REPRESENTATIONS IN SOLVING A WORD PROBLEM: THE INFORMAL DEVELOPMENT OF FORMAL METHODS

Nélia Amado[1], Susana Carreira[1], Sandra Nobre[2], João Pedro Ponte[3]

[1]Universidade do Algarve & CIEFCUL, [2]Escola EB [2, 3] Prof. Paula Nogueira, [3]IE, Universidade de Lisboa & CIEFCUL

The aim of this paper is to analyse different types of representations used by grade 7 and 8 students in solving a word problem. The problem was offered to finalists of a mathematics contest held outside the classroom. The analysis of the products of different students show the key role of the representations used to solve the problem and how they are critical in the development of a sustainable process for informal learning of formal methods of solving systems of linear equations.

INTRODUCTION

Research has revealed that students prefer to use arithmetical methods in solving algebraic word problems and show difficulties in setting up and using equations to solve such problems (Kieran, 2006). Moreover it is acknowledged that numerical thinking is beneficial while facilitating the stepping into algebraic thinking. There is also evidence that the most frequently used arithmetical processes are guess and trial and unwinding. Another important fact is that while assigning a value to a variable and verifying its accuracy, students are developing functional reasoning, as it entails recognising a relation between variables even if such relation is not always expressed in the formal language of algebra (Johanning, 2004): “Thus, algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter symbolic, but which can ultimately be used as a cognitive support for introducing and for sustaining the more traditional discourses of school algebra” (Kieran, 1996, pp. 274-275).

Students tend to use numerical and arithmetical processes mainly because they are used to perform operations and do procedural computations rather than to think about the operations they should consider, in order to represent the relations involved in a given problem. Problem solving has been emphasised in mathematics education for some decades, all over the world, and its presence in school mathematics curricula is widespread. In Portugal, the national curriculum adopted in 2007 for basic education, elects problem solving as a privileged curricular, together with mathematical reasoning and mathematical communication. This curricular model places the focus on students’ processes, including the algebraic ones, which are seen as fundamental in the mathematical preparation of all individuals (Ponte, 2006).

Students’ written representations, especially in the activity of problem solving, are powerful tools to be developed as their use comprises an essential part of learning mathematics. “The term *representation* refers both to process and to product – in

2010. In Pinto, M. F. & Kawasaki, T.F. (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 137-144. Belo Horizonte, Brazil: PME.

other words to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67). Students' ability to deal with representations as mathematical tools and to activate them when facing a particular situation is seen as decisive. In other words, when trying to solve a problem, it is desirable that students know how to decide for a certain representation system and take advantage of it. To acquire such ability, students need to have opportunities to evaluate the efficacy of different representations and to assimilate their meanings: "regardless of the specific objective envisaged concerning the use of representations, the overriding purpose should be that they are helpful to the child so that he learns to use them effectively" (Dufour-Janvier, Bednarz & Belanger, 1987, p. 121).

REPRESENTATION IN PROBLEM SOLVING

We may distinguish two major kinds of representations: "internal" and "external". The first corresponds to the internal images produced by an individual about a certain reality. The later refers to the external and symbolic organisations (symbols, images, diagrams, graphics, etc.) that are meant to represent or to encode a particular "mathematical reality". Internal representations are not directly observable and we may only infer them by means of the observable actions of individuals or through their interactions with external representations (Goldin, 2008).

Instead of using the term representations, Mason (1987) speaks of diverse modes of representing mathematical ideas. In this perspective, the activity of giving meaning to an idea is a result of the circuit established between "manipulation" and "expression". *Manipulation* includes the use of physical objects, the production of drawings and diagrams, either on paper or in the mind, as well as the use of symbols. Otherwise *expression* involves the act of articulating a certain experience that may appear as fuzzy, vague or ill-defined. Expression requires the intention to "say" something and to be able "to register" on paper the sense of what is said, through drawings, symbols or words, that is, by means of diverse modes of representation. Therefore, modes of representation can be used both as objects for *manipulation* and as means of *expression*.

Preston & Garner (2003) suggest that we may consider the following modes of representation: i) written natural language to explain reasoning and strategies, and to complement other modes of representation; ii) pictorial, through the use of drawings or images to present, coordinate and systematise information; iii) arithmetical, frequently associated with strategies of guess and trial, of unwind and also the creation of numerical tables; iv) graphical, involving the use of graphs of continuous and discrete variables as a way to clearly understand their behaviour; v) algebraic, which involves the use of symbolic language namely to make generalisations.

Representations can work as vehicles to understand and interpret mathematical ideas and tools for developing individual strategies in problem solving, as they may well operate as multiple instances of a concept. A certain concept or mathematical structure can be subsumed under different representations and the discovery of common properties in such representations contributes to the assimilation of the concept or structure. For instance, solving a system of two equations in two variables may be represented by finding the intersection of two straight lines.

INFORMAL AND FORMAL METHODS

In this study we are adopting the perspective of Realistic Mathematics Education (RME) in the sense that understanding mathematical concepts is to be seen as a process that evolves from informal methods, connected to real contexts and experiences, to strictly mathematical formal methods. The theory of RME is based on the assumption that doing mathematics is a human activity. The teacher, instead of trying to create bridges with an already set Mathematics, must help students build mathematics in a justified way, through *guided reinvention*. Given a contextual problem students have to model it, and this may include drawings, diagrams, or tables, informal notations or the use a more formal mathematical notation. Working on these models helps students to reinvent the intended more formal mathematics. Initially, the models relate to specific situations that are real experiences for students, and thus allowing them to come up with informal strategies. Later, when students are faced with similar problems, the models become more general and useful as a basis for mathematical reasoning and not solely as a way to represent a problem in its context. Thus a *model of* informal mathematical activities develops into a *model for* a mathematical reasoning, which means that there is a shift from thinking about modelling the situation in a particular context to thinking about mathematical relationships. The touchstone of this approach is that through contextual problem solving students move to higher levels of general understanding (Gravemeijer, 2005).

METHODOLOGY

SUB 14 is a problem solving competition, sponsored by the University of Algarve, addressing students of grades 7 and 8 (13-14 years old). During the qualifying, which takes place between January and June, problems are posted on a website, to which students answer by email, receiving feedback from the organization. The final, live, consists in solving a set of five problems, with paper and pencil. The analysis of students' written productions in such competitions is a way to understand how they think, interpret, conjecture and which methods and representations they use. It allows us also to have a perception of how they put into practice their school knowledge and informal knowledge outside the classroom. From the analysis of several solutions presented to an algebraic/numerical problem, we want to realise the presence of informal and formal methods and how they rely on certain representations.

The methodological approach is exploratory and draws on a qualitative and interpretative perspective of data analysis. All the answers to the problems given at the final were collected and reviewed, taking into account a set of items, including the representations displayed in the written papers. The problem is the following:

> Mr. José has three daughters who love to eat sweets: Joana, Josefina and Júlia. In the summer as they went to the beach they started to worry about their figure. So they decided to go on a diet and the three of them regularly weighed on a big scale their father keeps in his store. Before starting on diet they stepped on the scale, in pairs.
> Joana and Josefina both weighed 132 kg. Josefina and Júlia both weighed 151 kg. Júlia and Joana both weighed 137 kg. How much did each of the sisters weighed?

Figure 1: The proposed problem

This problem may be solved using a system of three equations. However, the students to whom it was proposed had not studied such linear systems at school yet. One thing, quite noticeable, that may facilitate in solving the problem is that the weight measurements were made in pairs, leading to only two unknowns in each equation.

The 64 solutions collected were examined and grouped according to the predominant mode of representation involved. Based on the theoretical framework of the study and in light of the data itself, we created three broad categories: arithmetic language, arithmetic/algebraic language, and symbolic language. The second category emerged from the data, as it was impossible to include some solutions in the other two categories. Therefore, it corresponds to hybrid solutions. Within the three categories, we distinguished correct, partially correct, and incorrect answers. For reasons of limited space we chose to make a detailed analysis of a selection of four correct answers that belong to all three categories.

RESULTS

Table 1 shows the type of answers given by the finalists. An arithmetic strategy is one that is strictly based on the use of elementary operations and on trial and error. In turn, an algebraic strategy corresponds to one in which students make use of algebraic symbolism – unknowns for writing equations that are solved afterwards by the usual rules. After analysing the participants' answers we found that some students began to write the three equations suggested by the problem but then diverged to the use of trial and error, thus generating another category – algebraic/arithmetic.

Mode of representation	Correct	Incomplete and/or with some errors	Blank
Arithmetic	14	11	-
Algebraic/arithmetic	2	5	-
Algebraic	11	2	-
Total	27	18	19

Table 1: Types of students' answers to the problem

Around 40% of students answered the problem correctly. Arithmetic strategies were the most used in all responses, which agrees with other research results. With regard to the representations used by the students we will then analyse what we consider to be some illustrations of the various modes of external representation obtained. Most students used natural language, in combination with mathematical language, through different modes of representation.

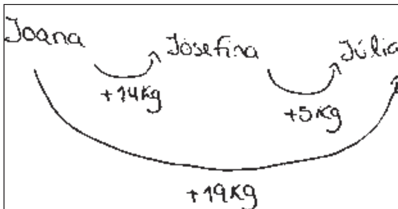
Mode of representation – Arithmetic Language

> Since the difference between the combined weight of Joana and Josefina and the combined weight of Joana and Júlia is 5 Kg (137-132=5), we know that Joana weighs 5 Kg more than Josefina. Therefore (151-5):2 = Weight of Josefina.
>
> So, we know that Josefina weighs 73 Kg and Júlia must weigh 78 Kg (73 + 5).
>
> Then Joana weighs 132-73=59 Kg …

Figure 2: Answer of participant nº. 270 (grade 8)

This participant uses a strategy of unwinding followed by explanations in natural language, which clarify the successive steps of the solution. Implicitly, this student is actually working with informal methods that match formal algebraic procedures to solve of a system of three simultaneous equations, using ordered addition.

Mode of representation – Arithmetic/Algebraic Language

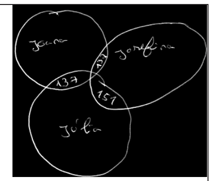


> … We can see that Joana is the lightest and Júlia is the heaviest:
>
> Assuming that Joana weighs 60 Kg, than Josefina would weigh 74 Kg, since 60+14=74. So, the sum of their weights would be 134 Kg. But that's not what the problem states. Instead they both weigh 132 Kg. This means that the guessed weight is larger than the real one. There is a difference of 2 Kg. So, if Joana weighed 59 Kg, than Josefina would weigh 73 Kg, since 59+14=73. And the sum of the two weighs is 59+73=132, as the problem states. This way, as Júlia weighs 5 Kg more than Josefina, her weight will be 78 Kg: 73+5=78. And to check that Josefina and Júlia both weigh 151 Kg: 73+78=151. And to check that Júlia and Joana both weigh 137 Kg: 78+59=137 …

Figure 3: Answer of participant n. º 180 (grade 8)

This is an example of a student who uses a diagram displaying information that complements the givens in the problem and which works as a means of controlling the results that come from guess and trial.

…First I draw three circles together, each with the name of each sister. I put the weight of them together in pairs. I noticed that the sisters Josefina and Júlia were fatter, and her sister Joana below 60 kg and the other two sisters with over 70 kg. Since the difference between the combined weights of Júlia and Josefina and the combined weights of each of the other pairs was more than 10 Kg: 151-132=19 and 151-137=14.



I assumed Joana weighs 59 kg, Josefina will weigh 59+19= 78 kg and Júlia will weigh 59+14=73 kg. To calculate the weights of the sisters we just add the weight of Joana with the difference of the following weights:

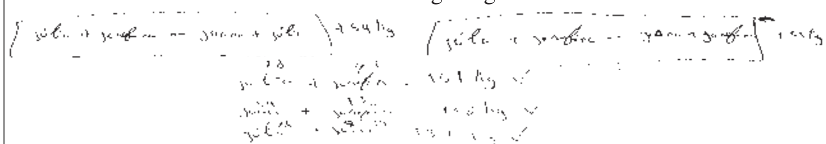


Figure 4: Answer of participant n. º 12 (grade 8)

This participant begins writing the combined weights of the sisters and then explains the procedure for the presentation of the circles diagram. This representation shows only the information in the problem, instead of the previous case that signalled the differences of weights between the two girls in each pair.

Mode of representation – Algebraic Language

Student **235** (Figure 5) begins writing the three equations suggested by the problem, and solves them with the usual rules.

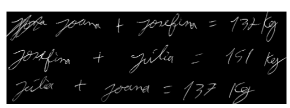
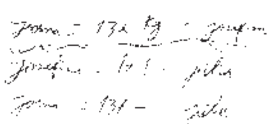
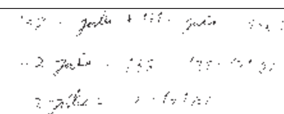
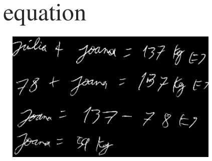
This problem is solved with equations as follows: 1. we translate everything to mathematical language  2. we write the expressions so that one girl is isolated in one of the sides:  3º We replace the girls in one of the equations so that we get one equation referring to only one girl:	 4. we solve the equation 5. now as we already know one of the girls, we find another one  6. as we know two of the girls, we get the last one Therefore, we find Júlia=78; Joana=59; Josefina=73

Figure 5: Answer of participant n. º 235 (grade 8)

Natural language is used to explain all the procedures. Using algebraic language, the student assigns the sisters names to the variables, still indicating some uncertainty with the symbolism of letters and a strong influence of the context in the solution of the problem.

CONCLUDING REMARKS

After examining the representations produced by the students we found that most of them, at the right time, made use of particular representations, so that their manipulation became an active mathematical way of finding the solution to the problem. This suggests that it is important that students develop positive attitudes towards seeking representations that help them to approach a problem. The spontaneous choice of alternative representations may be a strategy to overcome students' difficulties (Dufour-Janvier *et al.*, 1987).

The use of symbolic/algebraic representations in solving this problem proved to be important because most of the students who selected this kind of representation could find the solution. However those who used informal algebraic/arithmetic representations and unwind strategies or even trial and error also revealed the usefulness of such approaches. In fact, these representations as students' answers illustrate, provide a consistent basis for a more symbolic representation (Johanning, 2004). This is a very important aspect, given the role of informal methods to find solutions and to anticipate the formal procedures (Streefland, 1991).

The evolving process of mathematics learning in a problem solving environment emphasizes the fact that different systems of representation can be associated with some mathematical ideas. It is important to encourage students to represent their mathematical ideas in a way that makes sense for them, even if these representations are not conventional. However it is important that students learn conventional forms of representation to improve their learning and their communication about mathematical ideas. Problem solving competitions can be a good vehicle to encourage the use of a wide variety of representations. It allows making connections between different types of representation and the transition among them expanding students' mathematical knowledge (Dufour-Janvier *et al.*, 1987).

The modes of arithmetic and arithmetic/algebraic representation displayed in the data of this study give a clear idea that this problem offers an environment for students to work in relations of dependence between variables. From our point of view this is a manifestation of an algebraic thinking largely intuitive, but actually powerful and relevant to the learning of formal methods, particularly those of formulating and solving a system of linear equations.

References

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 109-122). Hillsdale, NJ: Lawrence Erlbaum.

Goldin, G. (2008). Perspectives on representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education* (2nd ed.). New York, NY: Routledge.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177.

Gravemeijer, K. P. E. (2005). What makes mathematics so difficult, and what can we do about it? In L. Santos, A. P. Canavarro, & J. Brocardo (Eds.), *Educação matemática: Caminhos e encruzilhadas* (pp. 83-101). Lisboa: APM.

Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *Journal of Mathematical Behavior*, *23*, 371-388.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. M. Alvarez, B. Hodgson, C. Laborde & A. Pérez (Eds.), *Eighth International Conference on Mathematics Education: Selected Lectures* (pp.271-290). Seville, Spain: SAEM Thales.

Kieran, C. (2006). Research on learning and teaching of algebra. In A. Guitierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future*. (pp. 11-49). Rotterdam: Sense.

Mason, J. (1987). Representing representing: Notes following the conference. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 207-214). Hillsdale, NJ: Erlbaum.

NCTM (2000). Principles and Standards for School Mathematics. Reston, VA: NCTM.

Ponte, J. P. (2006). Números e Álgebra no currículo escolar. In I. Vale *et al.* (Orgs.). *Números e Álgebra na aprendizagem da matemática e na formação de professores*. (pp.5-27). Lisboa: Secção de Educação Matemática da SPCE.

Preston, R., & Garner, A. (2003). Representation as a vehicle for solving and communication. *Mathematics Teaching in the Middle School, 9*, 38-43.

Streefland, L. (1991). Fractions, an integrated perspective. In L. Streefland (Ed.). *Realistic Mathematics Education in Primary School* (pp. 93-118). Utrecht: Freudenthal Institute.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making Sense of Word Problems*. Lisse: Swets & Zeitlinger.