, e^{-2\alpha_m} + \nu}\, \frac{S_m^- \cdot S_n^+ + S_n^+ \cdot S_m^-}{4} \right) \\
&+ \sum_{n=1}^{N} \frac{e^{\alpha_m+\alpha_n}}{\sinh(\alpha_m + \alpha_n)} \left(\frac{\xi\, e^{-2\alpha_m} + \nu}{\xi\, e^{2\alpha_m} + \nu}\, \frac{S_m^+ \cdot S_n^- + S_n^- \cdot S_m^+}{4} \right).
\end{aligned} \tag{6.3}$$

Also, observe that, in this case, the equation (5.1) takes the following from

$$\begin{aligned}
&\widetilde{H}_m \widetilde{\varphi}_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{E}_{m,M}\, \widetilde{\varphi}_M(\mu_1, \mu_2, \ldots, \mu_M) \\
&+ \sum_{j=1}^{M} \frac{\mathcal{A}(\mu_j)}{\mathcal{A}(\alpha_m)}\, \frac{\sinh^2(2\lambda)}{\sinh(\alpha_m + \mu_j)\sinh(\alpha_m - \mu_j)}\, \frac{\xi^2 + \nu^2 + 2\xi\nu\cosh(2\mu_j)}{\xi^2 + \nu^2 + 2\xi\nu\cosh(2\alpha_m)} \times \\
&\times \left(\rho(\mu_j) - \frac{2\xi\nu}{\xi^2 + \nu^2 + 2\xi\nu\cosh(2\mu_j)} - \sum_{k\neq j}^{M} \frac{1}{\sinh(\mu_j + \mu_k)\sinh(\mu_j - \mu_k)} \right) \times \\
&\times \operatorname*{Res}_{\lambda=\alpha_m} \widetilde{\varphi}_M(\lambda, \mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M),
\end{aligned} \tag{6.4}$$

where the eigenvalue $\mathcal{E}_{m,M}$ of the Gaudin Hamiltonian $\widetilde{H}_m$ is given by (5.2), the notation $\widehat{\mu}_j$ means that the argument μ_j is not present and

$$\operatorname*{Res}_{\lambda=\alpha_m} \widetilde{\varphi}_M(\lambda, \mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M) = \operatorname*{Res}_{\lambda=\alpha_m} \widetilde{\mathcal{C}}(\lambda)\, \widetilde{\varphi}_{M-1}(\mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M) \tag{6.5}$$

with

$$\operatorname*{Res}_{\lambda=\alpha_m} \widetilde{\mathcal{C}}(\lambda) = \frac{\mathcal{A}(\alpha_m)}{Z(\alpha_m)} \frac{\xi\, e^{2\alpha_m} + \nu}{\sinh(2\alpha_m)} S_m^- . \tag{6.6}$$

The interplay between the Bethe vectors of the Gaudin models and solutions to the corresponding Knizhnik-Zamolodchikov equations have attracted some attention [11,12,36,43–48]. The main objective of this section is to consider the Knizhnik-Zamolodchikov equations

$$\kappa\, \partial_{\alpha_m} \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \widetilde{H}_m\, \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)\ , \tag{6.7}$$

where the Hamiltonians $\widetilde{H}_m$ are given by (6.3). The goal is to find the functions (i.e. states) $\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ with the curious property that the action of the Gaudin Hamiltonians $\widetilde{H}_m$ on these states reduces to mere derivation with respect to parameters α_m.

Following a common approach [11,12,36,45,46], we seek solutions to the equations above in the form of contour integrals with respect to the variables $\mu_1, \mu_2, \ldots, \mu_M$

$$\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \oint \cdots \oint \Phi(\vec{\mu}|\vec{\alpha}) \cdot \widetilde{\varphi}_M(\vec{\mu}|\vec{\alpha})\, d\mu_1 \cdots d\mu_M, \tag{6.8}$$

where the integrating factor is a scalar function $\Phi(\vec{\mu}|\vec{\alpha})$ that we will seek in the form:

$$\Phi(\vec{\mu}|\vec{\alpha}) = \exp\left(\frac{S(\vec{\mu}|\vec{\alpha})}{\kappa}\right). \tag{6.9}$$

We choose the function $S(\vec{\mu}|\vec{\alpha})$ to be:

$$\begin{aligned} S(\vec{\mu}|\vec{\alpha}) = {} & \sum_{m=1}^{N} \frac{s_m(s_m+1)}{2} \ln(\sinh(2\alpha_m)) - \sum_{m=1}^{N} \frac{s_m}{2} \ln\left(\xi^2 + \nu^2 + 2\xi\nu\cosh(2\alpha_m)\right) \\ & + \sum_{m>n}^{N} s_m s_n \ \left(\ln(\sinh(\alpha_m - \alpha_n)) + \ln(\sinh(\alpha_m + \alpha_n))\right) \\ & - \sum_{j=1}^{M}\sum_{m=1}^{N} s_m \left(\ln(\sinh(\alpha_m - \mu_j)) + \ln(\sinh(\alpha_m + \mu_j))\right) \\ & + \sum_{j=1}^{M} \ln\left(\xi^2 + \nu^2 + 2\xi\nu\cosh(2\mu_j)\right) + \sum_{j>k}^{M}\Big(\ln(\sinh(\mu_j - \mu_k)) \\ & + \ln(\sinh(\mu_j + \mu_k))\Big), \end{aligned} \tag{6.10}$$

so that it becomes straightforward to check that the function $\phi(\vec{\mu}|\vec{\alpha})$, as defined above, satisfies the following system of linear partial differential equations

$$\kappa\, \partial_{\alpha_m} \Phi = \mathcal{E}_{m,M}\ \Phi, \tag{6.11}$$

$$\kappa\, \partial_{\mu_j} \Phi = \beta_M(\mu_j)\ \Phi, \tag{6.12}$$

with $\mathcal{E}_{m,M}$ given by (5.2) and

$$\beta_M(\mu_j) = -\sinh(2\mu_j)\Bigg(\rho(\mu_j) - \frac{2\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \sum_{k\neq j}^{M} \frac{1}{\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)}\Bigg). \tag{6.13}$$

Note that, up to multiplication by $-\sinh(2\mu_j)$, the functions $\beta_M(\mu_j)$ coincide with the left-hand sides of Bethe equations (4.5). Alternatively, we might have imposed Bethe equations in the form $\beta_M(\mu_j) = 0$.

For later convenience we rewrite the equation (6.4) as follows

$$\begin{aligned}\widetilde{H}_m \widetilde{\varphi}_M(\mu_1, \mu_2, \dots, \mu_M) &= \mathcal{E}_{m,M}\, \widetilde{\varphi}_M(\mu_1, \mu_2, \dots, \mu_M) \\ &+ \sum_{j=1}^{3} \frac{\sinh(2\alpha_m)}{\sinh(\mu_j+\alpha_m)\sinh(\mu_j-\alpha_m)} \frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{Z(\alpha_m)\left(\xi\, e^{-2\alpha_m}+\nu\right)} \\ &\times \frac{\mathcal{A}(\mu_j)}{\sinh(2\mu_j)}\, \beta_M(\mu_j)\cdot \widetilde{\varphi}^{(j,m)}_{M-1},\end{aligned} \tag{6.14}$$

where we have used the notation

$$\widetilde{\varphi}^{(j,m)}_{M-1} = S^-_m\, \widetilde{\varphi}_{M-1}(\mu_1, \dots, \widehat{\mu}_j, \dots, \mu_M), \tag{6.15}$$

once more, $\widehat{\mu}_j$ means that the argument μ_j is not present.

Now, for the KZ equations to be satisfied, the following must hold:

$$\partial_{\alpha_m}\widetilde{\varphi}_M = \sum_{j=1}^{M} \partial_{\mu_j}\Bigg(\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)} \frac{\mathcal{A}(\mu_j)}{\sinh(2\mu_j)} \times \frac{\sinh(2\alpha_m)}{Z(\alpha_m)\left(\xi\, e^{-2\alpha_m}+\nu\right)}\; \widetilde{\varphi}^{(j,m)}_{M-1}\Bigg). \tag{6.16}$$

The novel feature in the present case is that the functions $\mathcal{A}(\mu_j)$ of the rapidity parameter μ_j and $Z(\alpha_m)$ of the inhomogeneous parameter α_m appear on both sides of the equation (6.16). It is easy to check that, without these functions, equality (6.16) simply would not hold. But in their presence, we can try to determine these functions in such a way that the equation (6.16) is valid for an arbitrary natural number M.

With the aim of satisfying the equation (6.16) for arbitrary M, we write the same equation in the case when $M = 1$

$$\partial_{\alpha_m}\widetilde{\mathcal{C}}(\mu)\,\Omega_+ = \partial_\mu\Bigg(\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}{\sinh(\alpha_m+\mu)\sinh(\alpha_m-\mu)}\; \frac{\mathcal{A}(\mu)}{\sinh(2\mu)}\; \frac{\sinh(2\alpha_m)}{Z(\alpha_m)\left(\xi\, e^{-2\alpha_m}+\nu\right)} \times S^-_m\,\Omega_+\Bigg) \tag{6.17}$$

On the left hand side of the equation above we substitute the expression for the operator $\widetilde{\mathcal{C}}(\mu)$ (6.1) and we use the identity

$$\partial_{\alpha_m} \frac{\sinh(2\mu)}{\sinh(\mu-\alpha_m)\sinh(\mu+\alpha_m)} = \partial_\mu \left(\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}{\sinh(\alpha_m-\mu)\sinh(\alpha_m+\mu)} \right) \times \frac{\sinh(2\alpha_m)}{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}, \tag{6.18}$$

to obtain the following equation

$$\frac{Z(\alpha_m)\left(\xi\, e^{-2\alpha_m}+\nu\right)}{\sinh(2\alpha_m)}\, \partial_{\alpha_m} \frac{\xi\, e^{2\alpha_m}+\nu}{Z(\alpha_m)} = -\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}{\mathcal{A}(\mu)}\, \partial_\mu \frac{\mathcal{A}(\mu)}{\sinh(2\mu)}. \tag{6.19}$$

Evidently, in the equation above the variables are separated and therefore we have to solve

$$\frac{\partial_{\alpha_m} Z(\alpha_m)}{Z(\alpha_m)} - \frac{2\,\xi\, e^{2\alpha_m}}{\xi\, e^{2\alpha_m}+\nu} + \frac{C\,\sinh(2\alpha_m)}{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)} = 0, \tag{6.20}$$

and

$$\frac{\partial_\mu\, \mathcal{A}(\mu)}{\mathcal{A}(\mu)} - 2\coth(2\mu) + \frac{C\,\sinh(2\mu)}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)} = 0, \tag{6.21}$$

where C is an arbitrary constant. The general solution of the equations above is

$$Z(\alpha_m) = C_1\, \frac{\xi\, e^{2\alpha_m}+\nu}{\left(\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)\right)^{\frac{C}{4\xi\nu}}}, \tag{6.22}$$

$$\mathcal{A}(\mu) = C_2\, \frac{\sinh(2\mu)}{\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu)\right)^{\frac{C}{4\xi\nu}}}, \tag{6.23}$$

where C_1 and C_2 are additional arbitrary constants. One possible solution is obtained if we set $C=0$ and $C_1=C_2=1$. In this case, forms of the functions become extremely simple:

$$Z(\alpha_m) = \xi\, e^{2\alpha_m}+\nu, \quad \text{and} \quad \mathcal{A}(\mu) = \sinh(2\mu). \tag{6.24}$$

After substituting these expressions into (6.1) and (6.2), it is not difficult to confirm that (6.16) is valid for an arbitrary positive integer M. However, we will nevertheless choose $C = 2\xi\nu$ and $C_1 = C_2 = 1$, since these values will turn out to be far more suitable in the next section, when we discuss norms and scalar products of Bethe vectors. With these values, we obtain:

$$Z(\alpha_m) = \frac{\xi\, e^{2\alpha_m}+\nu}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}} \quad \text{and} \quad \mathcal{A}(\mu) = \frac{\sinh(2\mu)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}}. \tag{6.25}$$

It is straightforward to verify that (6.16) is valid also in this case.

For completeness, we proceed to show that the functions $\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ (6.8) are solutions to the Knizhnik-Zamolodchikov equations (6.7)

$$\kappa\, \partial_{\alpha_m} (\Phi \cdot \widetilde{\varphi}_M) = \left(\kappa\, \partial_{\alpha_m} \Phi\right) \cdot \widetilde{\varphi}_M + \Phi \cdot \left(\kappa\, \partial_{\alpha_m} \widetilde{\varphi}_M\right). \tag{6.26}$$

As the next step, in the first term, we substitute the right-hand-side of (6.11) and in the second term we use (6.16), with (6.25), to obtain

$$\kappa\,\partial_{\alpha_m}(\Phi\cdot\widetilde{\varphi}_M)=\mathcal{E}_{m,M}\;(\Phi\cdot\widetilde{\varphi}_M)+\Phi\cdot\kappa\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\right.$$
$$\left.\times\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\,\widetilde{\varphi}_{M-1}^{(j,m)}\right).\tag{6.27}$$

We use (6.14), with (6.25), to express the first term and we rewrite the second term of the equation above using the Leibniz rule

$$\kappa\,\partial_{\alpha_m}(\Phi\cdot\widetilde{\varphi}_M)=H_m\,(\Phi\cdot\widetilde{\varphi}_M)+\sum_{j=1}^{M}\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}$$
$$\times\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\times$$
$$\times\;\beta_M(\mu_j)\cdot\Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}+\kappa\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\right.$$
$$\left.\times\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\,\Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}\right)$$
$$-\kappa\sum_{j=1}^{M}\left(\partial_{\mu_j}\Phi\right)\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\,\widetilde{\varphi}_{M-1}^{(j,m)}.\tag{6.28}$$

Then, in the last term we use (6.12) to obtain

$$\kappa\,\partial_{\alpha_m}(\Phi\cdot\widetilde{\varphi}_M)=H_m\,(\Phi\cdot\widetilde{\varphi}_M)+\sum_{j=1}^{M}\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}$$
$$\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\times$$
$$\times\;\beta_M(\mu_j)\cdot\Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}+\kappa\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\right.$$
$$\left.\times\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\,\Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}\right)$$
$$-\sum_{j=1}^{M}\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\,\beta_M(\mu_j)\cdot\Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}.\tag{6.29}$$

Finally, we simplify the second and the last term to conclude

$$\kappa\ \partial_{\alpha_m}(\Phi\cdot\widetilde{\varphi}_M)=H_m(\Phi\cdot\widetilde{\varphi}_M)$$
$$+\kappa\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}}{\sinh(\alpha_m+\mu_j)\sinh(\alpha_m-\mu_j)}\frac{\sinh(2\alpha_m)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}}\ \Phi\cdot\widetilde{\varphi}_{M-1}^{(j,m)}\right). \tag{6.30}$$

Evidently, the terms in the second line of (6.30) will not contribute to the contour integrals with respect to variables μ_j, $j=1,2,\ldots,M$, in (6.8). Therefore, we have shown that every Bethe vector $\widetilde{\varphi}_M(\mu_1,\mu_2,\ldots,\mu_M)$ (6.2) yields a solution $\Psi(\alpha_1,\alpha_2,\ldots,\alpha_N)$ (6.8) to the Knizhnik-Zamolodchikov equations (6.7).

7. Norms and scalar products

In this section we will first study the on-shell norms of the Bethe vectors $\widetilde{\varphi}_M(\mu_1,\mu_2,\ldots,\mu_M)$ (6.2). With the choice we made for the function $Z(\alpha_m)$, the local realization of the generators is given by:

$$e(\lambda)=\sum_{m=1}^{N}\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}\ S_m^+}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}, \tag{7.1}$$

$$h(\lambda)=\sum_{m=1}^{N}\frac{S_m^3}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}, \tag{7.2}$$

$$f(\lambda)=\sum_{m=1}^{N}\frac{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\alpha_m)}\ S_m^-}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}. \tag{7.3}$$

As the first step, the norm of the Bethe vector

$$\widetilde{\varphi}_1(\mu)=\widetilde{\mathcal{C}}(\mu)\Omega_+=\frac{\sinh(2\mu)}{\sqrt{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}}\ \widetilde{f}(\mu)\Omega_+\ , \tag{7.4}$$

is calculated, when the Bethe equation (3.16) is imposed on the parameter μ, to be

$$\begin{aligned}\left\|\widetilde{\varphi}_1(\mu)\right\|^2&=-2\sinh(2\mu)\left(\rho'(\mu)+\frac{4\xi\nu\sinh(2\mu)}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu)}\ \rho(\mu)\right)\\&=2\ \frac{\partial\beta_1(\mu)}{\partial\mu}\Big|_{\beta_1(\mu)=0}=2\ \frac{\partial^2 S(\mu)}{\partial\mu^2}\Big|_{\beta_1(\mu)=0}.\end{aligned} \tag{7.5}$$

As the next step, we obtain the norm of the Bethe vector

$$\begin{aligned}\widetilde{\varphi}_2(\mu_1,\mu_2)&=\widetilde{\mathcal{C}}(\mu_1)\widetilde{\mathcal{C}}(\mu_2)\Omega_+\\&=\frac{\sinh(2\mu_1)}{\sqrt{\xi^2+\eta^2+2\xi\nu\cosh(2\mu_1)}}\ \frac{\sinh(2\mu_2)}{\sqrt{\xi^2+\eta^2+2\xi\nu\cosh(2\mu_2)}}\widetilde{f}(\mu_1)\widetilde{f}(\mu_2)\Omega_+\ ,\end{aligned} \tag{7.6}$$

when the Bethe equation (3.21) is imposed on the parameter μ_1 and μ_2, to be

$$\begin{aligned}\left\|\widetilde{\varphi}_2(\mu_1,\mu_2)\right\|^2 &= 2^2 \det\begin{pmatrix} \frac{\partial\beta_2(\mu_1)}{\partial\mu_1} & \frac{\partial\beta_2(\mu_2)}{\partial\mu_1} \\ \frac{\partial\beta_2(\mu_1)}{\partial\mu_2} & \frac{\partial\beta_2(\mu_2)}{\partial\mu_2}\end{pmatrix}\Bigg|_{\substack{\beta_2(\mu_1)=0\\ \beta_2(\mu_2)=0}} \\ &= 2^2 \det\begin{pmatrix} \frac{\partial^2 S}{\partial\mu_1^2} & \frac{\partial^2 S}{\partial\mu_1\partial\mu_2} \\ \frac{\partial^2 S}{\partial\mu_2\partial\mu_1} & \frac{\partial^2 S}{\partial\mu_2^2}\end{pmatrix}\Bigg|_{\substack{\beta_2(\mu_1)=0\\ \beta_2(\mu_2)=0}} .\end{aligned} \tag{7.7}$$

In general, the norm of the Bethe vector

$$\widetilde{\varphi}_M(\mu_1,\mu_2,\ldots,\mu_M) = \widetilde{\mathcal{C}}(\mu_1)\cdots\widetilde{\mathcal{C}}(\mu_M)\Omega_+ , \tag{7.8}$$

where M is an arbitrary natural number, is calculated to be

$$\left\|\widetilde{\varphi}_M(\mu_1,\mu_2,\ldots,\mu_M)\right\|^2 = 2^M \det\begin{pmatrix} \frac{\partial^2 S}{\partial\mu_1^2} & \frac{\partial^2 S}{\partial\mu_1\partial\mu_2} & \cdots & \frac{\partial^2 S}{\partial\mu_1\partial\mu_M} \\ \vdots & \ddots & & \vdots \\ \frac{\partial^2 S}{\partial\mu_M\partial\mu_1} & \frac{\partial^2 S}{\partial\mu_M\partial\mu_2} & \cdots & \frac{\partial^2 S}{\partial\mu_M^2}\end{pmatrix}\Bigg|_{\substack{\beta_M(\mu_1)=0\\ \vdots \\ \beta_M(\mu_M)=0}} , \tag{7.9}$$

when the Bethe equation (4.5) is imposed on the parameters $\mu_1,\ldots,\mu_M$. Therefore, we confirm that relations for norm, analogous to those in rational case [36], now hold also in the trigonometric case, thanks to the proper choices of $Z(\alpha_m)$ and $\mathcal{A}(\mu)$ functions.

Furthermore, with this choice for $Z(\alpha_m)$ and $\mathcal{A}(\mu)$, it is also possible to calculate more general expression for the off-shell scalar product of the Bethe vectors, again analogous as in the rational case [36]. It turns out that the scalar product

$$\begin{aligned}&\langle\widetilde{\varphi}_M(\mu_1,\mu_2,\ldots,\mu_M),\widetilde{\varphi}_M(\nu_1,\nu_2,\ldots,\nu_M)\rangle \\ &\quad = \langle\Omega_+,\ \widetilde{\mathcal{C}}^*(\mu_1)\cdots\widetilde{\mathcal{C}}^*(\mu_M)\widetilde{\mathcal{C}}(\nu_1)\cdots\widetilde{\mathcal{C}}(\nu_M)\ \Omega_+\rangle ,\end{aligned} \tag{7.10}$$

is proportional to

$$\langle\Omega_+,\ e(\mu_1)\cdots e(\mu_M) f(\nu_M)\cdots f(\nu_1)\ \Omega_+\rangle = 2^M \sum_{\sigma\in\mathcal{S}_M} \det\mathcal{M}^\sigma , \tag{7.11}$$

where $\mathcal{S}_M$ is the symmetric group of degree M and the entries of the $M\times M$ matrix $\mathcal{M}^\sigma$ are given by

$$\begin{aligned}\mathcal{M}^\sigma_{jj} = &-\frac{\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)\right)\rho(\mu_j) - \left(\xi^2+\nu^2+2\xi\nu\cosh(2\nu_{\sigma(j)})\right)\rho(\nu_{\sigma(j)})}{\sinh(\mu_j-\nu_{\sigma(j)})\sinh(\mu_j+\nu_{\sigma(j)})} \\ &-\frac{1}{2}\sum_{k\neq j}^{M}\frac{\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu_k)\right)+\left(\xi^2+\nu^2+2\xi\nu\cosh(2\nu_{\sigma(k)})\right)}{\sinh(\mu_j-\mu_k)\sinh(\mu_j+\mu_k)\sinh(\nu_{\sigma(j)}-\nu_{\sigma(k)})\sinh(\nu_{\sigma(j)}+\nu_{\sigma(k)})} ,\end{aligned} \tag{7.12}$$

$$\mathcal{M}^\sigma_{jk} = -\frac{1}{2}\frac{\left(\xi^2+\nu^2+2\xi\nu\cosh(2\mu_k)\right)+\left(\xi^2+\nu^2+2\xi\nu\cosh(2\nu_{\sigma(k)})\right)}{\sinh(\mu_j-\mu_k)\sinh(\mu_j+\mu_k)\sinh(\nu_{\sigma(j)}-\nu_{\sigma(k)})\sinh(\nu_{\sigma(j)}+\nu_{\sigma(k)})} , \tag{7.13}$$

where $j,k = 1,2,\ldots,M$.

8. Conclusions

We have demonstrated the implementation of the algebraic Bethe ansatz for the trigonometric Gaudin model with boundary terms, which we have conjectured in our previous paper [38]. The Bethe vectors are here defined by the recurrent relation (4.1). We have shown the action of the generating function $\tau(\lambda)$ on an arbitrary Bethe vector (4.3). The proof was based on mathematical induction. The key step in the proof is calculating the off-shell action of the commutator between the generating function and the relevant creation operator on the previous Bethe vector. This was been done by a straightforward calculation and the result is presented in Appendix D, equation (D.1), and is valid for an arbitrary natural number M.

The work we have presented here is based on the non-unitary classical r-matrix (A.5) which is given in the so-called homogeneous gradation, as opposed to our previous papers [37,38] where we have used the r-matrix in the principal gradation. As we have shown, the novel feature in this approach is the certain freedom we have found in the local realization of the generators of the generalized trigonometric $s\ell(2)$ Gaudin algebra as well as in the creation operators we have used in the algebraic Bethe ansatz. It is precisely this structure that supports the interplay with the corresponding Knizhnik-Zamolodchikov equations. As a result we have obtained not only the solutions to the Knizhnik-Zamolodchikov equations, each of which corresponding to every Bethe state we have constructed, but also some neat formulas for the on-shell norms and off-self scalar products of these Bethe states, in some sense, analogous to the ones we have obtained previously in the rational case [36].

CRediT authorship contribution statement

I. Salom: Formal analysis, Methodology, Software, Writing – review & editing. **N. Manojlović:** Conceptualization, Formal analysis, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Classical r-matrix

The study presented in this paper is based on the following classical r-matrix [20,49,50]

$$r(\lambda) = -\frac{\coth(\lambda)}{2}\,\sigma^3 \otimes \sigma^3 - \frac{1}{4\sinh(\lambda)}\left(e^{-\lambda}\,\sigma^+ \otimes \sigma^- + e^{\lambda}\,\sigma^- \otimes \sigma^+\right) , \tag{A.1}$$

where σ^α, with $\alpha = +, -, 3$ are the Pauli matrices

$$\sigma^\alpha = \begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix} .$$

This unitarity classical r-matrix is a solution to the (proper) classical Yang-Baxter equation and it has the following symmetry [4].

$$[\sigma_1^3 + \sigma_2^3, r_{12}(\lambda)] = 0 \,. \tag{A.2}$$

In this case, the relevant K-matrix is given by [20,49,50]

$$K(\lambda) = \begin{pmatrix} \xi\, e^{-2\lambda} + \nu & 2\psi \sinh(2\lambda) \\ 2\phi \sinh(2\lambda) & \xi\, e^{2\lambda} + \nu \end{pmatrix}, \tag{A.3}$$

where we use the standard parametrization of [51]. This K-matrix and the r-matrix (A.1) satisfy the classical reflection equation [35,52,53]

$$\begin{aligned} & r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = \\ & = K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu). \end{aligned} \tag{A.4}$$

Thus, the corresponding non-unitary r-matrix defined by [35]

$$r_{12}^K(\lambda,\mu) = r_{12}(\lambda-\mu) - K_2(\mu)r_{12}(\lambda+\mu)K_2^{-1}(\mu) \,, \tag{A.5}$$

satisfies the generalized classical Yang-Baxter equation [31–33,35,54–56]

$$[r_{32}^K(\nu,\mu), r_{13}^K(\lambda,\nu)] + [r_{12}^K(\lambda,\mu), r_{13}^K(\lambda,\nu)] + [r_{12}^K(\lambda,\mu), r_{23}^K(\mu,\nu)] = 0. \tag{A.6}$$

Moreover, it is straightforward to check the following useful identities

$$K(\lambda)K(-\lambda) = \det(K(\lambda))\,\mathbb{1}, \tag{A.7}$$

$$K(-\lambda) = \mathrm{tr}\,(K(\lambda))\,\mathbb{1} - K(\lambda). \tag{A.8}$$

When the parameters are chosen so that $\xi^2 + 4\psi\varphi \neq 0$, the K-matrix (A.3) admits two distinct eigenvalues

$$\epsilon_\pm(\lambda) = \xi \cosh(2\lambda) \pm \sqrt{\xi^2 + 4\psi\phi}\, \sinh(2\lambda) + \nu \,. \tag{A.9}$$

In this case, for $\phi \neq 0$, there exists the matrix

$$\mathcal{U} = \begin{pmatrix} \xi + \sqrt{\xi^2 + 4\psi\phi} & \xi - \sqrt{\xi^2 + 4\psi\phi} \\ -2\phi & -2\phi \end{pmatrix} \tag{A.10}$$

such that

$$K(\lambda) = \mathcal{U}\, D(\lambda)\, \mathcal{U}^{-1}, \quad \text{with} \quad D(\lambda) = \begin{pmatrix} \epsilon_-(\lambda) & 0 \\ 0 & \epsilon_+(\lambda) \end{pmatrix}. \tag{A.11}$$

Under the conditions above there exists the matrix

$$\mathcal{M} = \begin{pmatrix} \xi + \sqrt{\xi^2 + 4\psi\phi} & -2\phi \\ -2\phi & \xi + \sqrt{\xi^2 + 4\psi\phi} \end{pmatrix} \tag{A.12}$$

such that

$$\mathcal{M}^{-1}\, K(\lambda)\, \mathcal{M} = \begin{pmatrix} \epsilon_-(\lambda) & 2\,(\psi+\phi)\sinh(2\lambda) \\ 0 & \epsilon_+(\lambda) \end{pmatrix}. \tag{A.13}$$

An analogous diagonalization can be obtained for $\psi \neq 0$. However, when $\xi^2 + 4\psi\varphi = 0$, the matrix (A.3) cannot be diagonalized. In this case, its Jordan form is given by

$$K(\lambda) = \mathcal{V}(\lambda)\, J(\lambda)\, \mathcal{V}^{-1}(\lambda), \tag{A.14}$$

where

$$\mathcal{V}(\lambda) = \begin{pmatrix} -\xi \sinh(2\lambda) & 1 \\ 2\phi \sinh(2\lambda) & 0 \end{pmatrix}, \quad \text{and} \quad J(\lambda) = \begin{pmatrix} \xi \cosh(2\lambda) + \nu & 1 \\ 0 & \xi \cosh(2\lambda) + \nu \end{pmatrix}. \tag{A.15}$$

Appendix B. Hilbert space

It is well known that the Hilbert space of the inhomogeneous trigonometric $s\ell(2)$ Gaudin model with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ together with the corresponding inhomogeneous parameter α_m, is given by [20,37–39]

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{B.1}$$

The local spin operators S_m^α, here $\alpha = +, -, 3$, satisfy the usual commutation relations

$$[S_m^3, S_n^\pm] = \pm S_m^\pm\, \delta_{mn}, \quad [S_m^+, S_n^-] = 2S_m^3\, \delta_{mn}. \tag{B.2}$$

Moreover, for every $m \in \{1, \ldots, N\}$

$$\exists\, \omega_m \in V_m \, : \, S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \tag{B.3}$$

Then the vector Ω_+ is defined to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{B.4}$$

Appendix C. Linear bracket

The Lax operator

$$L_0(\lambda) = \sum_{m=1}^{N} \Big(\coth(\lambda - \alpha_m)\, \sigma_0^3 \otimes S_m^3 + \frac{1}{2\sinh(\lambda - \alpha_m)} \left(e^{-\lambda + \alpha_m} \sigma_0^+ \otimes S_m^- + e^{\lambda - \alpha_m} \sigma_0^- \otimes S_m^+ \right) \Big), \tag{C.1}$$

and the r-matrix (A.1) satisfy the so-called Sklyanin linear bracket [5,10–12]

$$[L_0(\lambda), L_{0'}(\mu)] = [r_{00'}(\lambda - \mu), L_0(\lambda) + L_{0'}(\mu)]. \tag{C.2}$$

Consequently, the entries of the Lax operator (C.1) generate the $s\ell(2)$ trigonometric Gaudin algebra.

In the study of the generalized trigonometric $s\ell(2)$ Gaudin algebra the relevant the Lax operator is given by

$$\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K_0^{-1}(\lambda), \tag{C.3}$$

where $L_0(\lambda)$ is the Lax operator (C.1) and $K_0(\lambda)$ the reflection K-matrix is defined in (A.3). The Lax operator (C.3) obeys the following linear bracket [31–33,35,54–56]

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]. \tag{C.4}$$

This linear bracket is obviously anti-symmetric and it obeys the Jacobi identity because the r-matrix (A.5) satisfies the classical Yang-Baxter equation (A.6).

The generating function of the Gaudin Hamiltonians with boundary terms is given by

$$\tau(\lambda) = \mathrm{tr}_0\left(\mathcal{L}_0^2(\lambda)\right), \tag{C.5}$$

and it generates an Abelian subalgebra since it commutes for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0\,. \tag{C.6}$$

It this formulation the trigonometric Gaudin Hamiltonians with boundary terms have slightly different form in terms of local operators

$$\begin{aligned}
H_m &= \frac{(\pm 1)}{4} \operatorname*{Res}_{\lambda=\pm\alpha_m} \tau(\lambda) \\
&= \sum_{n\neq m}^N \left(\coth(\alpha_m-\alpha_n)\, S_m^3\cdot S_n^3 + \frac{e^{\alpha_m-\alpha_n}\, S_m^-\cdot S_n^+ + e^{-(\alpha_m-\alpha_n)}\, S_m^+\cdot S_n^-}{2\sinh(\alpha_m-\alpha_n)}\right) \\
&+ \sum_{n=1}^N \coth(\alpha_m+\alpha_n)\left(\frac{S_m^3\cdot S_n^3 + S_n^3\cdot S_m^3}{2} - \frac{2\psi\sinh(2\alpha_m)}{\xi\, e^{2\alpha_m}+\nu}\,\frac{S_m^+\cdot S_n^3 + S_n^3\cdot S_m^+}{2}\right) \\
&+ \sum_{n=1}^N \frac{e^{-(\alpha_m+\alpha_n)}}{\sinh(\alpha_m+\alpha_n)}\left(\frac{2\psi\sinh(2\alpha_m)}{\xi\, e^{-2\alpha_m}+\nu}\,\frac{S_m^3\cdot S_n^+ + S_n^+\cdot S_m^3}{2}\right. \\
&\left. + \frac{\xi\, e^{2\alpha_m}+\nu}{\xi\, e^{-2\alpha_m}+\nu}\,\frac{S_m^-\cdot S_n^+ + S_n^+\cdot S_m^-}{4}\right) \\
&+ \sum_{n=1}^N \frac{e^{-(\alpha_m+\alpha_n)}}{\sinh(\alpha_m+\alpha_n)}\left(-\frac{2\psi^2\sinh^2(2\alpha_m)}{(\xi\, e^{2\alpha_m}+\nu)(\xi\, e^{-2\alpha_m}+\nu)}\,\frac{S_m^+\cdot S_n^+ + S_n^+\cdot S_m^+}{2}\right) \\
&+ \sum_{n=1}^N \frac{e^{\alpha_m+\alpha_n}}{\sinh(\alpha_m+\alpha_n)}\left(\frac{\xi\, e^{-2\alpha_m}+\nu}{\xi\, e^{2\alpha_m}+\nu}\,\frac{S_m^+\cdot S_n^- + S_n^-\cdot S_m^+}{4}\right).
\end{aligned} \tag{C.7}$$

Appendix D. Crucial formula

Obtained by direct calculation, the action of the commutator between $\tau(\lambda)$ and the creation operator on the Bethe vector $\varphi_{M-1}(\mu_2,\ldots,\mu_M)$ represents the crucial step in the proof of the off-shell action of the generating function

$$
\begin{aligned}
&[\tau(\lambda), C_1(\mu_1)]\,\varphi_{M-1}(\mu_2,\ldots,\mu_M) = -\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)} \times \\
&\times \left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \sum_{j=2}^{M}\frac{2}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\right) \\
&\times \varphi_M(\mu_1,\mu_2,\ldots,\mu_M) \\
&+\frac{\mathcal{A}(\mu_1)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \left(2\rho(\mu_1) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)} - \sum_{j=2}^{M}\frac{2}{\sinh(\mu_1+\mu_j)\sinh(\mu_1-\mu_j)}\right) \\
&\times \varphi_M(\lambda,\mu_2,\ldots,\mu_M) \\
&+\sum_{j=2}^{M}\frac{\mathcal{A}(\mu_j)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \frac{2}{\sinh(\mu_1+\mu_j)\sinh(\mu_1-\mu_j)}\,\varphi_M(\lambda,\mu_1,\mu_2,\ldots,\widehat{\mu}_j,\ldots,\mu_M) \\
&+\sum_{j=2}^{M}\frac{8\psi\,\mathcal{A}(\mu_j)\,\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\sinh(\mu_1+\mu_j)\sinh(\mu_1-\mu_j)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)} \times \\
&\times \left(2\rho(\lambda) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} - \sum_{k\neq 1,j}^{M}\frac{2}{\sinh(\lambda+\mu_k)\sinh(\lambda-\mu_k)}\right) \\
&\times \varphi_{M-1}(\mu_1,\mu_2,\ldots,\widehat{\mu}_j,\ldots,\mu_M) \\
&+\sum_{j=2}^{M}\frac{4\psi\,\mathcal{A}(\mu_j)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \sum_{k\neq 1,j}^{M}\frac{2\,\mathcal{A}(\mu_k)\,\sinh(\mu_1+\mu_k)\sinh(\mu_1-\mu_k)}{\sinh(\mu_j+\mu_1)\sinh(\mu_j-\mu_1)\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)} \\
&\times \varphi_{M-1}(\lambda,\mu_1,\mu_2,\ldots,\widehat{\mu}_j,\ldots,\widehat{\mu}_k,\ldots,\mu_M) \\
&-\sum_{j=2}^{M}\frac{4\psi\,\mathcal{A}(\mu_1)\,\mathcal{A}(\mu_j)}{\mathcal{A}(\lambda)}\,\frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_1)\sinh(\lambda-\mu_1)}\,\frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times \\
&\times \left(2\rho(\mu_1) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_1)} - \sum_{k\neq 1,j}^{M}\frac{2}{\sinh(\mu_1+\mu_k)\sinh(\mu_1-\mu_k)}\right) \\
&\times \varphi_{M-1}(\lambda,\mu_2,\ldots,\widehat{\mu}_j,\ldots,\mu_M)
\end{aligned}
$$

$$+\sum_{j=2}^{M} \frac{8\psi\, \mathcal{A}(\mu_j)\ \sinh^2(2\lambda)}{\sinh(\lambda+\mu_j)\sinh(\lambda-\mu_j)} \frac{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)}{\xi^2+\nu^2+2\xi\nu\cosh(2\lambda)} \times$$
$$\times\left(2\rho(\mu_j) - \frac{4\xi\nu}{\xi^2+\nu^2+2\xi\nu\cosh(2\mu_j)} - \sum_{k\neq 1,j}^{M} \frac{2}{\sinh(\mu_j+\mu_k)\sinh(\mu_j-\mu_k)}\right)$$
$$\times\varphi_{M-1}(\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M)\,. \tag{D.1}$$

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