Effective Computability of Solutions of Differential Inclusions
The Ten Thousand Monkeys Approach

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Abstract: In this paper we consider the computability of the solution of the initial-value problem for differential inclusions with semicontinuous right-hand side. We present algorithms for the computation of the solution using the "ten thousand monkeys" approach, in which we generate all possible solution tubes, and then check which are valid. In this way, we show that the solution of an upper-semicontinuous differential inclusion is upper-semicomputable, and the solution of a differential inclusion defined by a one-sided locally Lipschitz function is lower-semicomputable computable. We show that the solution of a locally Lipschitz differential equation is computable even if the function is not effectively locally Lipschitz. We also recover a result of Ruohonen, in which it is shown that if the solution is unique, then it is computable, even if the right-hand side is not locally Lipschitz. We also prove that the maximal interval of existence for the solution must be effectively enumerable open, and give an example of a computable locally Lipschitz function which is not effectively locally Lipschitz.
Key Words: Ordinary differential equations, differential inclusions, Lipschitz condition, computable analysis, semicomputability

1 Introduction

In this paper we study the computability of initial-value problems defined with ordinary differential equations and differential inclusions. Formally, let E be a domain in $\mathbb{R} \times \mathbb{R}^m$, $f : E \to \mathbb{R}^m$ be a continuous function and $F : E \rightrightarrows \mathbb{R}^m$ be a (semicontinuous) multivalued function. We consider the initial-value problem for a ordinary differential equation defined by

$$\dot{x} = f(t, x); \qquad x(0) = x_0, \tag{1}$$

and the initial value problem for differential inclusion

$$\dot{x} \in F(t, x); \qquad x(0) = x_0. \tag{2}$$

For simplicity of discussion, though not essential, we take the initial time to be $t_0 = 0$. The question of interest in this paper is to find conditions on f and F in order that one might be able to *effectively* compute the solution of (1) or (2).

The standard condition used in literature to study solutions of (1) is to assume that f is Lipschitz, in which case the problem (1) has a unique solution (see e.g. [5]). The proof of this result can be effectivised [13], [1], [3], [11] in order to show that solution can be computed from x_0 and f.

If the function f is continuous but not Lipschitz, then Peano's existence theorem guarantees the existence of at least one solution, but the solution need not be unique. In the case that the solution of (1) is not unique, it can happen that none of the solutions is a computable function [2], [14] even if f is a computable function. However, in [18] it was shown that if the solution of (1) is unique, then it can be computed from x_0 and f.

The results of [18] have the desirable characteristic of not demanding f to be globally Lipschitz on its domain. While the globally Lipschitz condition ensures that a solution for (1) exists and is unique, it may be too strong as a requirement. This has already been noticed by mathematicians when establishing existence and uniqueness results for solutions of (1) in unbounded domains, where a Lipschitz condition usually does not hold. To circumvent this problem, a possible approach is to require the Lipschitz condition to be satisfied only *locally*. Under these conditions, the solution of (1) can be shown unique — see e.g. [5].

Following this reasoning, in [9], the authors introduce a notion of effectively locally Lipschitz functions and show that if f in (1) has this property, then the solution of (1) can be computed from f and x_0 over the maximal interval of existence. The existence of such maximal interval follows from standard results from the theory of ODEs, that ensure that when computing the solution of (1) one can extend it until the solution becomes unbounded or reaches the boundary of the region where f is defined. In particular, the results in [9] yield the (expected) computability of the solution of the initial-value problem $\dot{x} = -x^2$, $x(0) = 1$ over $(0, +\infty)$. The effectivity in this definition is required in order that one might be able to pick for each compact set an appropriate Lipschitz constant to be used on the computation of the solution as soon as it reaches the aforementioned compact set.

The problem (2) for differential inclusions is typically studied for functions F which are convex-valued, and either Lipschitz lower-semicontinuous or upper-semicontinuous [7], [4]. The classical result is that if F is defined on the whole of $\mathbb{R} \times \mathbb{R}^m$ and is upper-semicontinuous with compact convex values and sublinear growth at infinity, then there is at least one solution defined for all times, and that the set of all solutions varies upper-semicontinuously with the initial condition. If F is locally Lipschitz continuous, then the solutions vary continuously with the initial conditions. It was recently shown [8] that the Lipschitz condition can be relaxed to a one-sided locally Lipschitz conditions. In the case that F is Lipschitz continuous, computability was shown in [16], [19].

In this paper we will prove that we can compute over-approximations of the solutions of (2) requiring only that F is upper-semicontinuous. We prove this result by giving an explicit algorithm based on the ten thousand monkey theorem — ten thousand monkeys[1] hitting keys at random on a typewriter keyboard will eventually type a particular chosen text. The idea is that we can approximate the solutions of $\dot{x} \in F(t,x)$, $x(0) \in X_0$ from above with a union of boxes, with arbitrary precision. These boxes satisfy some relations between them, which only apply to this kind of covering, which can be effectively checked. Now we just need to use the ten thousand monkey theorem to run computations over all the possible finite sequences of unions of boxes. We can check whether each sequence of boxes constitute a valid covering, and we know that such a covering approaching the solution of $\dot{x} = f(t,x)$, $x(0) \in X_0$ with an arbitrary preassigned precision exists. Since, as we will see, the solution set of $\dot{x} = f(t,x)$, $x(0) \in X_n$ will converge to the solution of (1), it is enough to keep computing these sequences until we get a finite sequence of boxes which cover the solution at time t with the desired precision. We do not need any conditions bounding the growth at infinity; instead, solutions are computable on their domains of definition.

We will also consider approximations to the solutions of (2) in the case that F is lower-semicontinuous. In this case, we do need an extra condition, namely the one-sided locally Lipschitz condition used in [8], in order to obtain lower-semicontinuity of the solution set. We obtain, for each time t, a collection of open boxes, each of which is guaranteed to contain a solution. Our results extend those of [8] since we do not require F to be continuous or even bounded; lower-semicontinuous with closed convex values suffices for one-sided locally Lipschitz functions. A further strengthening of [8] is that we obtain a continuous selection of the solution set through any given reference solution.

In the case of a Lipschitz differential equation, we give a simpler version of the algorithm in which the (necessarily unique) solution is contained in a single box at all times. However, we give a counterexample demonstrating that this algorithm is not powerful enough to compute the solution of a non-Lipschitz differential equation even if the solution is unique. Instead, in the non-Lipschitz case we need to allow the solution bound to be a finite union of boxes, giving an equivalent algorithm to the case of an upper-semicontinuous differential inclusion.

We now briefly sketch the contents of the paper. In Section 2 we present some standard results from ODEs, computable analysis, and Analysis that will

[1] In [6] we only used a thousand monkeys, but in the case of differential inclusions it is useful in practise to have more monkeys

support the main results of the paper. In Section 3 we show that our results are not a special case of those obtained in [9] since, as we will prove, there are computable functions which are locally Lipschitz (guaranteeing uniqueness of the solution), but not effectively locally Lipschitz. In Section 4 we prove that the solution of (1) can be computed provided f is continuous and the solution of (1) is unique. To achieve this result we present a first algorithm, that as we will see is not enough to compute the solution of (1), but will serve as a useful subroutine for a second algorithm that computes the desired solution. The results of this paper extend those of the earlier paper [6], and give more complete proofs.

2 Preliminaries

This section introduces concepts and results from the theory of differential equations, differential inclusions, computable analysis and real analysis. For more details the reader is referred to [5, 12] for ordinary differential equations, [7, 4] for differential inclusions, and [15, 11, 20] for computable analysis.

2.1 Standard results from differential equatoins/inclusions

We now recall some basic results concerning initial-value problems defined with ODEs and DIs.

Let E be a domain in $\mathbb{R} \times \mathbb{R}^m$ and $f : E \to \mathbb{R}^m$. Recall that f is *locally Lipschitz* in the second argument if for each compact $K \subset E$, there exists some constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L\,|x - y| \ \text{ whenever } (t,x),(t,y) \in K.$$

Let us consider the initial-value problem (1). The following theorem can be found in [10] (see also [5], [12]).

Theorem 1 (Existence and uniqueness of solutions of ODEs). *Let E be a domain over $\mathbb{R} \times \mathbb{R}^m$ and $f : E \to \mathbb{R}^m$ be a continuous function. Then the initial-value problem $\dot{x} = f(x)$; $x(0) = x_0$ has a solution y defined on some open interval containing 0. Every solution y can be extended to a solution on some maximal interval (α, β), and $y(t) \to \partial E$ as $t \to \alpha, \beta$. If f is locally Lipschitz in the second argument, then the maximal solution y is unique.*

Let us now consider the initial-value problem (2) for differential inclusions. It is a standard result that an absolutely continuous function is differentiable at almost every point in its domain.

Definition 2. A function $\xi : [0,T) \to \mathbb{R}^n$ is a solution of the differential inclusion $\dot{x} \in F(t,x)$ if ξ is absolutely continuous, and $\xi'(t) \in F(t, \xi(t))$ for almost every $t \in [0,T)$.

The *solution operator* of a differential inclusion $\dot{x} \in F(t,x)$ is the function $S_F : \mathbb{R}^m \rightrightarrows C(I, \mathbb{R}^m)$ taking each point x_0 to the set of solution curves $\xi : I \to \mathbb{R}^m$ with $\xi(0) = x_0$. The *flow operator* is the function $\Phi_F : \mathbb{R} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ taking (t, x_0) to $\{\xi(t) \mid \xi \in S_F(x_0)\}$.

The following result (see [7],[4]) is a classical existence theorem for solutions of upper-semicontinuous differential inclusions.

Theorem 3 (Solutions of upper-semicontinous differential inclusions). *Assume that a convex compact valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is upper-semicontinuous in $x \in \mathbb{R}^n$ and measurable in $t \in \mathbb{R}$. Assume moreover that F has sublinear growth at infinity, i.e. there is a constant c such that $|F(t,x)| \leq c(1 + |x|)$ for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.*

Then for every initial point x_0 there is a function $y : \mathbb{R} \to \mathbb{R}^n$ such that $y(0) = x_0$, $y(\cdot)$ is absolutely continuous and $\dot{y}(t) \in F(t, y(t))$ almost everywhere. Further, the flow operator $\Phi_F : \mathbb{R} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is continuous in the first argument and upper-semicontinuous in the second argument.

Let E be a domain in $\mathbb{R} \times \mathbb{R}^m$ and $F : E \rightrightarrows \mathbb{R}^m$. We say F is *one-sided Lipschitz* with constant L if for every $(t,x), (t,y) \in E$ and $f_x \in F(t,x)$, there exists $f_y \in F(t,y)$ such that

$$(x - y) \cdot (f_x - f_y) \leq L||x - y||^2. \tag{3}$$

The following theorem [8] generalises existence theorems for Lipschitz differential inclusions.

Theorem 4 (Solutions of lower-semicontinous differential inclusions). *Assume that a convex compact valued map $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and one-sided Lipschitz continuous with constant L. Assume moreover that F has a sublinear growth at infinity.*

Then there exists a continuous selection of the solution map $S_F : \mathbb{R}^n \rightrightarrows C(I, \mathbb{R}^n)$ corresponding to the differential inclusion $\dot{x} \in F(x)$.

2.2 Standard results from computability

We now continue our discussion with a presentation of fundamental concepts of computable analysis, which provides a notion of computability over the reals.

We define computability in terms of *open rational boxes*, which are sets of the form $(a_1, b_1) \times \cdots \times (a_m, b_m) \subset \mathbb{R}^m$ where $a_i, b_i \in \mathbb{Q}$ for $i = 1, \ldots, m$. We could equally well use open rational balls $B(a, r)$ where $a \in \mathbb{Q}^m$ and $r \in \mathbb{Q}$ with $r > 0$, or even rational convex polyhedra $P = \{x \in \mathbb{R}^m \mid Ax \leq b\}$ where $A \in \mathbb{Q}^{n \times m}$ and $b \in \mathbb{Q}^n$.

Definition 5.

1. A name for a point $x \in \mathbb{R}^m$ is a sequence of nested open rational boxes (I_n) such that $\cap_{n=1}^{\infty} I_n = \{x\}$.

2. A name for an open set $U \subset \mathbb{R}^m$ is a sequence of open rational boxes (I_n) such that $\overline{I}_n \subset U$ for all $n \in \mathbb{N}$, and $U = \bigcup_{n=1}^{\infty} I_n$.

3. A name for a function f is a list of all pairs of open rational boxes (I, J) such that $f(\overline{I}) \subseteq J$.

4. A name for a lower-semicontinuous multivalued function F is a list of all pairs of open rational boxes (I, J) such that $\overline{I} \subseteq F^{-1}(J)$.

5. A name for a upper-semicontinuous multivalued function F is a list of all tuples of open rational boxes $(I, J_1, \ldots, J_k)$ such that $F(\overline{I}) \subseteq \bigcup_{i=1}^{k} J_i$.

Definition 6. A point $x \in \mathbb{R}^m$ is computable if it has a computable name. An open set $U \subset \mathbb{R}^m$ is recursively-enumerable if it has a computable name. A function f is computable it has a computable name. A lower (upper) semicontinuous function F is lower (upper) semicomputable if it has a computable name.

If Y and Z are spaces with an associated naming system, then an operator $f : Y \to Z$ is computable if there is a computable function which associates each name of $y \in Y$ to a name of $f(y) \in Z$.

2.3 Effective Lipschitz properties

The following definition was introduced in [9], and gives a computable counterpart for the notion of a function which is locally Lipschitz in the second argument.

Definition 7. Let $E = \bigcup_{n=0}^{\infty} B(a_n, r_n) \subseteq \mathbb{R}^m$ be a recursively enumerable open set, where $a_n \in \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$ yield computable sequences satisfying $\overline{B(a_n, r_n)} \subseteq E$. A function $f : E \to \mathbb{R}^m$ is called effectively locally Lipschitz in the second argument if there exists a computable sequence $\{K_n\}$ of positive integers such that

$$|f(t, x) - f(t, y)| \leq K_n |y - x| \text{ whenever } (t, x), (t, y) \in \overline{B(a_n, r_n)}.$$

The following result was proved in [9].

Theorem 8. *Let $E \subseteq \mathbb{R}^{m+1}$ be a recursively enumerable open set and $f : E \to \mathbb{R}^m$ be an effectively locally Lipschitz function in the second argument. Let (α, β) be the maximal interval of existence of the solution $x(t)$ of the initial-value problem (1), where (t_0, x_0) is a computable point in E. Then:*

1. *The operator $(f, x_0) \mapsto (\alpha, \beta)$ is semicomputable (i.e. α can be computed from above and β can be computed from below), and*

2. *The operator $(f, x_0) \mapsto x(\cdot)$ is computable.*

2.4 Standard results from analysis

In this section we state three classical theorems that are the main tools in finding solutions of differential equations and inclusions.

Theorem 9 (Arzela-Ascoli). *Let X and Y be locally-compact metric spaces and ξ_n be a sequence of uniformly bounded uniformly equicontinuous functions $X \to Y$. i.e. $\forall \epsilon > 0,\ \exists \delta > 0\ \forall n \in \mathbb{N}, x, y \in X,\ d(x,y) < \delta \implies d(\xi_n(x), \xi_n(y)) < \epsilon$. Then there is a subsequence ξ_{n_k} which converges uniformly to a continuous function ξ_∞.*

Theorem 10 (Banach-Alaoglu). *Let X be a Banach space and X^* the dual space with the weak-* topology. i.e. the topology generated by open sets of the form $\{g \in X^* : |g(x) - f(x)| < \epsilon\}$ for $f \in X^*$, $x \in X$ and $\epsilon > 0$. Then X^* is locally compact.*

In particular, since L^∞ is the dual of L^1, the space L^∞ is locally compact, and the closed unit ball in L^∞ is compact.

Combining the above results, we obtain the following corollary.

Corollary 11. *Suppose f is a continuous function and ξ_ϵ are uniformly bounded absolutely continuous functions satisfying $||\dot{\xi}_\epsilon(t) - f(\xi_\epsilon(t))|| < \epsilon$ for almost all $t \in [0,T]$. Then the functions ξ_ϵ have at least one limit point, and any limit point of the functions ξ_ϵ as $\epsilon \to 0$ is a solution of the differential equation $\dot{x} = f(x)$.*

Proof. Suppose that there is a convergent sequence η_n corresponding to ϵ_n with limit η_∞. Then the sequence $f(\eta_n)$ converges uniformly to $f(\eta_\infty)$. By integration, we have $\dot{\eta}_\infty = f(\eta_\infty)$.

For differential inclusions we will need the following stronger corollary.

Corollary 12. *Suppose $F : \mathbb{R} \rightrightarrows \mathbb{R}$ is an upper-semicontinuous function with compact values and ξ_ϵ are absolutely continuous functions such that for almost all $t \in [0,T]$, there exists s with $|s-t| < \epsilon$ and $||\dot{\xi}_\epsilon(t) - F(\xi_\epsilon(s))|| < \epsilon$. Then any limit point of the functions ξ_ϵ as $\epsilon \to 0$ is a solution of the differential inclusion $\dot{x} \in F(x)$.*

Proof. Since the functions ξ_ϵ are uniformly-bounded and F is compact-valued, the derivatives $\dot{\xi}_\epsilon$ are uniformly-bounded almost everywhere, so the functions are uniformly equicontinuous (indeed, they are uniformly Lipschitz). By the Arzela-Ascoli theorem, any sequence ξ_{ϵ_n} has a convergent subsequence, and the convergence is uniform, so any limit is absolutely continuous.

Suppose that there is a convergent sequence ζ_n corresponding to ϵ_n with limit ζ_∞, and let C be a compact set such that $\dot{\zeta}_n(t) \in C$ for almost all $t \in [0,T]$. Since from the Banach-Alaoglu theorem L^∞ is locally-compact, by taking a subsequence of the ζ_n if necessary, there exists an L^∞ function δ_∞ such that $\zeta_n \to \delta_\infty$ as $n \to \infty$. Since the antidifferentiation operator is continuous from L^∞ to AC, we must have $\delta_\infty = \dot{\zeta}_\infty$. Fix t and suppose that $\dot{\zeta}_n(t)$ converges to $\dot{\zeta}_\infty(t)$. Take s_n and v_n such that $|s_n - t| < \epsilon_n$, $v_n \in F(\zeta_n(s_n))$ and $||\dot{\zeta}_n(t) - v_n|| < \epsilon_n$. Since the v_n remain in a bounded set, by passing to a subsequence if necessary, we can assume v_n converges. Then $\dot{\zeta}_\infty(t) = \lim_{n\to\infty} v_n$ and $\zeta_\infty(t) = \lim_{n\to\infty} \zeta_n(s_n)$ with $(\zeta_n(s_n), v_n) \in \text{graph}(F)$. Since F is upper-semicontinuous, graph(F) is closed, and so $\dot{\zeta}(t) \in F(\zeta(t))$. Since $\dot{\zeta}_n(t)$ converges to $\dot{\zeta}_\infty(t)$ for almost every t, the result follows.

The following result is useful in finding solutions of lower-semicontinuous differential inclusions.

Theorem 13 (Michael). *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be lower-semicontinuous with nonempty closed convex values. Suppose $x \in \mathbb{R}^n$ and $y \in F(x)$. Then there is neighbourhood U of x in $\mathbb{R}^n$ and a continuous function $f : U \to \mathbb{R}^n$ such that $f(x) = y$ and $f(w) \in F(w)$ for all $w \in U$.*

Definition 14. A function $\delta : \mathbb{Q}^+ \to \mathbb{Q}^+$ is a *modulus of continuity* on $U \subset X$ if for all $x, y \in U$, $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta(\epsilon)$.

Note that in the above definition, we take $\mathbb{Q}^+$ to be the set of strictly positive rationals.

3 A computable, non-effectively locally Lipschitz function

Before presenting our main result of this section, we start with a Lemma, which is an adaptation of a result from Radó [17]. We outline the proof, since it is needed for what follows.

Lemma 15. *There exists a function $S : \mathbb{N} \to \mathbb{N}$ and a computable function $u : \mathbb{N}^2 \to \mathbb{N}$ such that:*

1. *For any given computable function $f : \mathbb{N} \to \mathbb{N}$, there exist infinitely many $n \in \mathbb{N}$ for which $S(n) > f(n)$. In particular, S is non-computable.*
2. *For any given $n \in \mathbb{N}$, one has $\lim_{i\to+\infty} u(n, i) = S(n)$ and $u(n, i) \leq S(n)$ for every $i \in \mathbb{N}$.*
3. *For any $n, i \in \mathbb{N}$, one has $u(n, i) \leq i$.*

Proof. We recall the busy beaver game introduced by Radó in [17]. Consider a one-tape Turing machine M with binary alphabet $\{0, 1\}$, where 0 stands for the blank symbol. Suppose also that M has n non-halting states, plus one halting

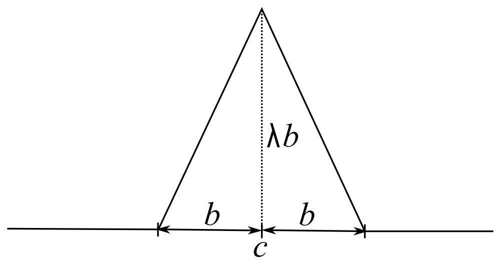

Figure 1: A spike function.

state. In this case we say that M has n operational states. Now run the machine with the tape initially filled only with blanks. We also assume that the head must always move either left or right in a transition. Now define:
(i) E_n as the set of n-operational state, 2-symbol Turing machines that halt when run on a blank tape.
(ii) $s(M) =$ the number of steps M takes before halting, for any $M \in E_n$.
(iii) $S(n) = \max\{s(M)|M \in E_n\}$.

Some remarks are in order. Notice that the set E_n is non-empty since, for every $n \in \mathbb{N}$, there is a n-operational state Turing machine which always halts (just take the Turing machine where every possible transition finishes on the halting state). Moreover E_n is finite, since the number of n-operational state, 2-symbol Turing machines is finite.

From all of the above, we conclude that we have defined a total function $S : \mathbb{N} \to \mathbb{N}$, the *maximum shift function*. Condition 1 of the Lemma was proved in [17].

For condition 2 of the Lemma, define $u(n, i)$ by the following algorithm:

1. Enumerate all n-operational state, 2-symbol Turing machines;
2. For every Turing machine obtained in Step 1, run it on a blank tape for i steps, or until it stops. From those machines who halted, count the number of steps needed to reach the halting configuration, and return the maximum of these numbers as $u(n, i)$.

It is easy to see that this function u satisfy conditions 2 and 3 of the Lemma.

We now provide a brief sketch of an example of a computable function which is locally Lipschitz, but not effectively so. First, for $\lambda \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, +\infty)$, we define a "spike" function $g_{\lambda,b,c} : \mathbb{R} \to \mathbb{R}$ as depicted in Fig. 1. The function g_λ is always 0, except in some interval $[c-b, c+b]$, where it increases with slope λ, until it reaches half-way of the interval, and then decreases with slope $-\lambda$

giving origin to a "spike", centered on the midpoint c of that interval. We now define a sequence of computable functions $\{f_i\}_{i\geq 1}$ such that, on each interval $[n, n+1)$, $n \geq 1$, f_i is constituted by $i+1$ spikes, in intervals of the format $[n, n+1/2], [n+1/2, n+3/4], \ldots, \left[n-1+\sum_{j=0}^{i} 2^{-j}, n-1+\sum_{j=0}^{i+1} 2^{-j}\right]$. A spike in the interval

$$\left[n+\sum_{j=0}^{m} 2^{-j}, n+\sum_{j=0}^{m+1} 2^{-j}\right] \tag{4}$$

with $n \in \mathbb{N}_0$, $0 \leq m \leq i$ is defined with the following parameters: $\lambda = u(n,m)$ (slope), $c = n + \sum_{j=0}^{m} 2^{-j} + 2^{-m-2}$ (midpoint, which is within distance 2^{-m-2} from the extremities of the interval (4)) and, if $u(n,m) \neq 0$ (and hence $u(n,m) \geq 1$), $b = 2^{-m-2}/u(n,m)$. With this definition the non-zero part of the spike is entirely contained in the interval (4), and its height (assuming that $u(n,m) \neq 0$. If it is $u(n,m) = 0$, the height is 0) is 2^{-m-2}.

Notice that f_i and f_{i+1} have the same first i spikes, the difference being that f_{i+1} has an extra spike with height at most 2^{-i-2}. This shows that the sequence $\{f_i\}_{i\geq 1}$ converges uniformly to a computable function f with the property that in the time interval $[n+1, n+2)$ it has infinitely many decreasing spikes, with slope $u(n,0), u(n,1), \ldots$.

The function f is locally Lipschitz: On an interval $[n, n+1]$, with $n \in \mathbb{N}$, it satisfies

$$|f(x) - f(y)| \leq K_n |x - y|$$

iff $K_n \geq \lim_{k\to+\infty} u(n,k) = S(n)$, where the function S is defined in Lemma 15. In general, $|f(x) - f(y)| \leq K|x-y|$ for a compact $B \subseteq \bigcup_{i=1}^{m} [a_i, a_i+1)$, with $a_i \in \mathbb{N}$, where

$$K \geq \max_{1\leq i\leq m} K_{a_i}.$$

Let us show that f is not effectively locally Lipschitz on $\mathbb{R}$. Suppose, for contradiction, that there are computable sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in \mathbb{Q}$ and $r_n \in \mathbb{Q}$ such that

$$\mathbb{R} = \bigcup_{n=0}^{\infty} B(a_n, r_n)$$

and a computable sequence $\{L_n\}$ of positive integers such that

$$|f(x) - f(y)| \leq L_n |x-y| \text{ whenever } x, y \in \overline{B(a_n, r_n)}.$$

Then we show that we will be able to present a computable function $g : \mathbb{N} \to \mathbb{N}$ which satisfies $g(n) \geq S(n)$ for all $n \in \mathbb{N}$, thus deriving the desired contradiction. How can we compute $g(n)$ for an arbitrary $n \in \mathbb{N}$? First notice that $[n, n+1]$ is a compact set. Thus there must exist $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$[n, n+1] \subseteq \bigcup_{j=1}^{k} B(a_{n_j}, r_{n_j}) \tag{5}$$

If we can compute the values $n_1, \ldots, n_k \in \mathbb{N}$ satisfying (5), then we can take

$$g(n) = \max_{1 \le j \le k} L_{n_j} \ge S(n).$$

The remaining issue is to compute the values $n_1, \ldots, n_k \in \mathbb{N}$ satisfying (5). We can do that with the following algorithm:

1. Start with $k = 0$.
2. Check if

$$[n, n+1] \subseteq \bigcup_{i=0}^{k} B(a_i, r_i).$$

If yes, return $0, \ldots, k$, else increment k and go to Step 2.

Notice that this algorithm always stops. Thus the result is proven. The above algorithm could certainly be improved to discard those balls which provably do not overlap $[n, n+1]$, but this is not necessary for our construction and therefore we avoid this step in order to prevent unneeded technical complications.

We thus have proved the following result.

Theorem 16. *There is a computable function $f : \mathbb{R} \to \mathbb{R}$ which is locally Lipschitz, but not effectively so.*

In particular, this obviously yields the following corollary, which is more related to Theorem 8 since it proves that it cannot always be used when f is locally Lipschitz in the second variable.

Corollary 17. *There is a computable function $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ which is locally Lipschitz in the second argument, but not effectively so.*

4 Computing the Solution of Differential Equations and Inclusions

We now consider the computation of the solution of an ordinary differential equation or differential inclusion. We give two algorithms to compute the solution of a differential equation, the former of which is simpler and the latter more general. We also give algorithms for lower-semicontinuous and upper-semicontinuous differential inclusions. All algorithms rely on an exhaustive enumeration of trial "runs" of the system; each run is then checked to see if it gives a valid bound for the solution set. In this way, we can compute bounds on the solution without a knowledge of Lipschitz constants or moduli of continuity. Of course, the resulting algorithms are highly inefficient in practice; the motivation for introducing them is their conceptual simplicity.

Without loss of generality, we suppose that the right hand side does not depend on the time t (if this is true, just encode t as a new variable τ by adding the extra component $\dot{\tau} = 1$, $\tau(0) = 0$ to the system).

4.1 The Thousand Monkeys Algorithm for locally Lipschitz differential equations

We now present a first algorithm which proves computability for ordinary differential equations in the case where f is locally Lipschitz, but is not sufficient to prove computability for non-locally Lipschitz equations with unique solutions.

The idea underlying this algorithm is to enclose the solution at times t_i by a box X_i. Since the solution of (1) is unique, there are covers which are arbitrarily close to the solution. In the time interval $[t_i, t_{i+1}]$ we enclose the solution curve by a box B_i, and the derivative vectors in a box C_i. We can enumerate all sequences of times t_i (actually, we use time differences $h_i = t_{i+1} - t_i$) and boxes X_i, B_i and C_i, and test if they contain the solution.

We use the notation $A \Subset B$ if $\overline{A} \subset B^\circ$ i.e. the closure of A is a subset of the interior of B.

Algorithm 18 *Enumerate all tuples of the form*

$$\big((X_i)_{i=0}^{l},\ (h_i)_{i=0}^{l-1},\ (B_i)_{i=0}^{l-1},\ (C_i)_{i=0}^{l-1}\big) \tag{6}$$

where $k \in \mathbb{N}$, the X_i, B_i and C_i are rational boxes and $h_i \in \mathbb{Q}$. Define $t_0 = 0$ and $t_i = \sum_{j=0}^{i-1} h_j$ for $i = 1, \ldots, l$. We call a tuple of the form (6) *a* run *of the algorithm.*

A run of the algorithm is said to be valid *if $x_0 \in X_0^\circ$ and for all $i = 0, \ldots, l-1$:*

1. $f(B_i) \Subset C_i$,
2. $X_i \cup X_{i+1} \subset B_i$, *and*
3. $X_i + h_i C_i \subset X_{i+1}$.

Note that condition (1) is effectively verifiable since the pairs (I, J) such that $f(\overline{I}) \subset J$ is enumerated in a name of f, and conditions (2,3) can be checked algebraically.

The algorithm works by launching computations of each of the countably many possible runs in parallel, and testing for validity. Whenever a run is shown to be valid, that run is written to the output.

Theorem 19. *Let f be a locally-Lipschitz continuous function, and suppose that the initial value problem has a (necessarily unique) solution ξ defined on $[0, T_{\max})$. Then*

1. *For any valid run of Algorithm 18, $\xi(t) \in B_i$ for all $t_i \leq t \leq t_{i+1}$.*
2. *For any $\epsilon > 0$ and $T < T_{\max}$ there is a run of Algorithm 18 such that $t_l > T$ and $\operatorname{diam}(B_i) < \epsilon$ for all i.*

The first part of the above theorem can be thought of as proving the "correctness" of the method; a computed run approximates a real solution. The second part can be seen as a convergence result; there exist runs computing arbitrarily accurate solutions.

Proof.

1. For any valid run we have $\xi(t_0) = x_0 \in X_0$. Suppose $\xi(t_i) \in X_i$. Let $h_{i,\max} = \sup\{h \leq h_i \mid \xi(t_i + h) \subset B_i\}$. Then $\dot{\xi}(t) \in C_i$ for $t \in [t_i, t_i + h_{i,\max}]$, so $\xi(t) \in X_i + [0, h_{i,\max}]C_i \subset B_i$. Therefore $h_{i,\max} = h_i$, so $\xi(t) \in B_i$ for all $t \in [t_i, t_{i+1}]$ and $\xi(t_{i+1}) \in X_i + h_i C_i \subset X_{i+1}$. The result follows by induction.

2. Let K be such that $||f(\xi(t))|| < K$ for all $t \in [0, T]$. Let L be a Lipschitz constant for f on a neighbourhood W of $\xi([0, T])$. Fix $\delta < L\epsilon/2(e^{LT} - 1)$ and $h < \delta/KL$, where ϵ is the precision to which we would like to compute the solution. Suppose X is such that $\mathrm{rad}(X) < r$. Then the solution over time step h lies in a box B of radius less than $r + hK$ of the centre x of X. Then $f(B)$ lies in a box C of radius less than $(r + hK)L$ in W. Then $X + hC$ lies in a box Y of radius r' less than $r + (r + hK)Lh = (1 + Lh)r + KLh^2$.

   Since $h < \delta/KL$, we have $r' < (1 + Lh)r + \delta h$. If we now take X_0 of radius r_0, we can find X_n of radius $r_n < r_0(1 + Lh)^n + \delta\big((1 + Lh)^n - 1\big)/L$. By taking $r_0 < \epsilon/2e^{LT}$, we have for $n \leq T/h$, that $r_n < \delta(e^{LT} - 1)/L < \epsilon$.

   Since the Thousand Monkeys Algorithm enumerates over all rational boxes and step sizes, we eventually find (X_i, h_i, B_i, C_i) such that $\mathrm{rad}(X_i) < r_i < \epsilon$ for all i, and hence $\mathrm{rad}(B_i) < \epsilon + h_i K$. Therefore for ϵ and h_i sufficiently small, we have $\xi(T) \in B_k \subset U$.

However, it is not true that the Thousand Monkeys algorithm can compute the solution of a non-Lipschitz ordinary differential equation, even if the solution is unique.

Example 1. Consider the ordinary differential equation $\dot{p} = f(p)$ in $\mathbb{R}^2$ defined in polar coordinates by

$$\begin{aligned} \dot{r} &= \sqrt{r}\,(\cos\theta - 1/2); \\ \dot{\theta} &= 1/\sqrt{r}. \end{aligned} \tag{7}$$

In Cartesian coordinates, the system becomes

$$\begin{aligned} \dot{x} &= (x^2 + y^2)^{1/4} \left(\frac{x^2}{x^2 + y^2} - \frac{y + x/2}{\sqrt{x^2 + y^2}} \right); \\ \dot{y} &= (x^2 + y^2)^{1/4} \left(\frac{xy}{x^2 + y^2} + \frac{x - y/2}{\sqrt{x^2 + y^2}} - \frac{y}{2\sqrt{x^2 + y^2}} \right). \end{aligned} \tag{8}$$

Since $(x^2 + y^2)^{1/4} \to 0$ as $x, y \to 0$, and the other factors in the expression for $\dot{x}$ and $\dot{y}$ are bounded, we see that the right-hand side is continuous. We claim that

(i) The initial value problem $\dot{p} = f(p)$; $p(0) = (0,0)$ has unique solution $p(t) = (0,0)$ for all t, and

(ii) For any run of the Thousand Monkeys Algorithm, the solution estimate $P(t)$ at time t contains the point $(t^2/4, 0)$.

For (i), suppose that the solution leaves the origin. Then it spirals round extremely rapidly, with r increasing if $|\theta| < \pi/3$ and decreasing otherwise. Since the average decrease of r per revolution exceeds the average increase, the state is pulled back to the origin immediately. Hence the only solution starting at the origin is the constant solution.

For (ii), suppose that an approximation to the solution is a box X containing the point $(x, 0)$. Then $f(X)$ contains the value $(\sqrt{x}/2, \dot{y})$ and also $(0,0)$, so the box C contains $(\sqrt{x}/2, 0)$. Since the initial box $P(0) = X_0$ contains a point $(x_0, 0)$, then the box at time t, $P(t)$, must contains a point $(x(t), 0)$ solving $\dot{x} = \sqrt{x}/2$, $x(0) = x_0$. From this we can show that the point $(t^2/4, 0)$ lies in the solution box $P(t)$, regardless of the value of $x_0 > 0$. In particular, at time $t = 1$, the solution estimate $P(1)$ must contain the point $(1/4, 0)$. Hence the solution computed by the Thousand Monkeys algorithm does not converge to the true solution.

Remark. The above counterexample relies crucially on the fact that the bounding sets are boxes. We do not know if this example remains valid if the bounding sets would be allowed to be general convex polytopes. We shall see in Section 4.2 that using finite unions of boxes is sufficient to compute a unique solution to arbitrary accuracy.

4.2 The Ten Thousand Monkeys Algorithm for Differential Equations

We now give an algorithm to compute the solution of any initial value problem with unique solutions. The main difference between this algorithm and the previous algorithm is that the enclosing sets X_i are now *unions* of boxes. We call the algorithm the *Ten Thousand Monkeys Algorithm* (TTM) since a practical implementation would require even more computational resources than the Thousand Monkeys algorithm.

Algorithm 20 *Enumerate all tuples of the form* $(X_{i,j}, h_i, B_{i,j}, C_{i,j}, Y_{i,j})$ *for* $i = 0, \ldots, l-1$, $j = 1, \ldots, m_i$, *where* $l, m_i \in \mathbb{N}$, $X_{i,j}$, $B_{i,j}$, $C_{i,j}$ *and* $Y_{i,j}$ *are rational boxes and* $h_i \in \mathbb{Q}$. *Such a tuple is a* run *of the algorithm.*

A run of the algorithm is said to be valid *if* $x_0 \in \operatorname{int}\big(\bigcup_{j=1}^{m_0} X_{0,j}\big)$, *and for all* $i = 0, \ldots, l-1$ *and* $j = 1, \ldots, m_i$, *we have*

1. $f(B_{i,j}) \Subset C_{i,j}$;

2. $X_{i,j} \cup Y_{i,j} \subset B_{i,j}$;

3. $X_{i,j} + hC_{i,j} \subset Y_{i,j}$;

4. $\bigcup_{j=1}^{m_i} Y_{i,j} \subset \bigcup_{j=1}^{m_{i+1}} X_{i+1,j}$.

Just as in Algorithm 18, we enumerate all runs and verify whether a run is valid. The output is the infinite sequence of all valid runs.

In order to simplify notation, we write X_i for $\bigcup_{j=1}^{m_i} X_{i,j}$, and use similar notation for B_i, C_i and Y_i.

Theorem 21. *Let f be a continuous function, and suppose that the initial value problem has a unique solution ξ on $[0, T_{\max})$. Then*

1. *For any valid run of Algorithm 20, $\xi(t) \in B_i$ for all $t_i \leq t \leq t_{i+1}$.*

2. *For any $\epsilon > 0$ and $T < T_{\max}$ there is a run of Algorithm 20 such that $t_l > T$ and $\operatorname{diam}(B_i) < \epsilon$ for all i.*

Proof.

1. Essentially the same as the proof of Theorem 19(1).

2. Let W be a bounded neighbourhood of $\xi([0,T])$, let K be such that $||f(x)|| < K$ for all $x \in \overline{W}$ and let $\delta(\cdot)$ be a modulus of continuity for f in $\overline{W}$.

   Fix $\epsilon > 0$, and time steps $h_i < \delta(\epsilon)/K$. We first show that the sets $X_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can be chosen such that $\operatorname{rad}(X_{i,j}) < \delta(\epsilon) - h_i K$, $\operatorname{rad}(B_{i,j}) < \delta(\epsilon)$ and $\operatorname{rad}(C_{i,j}) < \epsilon$, assuming all sets remain inside W. For a given X_i, the sets $X_{i,j}$ can be chosen to each have radius less than $\delta(\epsilon) - h_i K$, simply by taking the partition sufficiently small. Take $B_{i,j}$ to be the $h_i K$ neighbourhood of $X_{i,j}$, so $\operatorname{rad}(B_{i,j}) \leq \operatorname{rad}(X_{i,j}) + hK < \delta(\epsilon)$. By the assumption $B_{i,j} \subset W$, the set $C_{i,j}$ with $f(B_{i,j}) \Subset C_{i,j}$ can be chosen such that $\operatorname{rad}(C_{i,j}) < \epsilon$ and $||v|| < K$ for all $v \in C_{i,j}$. For $\delta(\cdot)$ is a modulus of continuity for f in $B_{i,j}$, so $d(f(w), f(b_{i,j})) < \epsilon$ for all $w \in B_{i,j}$, where $b_{i,j}$ is the centre of $B_{i,j}$, and $||f(x)|| < K$ for all $x \in B_{i,j}$. Then $X_{i,j} + h_i C_{i,j} \Subset B_{i,j}$ since $B_{i,j}$ is the hK-neighbourhood of $X_{i,j}$.

   Consider a run of the algorithm such that each B_i is a subset of W, the sets $C_{i,j}$ each have radius less than ϵ, and that $X_{i,j} + h_i C_{i,j} = Y_{i,j}$. Consider the set of functions $\eta : [0, t_l] \to X$ satisfying $\eta(t_0) \in X_0$, and for all $i < l$, there exists $j_i \leq m_i$ and $c_i \in C_{i,j}$ such that $\eta(t_i) \in X_{i,j}$ and $\eta(t) = \eta(t_i) + (t - t_i)c_i$ for all $t_i < t \leq t_{i+1}$. In other words, η is a piecewise-affine function whose derivative lies within bounds given by the run. Since $\eta(t) \in B_{i,j}$ for $t_i < t < t_{i+1}$, then $f(\eta(t)), \dot{\eta}(t) \in C_{i,j}$, so $||\dot{\eta}(t) - f(\eta(t)|| < \operatorname{diam}(C_{i,j}) < 2\epsilon$. Hence $||\dot{\eta}(t) - f(\eta(t))|| < 2\epsilon$ for almost every $t \leq t_l$.

Further, given any $x_i \in X_i$, we can construct such a function η with $\eta(t_i) = x_i$. For if $x_i \in X_i$, then $x_i = y_{i-1} \in Y_{i-1,j}$ for some j. Then since $X_{i-1,j} + h_{i-1}C_{i-1,j} = Y_{i-1,j}$, we can find $c_{i-1} \in C_{i-1,j}$ and $x_{i-1} \in X_{i-1,j}$ such that $x_{i-1} + h_{i-1}c_{i-1} = x_i$. Similarly, if $x_i \in X_{i,j}$, then we can take any $c_i \in C_{i,j}$ and set $x_{i+1} = x_i + h_i c_i$. We can then recursively construct the required function.

Suppose that for a given T there is a run of the algorithm with $t_l > T$, $\text{rad}(C_{i,j}) < \epsilon$ for all i, j and $B_i \subset W$ for all i. Then such a run also exists for any smaller ϵ. Suppose ϵ_n is a sequence of positive numbers with $\epsilon_n \to 0$, and ξ_n is a function of the form η for some run of the algorithm with the given ϵ_n and all $B_i \subset W$. Then by Corollary 11, we see that ξ_n converges uniformly to a solution of the differential equation $\dot{x} = f(x)$ on $[0, T]$. Since the solution of the equation is unique, and we can choose ξ with $\xi(t_i)$ an arbitrary point of X_i, we have $\text{rad}(X_i) \to 0$ as $\epsilon \to 0$. Taking also $h_i \to 0$ we have $\text{rad}(B_i) \to 0$.

It remains to show that T can be extended up to $T_{\max}$. This follows since if W is a ρ-neighbourhood of K, then by taking a run of the algorithm on $[0, t_k]$ which is accurate to within $\rho/2$, we can make the next time step at least $\delta(\rho/2)/K$ and still have $B_k \subset W$.

Remark. If f is one-sided Lipschitz and x and y are two solutions of the differential equation $\dot{x} = f(x)$, then it is easy to deduce that $||x(t) - y(t)|| \leq e^{Lt}||x(0) - y(0)||$. In particular, the solution is unique, so can be computed by Algorithm 20. We do not know if it can be computed by Algorithm 18.

We obtain from our algorithms the following slight extension of the main result of Ruohonen [18].

Theorem 22. *Consider the initial value problem*

$$\dot{x} = f(x); \quad x(0) = x_0,$$

where f is continuous on the open set E. Suppose there is a unique solution $y(\cdot)$, defined on the maximal interval (α, β), such that $y(t) \in E$ for each $t \in (\alpha, \beta)$. Then:

1. *The operator $(f, x_0) \mapsto (\alpha, \beta)$ is semicomputable (i.e. α can be computed from above and β can be computed from below), and*

2. *The operator $(f, x_0) \mapsto y(\cdot)$ is computable.*

In particular, if f is a computable function and x_0 a computable point, then (α, β) is a r.e. open set and the solution $y(\cdot)$ is a computable function.

4.3 The Ten Thousand Monkeys Algorithm for Lower-Semicontinuous Differential Inclusions

We now give an algorithm to solve a lower-semicontinuous one-sided Lipschitz differential inclusion. The runs of the algorithm will enclose solutions of the differential inclusion, and every solution will be included in an arbitrarily accurate run.

Algorithm 23 *A run of the algorithm is a tuple of the form* $(X_{i,j}, h_i, B_{i,j}, C_{i,j}, Y_{i,j})$ *for* $i = 0, \ldots, l-1$, $j = 1, \ldots, m_i$ *where* $l, m_i \in \mathbb{N}$, $X_{i,j}$, $B_{i,j}$, $C_{i,j}$ *and* $Y_{i,j}$ *are rational boxes and* $h_i \in \mathbb{Q}$.

A run of the algorithm is said to be valid *if* $x_0 \in X_0^\circ$ *and for all* $i = 0, \ldots, l-1$ *and* $j = 1, \ldots, m_i$, *we have*

1. $B_{i,j} \Subset F^{-1}(C_{i,j})$;
2. $X_{i,j} \cup Y_{i,j} \subset B_{i,j}$;
3. $X_{i,j} + hC_{i,j} \subset Y_{i,j}$;
4. $\bigcup_{j=1}^{m_i} Y_{i,j} \subset \bigcup_{j=1}^{m_{i+1}} X_{i+1,j}$.

Just as in Algorithm 18, we enumerate all runs and verify whether a run is valid.

Theorem 24. *Let* F *be a one-sided locally Lipschitz multifunction with closed convex values.*

1. *For any valid run of Algorithm 23, there is a solution* ξ *of the differential inclusion (2) such that* $\xi(t) \in B_i$ *for all* $t_i \leq t \leq t_{i+1}$.
2. *If* $\xi(\cdot)$ *is a solution of (2) defined on an interval* $[0, T_{\max})$, *then for any* $\epsilon > 0$ *and* $T < T_{\max}$ *there is a run of Algorithm 23 such that* $t_l > T$, $\xi(t) \in B_i$ *for all* $t_i < t < t_{i+1}$ *and* $\operatorname{diam}(B_i) < \epsilon$ *for all* i.

For the first part of the proof, we use the Michael Selection Theorem to construct piecewise-smooth solutions contained in the run. The second part is complicated by the fact that ξ need not satisfy $\dot{\xi}(t) \in F(\xi(t))$ everywhere. We partition the interval $[0, T_{\max})$ into sub-intervals $[t_i, t_{i+1}]$ such that for "good" intervals, there exists a time τ_i such that $\dot{\xi}(\tau_i)$ is a good estimate of $(\xi(t_{i+1}) - \xi(t_i))/(t_{i+1} - t_i)$, and such that the "bad" intervals have small total measure. The one-sided Lipschitz condition is used to control the growth of the error outside a small tube containing ξ.

Proof.

1. Suppose B is a compact box and C an open box such that $B \subset F^{-1}(C)$. Then for every $x \in B$, there exists $y \in C$ such that $y \in F(x)$. Using the Michael Selection Theorem (Theorem 13), we deduce that the compact box B can be covered by finitely many open subsets V_i such that on each $\overline{V}_i$ there is a continuous function f_i such that $f_i(\overline{V}_i) \subset C$ and $f_i(x) \in F(x)$ for all i. Now suppose X is a box such that $X + [0,h]C \subset B$. Then any solution ξ of the differential inclusion $\dot{x} \in C$ starting in X remains in B for times $t \in [0,h]$. In particular, any solution of the differential inclusion $\dot{x} \in \bigcup\{f_i(x) \mid x \in V_i\}$ starting in X remains in B. There exists $\delta > 0$ such that for each $x \in B$, there exists i such that $N_\delta(B) \subset V_i$. Further, there exists K such that $|f_i(\overline{V}_i)| \leq K$. Consequently, the solution lies in V_i for a time of at least δ/K. Since any ordinary differential equation with continuous right-hand side has a solution, for each $x_0 \in X$, there exists a piecewise-differentiable function ξ such that $\dot{\xi}(t) \in F(\xi(t)) \cap C$ at all points of differentiability of ξ.

2. Fix $\epsilon > 0$. Let W be a bounded neighbourhood of $\xi([0,T])$. Fix $K \in \mathbb{R}^+$ such that $\overline{W} \subset F^{-1}(N_K(0))$. Choose positive constants ρ, μ and λ such that $(\rho + K\mu)e^{LT} + (e^{LT} - 1)\lambda/L < \epsilon$ and $||\xi(t+h) - \xi(t)|| < \rho/2$ whenever $h < \mu$. The constant ρ is a bound on the size of the flow tube, the constant μ is a bound on the size of the "bad" time intervals, and the constant λ is a correction term in the lower-Lipschitz condition.

   For a full measure set $G \subset [0,T]$, the function ξ is differentiable at $\tau \in G$ and $\dot{\xi}(\tau) \in F(\xi(\tau))$. For each $h > 0$, let $E(\tau,h) = \xi([\tau - h, \tau + h])$, and $D(\tau,h)$ be the set of all $(\xi(t') - \xi(t))/(t' - t)$ for all t, t' with $t < \tau < t' \leq t + h$.

   Fix $\tau \in G$ and let $x = \xi(\tau)$ and $u = \dot{\xi}(\tau)$. For each $y \in \overline{W}$, there exists $v \in F(y)$ such that $(v - u) \cdot (y - x) \leq L||y - x||^2$. In other words, the one-sided Lipschitz condition (3) is satisfied with $f_x = \dot{\xi}(\tau)$ and $f_y = v(\tau, y)$. If $||y - x|| \geq \rho/2$, then there exists a box neighbourhood $C(\tau,y)$ of $v(\tau,y)$ such that $(v' - u) \cdot (y - x) < L||y - x||^2 + \lambda||y - x||$ for all $v' \in C(\tau,y)$. Further, there exists a box neighbourhood $B(\tau,y)$ of y such that $B(\tau,y) \Subset F^{-1}(C(\tau,y))$, $||y' - x|| > \rho/2$ for all $y' \in B(\tau,y)$ and $(v' - u) \cdot (y' - x) < L||y' - x||^2 + \lambda||y' - x||$ whenever $y' \in B(\tau,y)$ and $v' \in C(\tau,y)$. By taking $h(\tau,y)$ sufficiently small, we have that $y + h(\tau,y)C(\tau,y) \subset B(\tau,y)$ and that $(v' - u') \cdot (y' - x') < L||y' - x'||^2 + \lambda||y' - x'||$ whenever $x' \in E(\tau,h)$, $u' \in D(\tau,h)$, $y' \in B(\tau,y)$ and $v' \in C(\tau,y)$. Take a box neighbourhood $X(\tau,y)$ of y such that $X(\tau,y) + [0,h(\tau,y)]C(\tau,y) \subset B(\tau,y)$. If $||y - x|| < \rho/2$, choose $C(\tau,y)$, $B(\tau,y)$, $X(\tau,y)$ and $h(\tau,y)$ such that $||y' - x'|| < \rho$ whenever $x' \in E(\tau, h(\tau,y))$ and $y' \in B(\tau,y)$. Choose finitely many $y_j(\tau)$ such that the sets $X(\tau, y_j(\tau))$ cover $\overline{W}$ and let $h(\tau) = \min\{h(\tau, y_j(\tau))\}$. Let $I(\tau)$ be an open interval of radius at most $h(\tau)$ containing τ.

Choose an open cover of $[0,T]\setminus G$ by intervals of total size at most μ, and let $I(\tau)$ be an interval containing τ. Then the intervals $I(\tau)$ for $\tau\in[0.T]$ are an open cover of $[0,T]$. Let $I(\tau_0),I(\tau_1),\ldots,I(\tau_l)$ be a finite subcover with $\tau_i<\tau_{i+1}$; note that we can also assume $I(\tau_i)\cap I(\tau_{i+1})\neq\emptyset$. Choose t_i such that $[t_i,t_{i+1}]\subset I(\tau_i)$ and $t_i<\tau_i<t_{i+1}$, and let $h_i=t_{i+1}-t_i$.

We now construct a run of the algorithm such that $\mathrm{rad}(X_i)\leq r_i$, taking $\xi(0)\in X_0\subset N_\rho(\xi(0))$. Note that at any stage we can freely adjoin an extra box containing $\xi(t_i)$ of radius ρ to X_i in order to ensure that $\xi(t_i)\subset X_i$.

If $\tau_i\in G$, then we take boxes $X_{i,j}$ which are unions of boxes $X(\tau_i,y)$ with X_i, and $B_{i,j}$ and $C_{i,j}$ of the form $B(\tau_i,y)$ and $C(\tau_i,y)$. Then either $B_{i,j}\subset N_\rho(\xi(t))$ for all $t\in[t_i,t_{i+1}]$, or we have $(v-u)\cdot(y-x)\leq L||y-x||^2+\lambda||y-x||$ for all $x\in\xi([t_i,t_{i+1}])$, $y\in B_{i,j}$ and $v\in C(i,j)$, where $u=(\xi(t_{i+1})-\xi(t_i))/(t_{i+1}-t_i)$. Take $\eta(t)=y+(t-t_i)v$ now with $y\in X_{i,j}$, and $\zeta(t)=\xi(t_i)+(t-t_i)u$. Then using the formula $d||\phi(t)||/dt\leq\dot\phi(t)\cdot\phi(t)/||\phi(t)||$ with $\phi(t)=\eta(t)-\zeta(t)$, we find that $d||\eta(t)-\zeta(t)||\leq L||\eta(t)-\zeta(t)||+\lambda$. Hence by Gronwall's lemma, $||\eta(t+h)-\zeta(t+h)||\leq||eta(t)-\zeta(t)||e^{Lh}+(e^{Lh}-1)\lambda/L$. Therefore if $\mathrm{rad}(X_i)\leq r_i$ and $\rho\leq r_i$, then $\mathrm{rad}(X_{i+1})\leq r_ie^{Lh}+(e^{Lh}-1)\lambda/L$.

If $\tau_i\notin G$, then we can take $X_{i,1}=X_i$, $C_{i,1}=N_K(0)$ and $X_{i+1}=Y_{i,1}=X_{i,1}+hC_{i,1}$. If X_i contains an $\rho/2$-neighbourhood of $\xi(t_i)$, then by the condition $h<\mu$ we also have $\xi(t+i)\in X_i\subset X_{i+1}$. Hence $\mathrm{rad}(X_{i+1})\leq\mathrm{rad}(X_i)+Kh_i$. We therefore have bounds r_i on $\mathrm{rad}(X_i)$ which satisfy $r_0\leq\rho$, $r_{i+1}<r_ie^{Lh_i}+(e^{Lh_i}-1)\lambda/L$ if $\tau_i\in G$ and $r_{i+1}\leq r_i+h_iK$ if $\tau_i\notin G$. Since the errors for $\tau_i\in G$ are exponential and the errors for $\tau_i\notin G$ are additive, a worst-case bound on the error is then given by $r(T)<(\rho+\mu K)e^{LT}+(e^{LT}-1)\lambda/L$, which is less than ϵ as required.

We deduce the following generalisation of [8, Theorem 3.1]

Theorem 25. *Let F be a one-sided Lipschitz lower-semicontinuous multivalued function with closed convex values. Consider the initial value problem*

$$\dot{x}\in F(x);\quad x(0)=x_0,$$

where F is defined on some open domain $E\subset\mathbb{R}^n$. Then the solution operator $(F,x_0)\mapsto\Phi_F(x_0)\subset C_P(\mathbb{R},E)$ is lower-semicomputable in the following sense:

- *Given a name of F, it is possible to enumerate all triples (I,J,K) where I,K are open rational boxes and J is an open rational interval such that for every $x_0\in\overline{I}$ there is a solution ξ such that $\xi(0)=x_0$, and $\xi(\overline{J})\subset K$.*

Consequently, if $y(\cdot)$ is a solution defined on the maximal interval (α,β), such that $y(t)\in E$ for each $t\in(\alpha,\beta)$, then there exists a continuous selection of the solution map through y.

The following classical example shows that the solution set need not vary lower-semicontinuously with the initial conditions if the right-hand side of (2) is not one-sided locally-Lipschitz.

Example 2. Let $f(x) = 2\sqrt{x}$ for $x \geq 0$ and $f(x) = 0$ for $x \leq 0$. Then the solutions with $x(0) = 0$ are the functions

$$x(t) = \begin{cases} 0 \text{ for } t \leq c; \\ (t-c)^2 \text{ for } t \geq c. \end{cases}$$

In particular, the set of solution values $S(t, x_0)$ at time t starting at x_0 satisfy

$$S(1, x_0) = \begin{cases} \{0\} \text{ for } x_0 < 0; \\ [0,1] \text{ for } x_0 = 0; \\ \{(1-\sqrt{x_0})^2\} \text{ for } x_0 > 0. \end{cases}$$

This is an upper-semicontinuous function, but is not-lower-semicontinuous. Note that the best lower-semicontinuous under-approximation to $S(t, x_0)$ has $\underline{S}(1, 0) = \emptyset$.

4.4 The Ten Thousand Monkeys Algorithm for Upper-Semicontinuous Differential Inclusions

We now give an algorithm for computing the solution of an upper-semicontinuous differential inclusion. The idea of this algorithm is similar to Algorithm 20, but instead of enclosing the set $F(B_{i,j})$ by a single box, we use a finite number of boxes.

Algorithm 26 *A run of the algorithm is a tuple of the form* $(X_{i,j}, h_i, B_{i,j}, C_{i,j,k}, Y_{i,j,k})$ *for* $i = 0, \ldots, l-1$, $j = 1, \ldots, m_i$, $k = 1, \ldots, n_{i,j}$ *where* $l, m_i, n_{i,j} \in \mathbb{N}$, $X_{i,j}$, $B_{i,j}$, $C_{i,j,k}$ *and* $Y_{i,j,k}$ *are rational boxes and* $h_i \in \mathbb{Q}$.

A run of the algorithm is said to be valid *if* $x_0 \in X_0^\circ$ *and for all* $i = 0, \ldots, k-1$ *and* $j = 1, \ldots, l_i$, *we have*

1. $\operatorname{conv}(F(B_{i,j})) \Subset \bigcup_{k=1}^{n_{i,j}} C_{i,j,k}$;
2. $X_{i,j} \cup Y_{i,j,k} \subset B_{i,j}$;
3. $X_{i,j} + hC_{i,j,k} \subset Y_{i,j,k}$;
4. $\bigcup_{j=1,k=1}^{m_i,n_{i,j}} Y_{i,j,k} \subset \bigcup_{j=1}^{m_{i+1}} X_{i+1,j}$.

Just as in Algorithm 18, we enumerate all runs and verify whether a run is valid.

In order to simplify notation, we write $C_{i,j} = \bigcup_{k=1}^{n_{i,j}} C_{i,j,k}$ and $Y_{i,j} = \bigcup_{k=1}^{n_{i,j}} Y_{i,j,k}$.

Theorem 27. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper-semicontinuous function with compact convex values. Suppose that the initial value problem $\dot{x} \in F(x);\ x(0) = x_0$ has bounded solutions on $[0, T_{\max})$. Then:*

1. *For any solution ξ and any valid run of Algorithm 26, $\xi(t) \in B_i$ whenever $t_i \le t \le t_{i+1}$, $i < l$.*
2. *For any $\epsilon > 0$ and $T < T_{\max}$, there exists a valid run of Algorithm 26 with $t_l > T$ such that for any $y \in B_i$, there exists a solution ξ with $|\xi(t) - y| < \epsilon$ for all t with $t_i \le t \le t_{i+1}$.*

Proof.

1. If ξ is a solution to $\dot{x} \in F(x)$ with $\xi(0) \in X$, $\xi([0, h]) \subset B$, D is a convex set such that $F(B) \subset D$, and $D \subset C$, then $\xi(h) \in X + hC$ since $\xi(h) - \xi(0) = \int_0^h \xi'(t)\, dt \in hD$ by convexity. Note that the set $F(B)$ need not be convex even if B is convex and $F(x)$ is convex for all x, and ξ' is integrable since ξ is absolutely continuous by Definition 2. The rest of the proof follows that of Theorem 19(1). For if ξ is a solution and $\xi(t_i) \in X_i$, then $\xi(t_i) \in X_{i,j}$ for some j. Taking $D_{i,j} = \mathrm{conv}(F(B_{i,j}))$ we see that $\xi(t_{i+1}) \in X_{i,j} + h_i \bigcup_{k=1}^{n_{i,j}} C_{i,j,k} = X_{i,j} + h_i C_{i,j} \subset Y_{i,j} \subset X_{i+1}$, and also $\xi(t) \in X_{i,j} + [0, h_i] C_{i,j} \subset B_{i,j} \subset B_i$ for $t_i \le t \le t_{i+1}$
2. Consider runs of the algorithm such that for all i, j, k, $X_{i,j} + hC_{i,j,k} = Y_{i,j,k}$. Further, suppose that each h_i is less than δ, $C_{i,j,k}$ intersects $F(B_{i,j})$ and each $C_{i,j,k}$ has a diameter less than ϵ. As in the proof of Theorem 21, define functions η with $\eta(t_0) \in X_0$, and for $t_i < t \le t_{i+1}$ by $\eta(t) = \eta(t_i) + (t - t_i)c_i$ for some c_i such that $\eta(t_i) \in X_{i,j_i}$ and $c_i \in C_{i,j_i,k_i}$. Then η is a piecewise-affine function and, for almost every t, there exists s such that $|t - s| < \delta$ and $||\dot{\eta}(t) - F(\eta(s))|| < \epsilon$. In particular, the pair $(\eta(t), \dot{\eta}(t))$ lies within $\max(\delta, \epsilon)$ of the graph of F. Further, for any $y \in X_k$, we can find such an η such that $\eta(t_k) = y$.

   Taking any sequence of functions ξ_n corresponding to sequences $\delta_n, \epsilon_n \to 0$, by Corollary 12, we see that ξ_n converges uniformly to a solution of the differential inclusion $\dot{x} \in F(x)$. Hence for δ_n sufficiently small, every point in B_k is within ϵ of a solution as required.

Remark. Instead of letting $X_{i,j}$ and $B_{i,j}$ be boxes, and covering $\mathrm{conv}(F(B_{i,j}))$ by a finite union of boxes $\bigcup_{k=1}^{n_{i,j}} C_{i,j,k}$, we could equally well cover $F(B_{i,j})$ by a single convex polytope $C_{i,j}$. Condition (1) becomes $F(B_{i,j}) \Subset C_{i,j}$.

Note that Theorem 21 follows from Theorem 27, since an ordinary differential equation is simply a special kind of upper-semicontinuous differential inclusion. From Theorem 27 we obtain upper-semicomputability of the solution operator of upper-semicontinuous differential inclusions with compact convex values.

Theorem 28. *Let F be an upper-semicontinuous multivalued function with compact convex values. Consider the initial value problem*

$$\dot{x} \in F(x); \quad x(0) = x_0,$$

where F is defined on some open domain $E \subset \mathbb{R}^n$. Then the solution operator $(F, x_0) \mapsto \Phi_F(x_0) \subset C_P(\mathbb{R}, E)$ is computable in the following sense:

- *Given a name of F, it is possible to enumerate all tuples $(I, J, K_1, \ldots, K_m)$ where $I, K_1, \ldots, K_m$ are open rational boxes and J is an open rational interval such that for every $x_0 \in \overline{I}$ and every solution ξ such that $\xi(0) = x_0$, it holds that $\xi(\overline{J}) \subset \bigcup_{i=1}^{m} K_i$.*

5 Conclusion

We have shown that we can compute the solution of the initial-value problem for ordinary differential equations with continuous right-hand side, if the solution is assumed unique. We presented algorithms for the computation of the solution using the "thousand monkeys" approach. In this way, we have shown that the solution of a differential equation defined by a locally Lipschitz function is computable even if the function is not effectively locally Lipschitz, a situation which can happen, as we have seen.

One interesting direction for further research would be weakening the one-sided Lipschitz condition required to compute lower solutions to a differential inclusion. We have seen that a Lipschitz condition is necessary for the classically-defined solution set to be lower-semicontinuous in the initial conditions, but it would be interesting to see whether there is a reasonable notion of solution which is weaker than the classical notion, and for which the solutions are lower-semicomputable. Another interesting direction concerns minimal requirements on the class of sets used to compute the solution of a non-Lipschitz differential equation with a unique solution. We have seen that boxes are insufficient, but finite unions of boxes are sufficient. It would be interesting to determine whether a single zonotope or convex polytope is sufficient to bound the solution. A further direction for research is extending the algorithm to systems in which the right-hand side is only measurable in the time variable.

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