Appendix II - further properties of the elasticities

The H:D elasticities always add up to zero

Caswell (2001), based on Euler's theorem of homogeneous functions (Taylor, 1955), demonstrated that the elasticities of λ add up to 1 as λ is a homogeneous function of degree 1. Like the dominant eigenvalue (λ), the dominant eigenvector is also a homogeneous function of degree 1 (Equation 1a) but the H:D is a homogeneous function of degree 0 (Equation 1b).

$$\lambda.\begin{bmatrix} c.w_1 \\ c.w_2 \\ \ldots \\ c.w_n \end{bmatrix} = c^k.\lambda.\begin{bmatrix} w_1 \\ w_2 \\ \ldots \\ w_n \end{bmatrix} \quad \text{and} \quad k=1 \qquad \text{(eqn.1a)}$$

$$\lambda\left(\frac{c.w_2 + c.w_3 + c.w_4}{c.w_6 + c.w_7 + c.w_8}\right) = c^k.\lambda.\left(\frac{w_2 + w_3 + w_4}{w_6 + w_7 + w_8}\right) \quad \text{and} \quad k=0 \qquad \text{(eqn.1b)}$$

Euler's theorem on homogeneous functions states that if $f(x_1, \ldots, x_n)$ is a homogeneous function of degree k, then:

$$x_1 \frac{\partial f}{\partial x_1} + \ldots + x_n \frac{\partial f}{\partial x_n} = k.f(x_1,\ldots,x_n) \Leftrightarrow \frac{x_1}{f}\frac{\partial f}{\partial x_1} + \ldots + \frac{x_n}{f}\frac{\partial f}{\partial x_n} = k \Leftrightarrow \sum_n \frac{\partial f}{\partial x_n}\frac{x_n}{f} = k \qquad \text{(eqn.2)}$$

Substituting f by the H:D, x_n by the demographic model parameters and k=0, then:

$$\sum_n \frac{\partial H:D}{\partial p_n}\frac{p_n}{H:D} = 0 \qquad \text{(eqn.3)}$$

where each parcel of the sum is a H:D elasticity to model parameter p_n.

Dynamics of the implicit terms

The implicit terms relative to N_i and D_i of the H:D elasticites to parameter p can never reverse signs. This deduction is only presented below for the gametophyte terms of the equations as the deduction for the tetrasporophyte terms is similar. In order for N_1 and D_2 to reverse signs it would be necessary that:

$$2\lambda < St_2 + St_3 - G_1 \qquad \text{(eqn.4)}$$

But for any demographic matrix:

$$\lambda \geq \max(St_i ; St_i + CG_i) \qquad \text{(eqn.5)}$$

Then:

$$2\lambda \geq 2.\max(St_i ; St_i + CG_i) \geq 2.\max(St_i) \geq St_2 + St_3 \geq St_2 + St_3 - G_1 \qquad \text{(eqn.6)}$$

which allows to conclude that the equations of the implicit terms relative to N_1 and D_2 will never change sign. In the case of the N_3 implicit term, the sign can change if:

$$\begin{aligned} &2\lambda < St_2 + St_3 && \text{or} \\ &\lambda < St_4 + CG_4 && \text{or} \\ &\lambda < St_5 && \text{or} \\ &\lambda < St_6 && \text{or} \end{aligned} \qquad \text{(eqn.7)}$$

which, as just shown above, have empty space solutions.

Although the N_i and D_i implicit terms cannot reverse sign this does not apply to their sum, i.e. the Q_i implicit terms, as it is positive when the elasticity of N_i is bigger than the elasticity of D_i, and negative otherwise. The Q_1 quotient, after equaling the modules of N_1 and D_1, rearranging and cutting terms, can be expressed as:

$$\frac{1}{Q_1} . \frac{2\lambda + G_1 - St_2 - St_3}{2\lambda + G_4 - St_5 - St_6} \qquad \text{(eqn.8)}$$

If this expression equals 1, changes in λ do not affect Q_1. If it is bigger than 1, the N_1 term is bigger and changes in λ affect Q_1 by favoring the gametophyte growth in the numerator. If it is smaller than 1, the D_1 term is bigger and changes in λ affect Q_1 by favoring the tetrasporophyte growth in the denominator. This expression is the product of two components: the inverse of Q_1 and something that quite resembles to Q_1. The first is obviously always indirectly proportional to Q_1 and the second is usually directly proportional. If a component is bigger than 1 the other is usually smaller, but if it equals 1 the other will usually do the same. Thus, the two components tend to compensate each other, even though not exactly. The prevailing one will determine the direction of the implicit elasticity of Q_1 to parameter p. The Q_2 quotient, after equaling the modules of N_2 and D_2, rearranging and cutting terms, can be expressed as:

$$\frac{1}{Q_2^2}\cdot\frac{S_{tet}\left[Fec_4\left(2\lambda - St_5 - St_6\right) + Fec_5.G_4\right]}{S_{carp}\left[Fec_1\left(2\lambda - St_2 - St_3\right) + Fec_2.G_1\right]} \qquad \text{(eqn.9)}$$

Similarly to equation 8 this expression evaluates if λ is favoring tetrasporophyte or gametophyte fertility by being bigger, smaller or equal to 1. It is also the product of two components: the square of the inverse of Q_2 and something that quite resembles to Q_2. Hence the dynamics of the Q_2 implicit terms are similar to the dynamics of the implicit elasticities of Q_1. The Q_3 quotient, after equaling the modules of N_3 and D_3, rearranging and cutting terms, can be expressed as:

$$\frac{1}{Q_3^2}\cdot\frac{\left(2\lambda - St_5 - St_6\right)\left(\lambda - St_4 - CG_4\right) + \left(\lambda - St_5\right)\left(\lambda - St_6\right) - B_{56}.G_5 - G_4\left(B_{45} + CG_5\right)}{\left(2\lambda - St_2 - St_3\right)\left(\lambda - St_1 - CG_1\right) + \left(\lambda - St_2\right)\left(\lambda - St_3\right) - B_{23}.G_2 - G_1\left(B_{12} + CG_2\right)} \qquad \text{(eqn.10)}$$

Similarly to equations 8 and 9 this expression evaluates if λ is favoring tetrasporophyte or gametophyte looping paths by being bigger, smaller or equal to 1. Likewise, it is the product of two components: the square of the inverse of Q_3 and something that quite resembles to Q_3. Hence the dynamics of the implicit elasticities of Q_3 are similar to the dynamics of both the implicit elasticities of Q_1 and Q_2.