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Rational $so(3)$ Gaudin model with general boundary terms

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Abstract

We study the $so(3)$ Gaudin model with general boundary K-matrix in the framework of the algebraic Bethe ansatz. The off-shell action of the generating function of the $so(3)$ Gaudin Hamiltonians is determined. The proof based on the mathematical induction is presented on the algebraic level without any restriction whatsoever on the boundary parameters. The $so(3)$ Gaudin Hamiltonians with general boundary terms are given explicitly as well as their off-shell action on the Bethe states. The correspondence between the Bethe states and the solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations is established. In this context, the on-shell norm of the Bethe states is determined as well as their off-shell scalar product.

1. Introduction

The systems obtained as the quasi-classical limit of the Heisenberg spin chains [1] were first studied by Gaudin [2–4]. In the framework of the coordinate as well as the algebraic Bethe ansatz Gaudin has found the spectrum of the generating function of the corresponding Hamiltonians [2–4]. This system has been recasted in the framework of the quantum inverse scattering method [5–7] by exploring the so-called Sklyanin linear bracket using an $s\ell(2)$ invariant, unitary classical

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r-matrix [8]. This result enabled further generalisations based on other unitary solutions to the classical Yang-Baxter equation [9,10], prompting the interest in the Gaudin systems based on higher-rank simple Lie algebras [11–13] as well as Lie superalgebras [14–18]. The relation with the Knizhnik-Zamolodchikov equations of conformal field theory in two dimension [19] and the representation theory of the Kac-Moody algebras [20] was further strengthened when the connection between the Bethe states of the Gaudin model and the solution to the Knizhnik-Zamolodchikov equations was established [21,22,15–17,23]. It is also interesting to note that, in a somewhat more physical approach, the long-range interaction of these systems was studied in [24,25]. The Gaudin system on an elliptic curve was studied in [26], while the $s\ell(2)$ Gaudin with the Jordanian twist was studied in [27–29]. On the classical level, the Gaudin model corresponds to the so-called Schlesinger system in the theory of isomonodromic deformation [30–36].

In our considerations of the quantum Heisenberg spin chains with non-periodic boundary conditions we follow Sklyanin's approach where the boundary conditions are expressed in the form of the left and right reflection matrices [37]. The so-called reflection equation and the dual reflection equation represent the compatibility conditions between the bulk and the boundary of the system at the left and, respectively, right site of the system. The commutativity of the transfer matrix, in this case, is guaranteed on the one hand by the fact that the matrix form of the exchange relations between the entries of the Sklyanin monodromy matrix is analogous to the reflection equation and on the other hand, by the dual reflection equation [37–39].

Renewed interest has emerged in the implementation of algebraic Bethe ansatz on solvable Heisenberg chains with non-periodic boundary conditions [40–48]. As for alternative approaches, a review of the coordinate Bethe ansatz in this case is given in [49], the Bethe ansatz based on the functional relation between the eigenvalues of the transfer matrix and the quantum determinant, as well as the associated T-Q relation are studied in [50–52], functional relations for the eigenvalues of the transfer matrix based on fusion hierarchy were discussed in [53] and the Vertex-IRF correspondence in [54,55], while the Jordanian deformation of the open XXX chain is analysed in [56]. For the latest results, as well as an excellent review on the application of the separation of variables method on the 6-vertex model and the associate XXZ quantum chains see [57].

Our interest in open Heisenberg spin chains was twofold. On the one hand, we were interested in the implementation of the algebraic Bethe ansatz, and on the other hand, we wanted to consider the quasi-classical limit which yields the corresponding Gaudin model [58,59]. As it is well known [41,42,58], due to the symmetry of the R-matrix of the non-periodic XXX spin chain, the accomplishment of the algebraic Bethe ansatz does not imply any restriction on the boundary parameters. However, in the case of the open XXZ chain [60], the existence of the so-called vacuum vector requires the triangular form of the boundary K-matrix [43,44,59]. As for the quasi-classical limit, Hikami showed, in complete analogy with the periodic case [24,25], how the expansion of the XXZ transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the case when both reflection matrices are diagonal [61]. Similar expansion was done for the Jordanian deformation of the ration $s\ell(2)$ Gaudin model with generic boundaries [62]. The algebraic Bethe ansatz was applied to open Gaudin model in the context of the Vertex-IRF correspondence [63–65]. Also, results were obtained for the open Gaudin models based on Lie superalgebras [66]. Returning back to the quasi-classical limit, following the Sklyanin proposal for the periodic boundary conditions [8,67], we have derived the generating function of the Gaudin Hamiltonians both for the XXX [58] and the XXZ chain [59] as well as for the Jordanian deformation of the XXX Heisenberg spin chain [68]. Moreover, we have shown [69] how, in the context of the quasi-classical limit, the solutions to the classical Yang-Baxter equation [9,10] can be combined with the solutions to the classical reflection equation [70,71] to yield solutions to the so-called generalized classical Yang-Baxter equation [72–75]. These solutions are the non-unitary classical r-matrices [76–82]. In particular, the generic elliptic $s\ell(2)$ non-unitary r-matrix was studied in [83]. Also, we draw attention to the recent study of the generalized Gaudin and Richardson models based on a class of non-unitary r-matrices [84].

An approach to the implementation of the algebraic Bethe ansatz for the rational as well as the trigonometric $s\ell(2)$ Gaudin model based on the corresponding Maillet linear bracket was developed in [85–89]. Once a suitable set of generators of the relevant generalized Gaudin algebra is found, the local realization of these generators becomes compact and it naturally leads to the definition of the so-called creation operators. In both the rational and trigonometric case [86,89], these creation operators define Bethe states in such a way that the off-shell action of the generating function of the Gaudin Hamiltonians can be computed explicitly, and a completely algebraic proof of this action given.

This paper is centred on the application of the algebraic Bethe ansatz to the rational $so(3)$ Gaudin model with generic classical boundary K-matrix. We recall that this K-matrix can be obtained by the so-called fusion procedure [6,90,91], starting from the $s\ell(2)$ K-matrix [58,92–94]. The outline of this method in the trigonometric $so(3)$ case was given in [95]. Alternatively, one can use the so-called scaling limit [96], to obtain the K-matrix from the trigonometric $so(3)$ boundary K-matrix [95,97]. The non-unitary $so(3)$ classical r-matrix (B.8) is then obtained by combining the unitary $so(3)$ invariant classical r-matrix and the K-matrix. This non-unitary classical r-matrix defines the $so(3)$ Maillet linear bracket, for the suitable Lax operator, and provides an algebraic framework for our study of the non-periodic $so(3)$ Gaudin model. As an immediate consequence of the definition of the $so(3)$ Maillet bracket follows the mutual commutativity of the generating function for different values of the spectral parameter. However, as it will be confirmed in the following, the natural set of generators unfortunately turns out not to be adequate for the implementation of the algebraic Bethe ansatz. Thus we will here propose a new set of generators. Besides the relative simplicity of the local realization of the new generators, their most striking feature will be the compact form of their commutation relations. This is of great significance since it efficiently enables the algebraic proof of the off-shell action of the generating function on the Bethe states. Furthermore, it is important to stress that these results will be obtained without any restriction whatsoever on the boundary parameters. It is only when solving the generalized $so(3)$ Knizhnik-Zamolodchikov equations that the key identity in the proof will require one of four boundary parameters to be set to zero. However, in spite of this constraint we will retain a large improvement in generality over the previous studies: while the formulas that we here provide for the solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations, the on-shell norm of the Bethe vectors and the off-shell scalar product of the Bethe vectors do superficially look similar to the analogous formulae in the $s\ell(2)$ case [86], only one of the boundary parameters will be fixed here, instead of all four of them (as, for example, in [86]).

The paper is organised as follows. In Section 2 we study the $so(3)$ Maillet linear bracket which provides the algebraic framework for implementation of the Bethe ansatz. In the same section we propose the novel set of generators with simplified commutation relations and introduce Gaudin Hamiltonians. The implementation of the algebraic Bethe ansatz is the principal topic of the Section 3. There we will obtain the expression for the off-shell action of the generating function $\tau(\lambda)$, as well as for the off-shell action of the $so(3)$ Gaudin Hamiltonians with general boundary terms – and prove these formulas by mathematical induction. The solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations will be given in the Section 4. Our results will be summarised in the concluding Section 5. Fundamental definitions regarding the $so(3)$ Lie algebra, including the two $so(3)$ invariant operators in $\mathbb{C}^3 \otimes \mathbb{C}^3$ which generate the relevant Brauer algebra, are presented in the Appendix A. Finally, the cornerstone of our study – the non-unitary $so(3)$ classical r-matrix – is given in the Appendix B.

2. The $so(3)$ Maillet linear bracket

In this section we show how the non-unitary $so(3)$ classical r-matrix (B.8) helps define the $so(3)$ Maillet linear bracket (8) for the suitable Lax operator (7). Although this Maillet bracket provides an appropriate algebraic framework for studying the quantum $so(3)$ Gaudin model, yielding the generating function of the $so(3)$ Gaudin Hamiltonians with general boundary terms, it will be shown below that the natural set of generators unfortunately does not provide the most efficient way for implementing the algebraic Bethe ansatz in this case. Thus we will propose a new set off generators of the corresponding generalized $so(3)$ Gaudin algebra.

In our study we use the Lax operator

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\vec{\mathcal{S}}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} = \sum_{m=1}^{N} \frac{1}{\lambda - \alpha_m} \left(\mathcal{S}_0^3 \otimes S_m^3 + \frac{1}{2} \left(\mathcal{S}_0^+ \otimes S_m^- + \mathcal{S}_0^- \otimes S_m^+ \right) \right) , \tag{1}$$

where the spin operators S_m^{α}, with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$, are introduced in (A.12) and the matrices $\mathcal{S}_0^3$ and $\mathcal{S}_0^{\pm}$ in the auxiliary space $\mathbb{C}^3$ are specified by (A.1) and (A.3), respectively. The Lax operator (1) can also be represented in the following form

$$L_0(\lambda) = \vec{\mathcal{S}}_0 \cdot \vec{S}(\lambda) , \tag{2}$$

where the generators of the $so(3)$ Gaudin algebra are defined by [2–4]

$$S^3(\lambda) = \sum_{m=1}^{N} \frac{S_m^3}{\lambda - \alpha_m} , \qquad S^{\pm}(\lambda) = \sum_{m=1}^{N} \frac{S_m^{\pm}}{\lambda - \alpha_m} . \tag{3}$$

The so-called Sklyanin linear bracket [8,15,16] for the Lax operator (1) and the r-matrix (B.1)

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)] \tag{4}$$

yields nontrivial commutation relations for the generators (3)

$$\begin{aligned}
\left[S^3(\lambda), S^{\pm}(\mu) \right] &= \mp \frac{S^{\pm}(\lambda) - S^{\pm}(\mu)}{\lambda - \mu} , \\
\left[S^+(\lambda), S^-(\mu) \right] &= (-2) \frac{S^3(\lambda) - S^3(\mu)}{\lambda - \mu} .
\end{aligned} \tag{5}$$

The Lax operator corresponding to the generalized $so(3)$ Gaudin algebra is given by

$$\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K_0^{-1}(\lambda) , \tag{6}$$

where $L_0(\lambda)$ is the Lax operator (1) and $K_0(\lambda)$ is the reflection K-matrix defined in (B.4). This form of the Lax operator can be obtained by following a relatively general procedure of quasi-classical expansion of the Sklyanin monodromy [69]. By direct substitution we obtain

$$\mathcal{L}_0(\lambda) = \vec{\mathbb{S}}_0 \cdot \vec{S}(\lambda) - \left(K_0(\lambda)\vec{\mathbb{S}}_0 K_0^{-1}(\lambda)\right) \cdot \vec{S}(-\lambda)$$
$$= \begin{pmatrix} H(\lambda) & \frac{1}{\sqrt{2}}F(\lambda) & 0 \\ \frac{1}{\sqrt{2}}E(\lambda) & 0 & \frac{1}{\sqrt{2}}F(\lambda) \\ 0 & \frac{1}{\sqrt{2}}E(\lambda) & -H(\lambda) \end{pmatrix}. \tag{7}$$

The Lax operator (7) obeys the following $so(3)$ Maillet linear bracket [73–75,38,69]

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]. \tag{8}$$

This linear bracket is obviously anti-symmetric and it obeys the Jacobi identity because the r-matrix (B.6) satisfies the generalized classical Yang-Baxter equation (B.7).

The $so(3)$ Maillet bracket (8) implies the following commutation relations for the generators $E(\lambda)$, $F(\lambda)$ and $H(\lambda)$ (7)

$$[E(\lambda), E(\mu)] = \frac{-2\varphi^2}{\lambda+\mu}\left(\frac{\mu^2}{\xi^2-(\psi\varphi+\nu^2)\mu^2}H(\lambda) - \frac{\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}H(\mu)\right),$$
$$+\frac{2\varphi}{\lambda+\mu}\left(\frac{(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2}E(\lambda) - \frac{(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda}E(\mu)\right), \tag{9}$$

$$[F(\lambda), F(\mu)] = \frac{2\psi^2}{\lambda+\mu}\left(\frac{\mu^2}{\xi^2-(\psi\varphi+\nu^2)\mu^2}H(\lambda) - \frac{\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}H(\mu)\right)$$
$$+\frac{2\psi}{\lambda+\mu}\left(\frac{(\xi-\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2}F(\lambda) - \frac{(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda}F(\mu)\right), \tag{10}$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\lambda+\mu}\left(\frac{(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2}E(\lambda) - \frac{(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}E(\mu)\right)$$
$$+\frac{-\varphi}{\lambda+\mu}\left(\frac{(\xi-\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2}F(\lambda) - \frac{(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}F(\mu)\right), \tag{11}$$

and

$$[H(\lambda), E(\mu)] = \frac{\varphi}{\lambda+\mu}\left(\frac{\varphi\,\mu^2}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2}F(\lambda) - \frac{2\,(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}H(\mu)\right)$$
$$-\frac{1}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu)-\psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2}E(\lambda)\right.$$
$$\left.-\frac{2\left(\xi^2-\left(\psi\varphi\,\mu+\nu^2\lambda\right)\lambda\right)\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}E(\mu)\right), \tag{12}$$

$$[H(\lambda), F(\mu)] = \frac{-\psi}{\lambda+\mu}\left(\frac{\psi\,\mu^2}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2}E(\lambda) + \frac{2\,(\xi+\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}H(\mu)\right)$$
$$+\frac{1}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\mu)\,(\xi+\nu\lambda)-\psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2}F(\lambda)\right.$$

$$- \frac{2\left(\xi^2 - \left(\psi\varphi\,\mu + \nu^2\lambda\right)\lambda\right)\lambda}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}\, F(\mu)\Big)\,, \tag{13}$$

$$\begin{aligned}[F(\lambda), E(\mu)] = {} & \frac{-2}{\lambda+\mu}\left(\frac{\varphi\,(\xi+\nu\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\,\mu^2}\, F(\lambda) - \frac{\psi\,(\xi-\nu\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}\, E(\mu)\right) \\ & + \frac{2}{(\lambda-\mu)(\lambda+\mu)}\left(\frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu) - \psi\varphi\,(\lambda+\mu)\mu)\,\mu}{\xi^2-(\psi\varphi+\nu^2)\mu^2} H(\lambda)\right. \\ & \left. - \frac{(2(\xi-\nu\lambda)\,(\xi+\nu\mu) - \psi\varphi\,(\lambda+\mu)\lambda)\,\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} H(\mu)\right). \end{aligned} \tag{14}$$

Moreover, the Maillet linear bracket (8) yields the expression for the generating function of the $so(3)$ Gaudin Hamiltonians with general boundary terms in terms of the Lax operator (6)

$$\tau(\lambda) = \frac{1}{2}\,\mathrm{tr}_0\left(\mathcal{L}_0^2(\lambda)\right)\,. \tag{15}$$

Namely, using the Maillet bracket (8), it is straightforward to check that the operator $\tau(\lambda)$ commutes for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0\,. \tag{16}$$

From (7) it follows that

$$\tau(\lambda) = H^2(\lambda) + \frac{1}{2}\left(E(\lambda)F(\lambda) + F(\lambda)E(\lambda)\right)\,. \tag{17}$$

Our aim here is to obtain the spectrum and the corresponding states of the generating function $\tau(\lambda)$ by algebraic methods. To this end we would have to use the relations (9) – (14). As it is evident from the formulae above, these relations do not seem to be suitable to efficiently address this problem. Therefore, we propose a new set of generators

$$\begin{aligned}\mathcal{E}(\lambda) = {} & \frac{1}{2\psi\sqrt{\psi\varphi+\nu^2}}\left(\psi^2 E(\lambda) - \left(\psi\varphi + 2\nu\left(\nu - \sqrt{\psi\varphi+\nu^2}\right)\right) F(\lambda)\right. \\ & \left. + 2\psi\left(\nu - \sqrt{\psi\varphi+\nu^2}\right) H(\lambda)\right), \end{aligned} \tag{18}$$

$$\begin{aligned}\mathcal{F}(\lambda) = {} & \frac{1}{2\psi\sqrt{\psi\varphi+\nu^2}}\left(-\psi^2 E(\lambda) + \left(\psi\varphi + 2\nu\left(\nu + \sqrt{\psi\varphi+\nu^2}\right)\right) F(\lambda)\right. \\ & \left. - 2\psi\left(\nu + \sqrt{\psi\varphi+\nu^2}\right) H(\lambda)\right), \end{aligned} \tag{19}$$

$$\mathcal{H}(\lambda) = \frac{1}{2\sqrt{\psi\varphi+\nu^2}}\left(\psi E(\lambda) + \varphi F(\lambda) + 2\nu H(\lambda)\right)\,. \tag{20}$$

The commutation relations we obtain for the new generators are substantially simpler than the initial relations (12)–(11). In particular,

$$[\mathcal{E}(\lambda), \mathcal{E}(\mu)] = [\mathcal{F}(\lambda), \mathcal{F}(\mu)] = [\mathcal{H}(\lambda), \mathcal{H}(\mu)] = 0\,, \tag{21}$$

and the three non-trivial relations are

$$[\mathcal{H}(\lambda), \mathcal{E}(\mu)] = \frac{-2}{\lambda^2-\mu^2}\left(\mu \, \frac{\xi - \lambda\sqrt{\psi\varphi+\nu^2}}{\xi - \mu\sqrt{\psi\varphi+\nu^2}}\, \mathcal{E}(\lambda) - \lambda\, \mathcal{E}(\mu)\right), \tag{22}$$

$$[\mathcal{H}(\lambda), \mathcal{F}(\mu)] = \frac{2}{\lambda^2-\mu^2}\left(\mu \, \frac{\xi + \lambda\sqrt{\psi\varphi+\nu^2}}{\xi + \mu\sqrt{\psi\varphi+\nu^2}}\, \mathcal{F}(\lambda) - \lambda\, \mathcal{F}(\mu)\right), \tag{23}$$

$$[\mathcal{F}(\lambda), \mathcal{E}(\mu)] = \frac{4}{\lambda^2-\mu^2}\left(\mu \, \frac{\xi - \lambda\sqrt{\psi\varphi+\nu^2}}{\xi - \mu\sqrt{\psi\varphi+\nu^2}}\, \mathcal{H}(\lambda) - \lambda\, \frac{\xi + \mu\sqrt{\psi\varphi+\nu^2}}{\xi + \lambda\sqrt{\psi\varphi+\nu^2}}\, \mathcal{H}(\mu)\right). \tag{24}$$

Furthermore, the local realization of the new generators is:

$$\mathcal{E}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi+\nu^2}} \sum_{m=1}^{N} \frac{\xi - \alpha_m\sqrt{\psi\varphi+\nu^2}}{\xi - \lambda\sqrt{\psi\varphi+\nu^2}} \times \frac{2(\nu - \sqrt{\psi\varphi+\nu^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\,(\nu - \sqrt{\psi\varphi+\nu^2})}{\psi} S_m^-}{(\lambda-\alpha_m)(\lambda+\alpha_m)}, \tag{25}$$

$$\mathcal{F}(\lambda) = \frac{-\lambda}{\sqrt{\psi\varphi+\nu^2}} \sum_{m=1}^{N} \frac{\xi + \alpha_m\sqrt{\psi\varphi+\nu^2}}{\xi + \lambda\sqrt{\psi\varphi+\nu^2}} \times \frac{2(\nu + \sqrt{\psi\varphi+\nu^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\,(\nu + \sqrt{\psi\varphi+\nu^2})}{\psi} S_m^-}{(\lambda-\alpha_m)(\lambda+\alpha_m)}, \tag{26}$$

$$\mathcal{H}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi+\nu^2}} \sum_{m=1}^{N} \frac{2\nu S_m^3 + \psi S_m^+ + \varphi S_m^-}{(\lambda-\alpha_m)(\lambda+\alpha_m)}. \tag{27}$$

A straightforward but somewhat lengthy calculation shows that the generating function $\tau(\lambda)$ (17) has exactly the same form when expressed in terms of the new generators

$$\tau(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{2}\left(\mathcal{E}(\lambda)\mathcal{F}(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda)\right). \tag{28}$$

The explicit expressions for the $so(3)$ Gaudin Hamiltonians with general boundary terms are derived by substituting the local realization of the new generators (25) – (27) in the right-hand-side of (28)

$$\begin{aligned} \mathrm{H}_m =&(\pm) \operatorname*{Res}_{\lambda=\pm\alpha_m} \tau(\lambda) = \frac{1}{\xi^2 - \left(\psi\varphi+\nu^2\right)\alpha_m^2} \\ &\times \left(\frac{\xi^2 + \left(\psi\varphi-\nu^2\right)\alpha_m^2}{\alpha_m} (S_m^3)^2 - \frac{\alpha_m}{2}\left(\psi^2\,(S_m^+)^2 + \varphi^2\,(S_m^-)^2 \right.\right. \\ &\left.\left. +2\psi\nu\left(S_m^+S_m^3 + S_m^3S_m^+\right) + 2\varphi\nu\left(S_m^-S_m^3 + S_m^3S_m^-\right)\right)\right. \end{aligned}$$

$$
+ \frac{\xi^2 + \nu^2\alpha_m^2}{2\alpha_m}\left(S_m^+ S_m^- + S_m^- S_m^+\right)\Big)
$$

$$
+ \frac{\alpha_m}{\xi^2 - \left(\psi\varphi + \nu^2\right)\alpha_m^2} \sum_{n\neq m}^{N} \Big(\frac{4\left(\xi^2 - \psi\varphi\,\alpha_m\alpha_n - \nu^2\alpha_m^2\right)}{\alpha_m^2 - \alpha_n^2}\, S_m^3 S_n^3
$$

$$
- \frac{\alpha_m}{\alpha_m + \alpha_n}\big(\psi^2\; S_m^+ S_n^+ + \varphi^2\; S_m^- S_n^-
$$

$$
+ 2\psi\nu\left(S_m^+ S_n^3 + S_m^3 S_n^+\right) + 2\varphi\nu\left(S_m^- S_n^3 + S_m^3 S_n^-\right)\big)
$$

$$
+ \frac{2\left(\xi^2 - \left(\psi\varphi + \nu^2\right)\alpha_m\alpha_n\right) - \psi\varphi\,\alpha_m\left(\alpha_m - \alpha_n\right)}{\alpha_m^2 - \alpha_n^2}\left(S_m^- S_n^+ + S_m^+ S_n^-\right)\Big)
$$

$$
+ \frac{\xi\cdot\alpha_m}{\xi^2 - \left(\psi\varphi + \nu^2\right)\alpha_m^2} \sum_{n\neq m}^{N} \frac{1}{\alpha_m + \alpha_n}\Big(2\psi\left(S_m^+ S_n^3 - S_m^3 S_n^+\right)
$$

$$
+ 2\nu\left(S_m^- S_n^+ - S_m^+ S_n^-\right) + 2\varphi\left(S_m^3 S_n^- - S_m^- S_n^3\right)\Big)\,. \tag{29}
$$

Besides the formula above for the $so(3)$ Gaudin Hamiltonians with general boundary terms, our main result in this section is the new form of generators of the generalized $so(3)$ Gaudin algebra (25) – (27). Due to their strikingly simple commutation relations (21) – (24) they now provide a suitable framework for applying the algebraic Bethe ansatz without any restrictions on boundary parameters.

3. Implementation of the algebraic Bethe ansatz

Before we can proceed to find Bethe vectors and determine the off-shell action of the generating function $\tau(\lambda)$, we have to establish several intermediary results.

In the Hilbert space $\mathcal{H}$ (A.11) of the system we have to define the so-called vacuum vector $\Omega_+ \in \mathcal{H}$ together with the appropriate action of the generators (25) – (27) on it. To this purpose we observe that in every local space $V_m = \mathbb{C}^3$, $m \in \{1, \ldots, N\}$ there exists a vector $\omega_m \in V_m$ given by

$$
\omega_m = \begin{pmatrix} \psi^2 \\ -\sqrt{2}\,\psi\left(\nu - \sqrt{\psi\varphi + \nu^2}\right) \\ \left(\nu - \sqrt{\psi\varphi + \nu^2}\right)^2 \end{pmatrix} \in \mathbb{C}^3 = V_m\,, \tag{30}
$$

where the parameters ν, ψ and φ are the parameters of the boundary K-matrix (B.4). Then it is easy to check that

$$
\left(2\left(\nu - \sqrt{\psi\varphi + \nu^2}\right) S_m^3 + \psi\; S_m^+ - \frac{\psi\varphi + 2\nu\left(\nu - \sqrt{\psi\varphi + \nu^2}\right)}{\psi}\, S_m^-\right)\omega_m = 0\,, \tag{31}
$$

$$
\left(2\nu S_m^3 + \psi S_m^+ + \varphi S_m^-\right)\omega_m = 2\sqrt{\psi\varphi + \nu^2}\;\omega_m\,. \tag{32}
$$

Therefore the vacuum vector Ω_+, defined as

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H} \tag{33}$$

has the desired properties. Namely, it is annihilated by the generator $\mathcal{E}(\lambda)$ (25) and, at the same time, it is an eigenvector of the generator $\mathcal{H}(\lambda)$ (27), that is

$$\mathcal{E}(\lambda)\,\Omega_+ = 0 \quad \text{and} \quad \mathcal{H}(\lambda)\,\Omega_+ = \rho(\lambda)\,\Omega_+ \quad \text{with} \quad \rho(\lambda) = \sum_{m=1}^{N} \frac{2\lambda}{\lambda^2 - \alpha_m^2}\,. \tag{34}$$

Our next aim is to rewrite the formula for $\tau(\lambda)$ (28) in a more suitable way so that the action of the generating function $\tau(\lambda)$ on the vacuum vector Ω_+ (33) becomes more transparent. With this aim, we first note that the commutation relations (22) – (24) imply

$$[\mathcal{H}(\lambda), \mathcal{F}(\lambda)] = \frac{-\xi}{\lambda\left(\xi + \lambda\sqrt{\psi\varphi + \nu^2}\right)}\,\mathcal{F}(\lambda) + \mathcal{F}'(\lambda)\,, \tag{35}$$

$$[\mathcal{H}(\lambda), \mathcal{E}(\lambda)] = \frac{\xi}{\lambda\left(\xi - \lambda\sqrt{\psi\varphi + \nu^2}\right)}\,\mathcal{E}(\lambda) - \mathcal{E}'(\lambda)\,, \tag{36}$$

$$[\mathcal{F}(\lambda), \mathcal{E}(\lambda)] = 2\left(\frac{-1}{\lambda}\,\frac{\xi^2 + (\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}\,\mathcal{H}(\lambda) + \mathcal{H}'(\lambda)\right), \tag{37}$$

where prime denotes derivative with respect to parameter. Therefore we can express the generating function $\tau(\lambda)$ (28) as follows

$$\tau(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{\lambda}\,\frac{\xi^2 + (\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}\,\mathcal{H}(\lambda) - \mathcal{H}'(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda)\,. \tag{38}$$

Taking into account (34) and (38), it is evident that the vacuum vector Ω_+ (33) is an eigenvector of the generating function

$$\tau(\lambda)\,\Omega_+ = \chi_0(\lambda)\,\Omega_+ \quad \text{with} \quad \chi_0(\lambda) = \rho^2(\lambda) + \frac{\xi^2 + (\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}\,\frac{\rho(\lambda)}{\lambda} - \rho'(\lambda)\,. \tag{39}$$

In our approach, one fo the essential steps in the implementation of algebraic Bethe ansatz is to find the commutation relation between the generating function $\tau(\lambda)$ (38) and the generator $\mathcal{F}(\mu)$ (19). To this end, we will also need the following auxiliary result which follows from (23)

$$\begin{aligned}\left[\mathcal{H}'(\lambda), \mathcal{F}(\mu)\right] = \frac{2}{\lambda^2 - \mu^2}\Bigg(&\frac{\xi\,(\lambda - \mu)}{(\lambda + \mu)\left(\xi + \mu\sqrt{\psi\varphi + \nu^2}\right)}\,\mathcal{F}(\lambda) \\ &+ \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}\,(\mathcal{F}(\mu) - \mathcal{F}(\lambda)) + \mu\,\frac{\xi + \lambda\sqrt{\psi\varphi + \nu^2}}{\xi + \mu\sqrt{\psi\varphi + \nu^2}}\,\mathcal{F}'(\lambda)\Bigg)\,.\end{aligned} \tag{40}$$

Now we can compute the commutator by a straightforward calculation, based on the formulae (38), (23), (24) and (40):

$$
\begin{aligned}
[\tau(\lambda), \mathcal{F}(\mu)] = & -\frac{4}{\lambda^2-\mu^2}\, \mathcal{F}(\mu) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} \right) \\
& + \frac{4}{\lambda^2-\mu^2}\, \frac{\lambda}{\mu}\, \frac{\xi-\mu\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\, \mathcal{F}(\lambda) \left(\mu \mathcal{H}(\mu) + \frac{(\psi\varphi+\nu^2)\mu^2}{\xi^2-(\psi\varphi+\nu^2)\mu^2} \right).
\end{aligned}
\tag{41}
$$

The relative simplicity of the right hand side of the equation above has encouraged us to seek the commutator between the operator $\tau(\lambda)$ and the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)$ as the next step. In this case, an analogous direct calculation based on the previous formulae, leads to

$$
\begin{aligned}
& [\tau(\lambda), \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)] = \\
& \quad -\frac{4}{\lambda^2-\mu_1^2}\, \mathcal{F}(\mu_1)\mathcal{F}(\mu_2) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} - \frac{\lambda^2}{\lambda^2-\mu_2^2} \right) \\
& \quad -\frac{4}{\lambda^2-\mu_2^2}\, \mathcal{F}(\mu_1)\mathcal{F}(\mu_2) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} - \frac{\lambda^2}{\lambda^2-\mu_1^2} \right) \\
& \quad + \frac{4}{\lambda^2-\mu_1^2}\, \frac{\lambda}{\mu_1}\, \frac{\xi-\mu_1\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\, \mathcal{F}(\lambda)\mathcal{F}(\mu_2) \\
& \quad \times \left(\mu_1 \mathcal{H}(\mu_1) + \frac{(\psi\varphi+\nu^2)\mu_1^2}{\xi^2-(\psi\varphi+\nu^2)\mu_1^2} - \frac{2\mu_1^2}{\mu_1^2-\mu_2^2} \right) \\
& \quad + \frac{4}{\lambda^2-\mu_2^2}\, \frac{\lambda}{\mu_2}\, \frac{\xi-\mu_2\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\, \mathcal{F}(\mu_1)\mathcal{F}(\lambda) \\
& \quad \times \left(\mu_2 \mathcal{H}(\mu_2) + \frac{(\psi\varphi+\nu^2)\mu_2^2}{\xi^2-(\psi\varphi+\nu^2)\mu_2^2} - \frac{2\mu_2^2}{\mu_2^2-\mu_1^2} \right).
\end{aligned}
\tag{42}
$$

Evidently, the right hand side of (42) has extra lines and every line has extra terms in comparison with (41). While certain pattern is already visible, we will explicitly compute one more step before conjecturing the general case. The commutation relation between the generating function $\tau(\lambda)$ and the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3)$ is obtained in a similar manner, using the previous results,

$$
\begin{aligned}
& [\tau(\lambda), \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3)] = \\
& -\frac{4}{\lambda^2-\mu_1^2}\, \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} - \frac{\lambda^2}{\lambda^2-\mu_2^2} - \frac{\lambda^2}{\lambda^2-\mu_3^2} \right) \\
& -\frac{4}{\lambda^2-\mu_2^2}\, \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} - \frac{\lambda^2}{\lambda^2-\mu_1^2} - \frac{\lambda^2}{\lambda^2-\mu_3^2} \right) \\
& -\frac{4}{\lambda^2-\mu_3^2}\, \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3) \left(\lambda \mathcal{H}(\lambda) + \frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2} - \frac{\lambda^2}{\lambda^2-\mu_1^2} - \frac{\lambda^2}{\lambda^2-\mu_2^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4}{\lambda^2-\mu_1^2}\,\frac{\lambda}{\mu_1}\,\frac{\xi-\mu_1\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3)\\
&\qquad\times\left(\mu_1\mathcal{H}(\mu_1)+\frac{(\psi\varphi+\nu^2)\mu_1^2}{\xi^2-(\psi\varphi+\nu^2)\mu_1^2}-\frac{2\mu_1^2}{\mu_1^2-\mu_2^2}-\frac{2\mu_1^2}{\mu_1^2-\mu_3^2}\right)\\
&+ \frac{4}{\lambda^2-\mu_2^2}\,\frac{\lambda}{\mu_2}\,\frac{\xi-\mu_2\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\mu_1)\mathcal{F}(\lambda)\mathcal{F}(\mu_3)\\
&\qquad\times\left(\mu_2\mathcal{H}(\mu_2)+\frac{(\psi\varphi+\nu^2)\mu_2^2}{\xi^2-(\psi\varphi+\nu^2)\mu_2^2}-\frac{2\mu_2^2}{\mu_2^2-\mu_1^2}-\frac{2\mu_2^2}{\mu_2^2-\mu_3^2}\right)\\
&+ \frac{4}{\lambda^2-\mu_3^2}\,\frac{\lambda}{\mu_3}\,\frac{\xi-\mu_3\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\lambda)\\
&\qquad\times\left(\mu_3\mathcal{H}(\mu_3)+\frac{(\psi\varphi+\nu^2)\mu_3^2}{\xi^2-(\psi\varphi+\nu^2)\mu_3^2}-\frac{2\mu_3^2}{\mu_3^2-\mu_1^2}-\frac{2\mu_3^2}{\mu_3^2-\mu_2^2}\right). \qquad (43)
\end{aligned}
$$

In the general case, we conjecture validity of the following relation:

$$
\begin{aligned}
&[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)] = -\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\times\\
&\times\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k\neq j}^{M}\frac{\lambda^2}{\lambda^2-\mu_k^2}\right)\\
&+\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\,\frac{\lambda}{\mu_j}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots\mathcal{F}(\mu_M)\\
&\times\left(\mu_j\mathcal{H}(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j^2}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j^2}{\mu_j^2-\mu_k^2}\right), \qquad (44)
\end{aligned}
$$

where the notation $\widehat{\mathcal{F}(\mu_j)}$ means that the operator $\mathcal{F}(\mu_j)$ is omitted.

The proof of the formula above is by mathematical induction. Our initial hypothesis is that the equation (44) is valid for some natural number M. Thus, we have to show that the analogous equation is valid for $M+1$. To this end, we write

$$
\begin{aligned}
&[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})] =\\
&\qquad = [\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)]\,\mathcal{F}(\mu_{M+1})\\
&\qquad + \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\left[\tau(\lambda),\mathcal{F}(\mu_{M+1})\right] \qquad (45)\\
&\qquad = \left[[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)],\mathcal{F}(\mu_{M+1})\right]
\end{aligned}
$$

$$+\mathcal{F}(\mu_{M+1})\left[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\right]$$
$$+\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\left[\tau(\lambda),\mathcal{F}(\mu_{M+1})\right]. \tag{46}$$

It follows from (44) that the first term on the right hand side of (46) yields two type of terms. In the second term of (46) we can just substitute the right hand side of (44). Finally, in the last term of (46) we use (41). In this way we obtain

$$\begin{aligned}
&\left[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\right]=\\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\left[\mathcal{H}(\lambda),\mathcal{F}(\mu_{M+1})\right]\\
&+\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\times\\
&\quad\times\mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots\mathcal{F}(\mu_M)\left[\mathcal{H}(\mu_j),\mathcal{F}(\mu_{M+1})\right]\\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times\\
&\quad\times\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k\neq j}^{M}\frac{\lambda^2}{\lambda^2-\mu_k^2}\right)\\
&+\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\,\frac{\lambda}{\mu_j}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times\\
&\quad\times\left(\mu_j\mathcal{H}(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j^2}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j^2}{\mu_j^2-\mu_k^2}\right)\\
&+\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\left(-\frac{4}{\lambda^2-\mu_{M+1}^2}\,\mathcal{F}(\mu_{M+1})\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}\right)\right.\\
&+\frac{4}{\lambda^2-\mu_{M+1}^2}\,\frac{\lambda}{\mu_{M+1}}\,\frac{\xi-\mu_{M+1}\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)\times\\
&\quad\left.\times\left(\mu_{M+1}\mathcal{H}(\mu_{M+1})+\frac{(\psi\varphi+\nu^2)\mu_{M+1}^2}{\xi^2-(\psi\varphi+\nu^2)\mu_{M+1}^2}\right)\right).
\end{aligned}\tag{47}$$

In the first two terms on the right hand side of (47) we used the equation (23) and the remaining terms we rewrite in a more appropriate order

$$\begin{aligned}
&\left[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\right]=-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\times\\
&\quad\times\frac{2}{\lambda^2-\mu_{M+1}^2}\left(\mu_{M+1}\,\frac{\xi+\lambda\sqrt{\psi\varphi+\nu^2}}{\xi+\mu_{M+1}\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)-\lambda\,\mathcal{F}(\mu_{M+1})\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{M} \frac{4\lambda}{\lambda^2-\mu_j^2} \, \frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}} \, \mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots\mathcal{F}(\mu_M)\times \\
&\qquad \times \frac{2}{\mu_j^2-\mu_{M+1}^2}\left(\mu_{M+1}\,\frac{\xi+\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi+\mu_{M+1}\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\mu_j)-\mu_j\,\mathcal{F}(\mu_{M+1})\right) \\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times \\
&\qquad \times \sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k\neq j}^{M}\frac{\lambda^2}{\lambda^2-\mu_k^2}\right) \\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times \\
&\qquad \times \frac{4}{\lambda^2-\mu_{M+1}^2}\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}\right) \\
&+\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\,\frac{\lambda}{\mu_j}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times \\
&\qquad \times\left(\mu_j\mathcal{H}(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j^2}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j^2}{\mu_j^2-\mu_k^2}\right) \\
&+\frac{4}{\lambda^2-\mu_{M+1}^2}\,\frac{\lambda}{\mu_{M+1}}\,\frac{\xi-\mu_{M+1}\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\lambda)\times \\
&\qquad \times\left(\mu_{M+1}\mathcal{H}(\mu_{M+1})+\frac{(\psi\varphi+\nu^2)\mu_{M+1}^2}{\xi^2-(\psi\varphi+\nu^2)\mu_{M+1}^2}\right). \qquad (48)
\end{aligned}
$$

Now it is just a question of reordering the terms in a more suitable manner

$$
\begin{aligned}
&\left[\tau(\lambda),\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\right]= \\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times \\
&\qquad \times\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k\neq j}^{M+1}\frac{\lambda^2}{\lambda^2-\mu_k^2}\right) \\
&-\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\times \\
&\qquad \times\frac{4}{\lambda^2-\mu_{M+1}^2}\left(\lambda\mathcal{H}(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda^2}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k=1}^{M}\frac{\lambda^2}{\lambda^2-\mu_k^2}\right) \\
&+\sum_{j=1}^{M}\frac{4}{\lambda^2-\mu_j^2}\,\frac{\lambda}{\mu_j}\,\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\,\mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots
\end{aligned}
$$

$$\cdots \mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1}) \left(\mu_j \mathcal{H}(\mu_j) + \frac{(\psi\varphi + \nu^2)\mu_j^2}{\xi^2 - (\psi\varphi + \nu^2)\mu_j^2} - \sum_{k \neq j}^{M+1} \frac{2\mu_j^2}{\mu_j^2 - \mu_k^2} \right)$$

$$+ \frac{4}{\lambda^2 - \mu_{M+1}^2} \frac{\lambda}{\mu_{M+1}} \frac{\xi - \mu_{M+1}\sqrt{\psi\varphi + \nu^2}}{\xi - \lambda\sqrt{\psi\varphi + \nu^2}} \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\lambda) \times$$

$$\times \left(\mu_{M+1}\mathcal{H}(\mu_{M+1}) + \frac{(\psi\varphi + \nu^2)\mu_{M+1}^2}{\xi^2 - (\psi\varphi + \nu^2)\mu_{M+1}^2} - \sum_{k=1}^{M} \frac{2\mu_{M+1}^2}{\mu_{M+1}^2 - \mu_k^2} \right). \tag{49}$$

To obtain all the terms in the last sum (in the last line above) we had to use a generally valid, purely algebraic identity:

$$\begin{aligned} &\frac{-1}{(\lambda^2 - \mu_j^2)(\lambda^2 - \mu_{M+1}^2)} \frac{\xi^2 - (\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\mu_{M+1}^2} + \frac{1}{(\lambda^2 - \mu_j^2)(\mu_j^2 - \mu_{M+1}^2)} \times \\ &\times \frac{\xi^2 - (\psi\varphi + \nu^2)\mu_j^2}{\xi^2 - (\psi\varphi + \nu^2)\mu_{M+1}^2} = \frac{-1}{(\lambda^2 - \mu_{M+1}^2)(\mu_{M+1}^2 - \mu_j^2)}, \end{aligned} \tag{50}$$

which is valid for every $j = 1, 2, \ldots, M$.

Finally, to complete the proof we can simply combine together similar terms and obtain the desired result

$$\begin{aligned} &\left[\tau(\lambda), \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1})\right] = \\ &- \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1}) \times \\ &\quad \times \sum_{j=1}^{M+1} \frac{4}{\lambda^2 - \mu_j^2} \left(\lambda\mathcal{H}(\lambda) + \frac{(\psi\varphi + \nu^2)\lambda^2}{\xi^2 - (\psi\varphi + \nu^2)\lambda^2} - \sum_{k \neq j}^{M+1} \frac{\lambda^2}{\lambda^2 - \mu_k^2} \right) \\ &+ \sum_{j=1}^{M+1} \frac{4}{\lambda^2 - \mu_j^2} \frac{\lambda}{\mu_j} \frac{\xi - \mu_j\sqrt{\psi\varphi + \nu^2}}{\xi - \lambda\sqrt{\psi\varphi + \nu^2}} \mathcal{F}(\lambda)\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\widehat{\mathcal{F}(\mu_j)}\cdots \\ &\cdots\mathcal{F}(\mu_M)\mathcal{F}(\mu_{M+1}) \left(\mu_j\mathcal{H}(\mu_j) + \frac{(\psi\varphi + \nu^2)\mu_j^2}{\xi^2 - (\psi\varphi + \nu^2)\mu_j^2} - \sum_{k \neq j}^{M+1} \frac{2\mu_j^2}{\mu_j^2 - \mu_k^2} \right). \end{aligned} \tag{51}$$

This completes the proof by mathematical induction of the formula (44). It should be stressed that the right hand side is in an algebraically closed form. Thus the off-shell action of the generating function of the $so(3)$ Gaudin Hamiltonians with general boundary terms becomes a simple corollary of this result.

Namely, from (44) it follows that, for an arbitrary natural number M, the off-shel action of the generating function $\tau(\lambda)$ on the Bethe vectors

$$\Phi_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)\Omega_+, \tag{52}$$

is given by

$$\tau(\lambda)\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) = \chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M)$$
$$+\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\lambda\sqrt{\psi\varphi+\nu^2}}\left(\rho(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2-\mu_k^2}\right)\times$$
$$\times\,\Phi_M(\lambda,\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M)\,, \tag{53}$$

where the eigenvalue $\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)$ is given by

$$\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M) =$$
$$\chi_0(\lambda)-\sum_{j=1}^{M}\frac{4\lambda}{\lambda^2-\mu_j^2}\left(\rho(\lambda)+\frac{(\psi\varphi+\nu^2)\lambda}{\xi^2-(\psi\varphi+\nu^2)\lambda^2}-\sum_{k\neq j}^{M}\frac{\lambda}{\lambda^2-\mu_k^2}\right). \tag{54}$$

The unwanted terms on the right hand side of (53) are annihilated once the Bethe equations

$$\rho(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2-\mu_k^2}=0\,,\qquad j=1,2,\ldots,M\,, \tag{55}$$

are imposed on the parameters $\mu_1,\mu_2,\ldots,\mu_M$.

The off-shell action of the $so(3)$ Gaudin Hamiltonians with general boundary terms (29) on the Bethe vectors (52) is obtained by taking the residue, at $\lambda=\alpha_m$, of the equation (53)

$$\mathrm{H}_m\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{E}_{m,M}\,\Phi_M(\mu_1,\mu_2,\ldots,\mu_M)$$
$$+\sum_{j=1}^{M}\frac{4\alpha_m}{\alpha_m^2-\mu_j^2}\frac{\xi-\mu_j\sqrt{\psi\varphi+\nu^2}}{\xi-\alpha_m\sqrt{\psi\varphi+\nu^2}}\times$$
$$\times\left(\rho(\mu_j)+\frac{(\psi\varphi+\nu^2)\mu_j}{\xi^2-(\psi\varphi+\nu^2)\mu_j^2}-\sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2-\mu_k^2}\right)\times \tag{56}$$
$$\times\left(\frac{-2(\nu+\sqrt{\psi\varphi+\nu^2})S_m^3-\psi S_m^+ +\frac{\psi\varphi+2\nu\,(\nu+\sqrt{\psi\varphi+\nu^2})}{\psi}S_m^-}{2\sqrt{\psi\varphi+\nu^2}}\right)\times$$
$$\times\,\Phi_{M-1}(\mu_1,\ldots,\widehat{\mu}_j,\ldots,\mu_M)\,,$$

where

$$\mathrm{H}_m=\operatorname*{Res}_{\lambda=\alpha_m}\tau(\lambda) \tag{57}$$

and the eigenvalues $\mathcal{E}_{m,M}$ of the $so(3)$ Gaudin Hamiltonians are the residues of the eigenvalues $\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)$ (54) of the generating function $\tau(\lambda)$ at $\lambda=\alpha_m$,

$$\mathcal{E}_{m,M}=\operatorname*{Res}_{\lambda=\alpha_m}\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)$$
$$=\frac{2\xi^2}{\left(\xi^2-\left(\psi\varphi+\nu^2\right)\alpha_m^2\right)\alpha_m}+\sum_{n\neq m}^{N}\frac{4\alpha_m}{\alpha_m^2-\alpha_n^2}-\sum_{j=1}^{M}\frac{4\alpha_m}{\alpha_m^2-\mu_j^2}\,, \tag{58}$$

and

$$\underset{\lambda=\alpha_m}{\text{Res}}\, \Phi_M(\lambda, \mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M) = \underset{\lambda=\alpha_m}{\text{Res}}\, (\mathcal{F}(\lambda)) \cdot \Phi_{M-1}(\mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M)$$

$$= \left(\frac{-2(\nu + \sqrt{\psi\varphi + \nu^2}) S_m^3 - \psi S_m^+ + \frac{\psi\varphi + 2\nu\, (\nu + \sqrt{\psi\varphi + \nu^2})}{\psi} S_m^-}{2\sqrt{\psi\varphi + \nu^2}} \right) \cdot$$

$$\cdot\, \Phi_{M-1}(\mu_1, \ldots, \widehat{\mu}_j, \ldots, \mu_M)\,, \tag{59}$$

where the notation $\widehat{\mu}_j$ means that the argument μ_j is omitted.

As a closing remark for this section we must underline the complete gerality of these results: the formulae for the off-shell action of the generating function $\tau(\lambda)$ (53) and the $so(3)$ Gaudin Hamiltonians on the Bethe vectors (52) are obtained for an arbitrary natural number M and without any restriction whatsoever on all four boundary parameters. In this sense we can say that these formulae are as general as they can possibly be. In the next section we will establish a correspondence between the Bethe vectors (52) established here and the solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations.

4. Generalized $so(3)$ Knizhnik-Zamolodchikov equations

In this section we study solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations. To proceed further, we now have to set the parameter ξ to zero, i.e. $\xi = 0$. Consequently, the local realization of the generators (25) – (27) simplifies to

$$\widetilde{\mathcal{E}}(\lambda) = \sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{\psi\varphi + \nu^2}} \, \frac{2(\nu - \sqrt{\psi\varphi + \nu^2}) S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\, (\nu - \sqrt{\psi\varphi + \nu^2})}{\psi} S_m^-}{\lambda^2 - \alpha_m^2}\,, \tag{60}$$

$$\widetilde{\mathcal{F}}(\lambda) = \sum_{m=1}^{N} \frac{-\alpha_m}{\sqrt{\psi\varphi + \nu^2}} \, \frac{2(\nu + \sqrt{\psi\varphi + \nu^2}) S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2\nu\, (\nu + \sqrt{\psi\varphi + \nu^2})}{\psi} S_m^-}{\lambda^2 - \alpha_m^2}\,, \tag{61}$$

$$\mathcal{H}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi + \nu^2}} \sum_{m=1}^{N} \frac{2\nu S_m^3 + \psi S_m^+ + \varphi S_m^-}{\lambda^2 - \alpha_m^2}\,. \tag{62}$$

These generators have the following non-trivial commutation relations

$$\left[\mathcal{H}(\lambda), \widetilde{\mathcal{E}}(\mu)\right] = \frac{-2\lambda}{\lambda^2 - \mu^2} \left(\widetilde{\mathcal{E}}(\lambda) - \widetilde{\mathcal{E}}(\mu)\right)\,, \tag{63}$$

$$\left[\mathcal{H}(\lambda), \widetilde{\mathcal{F}}(\mu)\right] = \frac{2\lambda}{\lambda^2 - \mu^2} \left(\widetilde{\mathcal{F}}(\lambda) - \widetilde{\mathcal{F}}(\mu)\right)\,, \tag{64}$$

$$\left[\widetilde{\mathcal{E}}(\lambda), \widetilde{\mathcal{F}}(\mu)\right] = \frac{-4}{\lambda^2 - \mu^2} \left(\lambda\, \mathcal{H}(\lambda) - \mu\, \mathcal{H}(\mu)\right)\,. \tag{65}$$

Thus, in this case, the first Bethe vector is here defined by

$$\widetilde{\Phi}_1(\mu) = \Phi_1(\mu)\Big|_{\xi=0} = \widetilde{\mathfrak{F}}(\mu)\Omega_+ \,, \tag{66}$$

and, in general case, for the Bethe vectors $\widetilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M)$ we have

$$\widetilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M) = \Phi_M(\mu_1, \mu_2, \dots, \mu_M)\Big|_{\xi=0} = \widetilde{\mathfrak{F}}(\mu_1)\widetilde{\mathfrak{F}}(\mu_2)\cdots\widetilde{\mathfrak{F}}(\mu_M)\Omega_+ \,. \tag{67}$$

It is useful to explicitly write H_m for $\xi = 0$:

$$\begin{aligned}
\widetilde{\mathrm{H}}_m = \mathrm{H}_m\Big|_{\xi=0} &= \frac{1}{2\left(\psi\varphi + \nu^2\right)\alpha_m}\Big(2\left(\nu^2 - \psi\varphi\right)(S_m^3)^2 \\
&+ \psi^2\,(S_m^+)^2 + \varphi^2\,(S_m^-)^2 + 2\psi\nu\left(S_m^+ S_m^3 + S_m^3 S_m^+\right) \\
&+ 2\varphi\nu\left(S_m^- S_m^3 + S_m^3 S_m^-\right) - \nu^2\left(S_m^+ S_m^- + S_m^- S_m^+\right)\Big) \\
&+ \frac{1}{\psi\varphi + \nu^2}\sum_{n\neq m}^{N}\Bigg(\frac{4\left(\psi\varphi\,\alpha_n + \nu^2\alpha_m\right)}{\alpha_m^2 - \alpha_n^2}\, S_m^3 S_n^3 + \frac{1}{\alpha_m + \alpha_n}\times \\
&\times\Big(\psi^2\, S_m^+ S_n^+ + \varphi^2\, S_m^- S_n^- + 2\psi\nu\left(S_m^+ S_n^3 + S_m^3 S_n^+\right) + 2\varphi\nu\left(S_m^- S_n^3 + S_m^3 S_n^-\right)\Big) \\
&+ \frac{2\nu^2\alpha_n + \psi\varphi\left(\alpha_m + \alpha_n\right)}{\alpha_m^2 - \alpha_n^2}\left(S_m^- S_n^+ + S_m^+ S_n^-\right)\Bigg).
\end{aligned} \tag{68}$$

Therefore the off-shell action of these Hamiltonians reads

$$\begin{aligned}
&\widetilde{\mathrm{H}}_m\,\widetilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M) = \\
&\qquad \widetilde{\mathcal{E}}_{m,M}\,\widetilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M) + \sum_{j=1}^{M}\frac{(-2)\mu_j}{\alpha_m^2 - \mu_j^2}\,\beta_M(\mu_j)\,\widetilde{\Phi}_{M-1}^{(j,m)} \,,
\end{aligned} \tag{69}$$

where

$$\widetilde{\mathcal{E}}_{m,M} = \mathcal{E}_{m,M}\Big|_{\xi=0} = \sum_{n\neq m}^{N}\frac{4\alpha_m}{\alpha_m^2 - \alpha_n^2} - \sum_{j=1}^{M}\frac{4\alpha_m}{\alpha_m^2 - \mu_j^2} \,, \tag{70}$$

$$\beta_M(\mu_j) = -2\left(\rho(\mu_j) - \frac{1}{\mu_j} - \sum_{k\neq j}^{M}\frac{2\mu_j}{\mu_j^2 - \mu_k^2}\right), \tag{71}$$

and

$$\begin{aligned}
\widetilde{\Phi}_{M-1}^{(j,m)} &= \left(\frac{-2(\nu + \sqrt{\psi\varphi + \nu^2})S_m^3 - \psi\, S_m^+ + \frac{\psi\varphi + 2\nu\,(\nu + \sqrt{\psi\varphi + \nu^2})}{\psi}S_m^-}{2\sqrt{\psi\varphi + \nu^2}}\right)\cdot \\
&\quad\cdot\widetilde{\Phi}_{M-1}(\mu_1, \dots, \widehat{\mu}_j, \dots, \mu_M)\,.
\end{aligned} \tag{72}$$

Our main objective in this section is to show how to each Bethe vector (67) we can relate a solution to the generalized $so(3)$ Knizhnik-Zamolodchikov equations

$$\kappa\, \partial_{\alpha_m} \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \widetilde{\mathrm{H}}_m\ \Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)\,. \tag{73}$$

Within our approach [22,61,15–17,86,89] this correspondence is defined by a closed contour integration with respect to the variables $\mu_1, \mu_2, \ldots, \mu_M$

$$\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \oint \oint \cdots \oint\ \Upsilon\left(\vec{\mu}\,;\, \vec{\alpha}\right) \cdot \widetilde{\Phi}_M\left(\vec{\mu}\,;\, \vec{\alpha}\right) \mathrm{d}\mu_1\, \mathrm{d}\mu_2 \cdots \mathrm{d}\mu_M\,. \tag{74}$$

The scalar function $\Upsilon\left(\vec{\mu}\,;\, \vec{\alpha}\right)$ is defined by

$$\Upsilon\left(\vec{\mu}\,;\, \vec{\alpha}\right) = \exp\left(\frac{S(\vec{\mu}\,;\, \vec{\alpha}\,)}{\kappa}\right), \tag{75}$$

with the constant κ and the function $S(\vec{\mu}\,;\, \vec{\alpha}\,)$ specified by

$$S\left(\vec{\mu}\,;\, \vec{\alpha}\right) = \sum_{m=1}^{N} \left(\sum_{n\neq m}^{N} \ln\left(\alpha_n^2 - \alpha_m^2\right) - \sum_{j=1}^{M} 2\, \ln\left(\mu_j^2 - \alpha_m^2\right) \right) \tag{76}$$

$$+ \sum_{j=1}^{M} \left(\ln\left(\mu_j^2\right) + \sum_{k\neq j}^{M} \ln\left(\mu_j^2 - \mu_k^2\right) \right)\,. \tag{77}$$

It is straightforward to check that the function $\Upsilon\left(\vec{\mu}\,;\, \vec{\alpha}\right)$ satisfies the system

$$\kappa\, \partial_{\alpha_m} \Upsilon = \widetilde{\mathcal{E}}_{m,M}\, \Upsilon\,, \tag{78}$$

$$\kappa\, \partial_{\mu_j} \Upsilon = \beta_M(\mu_j)\, \Upsilon\,, \tag{79}$$

where $\widetilde{\mathcal{E}}_{m,M}$ and $\beta_M(\mu_j)$ are defined in (70) and (71), respectively.

The crucial identity in our approach is

$$\partial_{\alpha_m} \widetilde{\Phi}_M = \sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{2\mu_j}{\alpha_m^2 - \mu_j^2}\ \widetilde{\Phi}_{M-1}^{(j,m)} \right)\,. \tag{80}$$

It takes a few rather simple steps to confirm that the function $\Psi(\alpha_1, \alpha_2, \ldots, \alpha_N)$ (74) is a solution to the generalized $so(3)$ Knizhnik-Zamolodchikov equations (73). As the first step, using the Leibniz rule, we calculate

$$\kappa\, \partial_{\alpha_m} \left(\Upsilon \cdot \widetilde{\Phi}_M\right) = \left(\kappa\, \partial_{\alpha_m} \Upsilon\right) \cdot \widetilde{\Phi}_M + \Upsilon \cdot \left(\kappa\, \partial_{\alpha_m} \widetilde{\Phi}_M\right)\,. \tag{81}$$

Then we use the equation (78) in the first term on the right hand side of the equation above and the identity (80) in the second term

$$\kappa\, \partial_{\alpha_m} \left(\Upsilon \cdot \widetilde{\Phi}_M\right) = \mathcal{E}_{m,M} \left(\Upsilon \cdot \widetilde{\Phi}_M\right) + \Upsilon \cdot \kappa\, \sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{2\mu_j}{\alpha_m^2 - \mu_j^2}\ \widetilde{\Phi}_{M-1}^{(j,m)} \right)\,. \tag{82}$$

In the following step we use the equation (69) in the first term and the Leibniz rule in the second term on the right hand side of the equation above

$$\kappa\,\partial_{\alpha_m}\left(\Upsilon\cdot\widetilde{\Phi}_M\right)=\widetilde{\mathrm{H}}_m\left(\Upsilon\cdot\widetilde{\Phi}_M\right)+\sum_{j=1}^{M}\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\beta_M(\mu_j)\cdot\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}$$
$$+\kappa\,\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}\right)-\kappa\,\sum_{j=1}^{M}\left(\partial_{\mu_j}\Upsilon\right)\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\widetilde{\Phi}_{M-1}^{(j,m)}\,. \tag{83}$$

Now it remains to rewrite the last terms on the right hand side using the equation (79)

$$\kappa\,\partial_{\alpha_m}\left(\Upsilon\cdot\widetilde{\Phi}_M\right)=\widetilde{\mathrm{H}}_m\left(\Upsilon\cdot\widetilde{\Phi}_M\right)+\sum_{j=1}^{M}\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\beta_M(\mu_j)\cdot\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}$$
$$+\kappa\,\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}\right)-\sum_{j=1}^{M}\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\beta_M(\mu_j)\cdot\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}\,. \tag{84}$$

As the final step of this demonstration, we simplify the second and the last term in the equation above in order to obtain the desired result

$$\kappa\,\partial_{\alpha_m}\left(\Upsilon\cdot\widetilde{\Phi}_M\right)=\widetilde{\mathrm{H}}_m\left(\Upsilon\cdot\widetilde{\Phi}_M\right)+\kappa\,\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{2\mu_j}{\alpha_m^2-\mu_j^2}\,\Upsilon\cdot\widetilde{\Phi}_{M-1}^{(j,m)}\right). \tag{85}$$

This shows that the function $\Psi(\alpha_1,\alpha_2,\ldots,\alpha_N)$ (74) is a solution to the generalized $so(3)$ Knizhnik-Zamolodchikov equations (73) since the terms in the sum will not contribute to the closed contour integrals with respect to the variables μ_j, $j=1,2,\ldots,M$.

In the final part of this section we determine the on-shell norm as well as the off-shell scalar products of the Bethe vectors (67). In particular, the on-shell norm of the Bethe vector (66) is obtained to be

$$\left\|\widetilde{\Phi}_1(\mu)\right\|^2=\lim_{\nu\to\mu}\left\langle\Omega_+,\widetilde{\mathcal{E}}(\nu)\widetilde{\mathcal{F}}(\mu)\Omega_+\right\rangle=-2\left(\rho'(\mu)+\frac{\rho(\mu)}{\mu}\right)$$
$$=\frac{\partial\beta_1(\mu)}{\partial\mu}\Big|_{\beta_1(\mu)=0}=\frac{\partial^2S(\mu)}{\partial\mu^2}\Big|_{\beta_1(\mu)=0}\,. \tag{86}$$

Similarly, the norm of the Bethe vector

$$\widetilde{\Phi}_2(\mu_1,\mu_2)=\widetilde{\mathcal{F}}(\mu_1)\widetilde{\mathcal{F}}(\mu_2)\Omega_+\,, \tag{87}$$

when the Bethe equations are imposed on the parameters μ_1 and μ_2, is given by

$$\left\|\widetilde{\Phi}_2(\mu_1,\mu_2)\right\|^2=\lim_{\substack{\nu_1\to\mu_1\\ \nu_2\to\mu_2}}\left\langle\Omega_+,\;\widetilde{\mathcal{E}}(\nu_1)\widetilde{\mathcal{E}}(\nu_2)\widetilde{\mathcal{F}}(\mu_2)\widetilde{\mathcal{F}}(\mu_1)\Omega_+\right\rangle$$
$$=4\,\rho'(\mu_1)\,\rho'(\mu_2)+4\,\rho'(\mu_1)\left(\frac{1}{\mu_2^2}+\frac{2}{\mu_2^2-\mu_1^2}+\frac{4\,\mu_1^2}{\left(\mu_2^2-\mu_1^2\right)^2}\right)$$
$$+4\,\rho'(\mu_2)\left(\frac{1}{\mu_1^2}+\frac{2}{\mu_1^2-\mu_2^2}+\frac{4\,\mu_2^2}{\left(\mu_2^2-\mu_1^2\right)^2}\right)+12\,\frac{\left(\mu_1^2+\mu_2^2\right)^2}{\mu_1^2\mu_2^2\left(\mu_2^2-\mu_1^2\right)^2} \tag{88}$$

$$= \det \begin{pmatrix} \frac{\partial \beta_2(\mu_1)}{\partial \mu_1} & \frac{\partial \beta_2(\mu_2)}{\partial \mu_1} \\ \frac{\partial \beta_2(\mu_1)}{\partial \mu_2} & \frac{\partial \beta_2(\mu_2)}{\partial \mu_2} \end{pmatrix} \Bigg|_{\substack{\beta_2(\mu_1)=0 \\ \beta_2(\mu_2)=0}} = \det \begin{pmatrix} \frac{\partial^2 S}{\partial \mu_1^2} & \frac{\partial^2 S}{\partial \mu_1 \partial \mu_2} \\ \frac{\partial^2 S}{\partial \mu_2 \partial \mu_1} & \frac{\partial^2 S}{\partial \mu_2^2} \end{pmatrix} \Bigg|_{\substack{\beta_2(\mu_1)=0 \\ \beta_2(\mu_2)=0}} .$$

In the general case, for an arbitrary positive integer M, the norm of the Bethe vector $\widetilde{\Phi}_M(\mu_1, \mu_2, \ldots, \mu_M)$ (67), when the Bethe equations

$$\beta_M(\mu_j) = -2\left(\rho(\mu_j) - \frac{1}{\mu_j} - \sum_{k \neq j}^{M} \frac{2\mu_j}{\mu_j^2 - \mu_k^2}\right) = 0 , \quad j = 1, 2, \ldots, M , \tag{89}$$

are imposed on the parameter $\mu_1, \ldots, \mu_M$, is obtained to be

$$\left\| \widetilde{\Phi}_M(\mu_1, \mu_2, \ldots, \mu_M) \right\|^2 = \det \begin{pmatrix} \frac{\partial^2 S}{\partial \mu_1^2} & \frac{\partial^2 S}{\partial \mu_1 \partial \mu_2} & \cdots & \frac{\partial^2 S}{\partial \mu_1 \partial \mu_M} \\ \vdots & \ddots & & \vdots \\ \frac{\partial^2 S}{\partial \mu_M \partial \mu_1} & \frac{\partial^2 S}{\partial \mu_M \partial \mu_2} & \cdots & \frac{\partial^2 S}{\partial \mu_M^2} \end{pmatrix} \Bigg|_{\substack{\beta_M(\mu_1)=0 \\ \vdots \\ \beta_M(\mu_M)=0}} . \tag{90}$$

Finally, we also calculate the off-shell scalar products of the Bethe vectors $\widetilde{\Phi}_M(\mu_1, \mu_2, \ldots, \mu_M)$ (67). As our first step, we observe that in the case when $M = 1$ the scalar product is

$$\left\langle \widetilde{\Phi}_1(\mu), \widetilde{\Phi}_1(\nu) \right\rangle = 4\left(-\frac{\mu\, \rho(\mu) - \nu\, \rho(\nu)}{\mu^2 - \nu^2}\right) . \tag{91}$$

For $M = 2$, a straightforward calculation yields

$$\left\langle \widetilde{\Phi}_2(\mu_1, \mu_2), \widetilde{\Phi}_2(\nu_1, \nu_2) \right\rangle = 4^2 \sum_{\sigma \in \mathcal{S}_2} \det \mathcal{M}^{\sigma} = 16\left(\det \mathcal{M}^1 + \det \mathcal{M}^2\right) , \tag{92}$$

where $\mathcal{S}_2$ is the symmetric group of degree two and the two-by-two matrices $\mathcal{M}^1$ and $\mathcal{M}^2$ are given by

$$\begin{aligned} \mathcal{M}^1_{11} &= -\frac{\mu_1\, \rho(\mu_1) - \nu_1\, \rho(\nu_1)}{\mu_1^2 - \nu_1^2} - \frac{\mu_2^2 + \nu_2^2}{\left(\mu_1^2 - \mu_2^2\right)\left(\nu_1^2 - \nu_2^2\right)} , \\ \mathcal{M}^1_{12} &= -\frac{\mu_2^2 + \nu_2^2}{\left(\mu_1^2 - \mu_2^2\right)\left(\nu_1^2 - \nu_2^2\right)} , \\ \mathcal{M}^1_{22} &= -\frac{\mu_2\, \rho(\mu_2) - \nu_2\, \rho(\nu_2)}{\mu_2^2 - \nu_2^2} - \frac{\mu_1^2 + \nu_1^2}{\left(\mu_2^2 - \mu_1^2\right)\left(\nu_2^2 - \nu_1^2\right)} , \\ \mathcal{M}^1_{21} &= -\frac{\mu_1^2 + \nu_1^2}{\left(\mu_2^2 - \mu_1^2\right)\left(\nu_2^2 - \nu_1^2\right)} , \end{aligned} \tag{93}$$

and

$$
\begin{aligned}
\mathcal{M}^2_{11} &= -\frac{\mu_1\,\rho(\mu_1) - \nu_2\,\rho(\nu_2)}{\mu_1^2 - \nu_2^2} - \frac{\mu_2^2 + \nu_1^2}{\left(\mu_1^2 - \mu_2^2\right)\left(\nu_2^2 - \nu_1^2\right)}\,,\\
\mathcal{M}^2_{12} &= -\frac{\mu_2^2 + \nu_1^2}{\left(\mu_1^2 - \mu_2^2\right)\left(\nu_2^2 - \nu_1^2\right)}\,,\\
\mathcal{M}^2_{22} &= -\frac{\mu_2\,\rho(\mu_2) - \nu_1\,\rho(\nu_1)}{\mu_2^2 - \nu_1^2} - \frac{\mu_1^2 + \nu_2^2}{\left(\mu_2^2 - \mu_1^2\right)\left(\nu_1^2 - \nu_2^2\right)}\,,\\
\mathcal{M}^2_{21} &= -\frac{\mu_1^2 + \nu_2^2}{\left(\mu_2^2 - \mu_1^2\right)\left(\nu_1^2 - \nu_2^2\right)}\,.
\end{aligned}
\tag{94}
$$

In general case, for an arbitrary positive integer M, we have

$$
\left\langle \widetilde{\Phi}_M(\mu_1,\mu_2,\ldots,\mu_M),\ \widetilde{\Phi}_M(\nu_1,\nu_2,\ldots,\nu_M)\right\rangle = 4^M \sum_{\sigma\in\mathcal{S}_M} \det \mathcal{M}^\sigma\,, \tag{95}
$$

where S_M is the symmetric group of degree M and the matrix entries of the $M\times M$ matrix $\mathcal{M}^\sigma$ are given by

$$
\mathcal{M}^\sigma_{jj} = -\frac{\mu_j\,\rho(\mu_j) - \nu_{\sigma(j)}\,\rho(\nu_{\sigma(j)})}{\mu_j^2 - \nu^2_{\sigma(j)}} - \sum_{k\neq j} \frac{\mu_k^2 + \nu^2_{\sigma(k)}}{(\mu_j^2 - \mu_k^2)(\nu^2_{\sigma(j)} - \nu^2_{\sigma(k)})}\,, \tag{96}
$$

$$
\mathcal{M}^\sigma_{jk} = -\frac{\mu_k^2 + \nu^2_{\sigma(k)}}{(\mu_j^2 - \mu_k^2)(\nu^2_{\sigma(j)} - \nu^2_{\sigma(k)})}\,, \quad \text{for} \quad j,k = 1,2,\ldots,M\,. \tag{97}
$$

The off-shell scalar products of the Bethe vectors $\widetilde{\Phi}_M(\mu_1,\mu_2,\ldots,\mu_M)$ (67) in the $M=1$ case (91) and in the $M=2$ case (92) were derived by a direct, straightforward calculations. The formula (95) was obtained by symbolic computer calculations for $M=3,4,5$, for some values of N. In the general case, the proof of these formulae by induction would be very difficult, since it would require some highly non-trivial relations between a certain type of determinants of a different order. In this sense, the general formula (95), strictly speaking, remains a conjecture.

5. Conclusions

The cornerstone of our study of the non-periodic $so(3)$ Gaudin model was the $so(3)$ Maillet linear bracket (8) for the suitable Lax operator (7) and the non-unitary $so(3)$ classical r-matrix (B.8) constructed from the generic boundary K-matrix (B.4). Based on Maillet bracket we obtained the generating function (15) of the $so(3)$ Gaudin Hamiltonians with general boundary terms. However it turned out that the natural set of generators was not the most efficient choice for implementing the algebraic Bethe ansatz due to the cumbersome commutation relations (9) - (14). For this reason we proposed a new set of generators: (18) - (20). Not only that their commutation relations had a strikingly compact form (21) - (24) but also their local realization was fairly simple (25) - (27), yielding the explicit expression for the Gaudin Hamiltonians with all four boundary parameters (29).

There were several preceding objectives which we had to address before attempting to find the off-shell action of the generating function $\tau(\lambda)$ (28). In the first place, we had to define the so-called vacuum vector Ω_+ (30) - (33). Then we had to confirm the action of the generators on the vacuum vectors (34) and to show that the vector Ω_+ is the eigenvector of the generating function $\tau(\lambda)$ (39). As our next step, we have calculated the commutation relations between the generating function $\tau(\lambda)$ and the remaining generator $\mathcal{F}(\mu)$ of the generalized $so(3)$ Gaudin algebra (41). The idea of using the generator $\mathcal{F}(\mu)$ as the so-called creation operator prompted us to calculate the commutation relations between the generating function $\tau(\lambda)$ and the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)$ (42) as well as the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\mathcal{F}(\mu_3)$ (43). Hence, we have conjectured the formula (44), in the general case, for the commutator between the generating function $\tau(\lambda)$ and the product $\mathcal{F}(\mu_1)\mathcal{F}(\mu_2)\cdots\mathcal{F}(\mu_M)$, for an arbitrary natural number M. The proof of the formula (44) based on the mathematical induction was presented in (45) - (51). Once we have accordingly defined the Bethe vectors (52), the off-shell action (53) of the generating function $\tau(\lambda)$, including the formulae for the eigenvalues (54) and the Bethe equations (55), followed from (44). Moreover, the off-shell action (56) of the Gaudin Hamiltonians (29) on the Bethe vectors (52) was obtained by taking the residue, at $\lambda = \alpha_m$, of the left and the right hand side of (53). It should be stressed that the formulae of the off-shell action (53) and (56) have been obtained without any restriction whatsoever on any of the four boundary parameters and therefore we can say that these formulae are as general as they can possibly be.

Next, we found the solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations (73). In spite that the key identity (80) in the proof required the parameter ξ to be set to zero, the formulae we obtained for the solutions to the generalized $so(3)$ Knizhnik-Zamolodchikov equations (74), the on-shell norm of the Bethe vectors (90) and the off-shell scalar product of the Bethe vectors (95) – all possess higher degree of generality than the analogous formulae in the $s\ell(2)$ case [86] (here we have fixed only one of the four boundary parameters instead of all four).

In our future research we hope to address the remaining open problem of correlation functions for the $so(3)$ Gaudin model with general boundary, following Sklyanin approach in the periodic $s\ell(2)$ case [100].

CRediT authorship contribution statement

N. Manojlović: Conceptualization, Formal analysis, Methodology, Writing – original draft.
I. Salom: Formal analysis, Methodology, Software, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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Appendix A. Preliminaries

Some essential definitions regarding the $so(3)$ Lie algebra and its fundamental representation are given in the Appendix A. Namely, we consider the spin one operators

$$\mathcal{S}^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathcal{S}^y = \frac{\imath}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathcal{S}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.1}$$

acting in the space $V^{(1)} = \mathbb{C}^3$ with the commutation relations

$$[\mathcal{S}^x, \mathcal{S}^y] = \imath \mathcal{S}^z, \quad [\mathcal{S}^z, \mathcal{S}^x] = \imath \mathcal{S}^y, \quad [\mathcal{S}^y, \mathcal{S}^z] = \imath S^x$$

and the Casimir element

$$c_2 = \vec{\mathcal{S}} \cdot \vec{\mathcal{S}} = (\mathcal{S}^x)^2 + (\mathcal{S}^y)^2 + (\mathcal{S}^z)^2 = 2\mathbb{1}. \tag{A.2}$$

Introducing raising and lowering operators

$$\mathcal{S}^+ = \mathcal{S}^x + \imath \mathcal{S}^y = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{S}^- = \mathcal{S}^x - \imath \mathcal{S}^y = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{A.3}$$

the relations above can also be written as

$$[\mathcal{S}^z, \mathcal{S}^\pm] = \pm \mathcal{S}^\pm, \quad [\mathcal{S}^+, \mathcal{S}^-] = 2\mathcal{S}^z, \tag{A.4}$$

and

$$c_2 = (\mathcal{S}^z)^2 + \frac{1}{2}(\mathcal{S}^+\mathcal{S}^- + \mathcal{S}^-\mathcal{S}^+) = (\mathcal{S}^z)^2 + \mathcal{S}^z + \mathcal{S}^-\mathcal{S}^+. \tag{A.5}$$

It is useful to notice that the tensor Casimir operator can be expressed as follows

$$c_2^{\otimes}(1,2) = \vec{\mathcal{S}}_1 \cdot \vec{\mathcal{S}}_2 = \mathcal{P} - 3\mathcal{K}. \tag{A.6}$$

The permutation operator

$$\mathcal{P} = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{A.7}$$

the rank 1 projector

$$\mathcal{K}=\frac{1}{3}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{A.8}$$

and the identity operator $\mathbb{1}$ satisfy the relations

$$\mathcal{P}^2=\mathbb{1}\,, \quad \mathcal{K}^2=\mathcal{K}\,, \quad \mathcal{P}\mathcal{K}=\mathcal{K}\mathcal{P}=\mathcal{K}\,. \tag{A.9}$$

and therefore define the representation of the Brauer algebra in $\mathbb{C}^3\otimes\mathbb{C}^3$. Moreover these are the three invariant operators acting on $\mathbb{C}^3\otimes\mathbb{C}^3$

$$[\mathcal{S}^\alpha\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{S}^\alpha,\mathcal{P}]=0\,,\ [\mathcal{S}^\alpha\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{S}^\alpha,\mathcal{K}]=0\,, \tag{A.10}$$

here $\alpha=x,y,z$.

In our study of the $so(3)$ Gaudin model with N sites, characterised by the local space $V_m=\mathbb{C}^3$ together with the corresponding inhomogeneous parameter α_m, the Hilbert space is given by

$$\mathcal{H}=\bigotimes_{m=1}^{N} V_m=(\mathbb{C}^3)^{\otimes N}\,. \tag{A.11}$$

The local spin operators

$$S_m^\alpha=\mathbb{1}\otimes\cdots\otimes\underbrace{S^\alpha}_{m}\otimes\cdots\otimes\mathbb{1}\,, \tag{A.12}$$

with $\alpha=x,y,z$ and $m=1,2,\ldots,N$, are given by the matrices (A.3) and (A.1) in every local space $V_m=\mathbb{C}^3$. Evidently, they satisfy the usual commutation relations

$$[S_m^3,S_n^\pm]=\pm S_m^\pm\,\delta_{mn}\,, \quad [S_m^+,S_n^-]=2S_m^3\,\delta_{mn}\,. \tag{A.13}$$

Appendix B. The non-unitary $so(3)$ classical r-matrix

The cornerstone of our study presented in this paper is the non-unitary $so(3)$ classical r-matrix (B.8). Here, in the Appendix B, we recount how the r-matrix (B.8) can be obtained starting from the unitary, $so(3)$ invariant classical r-matrix

$$r(\lambda)=-\frac{\vec{\mathcal{S}}_1\cdot\vec{\mathcal{S}}_2}{\lambda}=-\frac{\mathcal{P}-3\mathcal{K}}{\lambda}\,, \tag{B.1}$$

where we have used the notation introduced in the Appendix A. In particular, the classical r-matrix (B.1) can be obtained as a quasi-classical limit of the $SO(3)$ quantum R-matrix [98,99,6]. Evidently, this classical r-matrix satisfies the classical Yang-Baxter equation [10]

$$[r_{12}(\lambda-\mu),r_{13}(\lambda-\nu)]+[r_{12}(\lambda-\mu),r_{23}(\mu-\nu)]+[r_{13}(\lambda-\nu),r_{23}(\mu-\nu)]=0, \tag{B.2}$$

and has the unitarity property

$$r_{21}(-\lambda)=-r_{12}(\lambda). \tag{B.3}$$

We also consider the following reflection matrix

$$K(\lambda) = \begin{pmatrix} (\xi - \nu\lambda)^2 & -\sqrt{2}\psi\,\lambda(\xi - \nu\lambda) & \psi^2\lambda^2 \\ -\sqrt{2}\varphi\lambda(\xi - \nu\lambda) & \xi^2 + (\psi\varphi - \nu^2)\,\lambda^2 & -\sqrt{2}\psi\lambda(\xi + \nu\lambda) \\ \varphi^2\lambda^2 & -\sqrt{2}\varphi\lambda(\xi + \nu\lambda) & (\xi + \nu\lambda)^2 \end{pmatrix}, \tag{B.4}$$

here ξ, ν, ψ, φ are arbitrary parameters. As it is well known, this K-matrix can be obtained by the so-called fusion procedure [6,90,91], starting from the $s\ell(2)$ K-matrix [58,92–94]. This method is outlined, in the trigonometric $so(3)$, in [95]. Alternatively, the so-called scaling limit [96] can be used to obtain the K-matrix (B.4) from the trigonometric $so(3)$ boundary K-matrix [95,97]. Evidently, this K-matrix satisfies the classical reflection equation

$$\begin{aligned} &r_{12}(\lambda - \mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda + \mu)K_2(\mu) = \\ &= K_2(\mu)r_{12}(\lambda + \mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda - \mu). \end{aligned} \tag{B.5}$$

It is worth mentioning that, while in the context of Heisenberg's open spin chain, one should also consider the dual reflection equation, this is not the case in the Gaudin model. Namely, as a consequence of long-range Gaudin model interactions, the "two ends of the chain" cannot have the same interpretation as in the case of Heisenberg's spin chain. In the Gaudin case, boundary parameters must be fixed in a way that the reflection equation and its dual effectively degenerate into a single equation.

Therefore, it follows that the corresponding non-unitary classical r-matrix, given by [69,76–82]

$$r^K_{12}(\lambda, \mu) = r_{12}(\lambda - \mu) - K_2(\mu)r_{12}(\lambda + \mu)K_2^{-1}(\mu)\,, \tag{B.6}$$

satisfies the generalized classical Yang-Baxter equation [69,72–75]

$$\left[r^K_{32}(\nu, \mu), r^K_{13}(\lambda, \nu)\right] + \left[r^K_{12}(\lambda, \mu), r^K_{13}(\lambda, \nu)\right] + \left[r^K_{12}(\lambda, \mu), r^K_{23}(\mu, \nu)\right] = 0. \tag{B.7}$$

The explicit form of this non-unitary $so(3)$ classical r-matrix is the following

$$r^K_{12}(\lambda, \mu) = -\left(\frac{\vec{\mathbb{S}}_1 \cdot \vec{\mathbb{S}}_2}{\lambda - \mu} - \frac{\vec{\mathbb{S}}_1 \cdot \left(K_2(\mu)\vec{\mathbb{S}}_2 K_2^{-1}(\mu)\right)}{\lambda + \mu}\right). \tag{B.8}$$

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